MULTIVARIABLE CALCULUS

MATH S-21A

Unit 13: Extrema

Lecture

13.1. In many applications we are led to the task to **maximize** or **minimize** a function f. As in single variable calculus we first search for points where the "derivative" is zero. This is the **Fermat principle** In one dimensions, like for $f(x) = 3x^5 - 5x^3$ we can use the second derivative test to classify extrema, like the local max at -1 and the local min at 1.

Definition: A point (a, b) in the plane is called a **critical point** of a function $f(x, y)$ if $\nabla f(a, b) = [0, 0]^T$.

13.2. The **Fermat principle** predates the discovery of calculus and says:

If $\nabla f(x, y)$ is not zero, then (x, y) is not a critical point.

13.3. Proof. Take the directional derivative in the direction $v = \nabla f / |\nabla f|$. Then $D_{\vec{v}} f = \nabla f \cdot \vec{v} = |\nabla f| > 0$. You can also see it with linear approximation. If $\nabla f \neq 0$, then the linear approximation L is not constant and f is neither maximal nor minimal. QED.

13.4. Note that in our definition we do **not** include points, where f or its derivative is not defined. Without stating otherwise, we always assume that a function can be differentiated arbitrarily often. Points where the function has no derivatives are not considered part of the domain and need to be studied separately. For the continuous function $f(x, y) = 1/\log(|xy|)$ for example, we would have to look at the points on the coordinate axes as well as the points $xy = 1$ separately.

13.5. In one dimension, we used the condition $f'(x) = 0, f''(x) > 0$ to get a local minimum and $f'(x) = 0, f''(x) < 0$ to assure a local maximum. If $f'(x) = 0, f''(x) = 0$, the nature of the critical point is undetermined and could be a maximum like for $f(x) = -x^4$, or a minimum like for $f(x) = x^4$ or a flat **inflection point** like for $f(x) = x^3$.

Definition: If $f(x,y)$ is a function of two variables with a critical point (a,b), the number $D = f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** of the critical point.

13.6. The discriminant can be remembered better if seen as the determinant of the **Hessian matrix** $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. As of default, we always assume that functions are twice continuously differentiable. Here is the **second derivative test**:

Theorem: Assume (a,b) is a critical point for $f(x,y)$.
If $D > 0$ and $f_{xx}(a,b) > 0$ then (a,b) is a local minimum.
If $D > 0$ and $f_{xx}(a,b) < 0$ then (a,b) is a local maximum.
If $D < 0$ then (a,b) is a saddle point.

13.7. In the case $D = 0$, we need higher derivatives or ad-hoc methods to determine the nature of the critical point.

13.8. To determine the maximum or minimum of $f(x,y)$ on a domain, determine all critical points **in the interior the domain**, and compare their values with maxima or minima **at the boundary**. We will see in the next unit how to get extrema on the boundary.
Sometimes, we want to find the overall maximum and not only the local ones.

Definition: A point (a,b) in the plane is called a **global maximum** of $f(x,y)$ if $f(x,y) \leq f(a,b)$ for all (x,y). For example, the point $(0,0)$ is a global maximum of the function $f(x,y) = 1 - x^2 - y^2$. Similarly, we call (a,b) a **global minimum**, if $f(x,y) \geq f(a,b)$ for all (x,y).

Examples

13.9. Find the critical points of $f(x,y) = x^4 + y^4 - 4xy + 2$. The gradient is $\nabla f(x,y) = [4(x^3 - y), 4(y^3 - x)]^T$ with critical points $(0,0), (1,1), (-1,-1)$.

13.10. $f(x,y) = \sin(x^2 + y) + y$. The gradient is $\nabla f(x,y) = [2x\cos(x^2 + y), \cos(x^2 + y) + 1]^T$. For a critical points, we must have $x = 0$ and $\cos(y) + 1 = 0$ which means $\pi + k2\pi$. The critical points are at $\ldots (0,-\pi), (0,\pi), (0,3\pi), \ldots$. There are infinitely many.

13.11. The graph of $f(x,y) = (x^2 + y^2)e^{-x^2-y^2}$ looks like a volcano. The gradient $\nabla f = [2x - 2x(x^2 + y^2), 2y - 2y(x^2 + y^2)]^T e^{-x^2-y^2}$ vanishes at $(0,0)$ and on the circle $x^2 + y^2 = 1$. This function has a continuum of critical points.

13.12. The function $f(x,y) = y^2/2 - g\cos(x)$ is the energy of the pendulum. The variable g is a constant. We have $\nabla f = (y, -g\sin(x)) = [(0,0]^T$ for

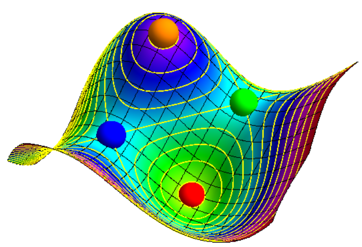
$$(x,y) = \ldots, (-\pi, 0), (0,0), (\pi, 0), (2\pi, 0), \ldots .$$

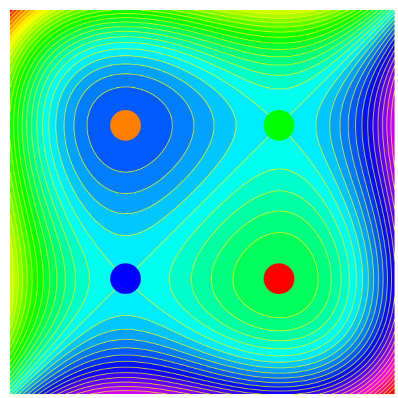
These points are equilibrium points, the angles for which the pendulum is at rest.

13.13. The function $f(x,y) = a\log(y) - by + c\log(x) - dx$ is a function which is invariant by the flow of the **Volterra-Lodka** differential equation $\dot{x} = ax - bxy, \dot{y} = -cy + dxy$. The point $(c/d, a/b)$ is a critical point of f and an equilibrium point of the system.

13.14. The function $f(x, y) = |x| + |y|$ is smooth on the first quadrant $\{x > 0, y > 0\}$. It does not have critical points there. The function has a minimum at $(0, 0)$ but it is not in the domain, where f and ∇f are defined. We have to look at the points on the coordinate axis separately. For $y = 0$, we see that $x = 0$ is a minimum. For $x = 0$ we see that $y = 0$ is a minimum. Indeed $(0, 0)$ is a minimum of f. This minimum was not detected using derivatives.

13.15. The function $f(x, y) = x^3/3 - x - (y^3/3 - y)$ has a graph which looks like a "napkin". It has the gradient $\nabla f(x, y) = [x^2 - 1, -y^2 + 1]^T$. There are 4 critical points $(1, 1)$,$(-1, 1)$,$(1, -1)$ and $(-1, -1)$. The Hessian matrix which includes all partial derivatives is $H = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$.
For $(1, 1)$ we have $D = -4$ and so a saddle point,
For $(-1, 1)$ we have $D = 4$, $f_{xx} = -2$ and so a local maximum,
For $(1, -1)$ we have $D = 4$, $f_{xx} = 2$ and so a local minimum.
For $(-1, -1)$ we have $D = -4$ and so a saddle point. The function has a local maximum, a local minimum as well as 2 saddle points.





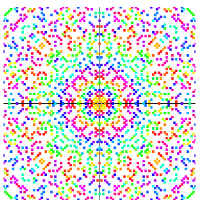
13.16. Find the maximum of $f(x, y) = 2x^2 - x^3 - y^2$ on $y \geq -1$. With $\nabla f(x, y) = 4x - 3x^2, -2y)$, the critical points are $(4/3, 0)$ and $(0, 0)$. The Hessian is $H(x, y) = \begin{bmatrix} 4 - 6x & 0 \\ 0 & -2 \end{bmatrix}$. At $(0, 0)$, the discriminant is -8 so that this is a saddle point. At $(4/3, 0)$, the discriminant is 8 and $H_{11} = 4/3$, so that $(4/3, 0)$ is a local maximum. We have now also to look at the boundary $y = -1$ where the function is $g(x) = f(x, -1) = 2x^2 - x^3 - 1$. Since $g'(x) = 0$ at $x = 0, 4/3$, where 0 is a local minimum, and 4/3 is a local maximum on the line $y = -1$. Comparing $f(4/3, 0), f(4/3, -1)$ shows that $(4/3, 0)$ is the global maximum.

13.17. Find the global maxima and minima of $f(x, y) = x^4 + y^4 - 2x^2 - 2y^2$ **Solution:** the function has no global maximum. This can be seen by restricting the function to the x-axis, where $f(x, 0) = x^4 - 2x^2$ is a function without maximum. The function has four global minima however. They are located on the 4 points $(\pm 1, \pm 1)$. The best way to see this is to note that $f(x, y) = (x^2 - 1)^2 + (y - 1)^2 - 2$ which is minimal when $x^2 = 1, y^2 = 1$.

Homework

This homework is due on Tuesday, 7/23/2019.

Problem 13.1: a) Find the critical points of $f(x,y) = (3x^4 - 8x^3 - 6x^2 + 24x) + (3y^4 - 8y^3 - 6y^2 + 24y)$.
b) Find all the extrema of the function $f(x,y) = xy + x^2y + xy^2$ and determine whether they are maxima, minima or saddle points.



Remark: b) is a **Gaussian Goldbach function** $f_n = \sum_{k+im \text{prime}, k,m \leq n} x^k y^m$. For $n = 2$ we sum over the Gaussian primes $1+i, 2+i, 1+2i$. (A complex integer $a+ib$ with $a,b \in \mathbb{N}$ is **prime** if a^2+b^2 is a usual prime. A 2D **Goldbach conjecture** claims that all partial derivatives $g^{(p,q)}(0,0)$ with $1 < p,q \leq n$ of $g(x,y) = f_{2n}^2(x,y)$ at $(0,0)$ are non-zero if $p+q$ is even. Equivalently, every Gaussian integer $a+ib$ with $a+b$ even and $a>1, b>1$ is a sum of two Gaussian primes in $Q = \{a+ib \mid a>0, b>0\}$.

Problem 13.2: Where on the parametrized surface $\vec{r}(u,v) = [1+u^3, v^2, uv]^T$ is the temperature $T(x,y,z) = 2 + x + 12y - 12z$ minimal? To find the minimum, look where the function $f(u,v) = T(\vec{r}(u,v))$ has an extremum. Find all local maxima, local minima or saddle points of f.

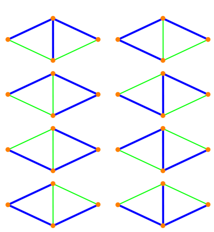
Problem 13.3: Find and classify all the extrema of the function $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$.

Problem 13.4: Find all extrema of the function $f(x,y) = 70 + x^3 + y^3 - 3x - 12y$ on the plane and characterize them. Do you find a global maximum or global minimum among them?

Problem 13.5: Graph theorists look at the **Tutte polynomial** $f(x,y)$ of a network. We work with the Tutte polynomial

$$f(x,y) = x + 2x^2 + x^3 + y + 2xy + y^2$$

of the **Kite network**. Classify using the second derivative test.



Remark. The polynomial is useful: $xf(1-x,0)$ tells in how many ways one can color the nodes of the network with x colors and $f(1,1)$ tells how many spanning trees there are. This picture illustrates that the number of spanning trees of the kite graph is $f(1,1) = 8$ as you see the 8 possible trees.

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MULTIVARIABLE CALCULUS

MATH S-21A

Unit 14: Lagrange

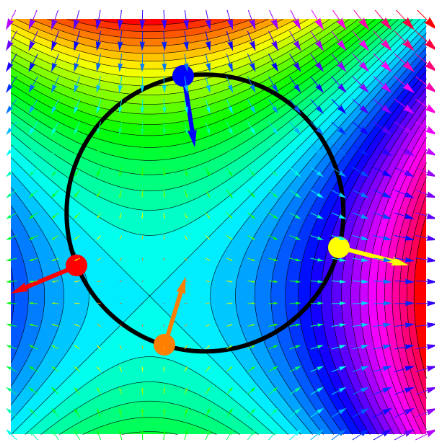
Lecture

14.1. When looking for maxima and minima of a function $f(x, y)$ in the presence of a **constraint** $g(x, y) = 0$, a necessary condition is that the gradients of f and g are parallel because otherwise, we can move along the curve $g = c$ and increase the value of f: the directional derivative of f in the direction tangent to the level curve is zero if and only if the tangent vector to g is perpendicular to the gradient of f. This can also include the case $\nabla g = [0, 0]^T$.

Definition: The system of equations $\nabla f(x, y) = \lambda \nabla g(x, y), g(x, y) = 0$ for the three unknowns x, y, λ are the **Lagrange equations**. The variable λ is a **Lagrange multiplier**.

Theorem: A maximum or minimum of $f(x, y)$ on the curve $g(x, y) = c$ is either a solution of the Lagrange equations or is a critical point of g.

Proof. The condition that ∇f is parallel to ∇g either means $\nabla f = \lambda \nabla g$ or $\nabla f = 0$ or $\nabla g = 0$. The case $\nabla f = 0$ can be included in the Lagrange equation case with $\lambda = 0$. The case $\nabla g = 0$ which would lead to $\lambda = \infty$ has to be included separately. QED.



14.2. In higher dimensions the statement is exactly the same: extrema of $f(\vec{x})$ under the constraint $g(\vec{x}) = c$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g, g = c$ or points where $\nabla g = \vec{0}$. But we also can have more than one constraint. For example:

Theorem: Extrema of $f(x,y,z)$ under the constraint $g(x,y,z) = c, h(x,y,z) = d$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d$ or solutions to $\nabla g = 0, \nabla f(x,y,z) = \mu \nabla h, h = d$ or solutions to $\nabla h = 0, \nabla f = \lambda \nabla g, g = c$ or solutions to $\nabla g = \nabla h = 0$.

14.3. Remarks.
1) The conditions in the Lagrange theorem are equivalent to $\nabla f \times \nabla g = 0$ in dimensions 2 or 3.
2) With $g(x,y) = 0$, the Lagrange equations can also be written as $\nabla F(x,y,\lambda) = 0$, where $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$.
3) The two conditions in the theorem are equivalent to "$\nabla g = \lambda \nabla f$ or f has a critical point".
4) Constrained optimization problems work also in higher dimensions.
5) Can we avoid Lagrange? Sometimes. It is often done in single variable calculus. In order to maximize xy under the constraint $2x + 2y = 4$ for example, we solve for y in the second equation and extremize the single variable problem $f(x, y(x))$. This needs to be done carefully and the boundaries must be considered. To extremize $f(x,y) = y$ on $x^2 + y^2 = 1$ for example we need to maximize $\sqrt{1 - x^2}$. We can differentiate to get the critical points but also have to look at the cases $x = 1$ and $x = -1$, where the actual minima and maxima occur. In general also, we can not do the substitution. To extremize $f(x,y) = x^2 + y^2$ with constraint $g(x,y) = x^4 + 3y^2 - 1 = 0$ for example, we solve $y^2 = (1 - x^4)/3$ and minimize $h(x) = f(x, y(x)) = x^2 + (1 - x^4)/3$. $h'(x) = 0$ gives $x = 0$. The find the maximum $(\pm 1, 0)$, we had to maximize $h(x)$ on $[-1, 1]$, which occurs at ± 1.
To extremize $f(x,y) = x^2 + y^2$ under the constraint $g(x,y) = p(x) + p(y) = 1$, where p is a complicated function in x which satisfies $p(0) = 0, p'(1) = 2$,the Lagrange equations $2x = \lambda p'(x), 2y = \lambda p'(y), p(x) + p(y) = 1$ can be solved with $x = 0, y = 1, \lambda = 1$. We can not solve $g(x,y) = 1$ however for y in an explicit way.
6) How do we determine whether a solution of the Lagrange equations is a maximum or minimum? Instead of using a second derivative test, we make a list of critical points and pick the maximum and minimum. A second derivative test can be designed using second directional derivative in the direction of the tangent.
7) The Lagrange method also works with more constraints. The constraints $g = c, h = d$ define a curve in space. The gradient of f must now be in the plane spanned by the gradients of g and h because otherwise, we could move along the curve and increase f:

Examples

14.4. Minimize $f(x,y) = x^2 + 2y^2$ under the constraint $g(x,y) = x + y^2 = 1$. **Solution:** The Lagrange equations are $2x = \lambda, 4y = \lambda 2y$. If $y = 0$ then $x = 1$. If $y \neq 0$ we can divide the second equation by y and get $2x = \lambda, 4 = \lambda 2$ again showing $x = 1$. The point $x = 1, y = 0$ is the only solution.

14.5. Find the shortest distance from the origin to the curve $x^6 + 3y^2 = 1$. **Solution:** Minimize the function $f(x,y) = x^2 + y^2$ under the constraint $g(x,y) = x^6 + 3y^2 = 1$. The gradients are $\nabla f = [2x, 2y]^T, \nabla g = [6x^5, 6y]^T$. The Lagrange equations $\nabla f = \lambda \nabla g$ lead to the system $2x = \lambda 6x^5, 2y = \lambda 6y, x^6 + 3y^2 - 1 = 0$. We get $\lambda = 1/3, x = x^5$, so that either $x = 0$ or 1 or -1. From the constraint equation $g = 1$, we obtain

$y = \sqrt{(1-x^6)/3}$. So, we have the solutions $(0, \pm\sqrt{1/3})$ and $(1,0), (-1,0)$. To see which is the minimum, just evaluate f on each of the points. $(0, \pm\sqrt{1/3})$ are the minima.

14.6. Which cylindrical soda cans of height h and radius r has minimal surface for fixed volume? **Solution:** The volume is $V(r,h) = h\pi r^2 = 1$. The surface area is $A(r,h) = 2\pi rh + 2\pi r^2$. With $x = h\pi, y = r$, you need to optimize $f(x,y) = 2xy + 2\pi y^2$ under the constrained $g(x,y) = xy^2 = 1$. Calculate $\nabla f(x,y) = (2y, 2x + 4\pi y), \nabla g(x,y) = (y^2, 2xy)$. The task is to solve $2y = \lambda y^2, 2x + 4\pi y = \lambda 2xy, xy^2 = 1$. The first equation gives $y\lambda = 2$. Putting that in the second one gives $2x + 4\pi y = 4x$ or $2\pi y = x$. The third equation finally reveals $2\pi y^3 = 1$ or $y = 1/(2\pi)^{1/3}, x = 2\pi(2\pi)^{1/3}$. This means $h = 0.54.., r = 2h = 1.08$. Remark: Other factors can influence the shape. For example, the can has to withstand a pressure up to 100 psi. A typical can of "Coca-Cola classic" with 3.7 volumes of CO_2 dissolve has at 75F an internal pressure of 55 psi, where PSI stands for pounds per square inch.

14.7. On the curve $g(x,y) = x^3 - y^2$ the function $f(x,y) = x$ obviously has a minimum $(0,0)$. The Lagrange equations $\nabla f = \lambda \nabla g$ have no solutions. This is a case where the minimum is a solution to $\nabla g(x,y) = 0$.

14.8. Find the extrema of $f(x,y,z) = z$ on the sphere $g(x,y,z) = x^2 + y^2 + z^2 = 1$. Solution: compute the gradients $\nabla f(x,y,z) = (0,0,1)$, $\nabla g(x,y,z) = (2x, 2y, 2z)$ and solve $(0,0,1) = \nabla f = \lambda \nabla g = (2\lambda x, 2\lambda y, 2\lambda z), x^2 + y^2 + z^2 = 1$. The case $\lambda = 0$ is excluded by the third equation $1 = 2\lambda z$ so that the first two equations $2\lambda x = 0, 2\lambda y = 0$ give $x = 0, y = 0$. The 4'th equation gives $z = 1$ or $z = -1$. The minimum is the south pole $(0,0,-1)$ the maximum the north pole $(0,0,1)$.

14.9. A dice shows k eyes with probability p_k with k in $\Omega = \{1,2,3,4,5,6\}$. A probability distribution is a non-negative function p on Ω which sums up to 1. It can be written as a vector $(p_1, p_2, p_3, p_4, p_5, p_6)$ with $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. The **entropy** of the probability vector $\vec{p}$ is defined as $f(\vec{p}) = -\sum_{i=1}^{6} p_i \log(p_i) = -p_1 \log(p_1) - p_2 \log(p_2) - ... - p_6 \log(p_6)$. Find the distribution p which maximizes entropy under the constrained $g(\vec{p}) = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$. **Solution:** $\nabla f = (-1 - \log(p_1), \dots, -1 - \log(p_n))$, $\nabla g = (1, \dots, 1)$. The Lagrange equations are $-1 - \log(p_i) = \lambda, p_1 + \cdots + p_6 = 1$, from which we get $p_i = e^{-(\lambda+1)}$. The last equation $1 = \sum_i \exp(-(\lambda+1)) = 6\exp(-(\lambda+1))$ fixes $\lambda = -\log(1/6) - 1$ so that $p_i = 1/6$. The distribution, where each event has the same probability is the distribution of maximal entropy. Maximal entropy means **least information content**. An unfair dice allows a cheating gambler or casino to gain profit. Cheating through asymmetric weight distributions can be avoided by making the dices transparent.

14.10. Assume that the probability that a chemical system is in a state k is p_k and the energy of the state k is E_k. Nature tries to minimize the **free energy** $f(p_1, \dots, p_n) = -\sum_i [p_i \log(p_i) - E_i p_i]$ if the energies E_i are fixed. The probability distribution p_i satisfying $\sum_i p_i = 1$ minimizing the free energy is called a **Gibbs distribution**. Find this distribution in general if E_i are given. **Solution:** $\nabla f = (-1 - \log(p_1) - E_1, \dots, -1 - \log(p_n) - E_n)$, $\nabla g = (1, \dots, 1)$. The Lagrange equation are $\log(p_i) = -1 - \lambda - E_i$, or $p_i = \exp(-E_i)C$, where $C = \exp(-1-\lambda)$. The constraint $p_1 + \cdots + p_n = 1$

gives $C(\sum_i \exp(-E_i)) = 1$ so that $C = 1/(\sum_i e^{-E_i})$. The Gibbs solution is $p_k = \exp(-E_k)/\sum_i \exp(-E_i)$.

15. Homework

This homework is due on Tuesday, 7/23/2019.

Problem 14.1: Find the cylindrical basket which is open on the top has has the largest volume for fixed area π. If x is the radius and y is the height, we have to extremize $f(x,y) = \pi x^2 y$ under the constraint $g(x,y) = 2\pi xy + \pi x^2 = \pi$. Use the method of Lagrange multipliers.

Problem 14.2: Find the extrema of the same function

$$f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$$

you have seen in the last homework set but now on the entire disc $\{x^2 + y^2 \leq 4\}$ of radius 2. Besides the already found extrema **inside** the disk, now find also the extrema on the boundary.

Problem 14.3: Motivated by the Disney movie "Tangled", we want to build a hot air balloon with a cuboid mesh of dimension x, y, z which together with the top and bottom fortifications uses wires of total length $g(x,y,z) = 6x + 6y + 4z = 32$. Find the balloon with maximal volume $f(x,y,z) = xyz$.

Problem 14.4: A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi rh + \pi r^2$. Doctor Manhattan designs a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes $f(h,r) = 3r^2 + 2rh$ under the constraint $g(h,r) = 2r^3 + 3r^2 h = 1$. Use the Lagrange method to find a local minimum of f under the constraint $g = 1$.

Problem 14.5: Which pyramid of height h over a square $[-a,a] \times [-a,a]$ with surface area is $4a\sqrt{h^2+a^2} + 4a^2 = 4$ has maximal volume $V(h,a) = 4ha^2/3$? By using new variables (x,y) and multiplying V with a constant, we get to the equivalent problem to maximize $f(x,y) = yx^2$ over the constraint $g(x,y) = x\sqrt{y^2+x^2} + x^2 = 1$. Use the later variables.

MULTIVARIABLE CALCULUS

MATH S-21A

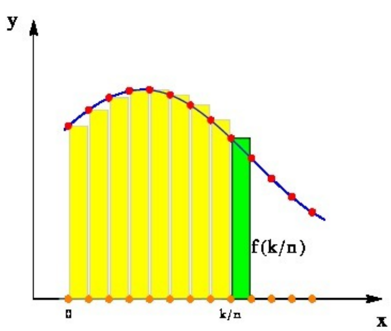
Unit 15: Double Integrals

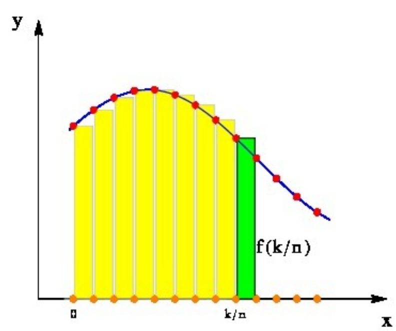
Lecture

15.1. If $f(x)$ is a continuous function, the **Riemann integral** $\int_a^b f(x)\, dx$ is defined as the limit of the **Riemann sums** $S_n f(x) = \frac{1}{n}\sum_{k/n\in[a,b]} f(k/n)$ for $n \to \infty$. The **derivative** of f is the limit of **difference quotients** $D_n f(x) = n[f(x+1/n) - f(x)]$ as $n \to \infty$. The integral $\int_a^b f(x)\, dx$ is the **signed area** under the graph of f and above the x-axes, where "signed" indicates that area below the x-axes has negative sign. The function $F(x) = \int_0^x f(y)\, dy$ is called an **anti-derivative** of f. It is determined up a constant. The **fundamental theorem of calculus** states

$$F'(x) = f(x), \int_0^x f(x) = F(x) - F(0) \ .$$

It allows to compute integrals by inverting differentiation so that **differentiation rules** become **integration rules**: the product rule leads to integration by parts, the chain rule becomes partial integration.





Definition: If $f(x,y)$ is continuous on a region R, the integral $\iint_R f(x,y)\, dxdy$ is defined as the limit of the Riemann sum

$$\frac{1}{n^2}\sum_{(\frac{i}{n},\frac{j}{n})\in R} f(\frac{i}{n},\frac{j}{n})$$

when $n \to \infty$. We write also $\iint_R f(x,y)\, dA$, where $dA = dxdy$ is a notation standing for "an area element".

15.2. Fubini's theorem allows to switch the order of integration over a rectangle if the function f is continuous:

Theorem: $\int_a^b \int_c^d f(x,y)\, dxdy = \int_c^d \int_a^b f(x,y)\, dydx$.

Proof. For every n, there is the "quantum Fubini identity"

$$\sum_{\frac{i}{n}\in[a,b]} \sum_{\frac{j}{n}\in[c,d]} f(\frac{i}{n},\frac{j}{n}) = \sum_{\frac{j}{n}\in[c,d]} \sum_{\frac{i}{n}\in[a,b]} f(\frac{i}{n},\frac{j}{n})$$

holding for all functions. Now divide both sides by n^2 and take the limit $n \to \infty$. This is possible for continuous functions. Fubini's theorem only holds for rectangles. We extend the class of regions now to so called Type I and Type II regions:

Definition: A **type I region** is of the form

$$R = \{(x,y) \mid a \le x \le b,\ c(x) \le y \le d(x)\ \} .$$

An integral over a type I region is called a **type I integral**

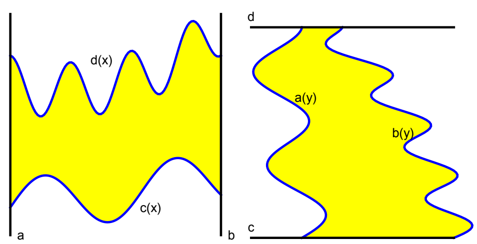
$$\iint_R f\ dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\ dydx .$$

A **type II region** is of the form

$$R = \{(x,y) \mid c \le y \le d, a(y) \le x \le b(y)\ \} .$$

An integral over such a region is called a **type II integral**

$$\iint_R f\ dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\ dxdy .$$



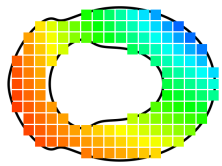
15.3. Similarly as we could see in one dimensions an integral as a signed area, one can interpret $\int\int_R f(x,y)\ dydx$ as the **signed volume** of the solid below the graph of f and above R in the xy plane. As in 1D integration, the volume of the solid below the xy-plane is counted negatively.

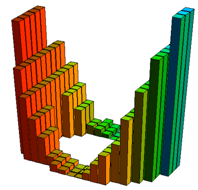
Examples

15.4. If we integrate $f(x,y) = xy$ over the unit square we can sum up the Riemann sum for fixed $y = j/n$ and get $y/2$. Now perform the integral over y to get $1/4$. This example shows how to reduce double integrals to single variable integrals.

15.5. If $f(x,y) = 1$, then the integral is the **area** of the region R. The integral is the limit $L(n)/n^2$, where $L(n)$ is the number of lattice points $(i/n, j/n)$ contained in R.

15.6. The value $\iint_R f(x,y)\,dA/\iint_R 1\,dA$ is the **average** value of f.





15.7. Integrate $f(x,y) = x^2$ over the region bounded above by $\sin(x^3)$ and bounded below by the graph of $-\sin(x^3)$ for $0 \le x \le \pi$. The value of this integral has a physical meaning. It is called **moment of inertia**.

$$\int_0^{\pi^{1/3}} \int_{-\sin(x^3)}^{\sin(x^3)} x^2\,dydx = 2\int_0^{\pi^{1/3}} \sin(x^3)x^2\,dx$$

We have now an integral, which we can solve by substitution $-\frac{2}{3}\cos(x^3)|_0^{\pi^{1/3}} = \frac{4}{3}$.

15.8. Integrate $f(x,y) = y^2$ over the region bound by the x-axes, the lines $y = x+1$ and $y = 1-x$. The problem is best solved as a type I integral. As you can see from the picture, we would have to compute 2 different integrals as a type I integral. To do so, we have to write the bounds as a function of y: they are $x = y-1$ and $x = 1-y$

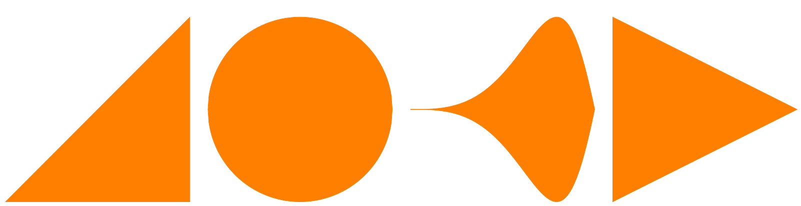
$$\int_0^1 \int_{x-1}^{1-x} y^2\,dy\,dx = 1/6\ .$$

15.9. Let R be the triangle $1 \ge x \ge 0, 0 \le y \le x$. What is

$$\int\int_R e^{-x^2}\,dxdy\ ?$$

The type II integral $\int_0^1 [\int_y^1 e^{-x^2}\ dx]dy$ can not be solved because e^{-x^2} has no anti-derivative in terms of elementary functions.
The type I integral $\int_0^1 [\int_0^x e^{-x^2}\ dy]\,dx$ however can be solved:

$$= \int_0^1 xe^{-x^2}\,dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2} = 0.316...\ .$$



15.10. The area of a disc of radius R is $\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1\ dydx = \int_{-R}^{R} 2\sqrt{R^2-x^2}\ dx$. Substitute $x = R\sin(u), dx = R\cos(u)$, to get $\int_{-\pi/2}^{\pi/2} 2\sqrt{R^2 - R^2\sin^2(u)}R\cos(u)\ du = \int_{-\pi/2}^{\pi/2} 2R^2\cos^2(u)\ du = R^2\pi$.

Homework

This homework is due on Tuesday, 7/23/2019.

Problem 15.1: a) (4 points) Find the iterated integral

$$\int_0^1 \int_0^2 3xy/\sqrt{x^2 + (y^2/2)}\, dy\, dx \; .$$

b) (4 points) Now compute

$$\int_0^1 \int_0^2 3xy/\sqrt{x^2 + y^2/2}\, dx\, dy \; .$$

c) (2 points) Wouldn't Fubini assure that a) and b) are the same? What change would be needed in b) to make the results agree?

Problem 15.2: Find the area of the region

$$R = \{(x,y) \mid 0 \leq x \leq 2\pi, \sin(x) - 1 \leq y \leq \cos(x) + 2\}$$

and use it to compute the average value $\int\int_R f(x,y)\ dxdy/\text{area}(R)$ of $f(x,y) = y$ over that region.

Problem 15.3: Find the volume of the solid lying under the paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2,2] \times [-3,6] = \{(x,y) \mid -2 \leq x \leq 2, -3 \leq y \leq 6\ \}$.

Problem 15.4: Calculate the iterated integral $\int_0^1 \int_x^{2-x} (x^2 - y)\ dydx$. Sketch the corresponding type I region. Write this integral as integral over a type II region and compute the integral again.

Problem 15.5: There is only one way to identify zombies: throw two difficult integrals at them and see whether they can solve them. Prove that you are not a zombie!
a) (6 points) Find the integral

$$\int_0^1 \int_{\sqrt{y}}^{y^2} \frac{3x^7}{\sqrt{x} - x^2}\, dx\, dy \; .$$

b) (4 points) Integrate

$$\int_0^1 \int_0^{\sqrt{1-y^2}} 11(x^2 + y^2)^{10}\ dxdy \; .$$

You might want to "time travel" one lecture forward, where polar coordinates are known to solve this problem.

MULTIVARIABLE CALCULUS

MATH S-21A

Unit 16: Surface Integration

Lecture

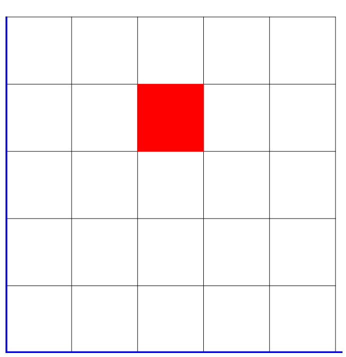
16.1. For certain regions, it is better to use different coordinate system. A reparametrization $(x, y) = \vec{r}(u, v)$ often helps. This works then also in higher dimensions, when suraces are parametrized as $(x, y, z) = \vec{r}(u, v)$. But first to the two dimensional case, where polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta)$ are an important example

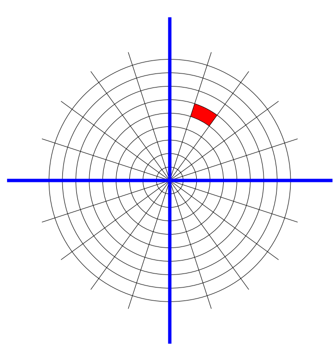
Definition: A **polar region** is a planar region bound by a simple closed curve given in polar coordinates as the curve $(r(t), \theta(t))$. The most common case is $\theta(t) = t$. In Cartesian coordinates the parametrization of the boundary of a polar region is $\vec{r}(t) = [r(t)\cos(\theta(t), r(t)\sin(\theta(t)]^T$, a **polar graph** like the spiral with $\theta(t) = t$.

Theorem: To integrate in polar coordinates, we evaluate the integral

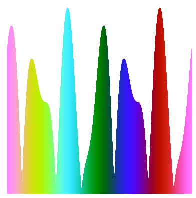
$$\iint_R f(x, y)\, dxdy = \iint_R f(r\cos(\theta), r\sin(\theta)r\, drd\theta$$

16.2. Why do we have to include the factor r, when we move to polar coordinates? The reason is that a small rectangle R with dimensions $d\theta dr$ in the (r, θ) plane is mapped by $T : (r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$ to a sector segment S in the (x, y) plane. It has the area $r\ d\theta dr$. If you have seen some linear algebra, note that the Jacobean matrix dT has the determinant r.

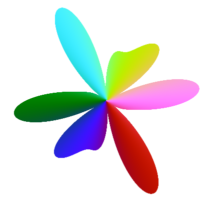




16.3. We can now integrate over type I or type II regions in the (θ, r) plane. like **flowers**: $\{(\theta, r) \,|0 \leq r \leq f(\theta)\}$ where $f(\theta)$ is a periodic function of θ.



A polar region shown in polar coordinates. It is a type I region.

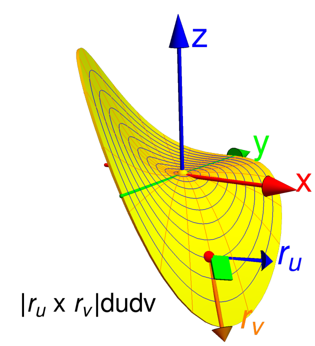


The same region in the xy coordinate system is not type I or II.

Theorem: A surface $\vec{r}(u, v)$ parametrized on a parameter domain R has the **surface area**

$$\int\int_R |\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \; dudv \; .$$

Proof. The vector $\vec{r}_u$ is tangent to the grid curve $u \mapsto \vec{r}(u, v)$ and $\vec{r}_v$ is tangent to $v \mapsto \vec{r}(u, v)$, the two vectors span a parallelogram with area $|\vec{r}_u \times \vec{r}_v|$. A small rectangle $[u, u + du] \times [v, v + dv]$ is mapped by $\vec{r}$ to a parallelogram spanned by $[\vec{r}, \vec{r} + \vec{r}_u]$ and $[\vec{r}, \vec{r} + \vec{r}_v]$ which has the area $|\vec{r}_u(u, v) \times \vec{r}_v(u, v)| \; dudv$.

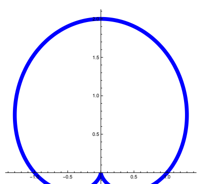


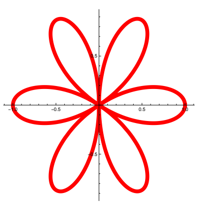
Examples

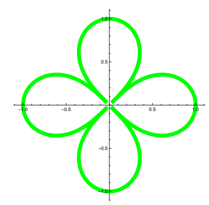
16.4. The polar graph defined by $r(\theta) = |\cos(3\theta)|$ belongs to the class of **roses** $r(t) = |\cos(nt)|$. Regions enclosed by this graph are also called **rhododenea**.

16.5. The polar curve $r(\theta) = 1 + \sin(\theta)$ is called a **cardioid**. It looks like a heart. It belongs to the class of **limacon** curves $r(\theta) = 1 + b\sin(\theta)$.

16.6. The polar curve $r(\theta) = |\sqrt{\cos(2t)}|$ is called a **lemniscate**.







16.7. Integrate

$$f(x,y) = x^2 + y^2 + xy \ ,$$

over the unit disc. We have $f(x,y) = f(r\cos(\theta), r\sin(\theta)) = r^2 + r^2\cos(\theta)\sin(\theta)$ so that $\iint_R f(x,y)\ dxdy = \int_0^1 \int_0^{2\pi} (r^2 + r^2\cos(\theta)\sin(\theta))r\ d\theta dr = 2\pi/4$.

16.8. We have earlier computed area of the disc $\{x^2 + y^2 \leq 1\ \}$ using substitution. It is more elegant to do this integral in polar coordinates:

$$\int_0^{2\pi} \int_0^1 r\ drd\theta = 2\pi r^2/2|_0^1 = \pi \ .$$

16.9. Integrate the function $f(x,y) = 1\ \{(\theta, r(\theta)) \mid r(\theta) \leq \ |\ \cos(3\theta)|\ \}$.

$$\int\int_R 1\ dxdy = \int_0^{2\pi} \int_0^{\cos(3\theta)} r\ dr\ d\theta = \int_0^{2\pi} \frac{\cos(3\theta)^2}{2}\ d\theta = \pi/2 \ .$$

16.10. Integrate $f(x,y) = y\sqrt{x^2+y^2}$ over the region $R = \{(x,y) \mid 1 < x^2 + y^2 < 4, y > 0\ \}$.

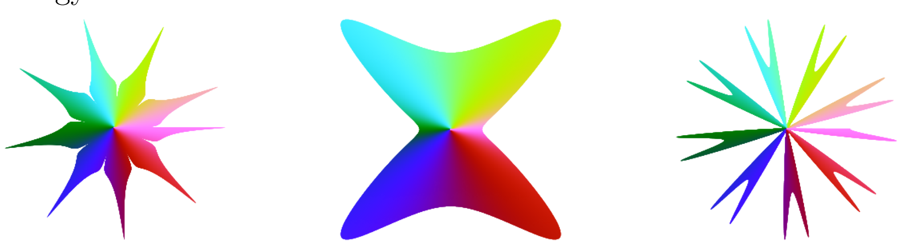
$$\int_1^2 \int_0^{\pi} r\sin(\theta) r\ r\ d\theta dr = \int_1^2 r^3 \int_0^{\pi} \sin(\theta)\ d\theta dr = \frac{(2^4 - 1^4)}{4} \int_0^{\pi} \sin(\theta)\ d\theta = 15/2$$

For integration problems, where the region is part of an annular region, or if you see function with terms $x^2 + y^2$ try to use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$.

16.11. The Belgian Biologist **Johan Gielis** defined in 1997 with the family of curves given in polar coordinates as

$$r(\phi) = (\frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n_2}}{b})^{-1/n_3}$$

This so called **super-curve** can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes". The super-curve generalizes the **super-ellipse** which had been discussed in 1818 by Lamé and helps to **describe forms** in biology. [1]



16.12. The parametrized surface $\vec{r}(u,v) = [2u, 3v, 0]^T$ is part of the xy-plane. The parameter region G just gets stretched by a factor 2 in the x coordinate and by a factor 3 in the y coordinate. $\vec{r}_u \times \vec{r}_v = [0,0,6]^T$ and we see for example that the area of $\vec{r}(G)$ is 6 times the area of G.

[1]"Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003).

16.13. The map $\vec{r}(u,v) = [L\cos(u)\sin(v), L\sin(u)\sin(v), L\cos(v)]^T$ maps the rectangle $G = [0,2\pi]\times[0,\pi]$ onto the sphere of radius L. We compute $\vec{r}_u \times \vec{r}_v = L\sin(v)\vec{r}(u,v)$. So, $|\vec{r}_u \times \vec{r}_v| = L^2|\sin(v)|$ and $\int\int_R 1\, dS = \int_0^{2\pi}\int_0^{\pi} L^2\sin(v)\, dvdu = 4\pi L^2$

16.14. For graphs $(u,v) \mapsto [u, v, f(u,v)]^T$, we have $\vec{r}_u = (1, 0, f_u(u,v))$ and $\vec{r}_v = (0, 1, f_v(u,v))$. The cross product $\vec{r}_u \times \vec{r}_v = (-f_u, -f_v, 1)$ has the length $\sqrt{1 + f_u^2 + f_v^2}$. The area of the surface above a region G is $\int\int_G \sqrt{1 + f_u^2 + f_v^2}\, dudv$.

16.15. Lets take a surface of revolution $\vec{r}(u,v) = [v, f(v)\cos(u), f(v)\sin(u)]^T$ on $R = [0,2\pi]\times[a,b]$. We have $\vec{r}_u = (0, -f(v)\sin(u), f(v)\cos(u))$, $\vec{r}_v = (1, f'(v)\cos(u), f'(v)\sin(u))$ and $\vec{r}_u \times \vec{r}_v = (-f(v)f'(v), f(v)\cos(u), f(v)\sin(u)) = f(v)(-f'(v), \cos(u), \sin(u))$. The surface area is $\int\int |\vec{r}_u \times \vec{r}_v|\, dudv = 2\pi\int_a^b |f(v)|\sqrt{1 + f'(v)^2}\, dv$.

Homework

This homework is due on Tuesday, 7/23/2019.

Problem 16.1: a) A city near the sea is modeled by a half disk $D = \{(x,y) \mid x^2 + y^2 \le 49, x \ge 0$ with center the origin and radius 7. What is the average distance of a point in D to the origin? in other words, what is the integral $\int\int_D \sqrt{x^2+y^2}\, dxdy / \int\int_D 1\, dxdy$.
b) The distance to the beach is x. Find the average distance $\int\int_D x\, dxdy / \int\int_D 1\, dxdy$ to the beach.

Problem 16.2: Find $\int\int_R (x^2 + y^2)^{44}\, dA$, where R is the part of the unit disc $\{x^2 + y^2 \le 1\}$ for which $y > x$.

Problem 16.3: What is the area of the region which is bounded by the following three curves, first by the polar curve $r(\theta) = \theta$ with $\theta \in [0, 2\pi]$, second by the polar curve $r(\theta) = 2\theta$ with $\theta \in [0, 2\pi]$ and third by the positive x-axis?

Problem 16.4: The average of a function f on a region is defined as

$$\frac{\int_R f\, dxdy}{\int_R 1\, dxdy}.$$

Find the average value of $f(x,y) = 2(x^2 + y^2)$ on the annular region $R : 1 \le |(x,y)| \le 2$.

Problem 16.5: Find the surface area of the part of the paraboloid $x = y^2 + z^2$ which is inside the cylinder $y^2 + z^2 \le 16$.

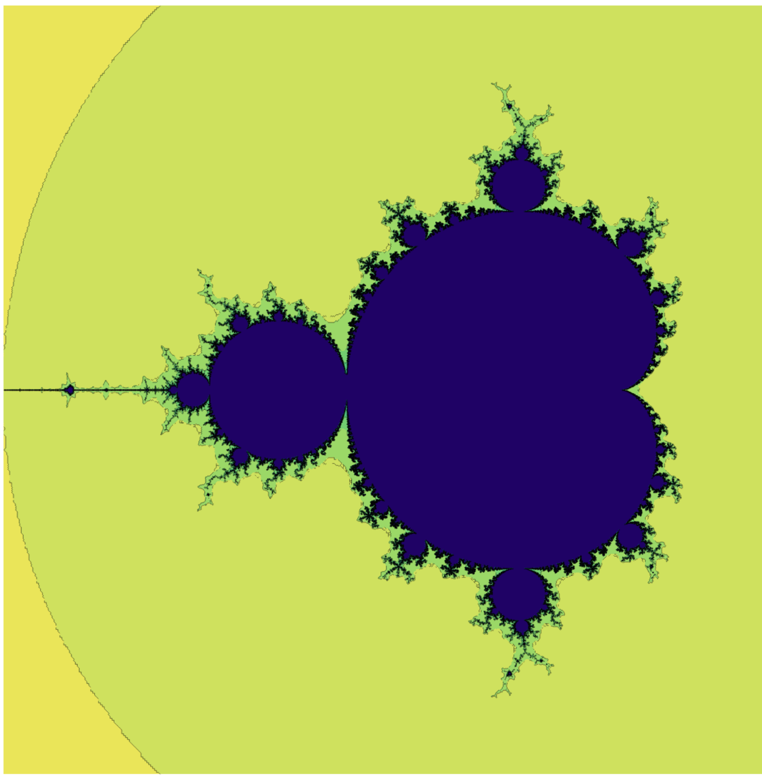
Postscript: Area of the Mandelbrot set

16.16. Often, when we deal with real data, we do not have analytic expressions for the region or function we want to integrate. We want to elaborate here on an example mentioned already in the text. It is the problem to find the area of Mandelbrot set

$$M = \{c = a + ib \in \mathbb{C} \in \mathbb{R}^2 \mid T_c(0)^n \text{ stays bounded } \} ,$$

where $T_c(z) = z^+c$ (as complex numbers, which is written out in real coordinates the map $T_c(x, y) = (x^2 - y^2 + a, 2xy + b)$.

16.17. Here is a picture: it can also be visualized as a function which is 1 on the Mandelbrot set and 0 else.



16.18. What is the area of the Mandelbrot set? We know it is contained in the rectangle $x \in [-2, 1]$ and $y \in [-3/2, 3/2]$. We now just randomly shoot into this rectangle and see whether we are in the Mandelbrot set or not after 1000 iterations. Here is some Mathematica code which allows you to compute things. When we ran it, it gave a value of about 1.515.... More accurate measurements reported hint for a slightly smaller value like 1.506.... Others have given bounds $[1.50311, 1.5613027]$.

```
M=Compile[{x,y},Module[{z=x+I y,k=0},
    While[Abs[z]<2.&&k<1000,z=N[z^2+x+I y];++k];Floor[k/1000]]];
9*Sum[M[-2+3 Random[],-1.5+3 Random[]],{1000000}]/1000000
```

How accurately can you compute the area of the Mandelbrot set? It is a data problem unless somebody comes up with a formula.

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