MULTIVARIABLE CALCULUS

MATH S-21A

Unit 9: Partial derivatives

Lecture

9.1. For functions of several variables we can differentiate to any of them:

Definition: If $f(x,y)$ is a function of the two variables x and y, the **partial derivative** $\frac{\partial}{\partial x} f(x,y)$ is defined as the derivative of the function $g(x) = f(x,y)$ with respect to x, where y is considered a constant. The partial derivative with respect to y is the derivative with respect to y where x is fixed.

9.2. The short hand notation $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$ is convenient. When iterating derivatives, the notation is similar: we write for example $f_{xy} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$. The number $f_x(x_0,y_0)$ gives the slope of the graph sliced at (x_0,y_0) in the x direction. The second derivative f_{xx} is a measure of concavity in that direction. The meaning of f_{xy} is the rate of change of the x-slope if you move the cut along the y-axis.

9.3. The notation $\partial_x f, \partial_y f$ was introduced by Carl Gustav Jacobi. Before that, Josef Lagrange used the term "partial differences". For functions of three or more variables, the partial derivatives are defined in the same way. We write for example $f_x(x,y,z)$ or $f_{xxz}(x,y,z)$.

Theorem: Clairaut's theorem: If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

9.4. Proof. Following Euler, we first look at the difference quotients and say that if the "Planck constant" h is positive, then $f_x(x,y) = [f(x+h,y) - f(x,y)]/h$. For $h = 0$, we mean the usual partial derivative f_x. Comparing the two sides of the equation for fixed $h > 0$ shows

$$hf_x(x,y) = f(x+h,y) - f(x,y) \qquad hf_y(x,y) = f(x,y+h) - f(x,y).$$
$$h^2 f_{xy}(x,y) = f(x+h,y+h) - f(x,y+h) - (f(x+h,y) - f(x,y)) \qquad h^2 f_{yx}(x,y) = f(x+h,y+h) - f(x+h,y) - (f(x,y+h) - f(x,y))$$

9.5. Without having taken any limits we established an identity which holds for all $h > 0$: the discrete derivatives f_x, f_y satisfy the relation $f_{xy} = f_{yx}$ for any $h > 0$. We could fancy it as **"quantum Clairaut" formula**. If the classical derivatives f_{xy}, f_{yx} are both continuous, it is possible to take the limit $h \to 0$. The classical Clairaut's theorem can be seen as a "classical limit". The quantum Clairaut holds however for **all** functions $f(x, y)$ of two variables. Not even continuity is needed. [1]

9.6. An equation for an unknown function $f(x, y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**. We abbreviate PDE. If only the derivative with respect to one variable appears, it is an **ordinary differential equation**, abbreviated ODE.

Examples

9.7. For $f(x, y) = x^4 - 6x^2y^2 + y^4$, we have $f_x(x, y) = 4x^3 - 12xy^2$, $f_{xx} = 12x^2 - 12y^2$, $f_y(x, y) = -12x^2y + 4y^3$, $f_{yy} = -12x^2 + 12y^2$ and see that $\Delta f = f_{xx} + f_{yy} = 0$. A function which satisfies $\Delta f = 0$ is also called **harmonic**. The equation $f_{xx} + f_{yy} = 0$ is a PDE:

Definition: A **partial differential equation** (PDE) is an equation for an unknown function $f(x, y)$ which involves partial derivatives with respect to more than one variables.

9.8.

The **wave equation** $f_{tt}(t, x) = f_{xx}(t, x)$ governs the motion of light or sound. The function $f(t, x) = \sin(x - t) + \sin(x + t)$ satisfies the wave equation.

The **heat equation** $f_t(t, x) = f_{xx}(t, x)$ describes diffusion of heat or spread of an epidemic. The function $f(t, x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}$ satisfies the heat equation.

The **Laplace equation** $f_{xx} + f_{yy} = 0$ determines the shape of a membrane. The function $f(x, y) = x^3 - 3xy^2$ is an example satisfying the Laplace equation.

The **advection equation** $f_t = f_x$ is used to model transport in a wire. The function $f(t, x) = e^{-(x+t)^2}$ satisfies the advection equation.

[1] For a full proof of Clairaut's theorem, see *www.math.harvard.edu/knill/teaching/math22a2018/handouts/lecture14.pdf*.

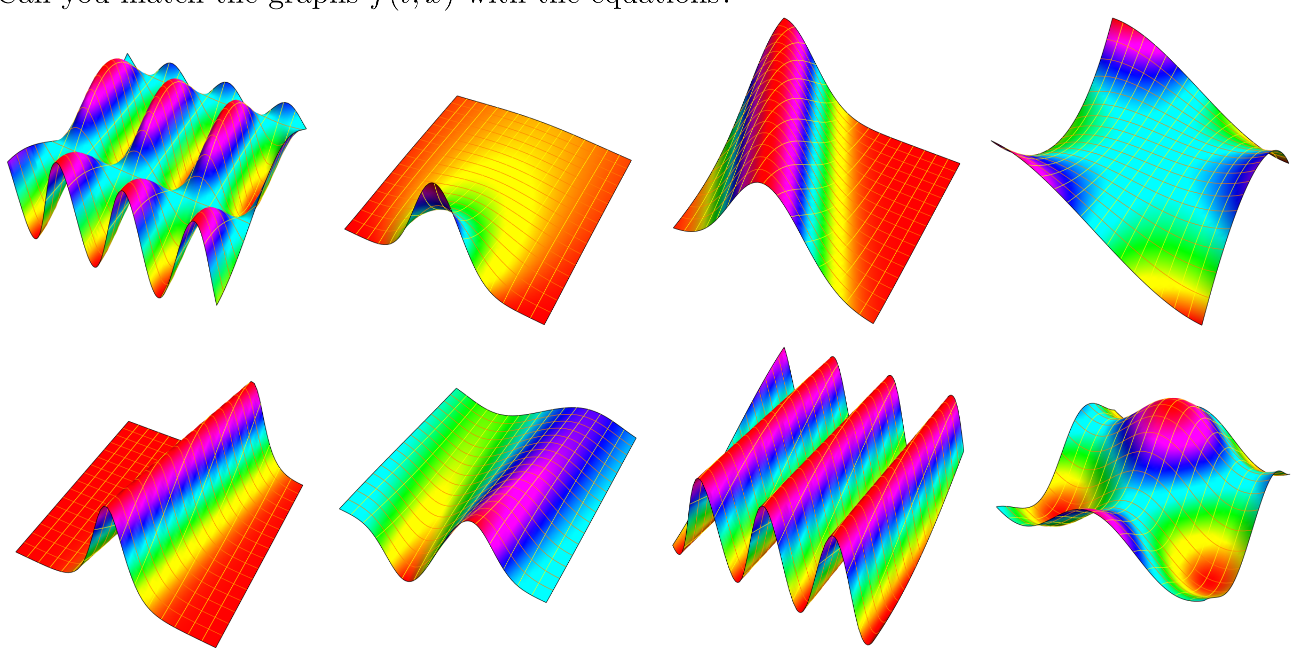
The **eiconal equation** $f_x^2 + f_y^2 = 1$ is used to see the evolution of wave fronts in optics. The function $\boxed{f(x,y) = \cos(x) + \sin(y)}$ satisfies the eiconal equation.

The **Burgers equation** $f_t + f f_x = f_{xx}$ describes waves at the beach which break. The function $\boxed{f(t,x) = \frac{x}{t}\frac{\sqrt{\frac{1}{t}}e^{-x^2/(4t)}}{1+\sqrt{\frac{1}{t}}e^{-x^2/(4t)}}}$ satisfies the Burgers equation.

The **KdV equation** $f_t + 6 f f_x + f_{xxx} = 0$ models **water waves** in a narrow channel. The function $\boxed{f(t,x) = \frac{a^2}{2}\cosh^{-2}(\frac{a}{2}(x - a^2 t))}$ satisfies the KdV equation.

The **Schrödinger equation** $f_t = \frac{i\hbar}{2m} f_{xx}$ is used to describe a **quantum particle** of mass m. The function $\boxed{f(t,x) = e^{i(kx - \frac{\hbar}{2m}k^2 t)}}$ solves the Schrödinger equation. [Here $i^2 = -1$ is the imaginary i and $\hbar$ is the **Planck constant** $\hbar \sim 10^{-34} Js$.]

Can you match the graphs $f(t,x)$ with the equations?



9.9. In all examples, we just see one possible solution to the partial differential equation. There are in general many solutions and additional initial or boundary conditions then determine the solution uniquely. If we know $f(0,x)$ for the Burgers equation, then the solution $f(t,x)$ is determined.

Homework

This homework is due on Tuesday, 7/16/2019.

Problem 9.1: Verify that $f(t,x) = \cos^2(t+x) + e^{e^{\sin(t+x)}}$ is a solution of the transport equation $f_t(t,x) = f_x(t,x)$.

Problem 9.2: a) Verify that $f(x,y) = \sin(x)(\cos(7y) + \sin(7y))$ satisfies the **Klein Gordon equation** $u_{xx} - u_{yy} = 48u$. This PDE is useful in quantum mechanics.
b) Verify that $4\arctan(e^{(x-t/2)2/\sqrt{3}})$ satisfies the **Sine Gordon equation** $u_{tt} - u_{xx} = -\sin(u)$. Use technology. (If you can do it without technology, show it to Oliver).

Problem 9.3: Verify that for any real constant b, the function $f(x,t) = e^{-bt}\cos(x+t)$ satisfies the driven transport equation $f_t(x,t) = f_x(x,t) - bf(x,t)$ This PDE is sometimes called the **advection equation** with damping b.

Problem 9.4: The differential equation

$$f_t = f - xf_x - x^2 f_{xx}$$

is a version of the **infamous Black-Scholes equation**. Here $f(x,t)$ is the prize of a **call option** and x the stock prize ad t is time. Find a function $f(x,t)$ solving it which depends both on x and t. The examples $f(x,y) = x$ or $f(x) = e^t$ do not qualify as they depend only on one variable.

Problem 9.5: The partial differential equation $f_t + ff_x = f_{xx}$ is called **Burgers equation** and describes waves at the beach. In higher dimensions, it leads to the Navier-Stokes equation which are used to describe the weather. Verify that

$$f(t,x)\frac{\left(\frac{1}{t}\right)^{3/2} xe^{-\frac{x^2}{4t}}}{\sqrt{\frac{1}{t}}e^{-\frac{x^2}{4t}} + 1}$$

solves the Burgers equation. You also here might want to get help with technology.

MULTIVARIABLE CALCULUS

MATH S-21A

Unit 10: Linearization

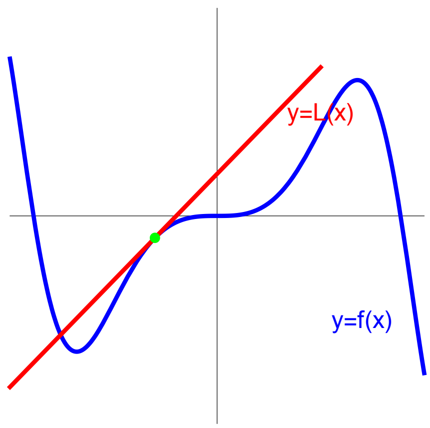
Lecture

10.1. In single variable calculus, you have seen the following definition for a differentiable function:

Definition: The **linear approximation** of $f(x)$ at a is the affine function

$$L(x) = f(a) + f'(a)(x-a) \ .$$

10.2. If you have seen **Taylor series**, this is the part of the series $f(x) = \sum_{k=0}^{\infty} f^{(k)}(a)(x-a)^k/k!$ where only the $k=0$ and $k=1$ term are considered. We think about the linear approximation L as a function and not as a graph because we will also look at linear approximations for functions of three variables, where we can not draw graphs.



10.3. The graph of the function L is close to the graph of f at a. What about higher dimensions?

Definition: The **linear approximation** of $f(x,y)$ at (a,b) is the affine function

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \ .$$

The **linear approximation** of a function $f(x,y,z)$ at (a,b,c) is

$$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) \ .$$

10.4. Using the **gradient**

$$\nabla f(x,y) = [f_x, f_y]^T, \qquad \nabla f(x,y,z) = [f_x, f_y, f_z]^T ,$$

the linearization can be written more compactly as

$$L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) .$$

10.5. How do we justify the linearization? If the second variable $y = b$ is fixed, we have a one-dimensional situation, where the only variable is x. Now $f(x,b) = f(a,b) + f_x(a,b)(x-a)$ is the linear approximation. Similarly, if $x = x_0$ is fixed y is the single variable, then $f(x_0,y) = f(x_0,y_0) + f_y(x_0,y_0)(y-y_0)$. Knowing the linear approximations in both the x and y variables, we can get the general linear approximation by $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$.

Examples

10.6. What is the linear approximation of the function $f(x,y) = \sin(\pi xy^2)$ at the point $(1,1)$? Answer: We have $[f_x(x,y), f_y(x,y)]^T = [\pi y^2 \cos(\pi xy^2), 2xy\pi\cos(\pi xy^2)]^T$ which is at the point $(1,1)$ equal to $\nabla f(1,1) = [\pi\cos(\pi), 2\pi\cos(\pi)]^T = [-\pi, -2\pi]^T$. The function is $L(x,y) = 0 + (-\pi)(x-1) - 2\pi(y-1)$.

10.7. Linearization can be used to estimate functions near a point. In the previous example,

$$f(1+0.01, 1+0.01) = -0.0095$$

$$L(1+0.01, 1+0.01) = -\pi 0.01 - 2\pi 0.01 = -3\pi/100 = -0.00942 .$$

10.8. Here is an example in three dimensions: find the linear approximation to $f(x,y,z) = xy+yz+zx$ at the point $(1,1,1)$. Since $f(1,1,1) = 3$, and $\nabla f(x,y,z) = (y+z, x+z, y+x)$, $\nabla f(1,1,1) = (2,2,2)$. we have $L(x,y,z) = f(1,1,1) + (2,2,2)\cdot(x-1,y-1,z-1) = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x+2y+2z-3$.

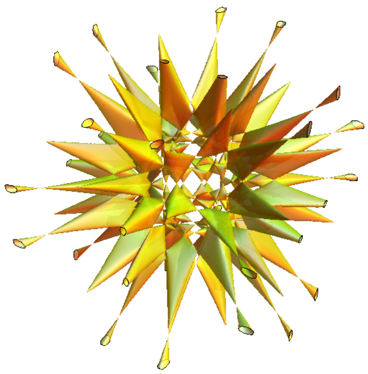
10.9. Estimate $f(0.01, 24.8, 1.02)$ for $f(x,y,z) = e^x\sqrt{y}z$.
Solution: take $(x_0,y_0,z_0) = (0,25,1)$, where $f(x_0,y_0,z_0) = 5$. The gradient is $\nabla f(x,y,z) = (e^x\sqrt{y}z, e^x z/(2\sqrt{y}), e^x\sqrt{y})$. At the point $(x_0,y_0,z_0) = (0,25,1)$ the gradient is the vector $(5, 1/10, 5)$. The linear approximation is $L(x,y,z) = f(x_0,y_0,z_0) + \nabla f(x_0,y_0,z_0)(x-x_0, y-y_0, z-z_0) = 5 + (5,1/10,5)(x-0, y-25, z-1) = 5x + y/10 + 5z - 2.5$. We can approximate $f(0.01, 24.8, 1.02)$ by $5 + (5,1/10,5)\cdot(0.01, -0.2, 0.02) = 5 + 0.05 - 0.02 + 0.10 = 5.13$. The actual value is $f(0.01, 24.8, 1.02) = 5.1306$, very close to the estimate.

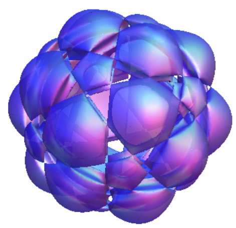
10.10. Find the tangent line to the graph of the function $g(x) = x^2$ at the point $(2,4)$.
Solution: the level curve $f(x,y) = y - x^2 = 0$ is the graph of a function $g(x) = x^2$ and the tangent at a point $(2, g(2)) = (2,4)$ is obtained by computing the gradient $[a,b]^T = \nabla f(2,4) = [-g'(2), 1]^T = [-4,1]^T$ and forming $-4x + y = d$, where $d = -4\cdot 2 + 1\cdot 4 = -4$. The answer is $\boxed{-4x + y = -4}$ which is the line $y = 4x - 4$ of slope 4.

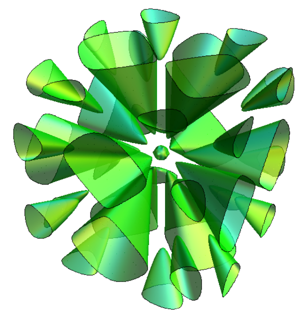
10.11. The **Barth surface** is defined as the level surface $f = 0$ of

$$\begin{aligned} f(x,y,z) &= (3+5t)(-1+x^2+y^2+z^2)^2(-2+t+x^2+y^2+z^2)^2 \\ &+ 8(x^2-t^4y^2)(-(t^4x^2)+z^2)(y^2-t^4z^2)(x^4-2x^2y^2+y^4-2x^2z^2-2y^2z^2+z^4) , \end{aligned}$$

where $t = (\sqrt{5}+1)/2$ is a constant called the **golden ratio**. If we replace t with $1/t = (\sqrt{5}-1)/2$ we see the surface to the middle. For $t = 1$, we see to the right the surface $f(x,y,z) = 8$. Find the tangent plane of the later surface at the point $(1,1,0)$.
Answer: We have $\nabla f(1,1,0) = [64,64,0]^T$. The surface is $x+y = d$ for some constant d. By plugging in $(1,1,0)$ we see that $x+y = 2$.



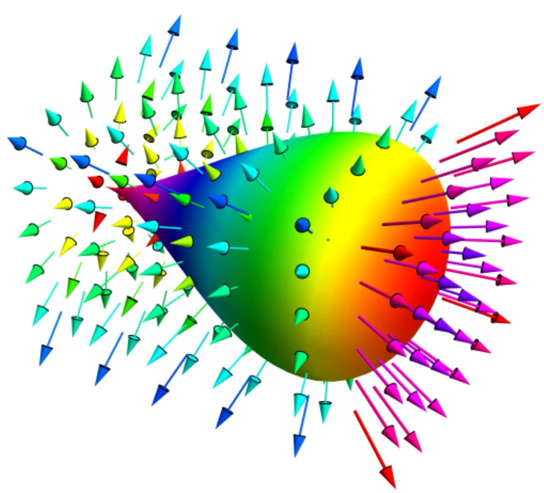




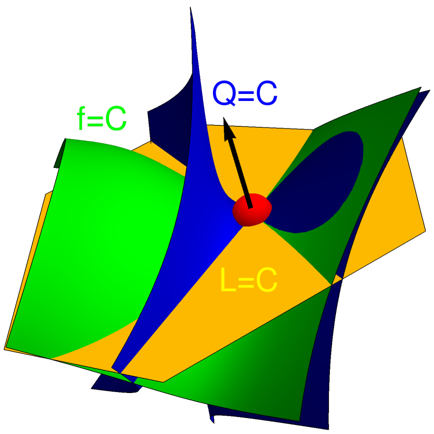
The quartic surface

$$f(x,y,z) = x^4 - x^3 + y^2 + z^2 = 0$$

is called the **piriform**. What is the equation for the tangent plane at the point $P = (2,2,2)$ of this pair shaped surface? We get $[a,b,c]^T = [20,4,4]^T$ and so the equation of the plane $20x+4y+4z = 56$, where we have obtained the constant to the right by plugging in the point $(x,y,z) = (2,2,2)$.



10.12. Finally, we want to point out that linearization is just the first step. One could do quadratic approximations for example. In one dimension, one has $Q(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!}$. In two dimensions, this becomes $Q(x,y) = L(x,y) + H(a,b)[x-a, y-b]^T \cdot [x-a, y-b]^T/2$, where H is the **Hessian matrix** $H(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$. We will see this matrix when we maximize or minimze functions.



Homework

This homework is due on Tuesday, 7/16/2019.

Problem 10.1: Estimate $200'000'000'000'000^{1/11}$ using linear approximation of $f(x) = x^{1/11}$ near $x_0 = 20^{11}$.

Problem 10.2: Given $f(x,y) = 3yx/\pi - \cos(x)$. Estimate $f(\pi+0.01, \pi-0.03)$ using linearization

Problem 10.3: Estimate $f(0.003, 0.9999)$ for $f(x,y) = \cos(\pi y) + \sin(x + \pi y)$ using linearization.

Problem 10.4: Find the linear approximation $L(x,y)$ of the function

$$f(x,y) = \sqrt{10 - x^2 - 5y^2}$$

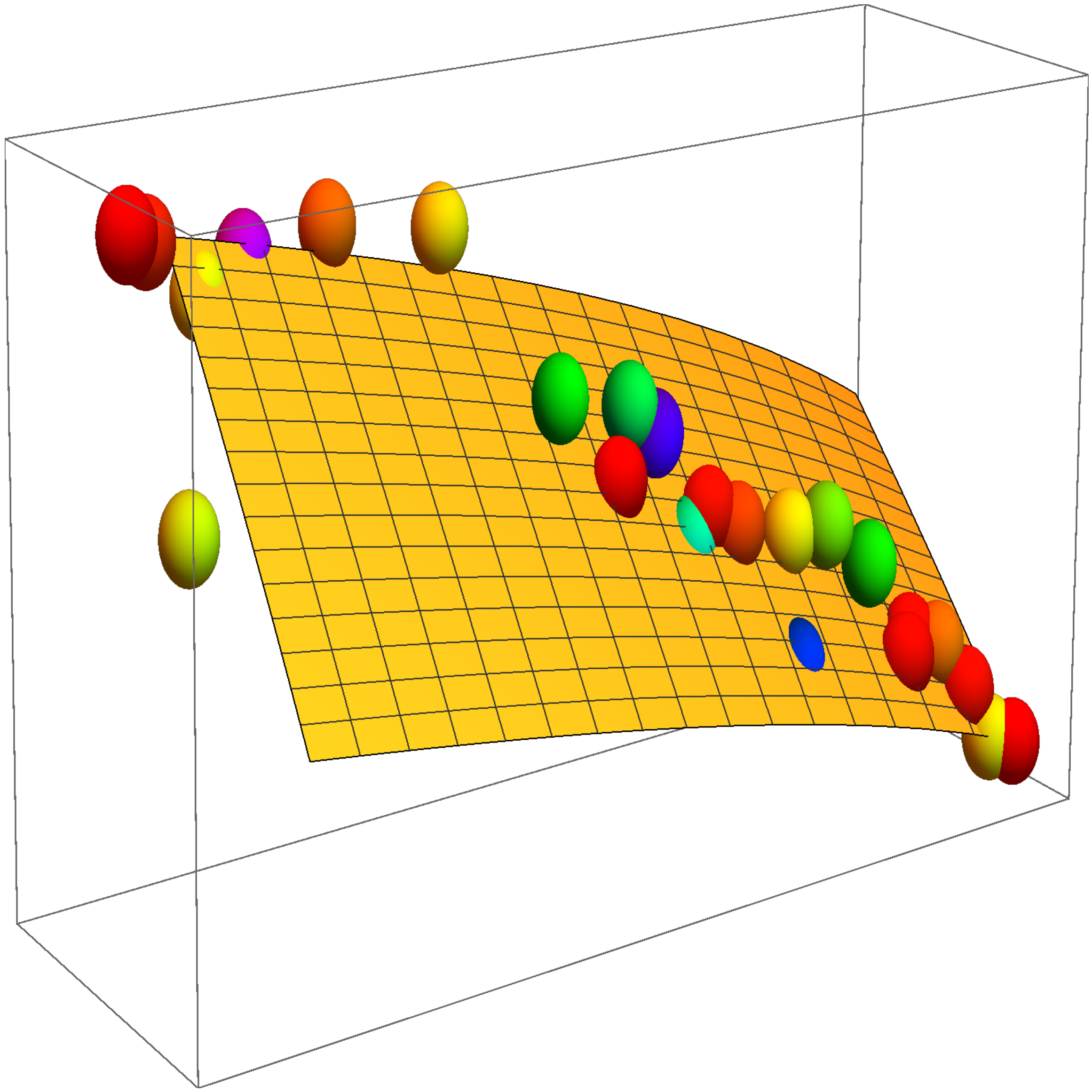
at $(2,1)$ and use it to estimate $f(1.95, 1.04)$.

Problem 10.5: Estimate $(99^3 * 101^2)$ by linearizing the function $f(x,y) = x^3y^2$ at $(100,100)$. What is the difference between $L(100,100)$ and $f(100,100)$?

Data illustration Cobb-Douglas

10.13. The mathematician and economist **Charles W. Cobb** at Amherst college and the economist and politician **Paul H. Douglas** who was also teaching at Amherst found in 1928 empirically a formula $F(K, L) = L^\alpha K^\beta$ which fits the **total production** F of an economic system as a function of the **capital investment** K and the **labor** L. The two authors used logarithms variables and assumed linearity to find α, β. Below are the data normalized so that the date for year 1899 has the value 100. On the website, we give you access to these historical data as well as to the original Nobel prize winning article which already made this assumption.

Year	K	L	P
1899	100	100	100
1900	107	105	101
1901	114	110	112
1902	122	118	122
1903	131	123	124
1904	138	116	122
1905	149	125	143
1906	163	133	152
1907	176	138	151
1908	185	121	126
1909	198	140	155
1910	208	144	159
1911	216	145	153
1912	226	152	177
1913	236	154	184
1914	244	149	169
1915	266	154	189
1916	298	182	225
1917	335	196	227
1918	366	200	223
1919	387	193	218
1920	407	193	231
1921	417	147	179
1922	431	161	240



The graph of $F(L,K) = L^{3/4}K^{1/4}$ fits pretty well that data set.

Oliver Knill, knill@math.harvard.edu, Math S-21a, Harvard Summer School, 2019

MULTIVARIABLE CALCULUS

MATH S-21A

Unit 11: Chain rule

Lecture

11.1. If f and g are functions of a single variable t, the **single variable chain rule** tells us that $d/dt f(g(t)) = f'(g(t))g'(t)$. For example, $d/dt \sin(\log(t)) = \cos(\log(t))/t$. The rule can be proven by linearizing the functions f and g and verifying the chain rule in the linear case. The **chain rule** is also useful:

11.2. To find $\arccos'(x)$ for example, we differentiate $x = \cos(\arccos(x))$ to get $1 = d/dx \cos(\arccos(x)) = -\sin(\arccos(x)) \arccos'(x) = -\sqrt{1-\cos^2(\arccos(x))} \arccos'(x) = -\sqrt{1-x^2} \arccos'(x)$ so that $\arccos'(x) = -1/\sqrt{1-x^2}$.

Definition: Define the **gradient** $\nabla f(x,y) = [f_x(x,y), f_y(x,y)]^T$ or $\nabla f(x,y,z) = [f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)]^T$.

11.3. If $\vec{r}(t)$ is curve and f is a function of several variables we get a function $t \mapsto f(\vec{r}(t))$ of one variable. Similarly, if $\vec{r}(t)$ is a parametrization of a planar curve f is a function of two variables, then $t \mapsto f(\vec{r}(t))$ is a function of one variable.

Theorem: $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$.

Proof. When written out in two dimensions, it is

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) \ .$$

The identity

$$\frac{f(x(t+h),y(t+h))-f(x(t),y(t))}{h} = \frac{f(x(t+h),y(t+h))-f(x(t),y(t+h))}{h} + \frac{f(x(t),y(t+h))-f(x(t),y(t))}{h}$$

holds for every $h > 0$. The left hand side converges to $\frac{d}{dt} f(x(t), y(t))$ in the limit $h \to 0$ and the right hand side to $f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$ using the single variable chain rule twice. Here is the proof of the later, when we differentiate f with respect to t and y is treated as a constant:

$$\frac{f(\text{ x(t+h) }) - f(x(t))}{h} = \frac{[f(\text{ x(t) + (x(t+h)-x(t)) }) - f(x(t))]}{\text{[x(t+h)-x(t)]}} \cdot \frac{\text{[x(t+h)-x(t)]}}{h} \ .$$

Write $H(t) =$ x(t+h)-x(t) in the first part on the right hand side.

$$\frac{f(x(t+h)) - f(x(t))}{h} = \frac{[f(x(t)+H) - f(x(t))]}{H} \cdot \frac{x(t+h) - x(t)}{h} .$$

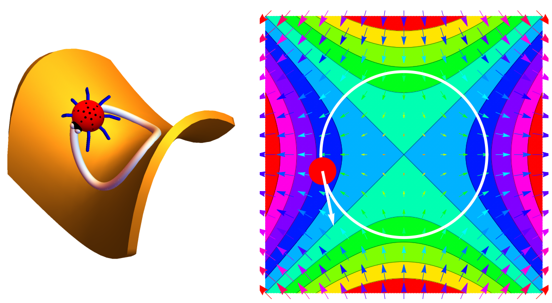
As $h \to 0$, we also have $H \to 0$ and the first part goes to $f'(x(t))$ and the second factor to $x'(t)$.

11.4. The chain rule is powerful because it implies other differentiation rules like the addition, product and quotient rule in one dimensions: $f(x,y) = x + y, x = u(t), y = v(t), d/dt(x+y) = f_x u' + f_y v' = u' + v'$.
$f(x,y) = xy, x = u(t), y = v(t), d/dt(xy) = f_x u' + f_y v' = vu' + uv'$.
$f(x,y) = x/y, x = u(t), y = v(t), d/dt(x/y) = f_x u' + f_y v' = u'/y - v'u/v^2$.

11.5. As in one dimensions, the chain rule follows from linearization. If f is a linear function $f(x,y) = ax + by - c$ and if the curve $\vec{r}(t) = [x_0 + tu, y_0 + tv]^T$ parametrizes a line. Then $\frac{d}{dt} f(\vec{r}(t)) = \frac{d}{dt}(a(x_0 + tu) + b(y_0 + tv)) = au + bv$ and this is the dot product of $\nabla f = (a,b)$ with $\vec{r}\,'(t) = (u,v)$. Since the chain rule only refers to the derivatives of the functions which agree at the point, the chain rule is also true for general functions.

Examples

11.6. A ladybug moves on a circle $\vec{r}(t) = [\cos(t), \sin(t)]^T$ on a table with temperature distribution $f(x,y) = x^2 - y^3$. Find the rate of change of the temperature $\nabla f(x,y) = (2x, -3y^2)$, $\vec{r}'(t) = (-\sin(t), \cos(t))$ $d/dt f(\vec{r}(t)) = \nabla T(\vec{r}(t)) \cdot \vec{r}'(t) = (2\cos(t), -3\sin(t)^2) \cdot (-\sin(t), \cos(t)) = -2\cos(t)\sin(t) - 3\sin^2(t)\cos(t)$.



11.7. From $f(x,y) = 0$, one can express y as a function of x, at least near a point where f_y is not zero. From $d/df(x, y(x)) = \nabla f \cdot (1, y'(x)) = f_x + f_y y' = 0$, we obtain $y' = -f_x/f_y$. Even so, we do not know $y(x)$, we can compute its derivative! Implicit differentiation works also in three variables. The equation $f(x,y,z) = c$ defines a surface. Near a point where f_z is not zero, the surface can be described as a graph $z = z(x,y)$. We can compute the derivative z_x without actually knowing the function $z(x,y)$. To do so, we consider y a fixed parameter and compute, using the chain rule

$$f_x(x,y,z(x,y))1 + f_z(x,y) z_x(x,y) = 0$$

so that $z_x(x,y) = -f_x(x,y,z)/f_z(x,y,z)$. This works at points where f_z is not zero.

11.8. The surface $f(x,y,z) = x^2 + y^2/4 + z^2/9 = 6$ is an ellipsoid. Compute $z_x(x,y)$ at the point $(x,y,z) = (2,1,1)$.
Solution: $z_x(x,y) = -f_x(2,1,1)/f_z(2,1,1) = -4/(2/9) = -18$.

Homework

This homework is due on Tuesday, 7/16/2019.

Problem 11.1: You know that $d/dt f(\vec{r}(t)) = 25$ at $t = 0$ if $\vec{r}(t) = [t, t]^T$ and $d/dt f(\vec{r}(t)) = 11$ at $t = 0$. $\vec{r}(t) = [t, -t]^T$. Find the gradient of f at $(0, 0)$.

Problem 11.2: The pressure in the space at the position (x, y, z) is $p(x, y, z) = x^2 + y^2 - z^3$ and the trajectory of an observer is the curve $\vec{r}(t) = [t, t, 1/t]^T$. Using the chain rule, compute the rate of change of the pressure the observer measures at time $t = 2$.

Problem 11.3: The chain rule is closely related to linearization. Lets get back to linearlization a bit: A farm costs $f(x, y)$, where x is the number of cows and y is the number of ducks. There are 10 cows and 20 ducks and $f(10, 20) = 1000000$. We know that $f_x(x, y) = 2x$ and $f_y(x, y) = y^2$ for all x, y. Estimate $f(12, 19)$.

Here is a song out of this:
"Old MacDonald had a million dollar farm, E-I-E-I-O,
and on that farm he had $x = 10$ cows, E-I-E-I-O,
and on that farm he had $y = 20$ ducks, E-I-E-I-O,
with $f_x = 2x$ here and $f_y = y^2$ there,
and here two cows more, and there a duck less,
how much does the farm cost now, E-I-E-I-O?"

Problem 11.4: Find, using implicit differentiation the derivative $d/dx\, \text{arctanh}(x)$, where

$$\tanh(x) = \sinh(x)/\cosh(x) .$$

The **hyperbolic sine** and **hyperbolic cosine** are defined as are $\sinh(x) = (e^x - e^{-x})/2$ and $\cosh(x) = (e^x + e^{-x})/2$. We have $\sinh' = \cosh$ and $\cosh' = \sinh$ and $\cosh^2(x) - \sinh^2(x) = 1$.

Problem 11.5: The equation $f(x, y, z) = e^{xyz} + z = 1 + e$ implicitly defines z as a function $z = g(x, y)$ of x and y. Find formulas (in terms of x,y and z) for $g_x(x, y)$ and $g_y(x, y)$. Estimate $g(1.01, 0.99)$ using linear approximation.

MULTIVARIABLE CALCULUS

MATH S-21A

Unit 12: Tangent spaces

Lecture

12.1. The notion of **gradient** is the derivative of a scalar function of many variables. It produces a vector. This vector is useful for example to compute tangent lines or tangent planes.

Definition: The **gradient** of a function $f(x, y)$ is defined as

$$\nabla f(x,y) = [f_x(x,y), f_y(x,y)]^T .$$

For functions of three variables, define

$$\nabla f(x,y,z) = [f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)]^T .$$

12.2. The symbol ∇ is spelled "Nabla" and named after an Egyptian or Assyrian harp. Early on, the name "Atled" was suggested. But the textbook of 1901 of Gibbs used Nabla was too persuasive. Here is a very important fact, which is true in any dimension. We only formulate it in dimension 2:

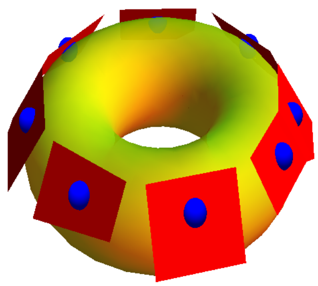
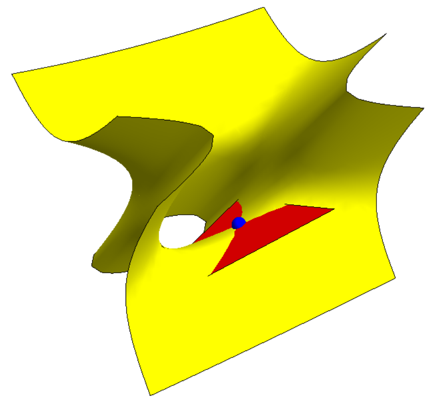
Theorem: **Gradient Theorem:** $\nabla f(x_0, y_0)$ is perpendicular to the level curve $\{(x, y) \mid f(x, y) = c\}$ containing (x_0, y_0).

Proof. Every curve $\vec{r}(t)$ on the level curve or level surface satisfies $\frac{d}{dt} f(\vec{r}(t)) = 0$. By the chain rule, $\nabla f(\vec{r}(t))$ is perpendicular to the tangent vector $\vec{r}'(t)$. QED.

12.3. Because $\vec{n} = \nabla f(p, q) = [a, b]^T$ is perpendicular to the level curve $f(x, y) = c$ through (p, q), the equation for the tangent line is $ax+by = d$, $a = f_x(p, q)$, $b = f_y(p, q)$, $d = ap + bq$. Compactly written, this is

$$\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0$$

and means that the gradient of f is perpendicular to any vector $(\vec{x} - \vec{x}_0)$ in the plane. It is one of the most important statements in multivariable calculus as it gives a crucial link between calculus and geometry. The just mentioned gradient theorem is also useful. We can immediately compute tangent planes and tangent lines, without linearization!

Definition: If f is a function of several variables and $\vec{v}$ is a unit vector then $D_{\vec{v}} f = \nabla f \cdot \vec{v}$ is called the **directional derivative** of f in the direction $\vec{v}$.

The name "directional derivative" is related to the fact that every unit vector gives a direction. If $\vec{v}$ is a unit vector, then the chain rule tells us $\frac{d}{dt} D_{\vec{v}} f = \frac{d}{dt} f(x + t\vec{v})$.

The directional derivative tells us how the function changes when we move in a given direction. Assume for example that $T(x, y, z)$ is the temperature at position (x, y, z). If we move with velocity $\vec{v}$ through space, then $D_{\vec{v}} T$ tells us at which rate the temperature changes for us. If we move with velocity $\vec{v}$ on a hilly surface of height $h(x, y)$, then $D_{\vec{v}} h(x, y)$ gives us the slope we drive on.

12.4. If $\vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and the speed is 1, then $D_{\vec{r}'(t)} f = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is the temperature change, one measures at $\vec{r}(t)$. The chain rule told us that this is $d/dt f(\vec{r}(t))$.

12.5. For $\vec{v} = (1, 0, 0)$, then $D_{\vec{v}} f = \nabla f \cdot v = f_x$, the directional derivative is a generalization of the partial derivatives. It measures the rate of change of f, if we walk with unit speed into that direction. But as with partial derivatives, it is a **scalar**.

12.6. The directional derivative satisfies $\boxed{|D_{\vec{v}} f| \leq |\nabla f||\vec{v}|}$ because

$$\nabla f \cdot \vec{v} = |\nabla f||\vec{v}||\cos(\phi)| \leq |\nabla f||\vec{v}| .$$

Definition: The direction $\vec{v} = \nabla f / |\nabla f|$ is the direction, where f **increases** most. It is the direction of **steepest ascent**.

12.7. If $\vec{v} = \nabla f / |\nabla f|$, then the directional derivative is $\nabla f \cdot \nabla f / |\nabla f| = |\nabla f|$. This means f **increases**, if we move into the direction of the gradient. The slope in that direction is $|\nabla f|$.

Definition: If f is a function of several variables and $\vec{v}$ is a unit vector then $D_{\vec{v}} f = \nabla f \cdot \vec{v}$ is called the **directional derivative** of f in the direction $\vec{v}$.

12.8. The name "directional derivative" is related to the fact that every unit vector gives a direction. If $\vec{v}$ is a unit vector, then the chain rule tells us $\frac{d}{dt} D_{\vec{v}} f = \frac{d}{dt} f(x + t\vec{v})$. The directional derivative tells us how the function changes when we move in a given direction. Assume for example that $T(x, y, z)$ is the temperature at position (x, y, z). If we move with velocity $\vec{v}$ through space, then $D_{\vec{v}} T$ tells us at which rate the temperature changes for us. If we move with velocity $\vec{v}$ on a hilly surface of height $h(x, y)$, then $D_{\vec{v}} h(x, y)$ gives us the slope we drive on.

12.9. If $\vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and the speed is 1, then $D_{\vec{r}'(t)} f = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is the temperature change, one measures at $\vec{r}(t)$. The chain rule told us that this is $d/dt f(\vec{r}(t))$.

12.10. For $\vec{v} = (1, 0, 0)$, then $D_{\vec{v}} f = \nabla f \cdot v = f_x$, the directional derivative is a generalization of the partial derivatives. It measures the rate of change of f, if we walk with unit speed into that direction. But as with partial derivatives, it is a **scalar**.

12.11. The directional derivative satisfies $\boxed{|D_{\vec{v}} f| \leq |\nabla f||\vec{v}|}$ because $\nabla f \cdot \vec{v} = |\nabla f||\vec{v}|| \cos(\phi)| \leq |\nabla f||\vec{v}|$.

Definition: The direction $\vec{v} = \nabla f / |\nabla f|$ is the direction, where f **increases** most. It is the direction of **steepest ascent**.

12.12. If $\vec{v} = \nabla f / |\nabla f|$, then the directional derivative is $\nabla f \cdot \nabla f / |\nabla f| = |\nabla f|$. This means f **increases**, if we move into the direction of the gradient. The slope in that direction is $|\nabla f|$.

12.13. The directional derivative has the same properties than any derivative: $D_v(\lambda f) = \lambda D_v(f)$, $D_v(f + g) = D_v(f) + D_v(g)$ and $D_v(fg) = D_v(f)g + f D_v(g)$.
We will see later that points with $\nabla f = \vec{0}$ are candidates for **local maxima** or **minima** of f. Points (x, y), where $\nabla f(x, y) = (0, 0)$ are called **critical points** and help to understand the function f.

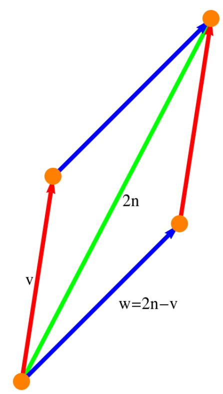
Examples

12.14. Compute the tangent plane to the surface $3x^2 y + z^2 - 4 = 0$ at the point $(1, 1, 1)$. **Solution:** $\nabla f(x, y, z) = [6xy, 3x^2, 2z]^T$. And $\nabla f(1, 1, 1) = [6, 3, 2]^T$. The plane is $6x + 3y + 2z = d$ where d is a constant. We can find the constant d by plugging in a point and get $6x + 3y + 2z = 11$.

12.15. Problem: reflect the ray $\vec{r}(t) = [1 - t, -t, 1]^T$ at the surface

$$x^4 + y^2 + z^6 = 6 \; .$$

Solution: $\vec{r}(t)$ hits the surface at the time $t = 2$ in the point $(-1, -2, 1)$. The velocity vector in that ray is $\vec{v} = [-1, -1, 0]^T$. The normal vector at this point is $\nabla f(-1, -2, 1) = [-4, -4, 6]^T = \vec{n}$. The reflected vector is $R(\vec{v} = 2\text{Proj}_{\vec{n}}(\vec{v}) - \vec{v}$. We have $\text{Proj}_{\vec{n}}(\vec{v}) = 8/68[-4, -4, 6]^T$. Therefore, the reflected ray is $\vec{w} = (4/17)[-4, -4, 6]^T - [-1, -1, 0]^T$.



12.16. You are on a trip in a air-ship over Cambridge at $(1,2)$ and you want to avoid a thunderstorm, a region of low pressure. The pressure is given by a function $p(x,y) = x^2 + 2y^2$. In which direction do you have to fly so that the pressure change is largest? **Solution:** The gradient $\nabla p(x,y) = [2x, 4y]^T$ at the point $(1,2)$ is $[2,8]^T$. Normalize to get the direction $[1,4]^T/\sqrt{17}$.

12.17. The "Dom" is a mountain in Switzerland with an altitude of 4'545 meters. In suitable units on the ground, the height $f(x,y)$ is approximated by the quadratic function $f(x,y) = 4000 - x^2 - y^2$. At height $f(-10,10) = 3800$, at the point $(-10,10,3800)$, you rest. The climbing route continues into the south-east direction $v = [1,-1]^T/\sqrt{2}$. Calculate the rate of change in that direction. We have $\nabla f(x,y) = [-2x,-2y]^T$, so that $[20,-20]^T \cdot [1,-1]^T/\sqrt{2} = 40/\sqrt{2}$. This is a place, with a ladder, where you climb $40/\sqrt{2}$ meters up when advancing 1m forward. The rate of change in all directions is zero if and only if $\nabla f(x,y) = 0$: if $\nabla f \neq \vec{0}$, we can choose $\vec{v} = \nabla f/|\nabla f|$ and get $D_{\nabla f} f = |\nabla f|$.

Dom as seen from the Alp Salmenfee in Switzerland

12.18. Assume we know $D_v f(1,1) = 3/\sqrt{5}$ and $D_w f(1,1) = 5/\sqrt{5}$, where $v = [1,2]^T/\sqrt{5}$ and $w = [2,1]^T/\sqrt{5}$. Find the gradient of f. Note that we do not know anything else about the function f. **Solution:** Let $\nabla f(1,1) = [a,b]^T$. We know $a + 2b = 3$ and $2a + b = 5$. This allows us to get $a = 7/3, b = 1/3$.

Homework

This homework is due on Tuesday, 7/16/2019.

Problem 12.1: Find the directional derivative $D_{\vec{v}}f(2,1) = \nabla f(2,1) \cdot \vec{v}$ into the direction $\vec{v} = [3,-4]^T/5$ for the function $f(x,y) = 7+x^5y+y^3+y$.

Problem 12.2: A surface $x^2+y^2-z=1$ radiates light away. It can be parametrized as $\vec{r}(x,y) = [x,y,x^2+y^2-1]^T$. Find the parametrization of the wave front $\vec{r}(x,y)+\vec{n}(x,y)$, which is distance 1 from the surface. Here $\vec{n}$ is a unit vector normal to the surface.

Problem 12.3: Assume $f(x,y) = 1-x^2+y^2$. Compute the directional derivative $D_{\vec{v}}(x,y)$ at $(0,0)$, where $\vec{v} = [\cos(t),\sin(t)]^T$ is a unit vector. Now compute

$$D_v D_v f(x,y)$$

at $(0,0)$, for any unit vector. For which values t is this **second directional derivative** positive?

Problem 12.4: The **Kitchen-Rosenberg formula** gives the curvature of a level curve $f(x,y)=c$ as

$$\kappa = \frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{(f_x^2+f_y^2)^{3/2}}$$

Use this formula to find the curvature of the ellipse $f(x,y) = x^2+2y^2 = 1$ at the point $(1,0)$.

This formula is useful in computer vision. If you want to derive the formula, you can check that the angle

$$g(x,y) = \arctan(f_y/f_x)$$

of the gradient vector has κ as the directional derivative in the direction $\vec{v} = [-f_y, f_x]^T/\sqrt{f_x^2+f_y^2}$ tangent to the curve.

Problem 12.5: One can find the maximum of a function numerically by moving in the direction of the gradient. This is called the **steepest ascent method**. You start at a point (x_0,y_0) then move in the direction of the gradient for some time c to be at $(x_1,y_1) = (x_0,y_0)+c\nabla f(x_0,y_0)$. Repeat to $(x_2,y_2) = (x_1,y_1)+c\nabla f(x_1,y_1)$ etc. It can be a bit difficult if the function has a flat ridge like in the **Rosenbrock function**

$$f(x,y) = 1-(1-x)^2-100(y-x^2)^2 .$$

Plot the contour map of this function on $-0.6 \leq x \leq 1, -0.1 \leq y \leq 1.1$, then and find the directional derivative at $(1/5,0)$ in the direction $(1,1)/\sqrt{2}$.