MULTIVARIABLE CALCULUS

MATH S-21A

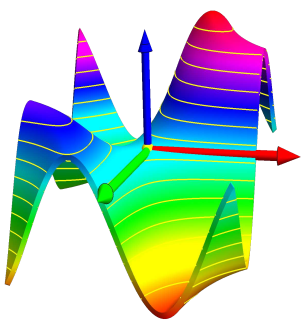
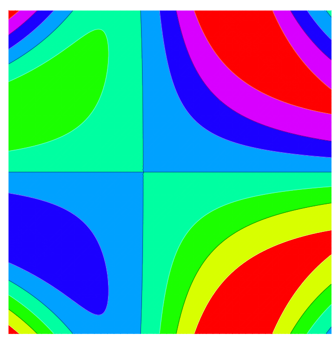
Unit 5: Functions

Lecture

5.1.

Definition: A **function of two variables** $f(x,y)$ is a rule which assigns to two numbers x, y a third number $f(x,y)$.

The function $f(x,y) = x^5y - 2y^2$ for example assigns to $(2,3)$ the number $96 - 18 = 78$.

5.2. In general, a function is assumed to be defined for all points (x,y) in $\mathbb{R}^2$. An example is $f(x,y) = |x|^7 + e^{\sin(xy)}$. Sometimes however it is required to restrict the function to a **domain** D in the plane. For example, if $f(x,y) = \log|y| + \sqrt{x}$, then (x,y) is only defined for $x > 0$ and for $y \neq 0$. The **range** of a function f is the set of values which the function f reaches. The function $f(x,y) = 3 + x^2/(1+x^2)$ for example takes all values $3 \leq z < 4$. While $z = 3$ is reached for $x = y = 0$, the value $z = 4$ is not attained.

5.3.

Definition: The set $\{(x,y,f(x,y)) \mid (x,y) \in D\} \subset \mathbb{R}^3$ is the **graph** of f.

Graphs are **surfaces** which allow to see the function visually. It is important not to mix up the graph of a function with the function itself. The function is a **rule** which assigns to (x,y) a third number, the graph is a **geometric object** in three dimensional space. The modern point of view of functions only came at the beginning of the 19th century. It might look irrelevant for you but it is like in computer science, where we distinguish different data structures, even so one can bring one into the other.

5.4. Here are some examples

Example $f(x,y)$	domain D of f	range = $f(D)$ of f
$\sqrt{9-x^2-y^2}$	closed disc $x^2+y^2 \leq 9$	$[0,3]$
$-\log(1-x^2-y^2)$	open unit disc $x^2+y^2<1$	$(0,\infty)$
$f(x,y)=x^2+y^3-xy+\cos(xy)$	plane R^2	the real line
$\sqrt{4-x^2-2y^2}$	$x^2+2y^2 \leq 4$	$[0,2]$
$1/(x^2+y^2-1)$	all except unit circle	$R \setminus (1,0]$
$1/(x^2+y^2)^2$	all except origin	positive real axis

5.5.

Definition: The set $\{(x,y) \mid f(x,y)=c=\text{const}\ \}$ is called a **contour curve** or **level curve** of f. A collection of contour curves is a **contour map**.

For example, for $f(x,y)=4x^2+3y^2$, the level curves $f=c$ are **ellipses** if $c>0$. Drawing several contour curves $\{f(x,y)=c\ \}$ simultaneously produces a **contour map** of the function f.

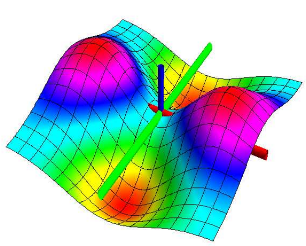
5.6. Level curves allow to visualize and analyze functions $f(x,y)$ without leaving the two dimensional space. The picture below for example shows the level curves of the function $\sin(xy)-\sin(x^2+y)$. Contour curves are everywhere: they appear as **isobars**=curves of constant pressure, or **isoclines**= curves of constant (wind) field direction, **isothermes**= curves of constant temperature or **isoheights** =curves of constant height.

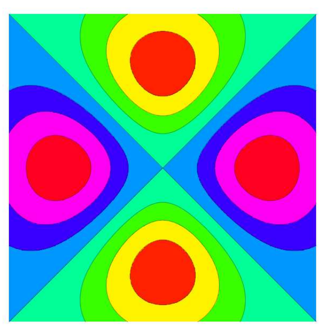
Examples

5.7. For $f(x,y)=x^2-y^2$, the set $x^2-y^2=0$ is the union of the lines $x=y$ and $x=-y$. The set $x^2-y^2=1$ consists of two hyperbola with with their "noses" at the point $(-1,0)$ and $(1,0)$. The set $x^2-y^2=-1$ consists of two hyperbola with their noses at $(0,1)$ and $(0,-1)$.

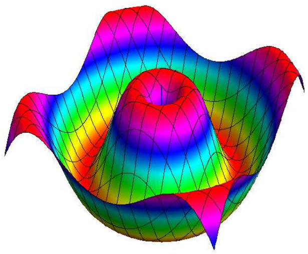
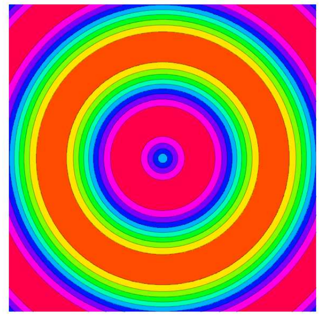
5.8. The function $f(x,y)=1-2x^2-y^2$ has contour curves $f(x,y)=1-2x^2+y^2=c$ which are ellipses $2x^2+y^2=1-c$ for $c<1$.

5.9. For the function $f(x,y)=(x^2-y^2)e^{-x^2-y^2}$, we can not find explicit expressions for the contour curves $(x^2-y^2)e^{-x^2-y^2}=c$. We can draw the traces curve $(x,0,f(x,0))$ or $(0,y,f(0,y)$ or then use a computer:





5.10. The surface $z = f(x,y) = \sin(\sqrt{x^2+y^2})$ has concentric circles as contour curves.

5.11. In applications, discontinuous functions can occur. The temperature of water in relation to pressure and volume is an example. One experiences then **phase transitions**, places where the function value can jumps. Mathematicians study singularities in a mathematical field called "catastrophe theory".

5.12.

Definition: A function $f(x,y)$ is called **continuous** at (a,b) if there is a finite value $f(a,b)$ with $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. This means that for any sequence (x_n, y_n) converging to (a,b), also $f(x_n, y_n) \to f(a,b)$. A function is **continuous** in $G \subset \mathbb{R}^2$ if it is continuous at every point (a,b) of G.

5.13. Continuity means that if (x,y) is close to (a,b), then $f(x,y)$ must be close to $f(a,b)$. Continuity for functions of more than two variables is defined in the same way. The bad news is that continuity is not always easy to check. The good news is that in general we do not have to worry about continuity. Lets look at some examples:

5.14. Example: For $f(x,y) = (xy)/(x^2+y^2)$, we have

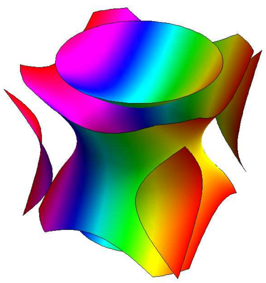
$$\lim_{(x,x)\to(0,0)} f(x,x) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

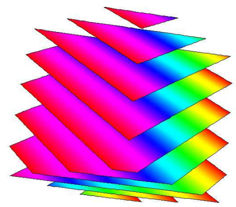
and $\lim_{(x,0)\to(0,0)} f(0,x) = \lim_{(x,0)\to(0,0)} 0 = 0$. The function is not continuous at $(0,0)$.

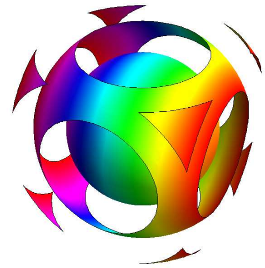
5.15. Example: The function $f(x,y) = (x^2y)/(x^2+y^2)$ is better described using polar coordinates: $f(r,\theta) = r^3\cos^2(\theta)\sin(\theta)/r^2 = r\cos^2(\theta)\sin(\theta)$. We see that $f(r,\theta) \to 0$ uniformly in θ if $r \to 0$. The function is continuous as we can extend it and extend the value to $f(0,0) = 0$. It is custom in mathematics to consider the above function **to be continuous**. The reason is that there is a **unique way** to give a function value at the undefined point.

5.16. A simpler example: the function $f(x,y) = (x^2-y^2)/(x+y)$ is continuous everywhere. Yes, the function is not defined a priori at $x+y=0$ but as it is outside this line equal to $f(x,y) = x-y$, there is a unique continuation to the entire plane and this continuation is $x-y$ whi

5.17. A function of three variables $g(x,y,z)$ assigns to three variables x,y,z a real number $g(x,y,z)$. The function $f(x,y,z) = x^2+y-z$ for example satisfies $f(3,2,1) = 10$. We can visualize a function by **contour surfaces** $g(x,y,z) = c$, where c is constant. It is an **implicit description** of the surface. The contour surface of $g(x,y,z) = x^2+y^2+z^2 = c$ is a sphere if $c > 0$. To understand a contour surface, it is helpful to look at the **traces**, the intersections of the surfaces with the coordinate planes $x = 0, y = 0$ or $z = 0$.



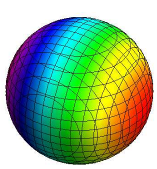




5.18. The function $g(x,y,z) = 2+\sin(xyz)$ could define a temperature distribution in space. We can no more draw the graph of g because that would be an object in 4 dimensions. We can however draw surfaces like $g(x,y,z) = 0$.

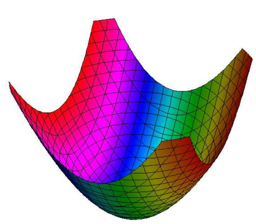
5.19. The level surfaces of $g(x,y,z) = x^2+y^2+z^2$ are spheres. The level surfaces of $g(x,y,z) = 2x^2+y^2+3z^2$ are ellipsoids. The equation $ax+by+cz = d$ is a plane. With $\vec{n} = [a,b,c]^T$ and $\vec{x} = [x,y,z]^T$, we can rewrite the equation $\vec{n}\cdot\vec{x} = d$. If a point $\vec{x}_0$ is on the plane, then $\vec{n}\cdot\vec{x}_0 = d$. so that $\vec{n}\cdot(\vec{x}-\vec{x}_0) = 0$. This means that every vector $\vec{x}-\vec{x}_0$ in the plane is orthogonal to $\vec{n}$. For $f(x,y,z) = ax^2+by^2+cz^2+dxy+exz+fyz+gx+hy+kz+m$ the surface $f(x,y,z) = 0$ is called a **quadric**.

Sphere



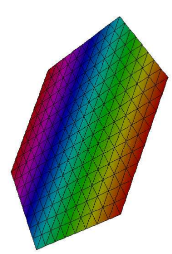
$(x-a)^2+(y-b)^2+(z-c)^2 = r^2$

Paraboloid



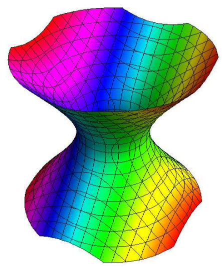
$(x-a)^2+(y-b)^2-c = z$

Plane



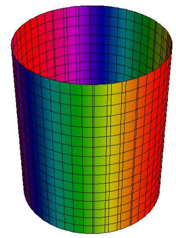
$ax+by+cz = d$

One sheeted hyperboloid



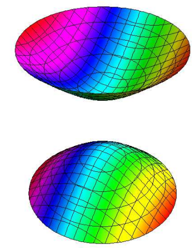
$$(x-a)^2+(y-b)^2-(z-c)^2=r^2$$

Cylinder



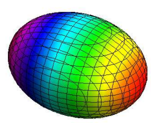
$$(x-a)^2+(y-b)^2=r^2$$

Two sheeted hyperboloid



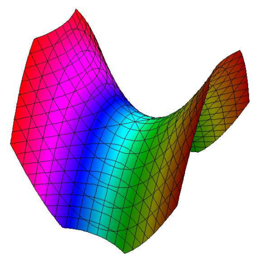
$$(x-a)^2+(y-b)^2-(z-c)^2=-r^2$$

Ellipsoid



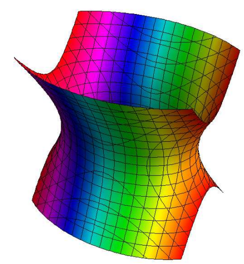
$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

Hyperbolic paraboloid



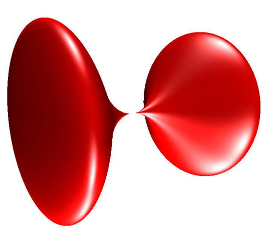
$$x^2-y^2+z=1$$

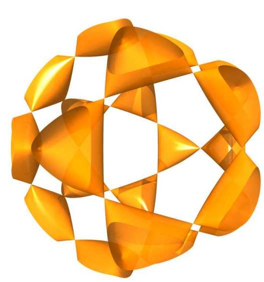
Elliptic hyperboloid

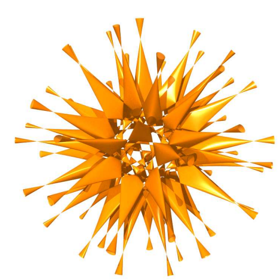


$$x^2/a^2+y^2/b^2-z^2/c^2=1$$

Problem: Higher order polynomial surfaces can be intriguingly beautiful and are sometimes difficult to describe. If f is a polynomial in several variables and $f(x,x,x)$ is a polynomial of degree d, then f is called a **degree d polynomial surface**. Degree 2 surfaces are **quadrics**, degree 3 surfaces **cubics**, degree 4 surfaces **quartics**, degree 5 surfaces **quintics**, degree 10 surfaces **decics** and so on.







Homework

This homework is due on Tuesday, 7/9/2019.

Problem 5.1: Plot the graph of the function $f(x, y) = \sin(x^2/3) - \sin(y^2/3)$ on the region $-\pi \leq x \leq \pi, -\pi \leq y \leq \pi$. How many mountain peaks do you count inside that region? How many sinks (local minima) do you count? Do not look at points on the boundary. You can do it by hand, or Wolfram alpha or Mathematica or a graphing calculator to check your picture. Do not compute the extrema analytically yet. We will do that later. This is a graphing problem.

Problem 5.2: a) Determine the domain and range of the **logarithmic mean**

$$f(x, y) = \frac{(y - x)}{\log(y) - \log(x)}$$

where log the natural logarithm.
b) The function is not defined at $x = y$ but one can define $f(x, y)$ on the diagonal $x = y$. Use Hôpital, show that the limit $\lim_{x \to 2} f(x, 2)$ exists.
c) The function is also not defined at first if $x = 0$ or $y = 0$. Show that the limit $\lim_{x \to 0} f(x, 2)$ exists.

Problem 5.3: a) Use the computer to draw the level surface $x^2 - y^2 + z^2 - x^4y^4z^4 - x^2y^2z^2 = 0$ with x, y, z all in $[-2, 2]$
b) Do the same for the $((x^2 + y^2)^2 - x^2 - y^2)^2 + z^2 = 0.05$ with with x, y, z all in $[-2, 2]$

Problem 5.4: a) Sketch the graph and contour map of
$f(x, y) = \cos(1 + 2x^2 + y^2)/(1 + 2x^2 + y^2)$.
b) Sketch the contour map of $g(x, y) = 2|x| - 5|y|$.
c) Sketch the contour map of $h(x, y) = (x^2 + y^2)^2 - x^2 + y^2$.

Problem 5.5: a) Verify that the line $\vec{r}(t) = [1, 3, 2]^T + t[1, 2, 1]^T$ is part of the hyperbolic paraboloid $z^2 - x^2 - y = 0$.
b) Verify that the line $\vec{r}(t) = [1 + t, 1 - t, t]^T$ is part of the one sheeted hyperboloid $x^2 + y^2 - 2z^2 = 2$.
c) As also the line $\vec{r}(s) = [1 - s, 1 + s, s]^T$ is part of the same hyperboloid, what is the intersection of the hyperboloid with the plane $\vec{r}(t, s) = [1 + t - s, 1 - t + s, t + s]^T$?

MULTIVARIABLE CALCULUS

MATH S-21A

Unit 6: Parametrized Surfaces

LECTURE

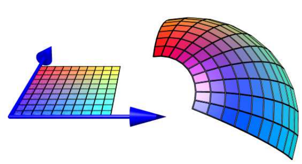
6.1. Surfaces can be described in two fundamentally different ways: first as **level surfaces** $g(x,y,z) = c$ and then through **parametrization**. What we have seen already for planes can be done more generally for other surfaces. Let us first look at the general setup:

Definition: A **parametrization** of a surface is a vector-valued function

$$\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)]^T ,$$

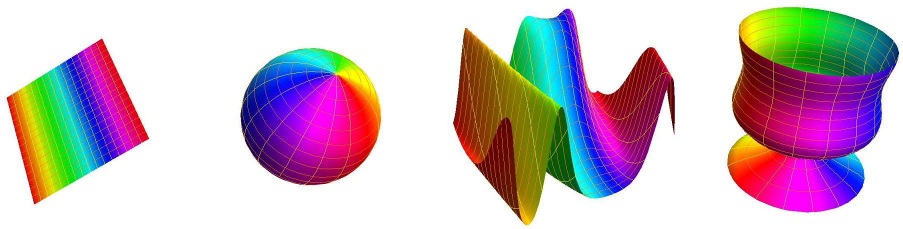
where $x(u,v), y(u,v), z(u,v)$ are three functions of two variables. The **parameters** u, v serve as coordinates on the surface. If we plug in concrete values like $u = 3, v = 2$ for example in a function $\vec{r}(u,v) = [u-2, v^2, u^3 - v]^T$, we get a concrete point $\vec{r}(3,2) = [1,4,25]^T$ in $\mathbb{R}^3$.

6.2. Because two parameters u and v are involved, the map $\vec{r}$ is also called uv**-map**. And like uv-light, it looks cool. If you like a fancy description, a parametrization is a map from $\mathbb{R}^2$ to $\mathbb{R}^3$. A **parametrized surface** is the image of the uv-map. The domain R of the uv-map is called the **parameter domain**. The parametrization is what you are **doing**, the surface itself is something you **see**. There are many different parametrizations of the same surface.



Definition: If the first parameter u is kept constant, then $v \mapsto \vec{r}(u,v)$ is a curve on the surface. Similarly, if v is constant, then $u \mapsto \vec{r}(u,v)$ traces a curve the surface. These curves are called **grid curves**.

Parametric surfaces can become complex. In that case, it is better to explored them with the help of a computer. The following four examples are important building blocks for more general surfaces.



Definition: A point $(x, y) \neq (0, 0)$ in the plane has the **polar coordinates** $r = \sqrt{x^2 + y^2}, \theta$, where θ is the angle from the positive x-axes to the point in counter clockwise direction. For $x > 0, y > 0$ it is $\arctan(y/x)$. In general $(x, y) = (r\cos(\theta), r\sin(\theta))$. A common choice is to take $\theta \in [0, 2\pi)$. The point $((0, -1)$ has then the polar coordinates $(r, \theta) = (1, 3\pi/2)$.

6.3. Note that the formula $\theta = \arctan(y/x)$ defines the angle θ only up to an addition of an integer multiple of π. The points $(1, 2)$ and $(-1, -2)$ for example have the same θ value. In order to get the correct θ value one can could take $\arctan(y/x)$ in $(-\pi/2, \pi/2]$, where $\pi/2$ is the limit when $x \to 0^+$, then add π if $x < 0$ or if $x = 0$ and $y < 0$.

Definition: The coordinate system obtained by representing points in space as

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$

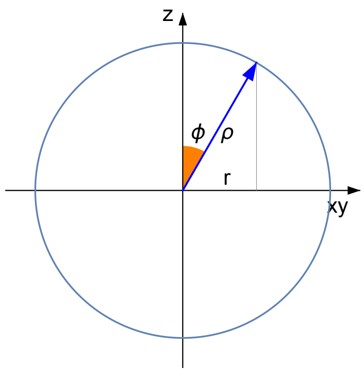
is called the **cylindrical coordinate system**.

Definition: **Spherical coordinates** use the distance ρ to the origin as well as two angles θ and ϕ called **Euler angles**. The first angle θ is the angle we have used in polar coordinates. The second angle, ϕ, is the angle between the vector $\vec{OP}$ and the z-axis. A point has the **spherical coordinate**

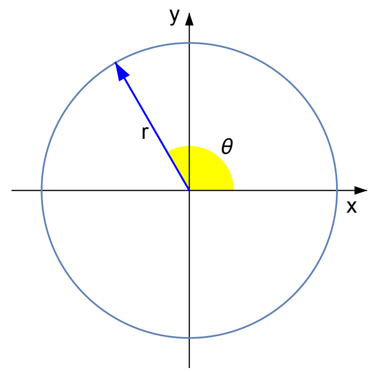
$$(x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) \ .$$

We always use $0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi, \rho \geq 0$.

The following figures allow you to derive the formulas. The distance to the z axes is $r = \rho\sin(\phi)$ and the height $z = \rho\cos(\phi)$ can be read off by the left picture, the coordinates $x = r\cos(\theta), y = r\sin(\theta)$ can be seen in the right picture.



$$x = \rho\cos(\theta)\sin(\phi),$$
$$y = \rho\sin(\theta)\sin(\phi),$$
$$z = \rho\cos(\phi)$$



Examples

6.4. A plane has the parametrization $\vec{r}(s,t) = \vec{OP} + s\vec{v} + t\vec{w}$ and the implicit equation $ax + by + cz = d$. To get from parametric to implicit, find the normal vector $\vec{n} = \vec{v} \times \vec{w}$. To get from implicit to parametric, find two vectors $\vec{v}, \vec{w}$ normal to the vector $\vec{n}$. For example, find three points P, Q, R on the surface and form $\vec{u} = \vec{PQ}, \vec{v} = \vec{PR}$.

6.5. The **sphere** $\vec{r}(u,v) = [a,b,c]^T + [\rho\cos(u)\sin(v), \rho\sin(u)\sin(v), \rho\cos(v)]^T$ can be brought into the implicit form by finding the center and radius $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$.

6.6. The parametrization of a graph is $\vec{r}(u,v) = [u, v, f(u,v)]^T$. It can be written in implicit form as $z - f(x,y) = 0$.

6.7. The surface of revolution is in parametric form given as $\vec{r}(u,v) = [g(v)\cos(u), g(v)\sin(u), v]^T$. It has the implicit description $\sqrt{x^2+y^2} = r = g(z)$ which can be rewritten as $x^2 + y^2 = g(z)^2$.

6.8. Here are some level surfaces in cylindrical coordinates:

Problem: $r = 1$ is a **cylinder**, $r = |z|$ is a **double cone**, $r^2 = z$ **elliptic paraboloid**, $\theta = 0$ is a **half plane**, $r = \theta$ is a **rolled sheet of paper**.

Problem: $r = 2 + \sin(z)$ is an example of a **surface of revolution**.

6.9. Here are some level surfaces described in spherical coordinates:

Problem: $\rho = 1$ is a **sphere**, the surface $\phi = \pi/4$ is a **single cone**, $\rho = \phi$ is an **apple shaped surface** and $\rho = 2 + \cos(3\theta)\sin(\phi)$ is an example of a **bumpy sphere**.

Homework

This homework is due on Tuesday, 7/9/2019.

Problem 6.1: Find a parametrization for the plane which contains the three points $P = (4, 7, 1)$,$Q = (2, 2, 1)$ and $R = (1, 3, 5)$.

Problem 6.2: Plot the surface with the parametrization

$$\vec{r}(u, v) = [\cos(t)\sin(s), \sin(t)\sin(s) + (\cos(t)\sin(s))^4, \cos(s)/2]^T ,$$

where $0 \leq u \in \pi$ and $0 \leq v \leq 2\pi$. You can use technology if you like.

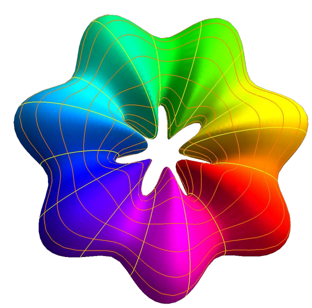
Problem 6.3: a) Find a parametrisations of the lower half of the ellipsoid $x^2/25 + y^2/16 + (z - 3)^2 = 1, z \geq 3$ by using that the surface is a graph $z = f(x, y)$ on a suitable domain.
b) Find a second parametrization but use angles ϕ, θ similarly as for the sphere.

Problem 6.4: Find a parametrisation of the **torus candy** (www.math-candy.com) given as the set of points which have distance $6 + 2\cos(7\theta)$ from the circle

$$[10\cos(\theta), 10\sin(\theta), 0]^T ,$$

where θ is the angle occurring in cylindrical and spherical coordinates. We can assure you that the candy melts wonderfully on your tongue.

Hint: Use r, the distance of a point $(x, y,)$ to the z-axis. This distance is $r = (10 + (6 + 2\cos(7\theta))\cos(\psi))$ if ψ is the angle for the circle winding around the candy. You can also use that $z = (6 + 2\cos(7\theta))\sin(\psi)$. To finish the parametrization problem, translate back to Cartesian coordinates.



Problem 6.5: a) What is the equation for the surface $x^2 + y^2 = 3x + z^2$ in cylindrical coordinates?
b) Describe in words or draw a sketch of the surface whose equation is $\rho = |\sin(4\phi)|$ in spherical coordinates (ρ, θ, ϕ).

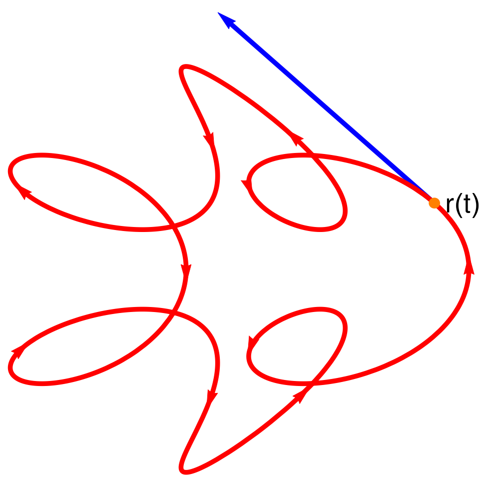
MULTIVARIABLE CALCULUS

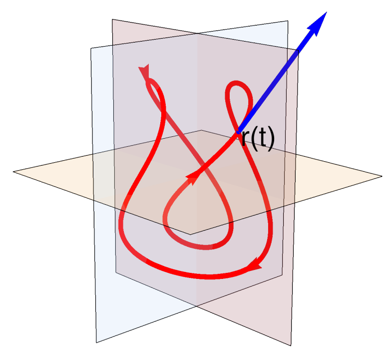
MATH S-21A

Unit 7: Parametrized curves

Lecture

Definition: A **parametrization** of a planar curve is a map $\vec{r}(t) = [x(t), y(t)]^T$ from a **parameter interval** $R = [a, b]$ to the plane. The functions $x(t)$ and $y(t)$ are called **coordinate functions**. The image of the parametrization is called a **parametrized curve** in the plane. Similarly, the parametrization of a **space curve** is $\vec{r}(t) = [x(t), y(t), z(t)]^T$. The image of $\vec{r}$ is called a **parametrized curve** in space.





7.1. We think of the **parameter** t as **time** and the parametrization as a **drawing process**. The curve is the result what you **see**. For a fixed time t, we have a vector $[x(t), y(t), z(t)]^T$ in space. As t varies, the end point of this vector moves along the curve. The parametrization contains **more information** about the curve then the curve itself. It tells for example how fast the curve was traced.

7.2. Curves can describe the paths of particles, celestial bodies, or other quantities which change in time. Examples are the motion of a star moving in a galaxy, or economical data changing in time. Here are some more places, where curves appear:

Strings or knots	are closed curves in space.
Molecules	like RNA or proteins.
Graphics:	grid curves produce a mesh of curves.
Typography:	fonts represented by Bézier curves.
Relativity:	curve in space-time describes the motion of an object
Topology:	space filling curves, boundaries of surfaces or knots.

Definition: Any vector parallel to the velocity $\vec{r}\,'(t)$ is called **tangent** to the curve at $\vec{r}(t)$.

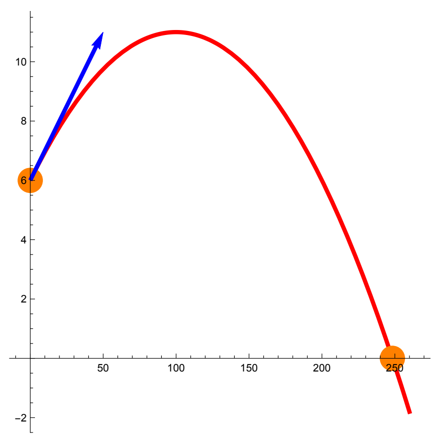
You know from single variable the **addition rule** $(f+g)' = f'+g'$, the **scalar multiplication rule** $(cf)' = cf'$ and the **Leibniz rule** $(fg)' = f'g + fg'$ as well as the **chain rule** $(f(g))' = f'(g)g'$. They generalize to vector-valued functions.

$(\vec{v}+\vec{w})' = \vec{v}' + \vec{w}'$, $(c\vec{v})' = c\vec{v}'$, $(\vec{v}\cdot\vec{w})' = \vec{v}'\cdot\vec{w} + \vec{v}\cdot\vec{w}'$ $(\vec{v}\times\vec{w})' = \vec{v}'\times\vec{w} + \vec{v}\times\vec{w}'$ $(\vec{v}(f(t)))' = \vec{v}'(f(t))f'(t)$.

The process of differentiation of a curve can be reversed using the **fundamental theorem of calculus**. If $\vec{r}'(t)$ and $\vec{r}(0)$ is known, we can figure out $\vec{r}(t)$ by **integration** $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}'(s)\, ds$.

Assume we know the acceleration $\vec{a}(t) = \vec{r}''(t)$ at all times as well as initial velocity and position $\vec{r}'(0)$ and $\vec{r}(0)$. Then $\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) + \vec{R}(t)$, where $\vec{R}(t) = \int_0^t \vec{v}(s)\, ds$ and $\vec{v}(t) = \int_0^t \vec{a}(s)\, ds$.

The **free fall** is the case when acceleration is constant. The direction of the constant force defines what is "down". If $\vec{r}''(t) = [0,0,-10]^T$, $\vec{r}'(0) = [0,1000,2]^T$, $\vec{r}(0) = [0,0,h]^T$, then $\vec{r}(t) = [0,1000t, h + 2t - 10t^2/2]^T$.

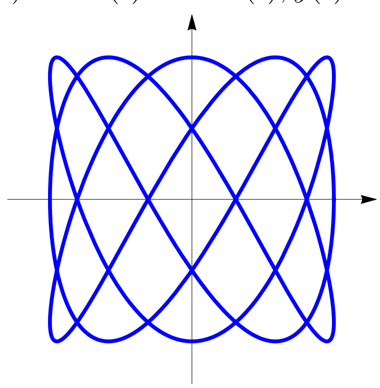
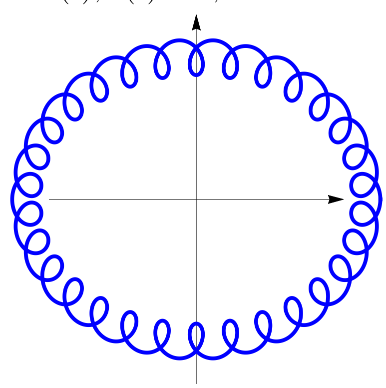
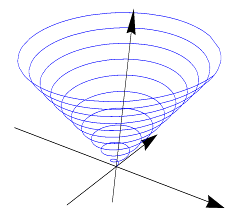


If $r''(t) = \vec{F}$ is constant, then $\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) - \vec{F}t^2/2$.

Examples

7.3. Examples:
1) The parametrization $\vec{r}(t) = [1+2\cos(t), 3+5\sin(t)]^T$ is the ellipse $(x-1)^2/4 + (y-3)^2/25 = 1$. The parametrization $\vec{r}(t) = [\cos(3t), \sin(5t)]^T$ is an example of a

Lissajous curve.
2) If $x(t) = t$, $y(t) = f(t)$, the curve $\vec{r}(t) = [t, f(t)]^T$ traces the **graph** of the function $f(x)$. For example, for $f(x) = x^2 + 1$, the graph is a parabola. 3) With $x(t) = t\cos(t), y(t) = t\sin(t), z(t) = t$ we get the parametrization of a **space curve** $\vec{r}(t) = [t\cos(t), t\sin(t), t]^T$ which traces a spiral on a cone $x^2 + y^2 = z^2$. 4) For $x(t) = 2t\cos(2t), y(t) = 2t\sin(2t), z(t) = 2t$ traces the same curve but twice as fast.
5) If $P = (a, b, c)$ and $Q = (u, v, w)$ are points in space, then $\vec{r}(t) = [a + t(u - a), b + t(v - b), c + t(w - c)]^T$ with $t \in [0, 1]$ is a **line segment** connecting P with Q. For example, $\vec{r}(t) = [1+t, 1-t, 2+3t]^T$ connects the points $P = (1, 1, 2)$ with $Q = (2, 0, 1)$.
6) For $x(t) = t\cos(t), y(t) = t\sin(t), z(t) = t$, then

The computation is done coordinate wise:

$$
\begin{array}{lll}
\text{Position} & \vec{r}\,(t) & = [\cos(3t), \sin(2t), 2\sin(t)]^T \\
\text{Velocity} & \vec{r}\,'(t) & = [-3\sin(3t), 2\cos(2t), 2\cos(t)]^T \\
\text{Acceleration} & \vec{r}\,''(t) & = [-9\cos(3t), -4\sin(2t), -2\sin(t)]^T \\
\text{Jerk} & \vec{r}\,'''(t) & = [27\sin(3t), 8\cos(2t), -2\cos(t)]^T
\end{array}
$$

7.4. Lets look at some examples of velocities and accelerations:

Example	**Velocity**
Hair growth:	0.000000005 m/s
Garden Snail	0.013 m/s
Signals in nerves:	40 m/s
Sound in air:	340 m/s
Speed of bullet:	1200-1500 m/s
Earth in solar system	30'000 m/s
Sun in galaxy:	200'000 m/s
Light in vacuum:	299'792'458 m/s

Example	**Acceleration**
Train:	0.1-0.3 m/s^2
Sprinter (100 m Dash):	3 m/s^2
Car:	3-8 m/s^2
Free fall:	1G = 9.81 m/s^2
Space X BFR:	4G m/s^2
Combat plane F35A:	9G m/s^2
Ejection from F35A:	14G m/s^2.
Electron in vacuum:	10^{15} m/s^2

Homework

This homework is due on Tuesday, 7/9/2019.

Problem 7.1: a) Sketch the plane curve

$$\vec{r}(t) = [x(t), y(t)]^T = [\cos(t) + \sin(2t), \sin(t) - \cos(2t)]^T ,$$

for $t \in [0, 2\pi]$ by plotting the points for different values of t. Calculate its velocity $\vec{r}\,'(t)$ as well as the acceleration $\vec{r}\,''(t)$ at $t = 0$.
b) Sketch the space curve

$$\vec{r}(t) = [(10 + 3\cos(17t))\cos(t), (10 + 3\cos(17t))\sin(t), 4t + 3\sin(17t)]^T$$

with $t \in [0, 5\pi]$.

Problem 7.2: Your cellphone app measures the acceleration

$$\vec{r}\,''(t) = [\cos(t), -\cos(9t), \sin(t)]^T$$

while you are riding a roller coaster. Assume you were at $(0, 0, 0)$ at time $t = 0$ with velocity $(1, 0, 0)$ at $t = 0$, what is its position $\vec{r}(t)$ at time t?

Problem 7.3: a) Two particles travel along space curves. The first is

$$\vec{r}_1(t) = [t, t^2, t^3]^T .$$

The second is

$$\vec{r}_2(t) = [1 + 2t, 1 + 6t, 1 + 14t]^T .$$

Do the particles collide? Do the particle paths intersect?
b) If $\vec{r}(t) = [\cos(t), 2\sin(t), 4t]^T$, find $\vec{r}\,'(0)$ and $\vec{r}\,''(0)$. Then compute $|\vec{r}\,'(0) \times \vec{r}\,''(0)|/|\vec{r}\,'(0)|^3$. We will later call this the curvature.

Problem 7.4: Find the parameterization $\vec{r}(t) = [x(t), y(t), z(t)]^T$ of the curve obtained by intersecting the elliptical cylinder $x^2/16 + y^2/25 = 1$ with the surface $z = x^2 y$. Find the velocity vector $\vec{r}\,'(t)$ at the time $t = \pi/2$.

Problem 7.5: Consider the curve

$$\vec{r}(t) = [x(t), y(t), z(t)]^T = [t^2, 1 + t, 1 + t^3]^T .$$

Check that it passes through the point $(1, 0, 0)$ and find the velocity vector $\vec{r}\,'(t)$, the acceleration vector $\vec{r}\,''(t)$ as well as the jerk vector $\vec{r}\,'''(t)$ at this point.

MULTIVARIABLE CALCULUS

MATH S-21A

Unit 8: Arc length and Curvature

Lecture

Definition: If $t \in [a,b] \mapsto \vec{r}(t)$ is a parametrized curve with velocity $\vec{r}\,'(t)$ and speed $|\vec{r}\,'(t)|$, then the number $L = \int_a^b |\vec{r}\,'(t)|\, dt$ is called the **arc length** of the curve.

We justify in class why this formula is reasonable if $\vec{r}$ is differentiable. Written out, the formula is $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$.

7.1. Because a parameter change $t = t(s)$ corresponds to a **substitution** in the integration which does not change the integral, we immediately see "path independence of arc length":

The arc length is independent of the parameterization of the curve.

Definition: Define the **unit tangent vector** $\vec{T}(t) = \vec{r}\,'(t)|/|\vec{r}\,'(t)|$ **unit tangent vector**.

Definition: The **curvature** of a curve at the point $\vec{r}(t)$ is defined as $\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}$.

7.2. The curvature is the length of the acceleration vector if $\vec{r}(t)$ parametrizes the curve with constant speed 1. A large curvature at a point means that the curve is strongly bent. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration. We can measure curvature at a point only if $\vec{r}\,'(t)$ is not zero.

The curvature does not depend on the parametrization.

Proof. Let $s(t)$ be an other parametrization, then by the chain rule $d/dtT'(s(t)) = T'(s(t))s'(t)$ and $d/dtr(s(t)) = r'(s(t))s'(t)$. We see that the s' cancels in T'/r'.

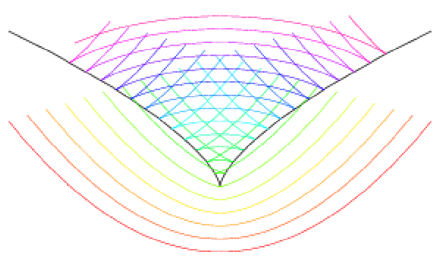
Especially, if the curve is parametrized by arc length, meaning that the velocity vector $r'(t)$ has length 1, then $\kappa(t) = |T'(t)|$. It measures the rate of change of the unit tangent vector.

Definition: If $\vec{r}(t)$ is a curve which has nonzero speed at t, then we can define $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, the **unit tangent vector**, $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$, the **normal vector** and $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ the **bi-normal vector**. The plane spanned by $\vec{N}$ and $\vec{B}$ is called the **normal plane**. It is perpendicular to the curve. The plane spanned by T and N is called the **osculating plane**.

7.3. If we differentiate $\vec{T}(t) \cdot \vec{T}(t) = 1$, we get $\vec{T}'(t) \cdot \vec{T}(t) = 0$ and see that $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$. Because $\vec{B}$ is automatically normal to $\vec{T}$ and $\vec{N}$, we have shown:

The three vectors $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ are unit vectors orthogonal to each other.

7.4. Here is an application of curvature: if a curve $\vec{r}(t)$ represents a **wave front** and $\vec{n}(t)$ is a **unit vector normal** to the curve at $\vec{r}(t)$, then $\vec{s}(t) = \vec{r}(t) + \vec{n}(t)/\kappa(t)$ defines a new curve called the **caustic** of the curve. Geometers call it the **evolute** of the original curve.



A useful formula for curvature is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

7.5. We prove this in class. Finally, lets mention that curvature is important also in **computer vision**. If the gray level value of a picture is modeled as a function $f(x, y)$ of two variables, places where the level curves of f have maximal curvature corresponds to **corners** in the picture. This is useful when **tracking** or **identifying** objects.

In this picture of John Harvard, software was looking for level curves for each color and started to draw at points where the curvature of the curves is large then follow the level curve.

Examples

Problem: The arc length of the **circle** $\vec{r}(t) = [R\cos(t), R\sin(t)]^T$ is $2\pi R$. The speed $|\vec{r}'(t)|$ is constant and equal to R.

Problem: The **helix** $\vec{r}(t) = [\cos(t), \sin(t), t]^T$ has velocity $\vec{r}\,'(t) = [-\sin(t), \cos(t), 1]^T$ and constant speed $|\vec{r}\,'(t)| = |[-\sin(t), \cos(t), 1]^T| = \sqrt{2}$.

Problem: What is the arc length of the curve

$$\vec{r}(t) = [\sqrt{2}t, \log(t), t^2/2]^T$$

for $1 \leq t \leq 2$? Answer: Because $\vec{r}\,'(t) = [\sqrt{2}, 1/t, t]^T$, we have $\vec{r}'(t) = \sqrt{2 + \frac{1}{t^2} + t^2} = |\frac{1}{t} + t|$ and $L = \int_1^2 \frac{1}{t} + t \; dt = \log(t) + \frac{t^2}{2}|_1^2 = \log(2) + 2 - 1/2$. This curve does not have a name. But because it is constructed in such a way that the arc length can be computed, we an call it "opportunity".

Problem: Find the arc length of the curve $\vec{r}(t) = [3t^2, 6t, t^3]^T$ from $t = 1$ to $t = 3$.

Problem: What is the arc length of the curve $\vec{r}(t) = [\cos^3(t), \sin^3(t)]^T$? Answer: We have $|\vec{r}\ '(t)| = 3\sqrt{\sin^2(t)\cos^4(t) + \cos^2(t)\sin^4(t)} = (3/2)|\sin(2t)|$. Therefore, $\int_0^{2\pi}(3/2)\sin(2t)\ dt = 6$.

Problem: Find the arc length of $\vec{r}(t) = [t^2/2, t^3/3]^T$ for $-1 \leq t \leq 1$. This cubic curve satisfies $y^2 = x^3 8/9$ and is an example of an **elliptic curve**. Because $\int x\sqrt{1+x^2}\ dx = (1+x^2)^{3/2}/3$, the integral can be evaluated as $\int_{-1}^1 |x|\sqrt{1+x^2}\ dx = 2\int_0^1 x\sqrt{1+x^2}\ dx = 2(1+x^2)^{3/2}/3|_0^1 = 2(2\sqrt{2}-1)/3$.

Problem: The arc length of an **epicycle** $\vec{r}(t) = [t + \sin(t), \cos(t)]^T$ parameterized by $0 \leq t \leq 2\pi$. We have $|\vec{r'}(t)| = \sqrt{2 + 2\cos(t)}$. so that $L = \int_0^{2\pi}\sqrt{2+2\cos(t)}\ dt$. A **substitution** $t = 2u$ gives $L = \int_0^{\pi}\sqrt{2+2\cos(2u)}\ 2du = \int_0^{\pi}\sqrt{2+2\cos^2(u) - 2\sin^2(u)}\ 2du = \int_0^{\pi}\sqrt{4\cos^2(u)}\ 2du = 4\int_0^{\pi}|\cos(u)|\ du = 8$.

Problem: Find the arc length of the **catenary** $\vec{r}(t) = [t, \cosh(t)]^T$, where $\cosh(t) = (e^t + e^{-t})/2$ is the **hyperbolic cosine** and $t \in [-1, 1]$. We have

$$\cosh^2(t)^2 - \sinh^2(t) = 1\ ,$$

where $\sinh(t) = (e^t - e^{-t})/2$ is the **hyperbolic sine**. Solution: We have $|\vec{r}\,'(t)| = \sqrt{1 + \sinh^2(t)} = \cosh(t)$ and $\int_{-1}^1 \cosh(t)\ dt = 2\sinh(1)$.

Problem: Often, there is no closed formula for the arc length of a curve. For example, the **Lissajous figure** $\vec{r}(t) = [\cos(3t), \sin(5t)]^T$ leads to the arc length integral $\int_0^{2\pi}\sqrt{9\sin^2(3t) + 25\cos^2(5t)}\ dt$ which can only be evaluated numerically.

Problem: The curve $\vec{r}(t) = [t, f(t)]^T$, which is the graph of a function f has the velocity $\vec{r}\,'(t) = (1, f'(t))$ and the unit tangent vector $\vec{T}(t) = (1, f'(t))/\sqrt{1 + f'(t)^2}$. After some simplification we get

$$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = |f''(t)|/\sqrt{1 + f'(t)^2}^3$$

For example, for $f(t) = \sin(t)$, then $\kappa(t) = |\sin(t)|/|\sqrt{1 + \cos^2(t)}^3$.

HOMEWORK

This homework is due on Tuesday, 7/9/2019.

Problem 8.1: a) Find the arc length of the curve

$$\vec{r}(t) = [t^2/2, t^3/3, 1]^T$$

from $t = -2$ to $t = 2$.
b) Find the arc length of

$$\vec{r}(t) = [4t, 4\sin(3t), 4\cos(3t), 2]^T ,$$

with $0 \leq t \leq \pi$.

Problem 8.2: Find the curvature of

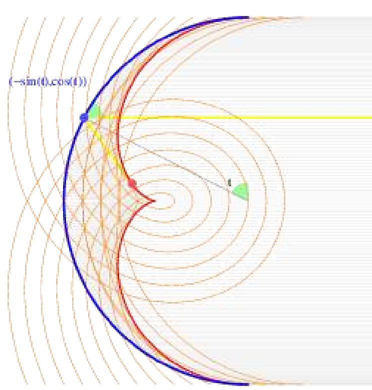
$$\vec{r}(t) = [e^t \cos(t), e^t \sin(t), t]^T$$

at the point $(1, 0, 0)$.

Problem 8.3: Find the vectors $\vec{T}(t), \vec{N}(t)$ and $\vec{B}(t))$ for the curve $\vec{r}(t) = [t^2, t^3, 0]^T$ for $t = 2$. Explore whether the vectors depend continuously on t for all t.

Problem 8.4: Let $\vec{r}(t) = [t, t^2]^T$. Find the equation for the **caustic** $\vec{s}(t) = \vec{r}(t) + \frac{\vec{N}(t)}{\kappa(t)}$. It is known also as the **evolute** of the curve.

Problem 8.5: If $\vec{r}(t) = [-\sin(t), \cos(t)]^T$ is the boundary of a coffee cup and light enters in the direction $[-1, 0]^T$, then light focuses inside the cup on a curve which is called the **coffee cup caustic**. The light ray travels after the reflection for length $\sin(\theta)/(2\kappa)$ until it reaches the caustic. Find a parameterization of the caustic.

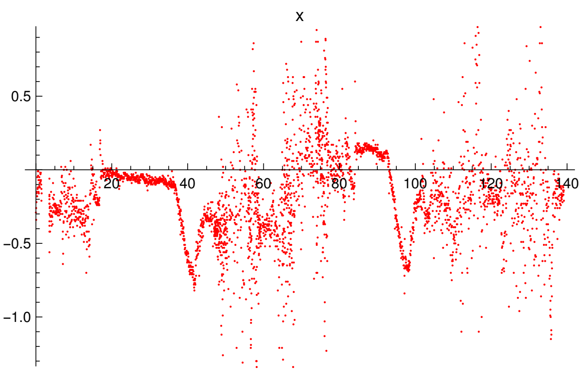


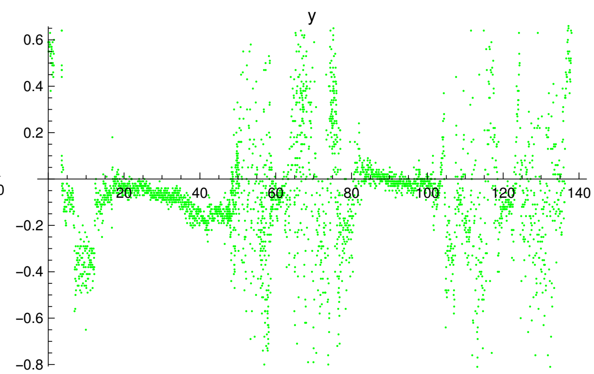
Postscript: Data Visualization

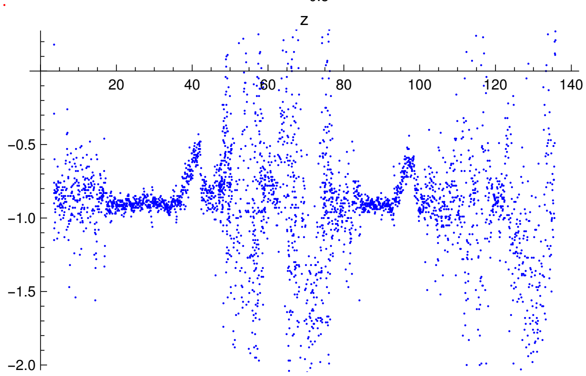
One can use measured acceleration data of a roller coaster to reconstruct the curve $\vec{r}(t) = [x(t), y(t), z(t)]$ which describes the ride. In the blog

$$http://blog.robindeits.com/2013/11/11/roller-coaster-visualizations\ ,$$

Robin Deits has used a smartphone to measure the accelerations during a roller coaster ride on **Cedar point**, the foremost American roller coaster park. Robin made the data available on the website *https://github.com/rdeits/coasters*.
Here are the **x-acceleration**, the **y-acceleration** and the **z-acceleration** data plotted for the Mine ride roller coaster:







One can visualize the actual shape of the roller coaster from these data.

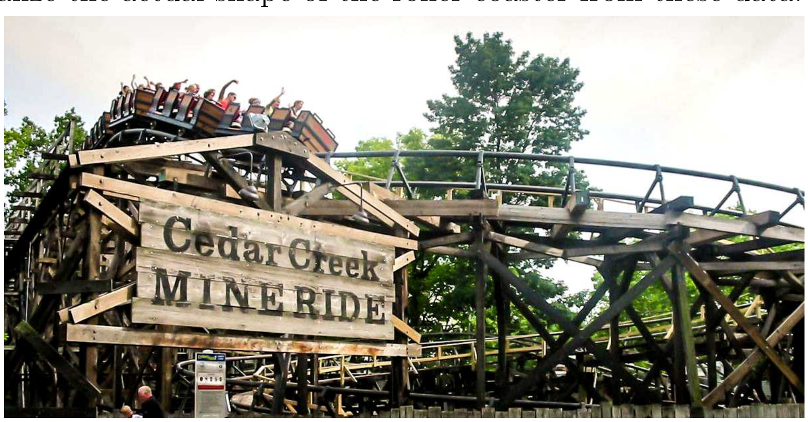


Image source: https://coasterforce.com/mine-train