8/7/2014 PRACTICE EXAM III **Maths 21a, O. Knill, Summer 2014**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

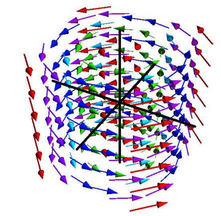
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are necessary

1)	T	F	The lines $\vec{r_1}(t) = \langle t, t, -t \rangle$ and $\vec{r_2}(t) = \langle 1+t, 1+t, 1-t \rangle$ do not intersect.
2)	T	F	The quadratic surface $x^2 - y^2 = z^2$ is a hyperbolic paraboloid.
3)	T	F	If $\vec{T}(t), \vec{B}(t), \vec{N}(t)$ are the unit tangent, normal and binormal vectors of a curve with $\vec{r}\,'(t) \neq 0$ everywhere, then they span a parallelepiped of volume 1.
4)	T	F	If $\vec{u} \cdot \vec{v} = 0$, then $\text{Proj}_{\vec{v}}(\vec{u}) = \vec{0}$.
5)	T	F	There is a vector field $\vec{F}(x, y)$ which has the property $\text{curl}(\vec{F}) = -\text{div}(\vec{F})$, where $\text{curl}(\vec{F})(x, y) = Q_x(x, y) - P_y(x, y)$ and $\text{div}(\vec{F})(x, y) = P_x(x, y) + Q_y(x, y)$.
6)	T	F	The acceleration vector $\vec{r}\,''(t) = \langle x(t), y(t), z(t) \rangle$ is always a unit vector if the velocity vector $\vec{r}\,'(t)$ is a unit vector.
7)	T	F	The grid curves $t \to \vec{r}(t, \phi)$ with fixed $0 < \phi < \pi$ for the standard parametrization of the unit sphere have curvature $1/\sin(\phi)$.
8)	T	F	Any smooth function $f(x, y)$ has a local maximum somewhere in the plane.
9)	T	F	The linearization $L(x, y)$ of constant function $f(x, y) = 3$ is $L(x, y) = 3$.
10)	T	F	A gradient field is incompressible: it satisfies $\text{div}(F) = 0$ everywhere.
11)	T	F	If $f(x, y)$ has a maximum under the constraint $g(x, y) = 1$, then $\nabla f = \langle 0, 0 \rangle$ at this point.
12)	T	F	Assume a vector field $\vec{F}(x, y, z)$ is the curl of a vector field $\vec{G}$ then the flux of the field F through the ellipsoid $x^2 + y^2 + 5z^2 \leq 1$ is zero.
13)	T	F	If the divergence of a field $\vec{F}$ are zero everywhere, then any line integral along a closed curve is zero.
14)	T	F	The gradient of the divergence of a field is always the zero field.
15)	T	F	The vector field $\vec{F}(x, y, z) = \langle x^2, y^2, z^3 \rangle$ is a gradient field.
16)	T	F	The volume of a solid can be computed as the flux of the field $\langle 0, y, 0 \rangle$ through the boundary surface.
17)	T	F	The curvature of a line is zero.
18)	T	F	The distance between the unit sphere centered at $(0, 0, 0)$ and the plane $z = 5$ is equal to 4.
19)	T	F	The partial differential equation $u_t = u_x$ is called heat equation.
20)	T	F	The point $(1, -1, \sqrt{2})$ in spherical coordinates is $(\rho, \phi, \theta) = (2, \pi/4, 3\pi/2)$.

Problem 2) (10 points) No justifications are necessary.

a) (4 points) Match the objects with the definitions.



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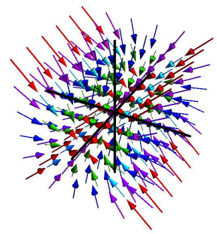

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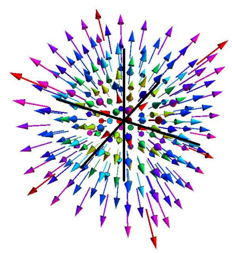

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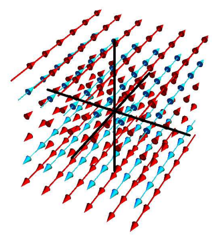
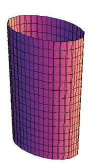


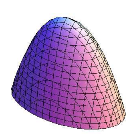
4

enter 1-4	vector field
	$\vec{F}(x,y,z) = \langle x, y, z\rangle$
	$\vec{F}(x,y,z) = \langle -y, x, 0\rangle$
	$\vec{F}(x,y,z) = \langle 0, z, 0\rangle$
	$\vec{F}(x,y,z) = \langle -x, 0, -z\rangle$

b) (3 points) Match the surfaces with their names: (put O if no match)



1



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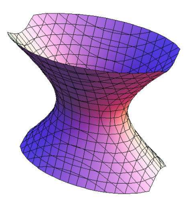

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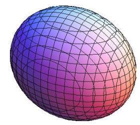
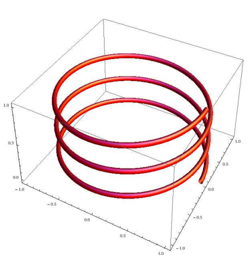


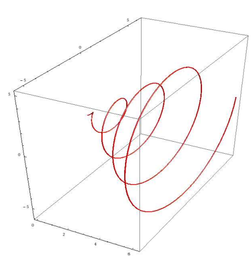
4

enter 1-4	surface
	$x^2 + y^2 + 3z = 0$
	$x^2 + y^2 - 3z^2 = 1$
	$x^2 + y^2 + 3z^2 = 1$
	$x^3 + 3y^2 = 1$

c) (3 points) Match the space curves



1



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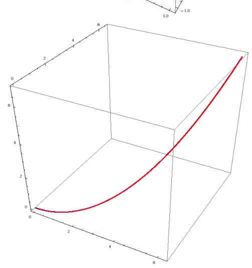

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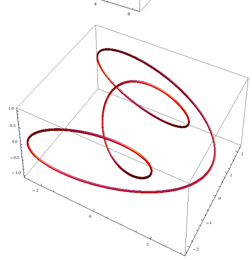
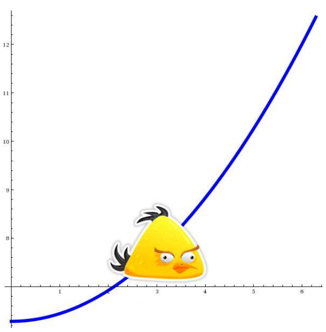


4

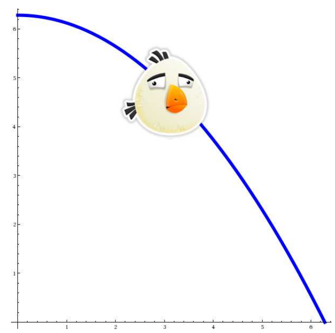
1-4	parametrized curve
	$\vec{r}(t) = \langle t, t^2, t^3\rangle$
	$\vec{r}(t) = \langle \cos(3t), \sin(3t), t\rangle$
	$\vec{r}(t) = \langle (2+\cos(t))\cos(3t),$ $(2+\cos(t))\sin(t),$ $\sin(3t)\rangle$
	$\vec{r}(t) = \langle t, t\cos(3t), t\sin(3t)\rangle$

Problem 3) (10 points) No justifications are necessary

a) (5 points) We watch "angry birds" attacking on curves with acceleration $\vec{r}\,''(t)$. (The pictures show the $xz-$ planes and the birds start with a constant velocity $\langle 1,0,0\rangle$.) Match the displayed curves $\vec{r}(t)$ with the formulas for accelerations.



1



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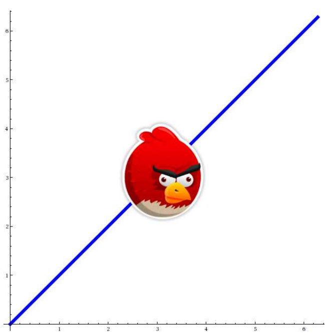
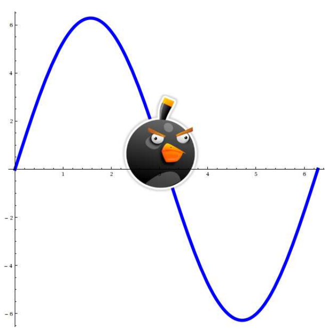


3

acceleration	enter curve 1-5
$\vec{r}\,''(t) = \langle 0,0,\sin(t)\rangle$	
$\vec{r}\,''(t) = \langle 0,0,-10\rangle$	
$\vec{r}\,''(t) = \langle 0,0,10\rangle$	
$\vec{r}\,''(t) = \langle 0,0,-\sin(t)\rangle$	
$\vec{r}\,''(t) = \langle 0,0,0\rangle$	



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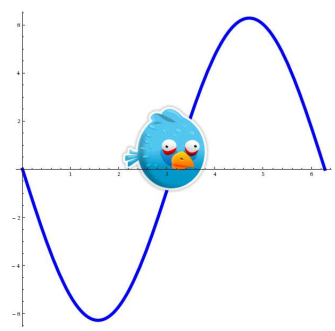

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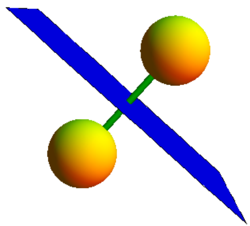
b) (5 points) Match the formulas: (put O if no match)

label	formula
A	$\vec{r}\,'(t)$
B	$\int_0^1 \lvert\vec{r}\,'(t)\rvert\, dt$
C	$\vec{r}\,'(t)/\lvert\vec{r}\,'(t)\rvert$
D	$\vec{T}\,'(t)/\lvert\vec{T}\,'(t)\rvert$
E	$\lvert\vec{r}\,'(t)\times\vec{r}\,''(t)\rvert/\lvert\vec{r}\,'(t)\rvert^3$

expression	enter A-E
Curvature	
Unit tangent vector	
Unit normal vector	
Velocity	
Arc length	

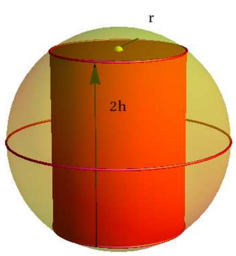
Problem 4) (10 points)



a) (5 points) Find a parametrization of the line L through the center of the two spheres $x^2 + (y-1)^2 + z^2 = 1$, $(x-5)^2 + y^2 + z^2 = 1$.

b) (5 points) Find the plane perpendicular to the line L for which the distances to the spheres are the same.

Problem 5) (10 points)

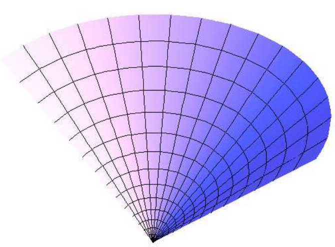


Johannes Kepler asked which cylinder or radius r and height $2h$ inscribed in the unit sphere has maximal volume. To solve his problem, use the Lagrange method and maximize the volume

$$f = 2\pi r^2 h$$

under the constraint that $r^2 + h^2 = 1$.

Problem 6) (10 points)

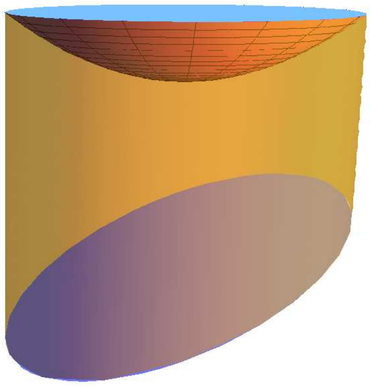


a) (6 points) Find the surface area of the surface

$$r(u,v) = \langle v^2 \cos(u), v^2 \sin(u), v^2 \rangle, 0 \leq u \leq \pi, 0 \leq v \leq 1 .$$

b) (4 points) Find the arc length of the boundary curve $\vec{r}(u,1)$ where $0 \leq u \leq \pi$.

Problem 7) (10 points)

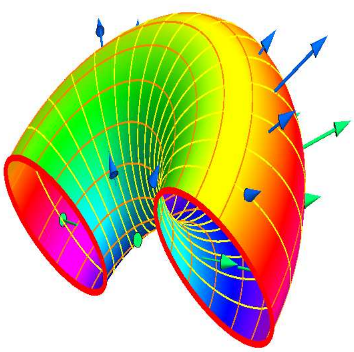


Find the volume of the solid inside the cylinder

$$x^2 + y^2 \leq 2$$

sandwiched between the graphs of $f(x, y) = x - y$ and $g(x, y) = x^2 + y^2 + 4$.

Problem 8) (10 points)



Find the flux of the curl of the vector field

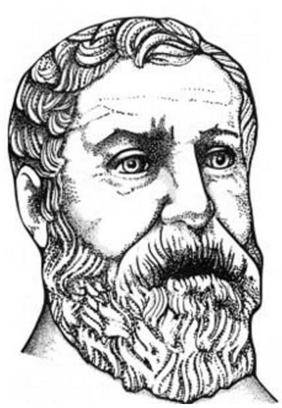
$$\vec{F}(x, y, z) = \langle x, y, z + \sin(\sin(y^2)) \rangle$$

through the torus

$$\vec{r}(s, t) = \langle (2 + \cos(s)) \cos(t), (2 + \cos(s)) \sin(t), \sin(s) \rangle$$

with $0 \leq t \leq \pi$ and $0 \leq s < 2\pi$.

Problem 9) (10 points)



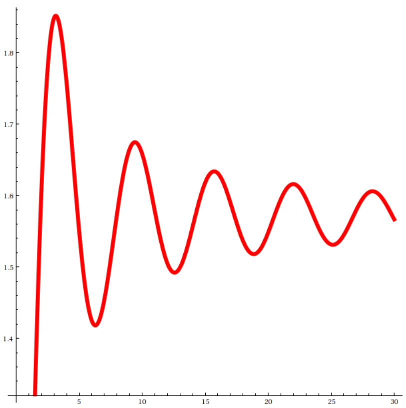
Heron's formula for the area A of a triangle of side length $x, y, 1$ satisfies $16A^2 = f(x, y)$, where

$$f(x, y) = -1 + 2x^2 - x^4 + 2y^2 + 2x^2y^2 - y^4 .$$

Classify all the critical points of f. Is there a global maximum of f and so for the area?

Remark not to worry about: The formula follows directly from Heron's formula $s = (a + b + 1)/2; A = \sqrt{s(s-a)(s-b)(s-1)}$.

Problem 10) (10 points)



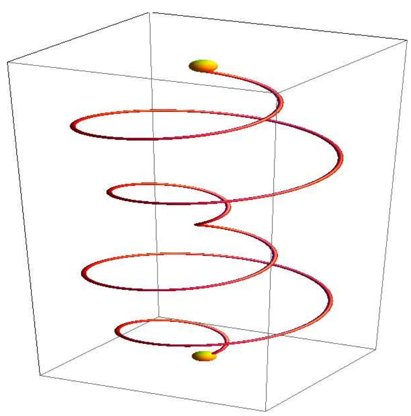
The anti derivative of the **sinc** function

$$\frac{\sin(x)}{x}$$

is called the **sine integral** $\mathrm{Si}(x)$. It can not be expressed in terms of known functions. Still we can compute the following double integral

$$\int_0^{\pi} \int_x^{\pi} \frac{\sin(y)}{y} \, dy \, dx \; .$$

Problem 11) (10 points)

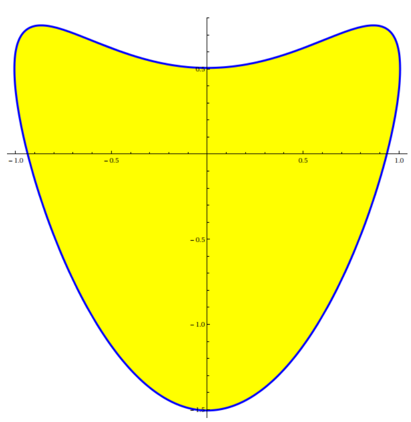


Find the line integral of the vector field

$$\vec{F}(x,y,z) = \langle -x^{10}, \sin(y), z^3 \rangle$$

along the curve $\vec{r}(t) = \langle \sin(t)\cos(5t), \sin(t)\sin(5t), t \rangle$ where $0 \leq t \leq 2\pi$.

Problem 12) (10 points)

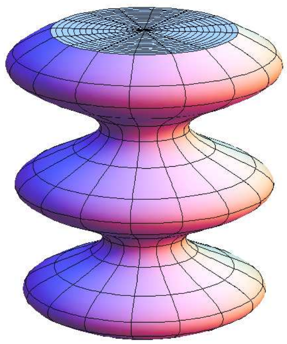


Find the area of the region enclosed by the curve

$$\vec{r}(t) = \langle \cos(t), \sin(t) + \cos(2t)/2 \rangle \; ,$$

where $0 \leq t < 2\pi$.

Problem 13) (10 points)



Find the flux of the vector field

$$\vec{F}(x,y,z) = \langle x^3/3, y^3/3, \sin(xy^5) \rangle$$

through the boundary surface of the solid bound by the surface of revolution $\vec{r}(t,z) = \langle (2+\sin(z))\cos(t), (2+\sin(z))\sin(t), z \rangle$ and the planes $z=0$, $z=3$. The surface is oriented so that the normal vector points outwards.