

HISTORICAL PAPER

On the Genesis of the Lagrange Multipliers

P. BUSSOTTI¹

Communicated by F. Giannessi

Abstract. The genesis of the Lagrange multipliers is analyzed in this work. Particularly, the author shows that this mathematical approach was introduced by Lagrange in the framework of statics in order to determine the general equations of equilibrium for problems with constraints. Indeed, the multipliers allowed Lagrange to treat the questions of maxima and minima in differential calculus and in calculus of variations in the same way as problems of statics: if the equilibrium of a point or a system of points is required, there is an analogy between statics and differential calculus; if the equilibrium of a rigid body is required, there is an analogy between statics and calculus of variations.

Key Words. Lagrange multipliers, constrained maxima and minima.

1. Introduction

The paper aims to analyze the origin of the Lagrange multipliers. The question will be embedded in the theoretical framework conceived by the Turinese mathematician. We will see that some questions of statics, connected with the principle of virtual velocities, suggested to Lagrange the introduction of the multipliers.

It is necessary to point out that every scientific concept has a gestational phase, so to speak. The multipliers do not escape this rule. Indeed, before Lagrange, other authors conceived solutions of some problems where an order of ideas similar to that of Lagrange is present. The most significant example is represented by the results of Eulero concerning the calculus of variations. However, it is necessary to separate the gestational phase of a

¹Research Associate, Department of Mathematics, University of Pisa, Pisa, Italy; Von Humboldt Researcher, University of Munich, Munich, Germany.

concept from the phase of real birth, because Lagrange was the first one to fully conceive the value of this method in order to solve many problems. So, he must be considered as the inventor of this mathematical theory.

Since the introduction of the multipliers takes place in the context of statics, it is obvious that it will be necessary to revisit, in a concise way, the concepts that Lagrange poses as the basis for statics. Since the literature about this argument is quite extended and deep, the author does not claim to be original and the references to statics will have only the aim to present the framework in which the treatment of the multipliers is embedded.

2. Brief Look at Lagrange Statics

At the beginning of his *Mécanique Analytique* (Ref. 1), Lagrange tackles statics and poses three principles as the foundations for the subject: (i) the principle of the lever; (ii) the principle of the composition of forces; (iii) the principle of virtual velocities.

In the first fifteen paragraphs of Section 1, the Turinese mathematician presents an historical picture, that is quite deep and complete, where he analyzes the evolution of the first two principles. We do not tackle the questions related to the first two principles because they are not strictly connected with the introduction of the multipliers. On the contrary, the principle of virtual velocities has a great importance. So, Lagrange writes (Ref. 1, pages 17–18):

“By virtual velocity, it has to be meant the one that a body in an equilibrium condition would receive if the equilibrium was interrupted; namely, the velocity that the body would really assume in the first instant of movement; the principle consists in this: the forces are in equilibrium if they are in the inverse proportion to their virtual velocities. . . .”

Lagrange further remarks that Galilei defines the product of a force and the virtual velocity, eventually multiplied by a constant, as the moment of a force (Ref. 1, page 18). Today, we call “power” this quantity; hence, we may speak of virtual power. Also, Wallis uses the word “moment” with the same meaning. Lagrange specifies that, in the physics of his time, the word “moment” denotes the product of the force and the arm. But in his opinion, this definition is less general and useful; so, he follows the Galilean meaning of the word. A body will be in equilibrium iff the (algebraic) sum of the virtual powers is zero.

In the first pages of the *Mécanique Analytique*, Lagrange does not employ mathematical symbols; however, for a more clear interpretation of the Lagrangian thought, a symbolic transcription can prove useful. So, if we consider a material point to which only two forces P and Q are applied

and if dp/dt and dq/dt (p and q being the directions, respectively, of P and Q) are the virtual velocities that the body receives, respectively, by the forces P and Q , then the body is in equilibrium if

$$Pdp/dt + Qdq/dt = 0.$$

This equation represents the principle of the virtual velocities. The quantity Pdp/dt is the moment of the force P .

At the beginning of Section 2 (Ref. 1, paragraph 1), when the real mathematical treatment begins, Lagrange replaces dp/dt with dp . He justifies the substitution in this way (Ref. 1, page 24):

“In order to express this principle by means of formulas, we suppose that the forces P, Q, R are directed along given lines and that they are in equilibrium; let p, q, r , etc. denote the lines that are directions of the forces P, Q, R . We indicate the variations or the differences (now called differentials) of these lines with dp, dq, dr , etc., and suppose that they are caused by an arbitrary infinitesimal change in the position of the different bodies or points of the system. It is clear that these differences will express the spaces that are covered in the same instant by the action of the forces P, Q, R along their respective directions . . . The differences dp, dq, dr will be proportional to the virtual velocities of the forces P, Q, R . In order to simplify the treatment, it will be possible to consider these differences instead of the velocities.”

So, the general equation of the equilibrium of a point on which three forces are applied will be

$$Pdp + Qdq + Rdr = 0.$$

Lagrange continues to use the name of principle of virtual velocities. Indeed, Pdp, Qdq, Rdr are works that are virtual iff the system is in equilibrium. For this reason, after the work of Lagrange, written in this form, the principle was called principle of virtual works.

In Section 3, Lagrange applies the principle to the study of the equilibrium of a system of points and he analyzes the conditions of translational and rotational equilibrium.

At the end of Section 2 (Ref. 1, paragraph 3, page 27), he points out that it is important to reduce Mechanics to purely analytical operations and to free it from intuitive geometrical considerations. For this reason, in Section 4, he introduces the method of the multipliers (*Méthode des multipliers* is the title of paragraph 1 of Section 4). In this way, Lagrange is able to solve the same problem as in Section 3, but in a more general and in a simpler way. Particularly, he studies in detail the equilibrium under constraints.

3. Multipliers and Statics of the Point

Lagrange examines, in a first instance, the equilibrium of a material point or of a system of material points that are submitted to constraints. If the constraining equations are $L = 0$, $M = 0, \dots$ and if L , M are functions of several variables, it follows necessarily that $dL = 0$, $dM = 0$, etc. Consequently, if λ , μ are arbitrary real numbers [Lagrange calls them *quantités indéterminées* or *coefficients indéterminées* (Ref. 1, page 70)], we have $\lambda dL = 0$, $\mu dM = 0$, etc. Then, Lagrange explains (Ref. 1, page 70):

“We will consider the sum of all the moments of the forces that must be in equilibrium, we will add the different differential functions that must be zero, on the basis of the conditions of the problem, after having multiplied every function by an indeterminate coefficient; we will make the whole equal to zero and so we will have a differential equation that we will treat as an ordinary equation *de maximis et minimis*.”

The general form of this differential equation is

$$Pdp + Qdq + \dots + \lambda dL + \mu dM \dots = 0.$$

Such equation is the general equation of equilibrium. Considering each variable of an orthogonal reference frame, we will have a particular equation of equilibrium. For example, in the direction x , we will have

$$Pdp/dx + Qdq/dx + \dots + \lambda dL/dx + \mu dM/dx \dots = 0.$$

From a physical point of view, the reaction of the constraint is equated to an agent force; indeed, λ and μ are treated as forces and the expressions λdL and μdM represent the virtual works that are realized by these forces (Lagrange continues to call them “moments”). The advantage of this method is that it allows the treatment of the problems with constraint in the same way as free problems, adding the virtual works realized by the reaction of the constraints to the virtual works realized by the agent forces.

A first extremely important consequence of such an approach is that the analytical and numerical treatment of the constraints is postponed to the writing of the equilibrium condition. Many other important consequences will be found by successive scholars and nowadays the discoveries still continues.

From a mathematical point of view, Lagrange points out that the difficulty consists in determining the values of λ , μ , etc. Actually, this difficulty can be overcome easily (Ref. 1, paragraph 4, Section 4, page 71), since in fact the use of the constraints allows having a system where the number of the equations and the number of the variables is the same.

An example can be useful to interpret and clarify the Lagrange reasoning: if we have two agent forces P and Q and a constraint represented by

the equation $L = 0$, we will have (imagining it to be in the ordinary space) the three unknown quantities x , y , z and the unknown quantity λ , and we will have four equations:

$$Pdp/dx + Qdq/dx + \lambda dL/dx = 0,$$

$$Pdp/dy + Qdq/dy + \lambda dL/dy = 0,$$

$$Pdp/dz + Qdq/dz + \lambda dL/dz = 0,$$

$$L = 0.$$

If the multiplier is interpreted as a new coordinate, the last equation can be replaced by another one that must be valid in the case of equilibrium:

$$Pdp/d\lambda + Qdq/d\lambda + \lambda dL/d\lambda = 0.$$

In this way, the solution of the problem of the equilibrium of a material point or of a system of material points appears to be, from a mathematical point of view, analogous to the problem of determining the maximum or the minimum of a function under constraints.

4. Multipliers and Statics of a Rigid Body

In order to analyze the problem of the equilibrium for a rigid body of a given mass M , the situation is different from the one that is valid for the material point or for a system of material points. Indeed, if a force acts on a point of a rigid body, its action will be directed to all points of the body and not only to the point of application. If P indicates this force, Lagrange denotes by δp the infinitesimal variation of the position that the force tends to induce in the body. Then, he states (Ref. 1, paragraph 10, page 75–76) that this variation must be distinguished from the ordinary differential dp .

If P , Q , etc. are the forces that act on the body, their virtual works (that the Turinense mathematician still calls “moments”) are represented by the expression $P\delta p + Q\delta q$. In order to obtain the condition of equilibrium of the body, Lagrange writes (Ref. 1, paragraph 11, page 76):

“I remark that it is possible to consider the given mass [of the body] as a set of an infinity of contiguous points; it will follow, according to the spirit of the infinitesimal calculus, that [the mass] can be seen as composed by infinitely small elements. The sum of all these elements is of the same order of size of the whole mass . . .”

In order to obtain the equilibrium of the body, the variation of the potential must be zero. If dm denotes the infinitesimal element of the mass, the variation of the potential is given by the integral of all the virtual works

extended to the whole mass. Lagrange employs the letter S to indicate this definite integral, while he uses the ordinary notation \int for the indefinite integral. The fact that the variation of potential becomes zero is represented by the equation

$$S(P\delta p + Q\delta q) = 0.$$

Nowadays, we write

$$\int_M (P\delta p + Q\delta q) = 0.$$

This is the equation of equilibrium if there is no constraint (Ref. 1, page 76).

If there are constraints, expressed by the equations $L = 0$, $M = 0$, etc., then the condition $\lambda\delta L = 0$, $\mu\delta M = 0$, etc. will be valid (λ , μ , etc being multiplicative constants), as in the case of the equilibrium of a point or of a system of points. Therefore, Lagrange writes that the integral $S(\lambda\delta L + \mu\delta M + \dots)$ will be considered; in this integral, the differentials of integration are not specified because they depend on the variables of which L , M , \dots are a function. Therefore, if the number of those variables is one, we will have a simple integral; if it is two, we will have a double integral, and so on.

Beside the forces that act on the whole mass and the reaction of the constraints, it may happen, in a particular physical context, that there are forces that act on single points which are, once more, subjected to further constraints. For this reason, if P' , Q' , etc. are such forces and if $A = 0$, $B = 0$ are the constraints, then the general equation of equilibrium will be

$$S(P\delta p + Q\delta q) dm + S(\lambda\delta L + \mu\delta M + \dots) \\ + P'\delta p' + Q'\delta q' + \dots + \alpha\delta A + \beta\delta B + \dots = 0.$$

In this way, the problem of determining the conditions of equilibrium for a rigid body is reduced to a problem of maximum or minimum in the calculus of variations.

5. Conclusions

With the introduction of the multipliers and with the use of the principle of virtual works, Lagrange is able to reduce the problems of equilibrium to the determination of the necessary conditions in order that a

function (case of a point or a system of points) or a functional (case of a rigid body) has a maximum or a minimum.

Reference

1. LAGRANGE, J. L., *Mécanique Analytique*, Complete Edition, joining the notes of the 3rd Edition revised, corrected, and annotated by Joseph Bertrand, and those of the 4th Edition published under the direction of Gaston Darboux; Albert Blanchard, Paris, France, 1965.