LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22A

Unit 21: Island mathematics

Seminar

21.1. With an "island" we mean a region in the plane $\mathbb{R}^2$ which is bound by a simple closed curve C which is continuous everywhere and differentiable everywhere except at a finite set of points. So, simple polygons are allowed. What island does have the maximal area if the length of the boundary is fixed? This is called the **isoperimetric problem**. If we look at the problem restricted to polygons with a fixed number n of vertices, then we have a nice finite dimensional Lagrange problem.

21.2. Let us look at a **triangular island** $T(x, y)$ with vertices $(-1, 0), (1, 0), (x, y)$.

> **Problem A:** Assume the circumference $g(x, y)$ of the triangle is 3. What is the maximal area $f(x, y) = y/2$ we can get? Set up the Lagrange equations and solve them.

21.3. Here is a side problem from good old Euclidean geometry. If you should not know, look up "string method pins".

> **Problem B:** What points (x, y) in the plane satisfy $g(x, y) = 3$.

21.4. Solving the problem to find the n-gon with maximal area is a messy Lagrange problem. It can be done by a computer but there is a more elegant way:

> **Problem C:** Use the computation in problem A to show that for a maximal polygon containing vertices ..., P, Q, R, ... in a row, the distance between P and Q is the same as the distance between Q and R.

> **Problem D:** Conclude that a polygon with n vertices and maximal area must be a regular polygon.

21.5. You are on your treasure island G and have two locations A, B in G. The problem to find the shortest connection between A and B can be quite complex in general. An example is when G is bound by a **Gosper curve**. For the following let us assume that the boundary of G is a **convex curve**: this means that for any two points A, B in G, the line segment through A, B is contained in G. A triangle A, B, C for which all three points A, B, C are on the boundary is called a "shore triangle".

Problem E: Verify that for a shore triangle, the **billiard law of reflection at the boundary** holds.

21.6. Hint: to see that the incoming angle is the same as the outgoing angle, take a minimal triangle A, B, C, where B is on the island shore, then replace the curve with the tangent curve L at B. Now reflect C at L to get a point C'. Verify that the shortest billiard path ABC has the same length than the straight line connecting A with C'.

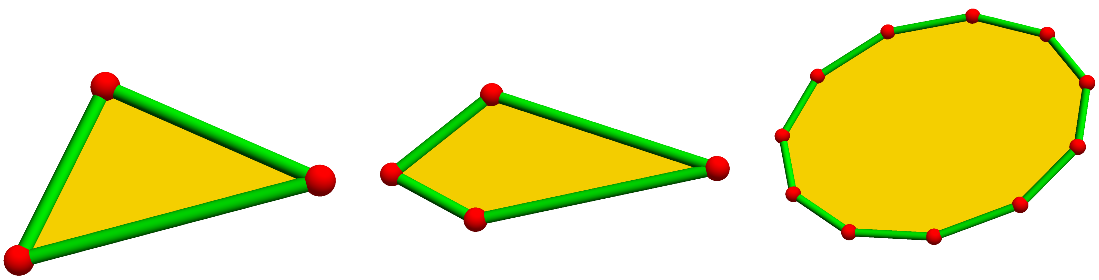

Figure 1. What polygon with fixed circumference has maximal area?

21.7. The next time you are cast away on an island, count the number m of mountain peaks, the number s of sinks and the number p of mountain passes. Make some experiments. You notice the following rule which is known as a special case of the Poincaré-Hopf theorem:

Theorem: maxima + minima − saddles = 1.

Problem F: Find an example where this equality holds, in which we have maxima = 3, minima = 1 and saddles = 3.

21.8. If you want to challenge yourself, see whether you can prove the island theorem by deformation. (This is probably too hard. Just enjoy the struggle!)

21.9. Assume now that our island is an atoll, a ring shaped reef.

Problem G: By looking at examples, what is the island number maxima + minima − saddles on an atoll?

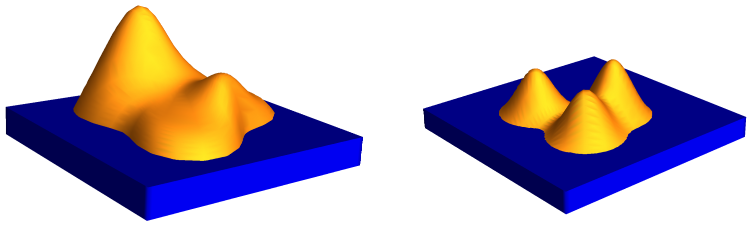

FIGURE 2. First an island with 2 mountain peaks and with 1 mountain pass. Then an island with 3 mountain peaks and 2 mountain passes. We see maxima + minima − saddles = 1.

FIGURE 3. The Atafu atoll. Picture by NASA Johnson Space Center, 2009.

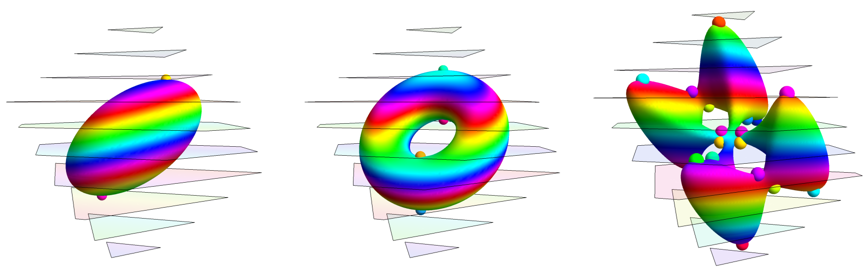

FIGURE 4. If we place a surface $S : g = c$ in space and look at the restriction of a function $f(x, y, z)$ on S, we solve a Lagrange problem. In a Morse situation, the numbers maxima + minima − saddles add up to a number which only depends on the number of holes.

21.10. Let us look at the one-dimensional case, where we prove things easier. Assume the island is the interval $[a, b]$. Let f be a smooth function on $[a, b]$ which has the property that f is zero for $x \geq b$ and for $x \leq a$. We look at critical points of f in the interior (a, b) which are Morse, (meaning $f''(x) \neq 0$ at critical points), so that we only have only local maxima and minima as critical points. Let m be the number of maxima and s the number of minima (sinks). In order to prevent the island to be flooded, we also assume that the function f is positive for $x > a$, close to a and $x < b$ close to b.

Theorem: maxima − minima = 1.

Problem H: Verify that there is an odd number of critical points for a Morse function f which has as a support a finite interval $[a, b]$.

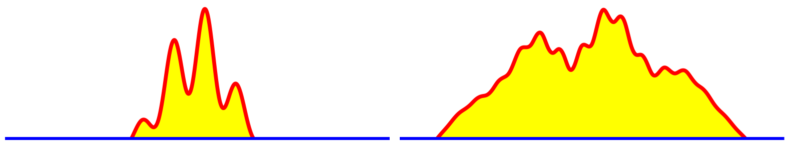

Figure 5. One-dimensional islands.

Problem I: Use a deformation argument to show that if there are $2k+1$ critical points, we can reduce them to $2k-1$ by merging a pair of neighboring maxima and minima

Homework

21.1 A spherical triangle A, B, C on the unit sphere has angles α, β, γ in $[0, \pi]$. What is the largest area that such a triangle can have? You can use the fact that $\alpha + \beta + \gamma - \pi$ is the area. The result might look a bit strange for a triangle.

21.2 Find an example of a non-Morse function $f(x, y, z)$ with a maximum. Similarly find an example with a minimum and an example of a non-Morse function where the critical point is neither a maximum, nor a minimum.

21.3 If we look at maxima, minima and saddle points for a function $f(x, y)$ defined on a doughnut. By looking at examples, find the island number maxima + minima − saddles there.

21.4 If we look at maxima, minima and saddle points for a function $f(x, y)$ defined on a sphere. By looking at examples, what is the island number maxima + minima − saddles there.

21.5 Assume $f : \mathbb{R} \to \mathbb{R}$ is a single variable Morse function which is 2π periodic. What is the relation between the number m of maxima on $[0, 2\pi)$ and the number of minima on $[0, 2\pi)$? Prove this.

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