Complex Chern-Simons invariants of 3-manifolds via abelianization

Dan Freed
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Joint work with Andy Neitzke

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Harmonic Spinors

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INTRODUCTION

With the introduction of general relativity, it became necessary to express the differential operators of mathematical physics in a coordinate-free form. This made it possible to define those operators on an arbitrary Riemannian manifold—the grads, divs, and curls got translated into the $d + d^*$ operator on the bundle of exterior forms. This particular operator found fruitful application in the theorem of Hodge which expressed the dimension of the null space (the space of harmonic forms) on a compact manifold in terms of topological invariants—the Betti numbers.

Another operator—the Dirac operator—made a later appearance in Riemannian geometry. It was used by Atiyah and Singer to explain the integrality of the $\hat{A}$-genus of a spin manifold, and then Lichnerowicz proved a strong vanishing theorem — if a spin manifold has positive scalar curvature, the null space of the Dirac operator (the space of

Linear field equations on self-dual spaces

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In a Riemannian context, a description is given of the Penrose correspondence between solutions of the anti-self-dual zero rest-mass field equations in a self-dual Yang–Mills background on a self-dual space X, and the sheaf cohomology groups $H^1(Z, \mathcal{O}F(n))$, for $n \leqslant -2$, of its twistor space Z. The case $n = -2$ is fundamental for the construction of instantons on Euclidean space. It is further shown how $H^1(Z, \mathcal{O}F(-1))$ corresponds to solutions of the self-dual Dirac equation, and an interpretation for $H^1(Z, \mathcal{O}F(n))$, for $n \geqslant 0$, is given in terms of the cohomology of an elliptic complex on X.

INTRODUCTION

In an earlier paper (Atiyah, Hitchin & Singer 1978) we gave an account of the ideas of R. Penrose connecting the Riemannian geometry of certain four-dimensional spaces with complex three-dimensional geometry. We were basically interested

formal weight $w \in \mathbb{R}$ so that $a \in CO(n)$ acts as a^w.

Instead of considering just $CO(n)$, we can look at the group of automorphisms of the 1-jet of the conformal structure at X. Since it takes curvature and second derivatives to distinguish any two conformal structures, this group is the same as for the flat case: it is just the group of conformal transformations of the n-sphere S^n which leave fixed a point. Taking this point as infinity, it is easy to see that this is the group $CE(n)$ generated by the Euclidean transformations of $\mathbb{R}^n$ and dilations: a semi-direct product of $CO(n)$ with the translation group $\mathbb{R}^n$.

Now suppose E is a bundle over X associated to a representation space $\boldsymbol{E}$ of $CO(n)$, then $J_1(E)_x$, the space of 1-jets of sections of E at x, is defined by the 1-jet of the conformal structure and thus is acted upon by $CE(n)$. A subspace $R_x \subset J_1(E)_x$ which is invariant by $CE(n)$ will then define a conformally invariant differential equation as x varies. We shall use this algebraic approach to characterize the Dirac equation and the corresponding zero rest-mass field equations for higher spin. This is the point of view adopted by Fegan (1976) and details for what follows may be found in his paper.

The first thing we need to know is the action of $CE(n)$ on $J_1(E)_x$. If we choose a

The enhanced Rogers dilogarithm

Begin with power series defined for $|z| < 1$:
(analytically continue to $\mathbb{C}\backslash[1,\infty)$)

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad \text{(Spence)}$$

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$\widehat{\mathcal{M}}_T = \mathbb{C}^2$ universal $(\mathbb{Z}\times\mathbb{Z})$ cover of $\mathcal{M}_T$

$\mathcal{M}'_T = \{(\mu_1,\mu_2) \in \mathcal{M}_T : \mu_1 + \mu_2 = 1\}$ (complex) curve in $\mathcal{M}_T$, $\quad \mathcal{M}_T \approx \mathbb{CP}^1\backslash\{0,1,\infty\}$

$\widehat{\mathcal{M}}'_T = \{(u_1,u_2) \in \widehat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\}$ $\mathbb{Z}\times\mathbb{Z}$ cover of $\mathcal{M}'_T$ $\qquad (u_1,u_2) \longmapsto (e^{u_1}, e^{u_2})$

$\mathbb{Z}(1) = 2\pi\sqrt{-1}\mathbb{Z}$ Tate twist

$\mathbb{Z}(2) = \mathbb{Z}(1)^{\otimes 2} = 4\pi^2\mathbb{Z}$

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Differentiate the power series: $\qquad d\,\mathrm{Li}_2(z) = -\frac{\log(1-z)}{z}\,dz = -u_2\,du_1 = -\frac{u_2\,du_2}{1-e^{-u_2}}$

$(z = \mu_1)$ $\qquad$ (meromorphic 1-form on the u_2-line with simple poles at $\mathbb{Z}(1) \subset \mathbb{C}$ and residues in $\mathbb{Z}(1)$)

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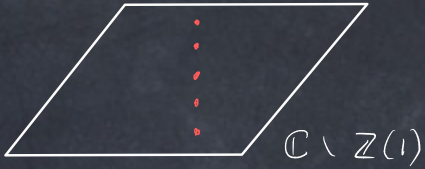
$$\begin{aligned} F: \quad \widehat{\mathcal{M}}'_T &\longrightarrow \mathbb{C}\setminus\mathbb{Z}(1) \longrightarrow \mathbb{C}/\mathbb{Z}(2) \\ (u_1, u_2) &\longmapsto u_2 \longmapsto \mathrm{Li}_2(1 - e^{u_2}) \end{aligned}$$



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Rogers enhanced dilogarithm

$$\begin{array}{rccc} R: & \widehat{\mathcal{M}}'_T & \longrightarrow & \mathbb{C}/\mathbb{Z}(2) \end{array}$$

$$(u_1, u_2) \longmapsto F(u_2) + \frac{u_1 u_2}{2} \qquad \mathrm{mod}\ \mathbb{Z}(2)$$

satisfies the differential equation

$$dR = \frac{u_1 du_2 - u_2 du_1}{2}$$

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- The Rogers enhanced dilogarithm $R \colon \widehat{\mathcal{M}}'_T \to \mathbb{C}/\mathbb{Z}(2)$ is essentially the function

$$\mathrm{Li}_2(z) + \frac{1}{2}\log(z)\log(1-z) \quad \mod \mathbb{Z}(2)$$

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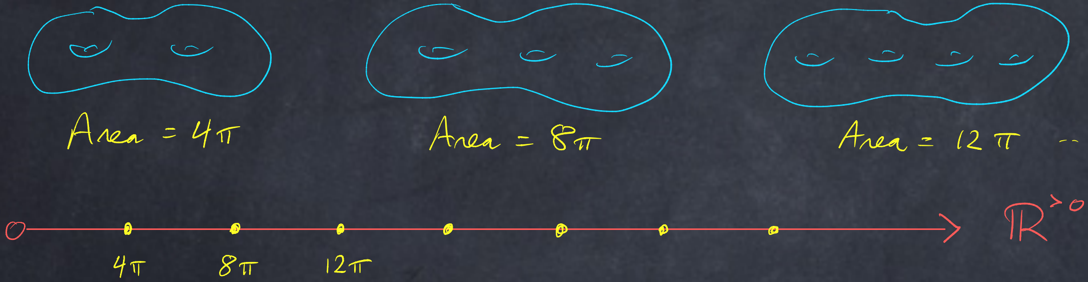
- It has known values at special z and satisfies several identities
- The most important is the 5-term identity

$$R(u_1, v_1) + \cdots + R(u_5, v_5) = \text{constant}, \qquad v_i = u_{i-1} + u_{i+1}$$

Volumes of Hyperbolic 3-Manifolds

Σ^2 complete Riemannian hyperbolic ($K_\Sigma = -1$) with Area$(\Sigma) < \infty$. Then (Gauss-Bonnet) Area$(\Sigma) = -2\pi\,$Euler(Σ) is a *topological* invariant.



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M^3 complete oriented Riemannian hyperbolic with $\text{Vol}(\Sigma) < \infty \implies M$ is the interior of a compact manifold with $\partial M \approx \bigsqcup S^1 \times S^1$. Then (Mostow) $\text{Vol}(M)$ is a *topological* invariant.

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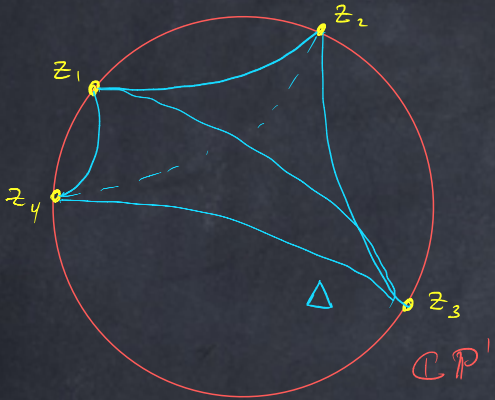
Intuitively, $\mathrm{Vol}(M)$ orders hyperbolic manifolds by complexity.

Tetrahedra and Dilogarithms

$\Delta \subset \mathbb{H}^3$ ideal tetrahedron

$Z_0, Z_1, Z_2, Z_3 \in \mathbb{CP}^1$ vertices

$$z = \frac{(Z_0 - Z_2)(Z_1 - Z_3)}{(Z_0 - Z_3)(Z_1 - Z_2)}$$



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Volume Formula (Lobachevsky): $\mathrm{Vol}(\Delta) = D(z)$

It was written in this form in work of Dupont-Sah and Neumann-Zagier:

$$D\colon \mathbb{CP}^1 \backslash \{0, 1, \infty\} \longrightarrow \mathbb{R}$$

$$D(z) = \mathrm{Im}\, Li_2(z) + \log|z| \arg(1 - z),$$

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After Dehn surgeries on cusps, write hyperbolic $M^3 = \bigcup_\nu \Delta(z_\nu)$ and

$$\mathrm{Vol}(M) = \sum_\nu D(z_\nu)$$

Complex Chern-Simons invariant

Combine Levi-Civita connection and soldering form into a complex (Cartan) connection Θ:

$$\begin{tikzcd} \mathcal{B}_{\mathrm{SO}}(M) \arrow[rr, hook] \arrow[dr, "\mathrm{SO}_3"'] & & P, \ \Theta = \Theta_{LC} + \sqrt{-1}\,\theta \arrow[dl, "\mathrm{PSL}_2\mathbb{C}"] \\ & M & \end{tikzcd}$$

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These invariants and closely related topics have been extensively studied in works of Dimofte, Dupont, Fock, Gabella, Garoufalidis, Goette, Goncharov, Meyerhoff, Neumann, Sah, D. Thurston, Yang, Yoshida, Zagier, Zickert, . . .

Our work

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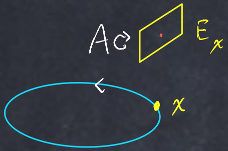
On a triangulated 2-manifold we similarly construct a line bundle over the character variety

(1) Stratified abelianization

Dimension 1

$$\left(A \in SL_2\mathbb{C}\right) \overset{\text{holonomy}}{\rightsquigarrow} \left(E \longrightarrow S^1 \ \text{ rank 2 flat vector bundle}\right)$$



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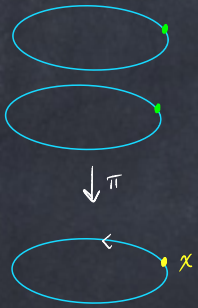

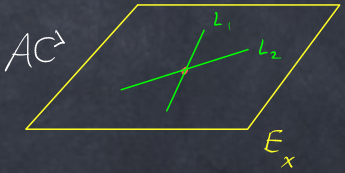

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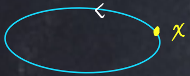
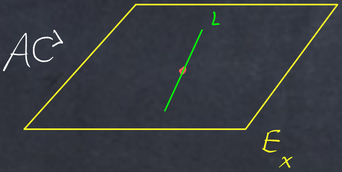
unique flat line subbundle $L \longrightarrow S^1$





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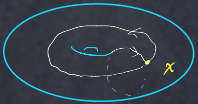
$A_1, A_2 \circlearrowright$

E_x



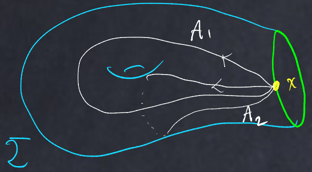
Dimension 2

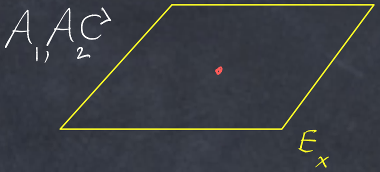
$$(A_1, A_2 \in SL_2\mathbb{C}, \quad A_1 A_2 = A_2 A_1) \xrightarrow{\text{holonomy}}$$

$$(E \longrightarrow S^1 \times S^1 \ \text{ rank 2 flat vector bundle})$$

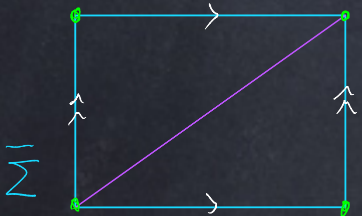
$$(A_1, A_2 \in SL_2\mathbb{C}, \quad A_1A_2A_1^{-1}A_2^{-1} \neq 1) \overset{\text{holonomy}}{\rightsquigarrow}$$

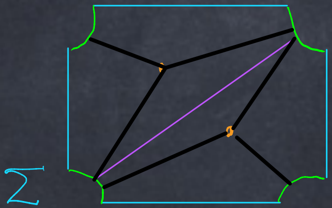
$$(E \longrightarrow \Sigma = S^1 \times S^1 \setminus D^2 \ \ \text{rank 2 flat vector bundle})$$



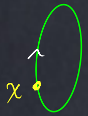


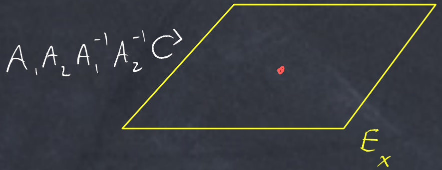
$\Sigma = \Sigma_0 \cup \Sigma_{-1} \cup \Sigma_{-2}$ ideal triangulation: abelianize on Σ_0



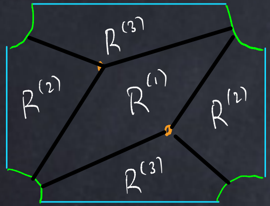


K flat section of $\mathbb{P}E\big|_{\partial\Sigma} \longrightarrow \partial\Sigma$





Parallel transport: two parallel sections K_i, K_j of $\mathbb{P}E\big|_{R^{(s)}} \longrightarrow R^{(s)}$



$\Sigma = \Sigma_0 \cup \Sigma_{-1} \cup \Sigma_{-2}$

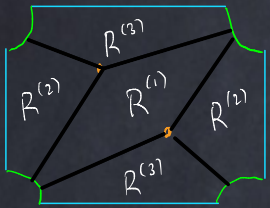
Genericity assumption: these sections are distinct.

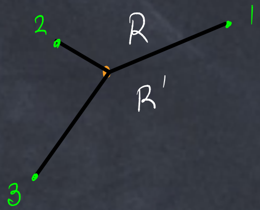
Abelianization: $\pi\colon \widetilde{\Sigma}_0 \longrightarrow \Sigma_0$ double cover

$L \longrightarrow \widetilde{\Sigma}_0$ line bundle

$E\big|_{\Sigma_0} \cong \pi_* L$

Extend double cover over Σ_{-1} to obtain $\pi\colon \widetilde{\Sigma}_{\geqslant -1} \longrightarrow \Sigma_{\geqslant -1}$

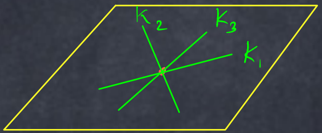




Unipotent automorphism of $E\big|_{\Sigma_{-1}}$

$$K_1 \oplus K_2 \longrightarrow K_1 \oplus K_3$$
$$k_1 \; + \; k_2 \longmapsto k_1 \; + \; \mathrm{proj}_{K_1}(k_2)$$

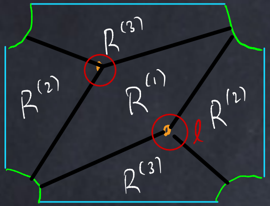
$$\mathrm{proj}_{K_1} : K_2 \xrightarrow{K_1} K_3$$

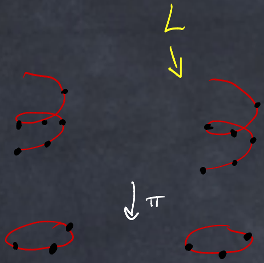


$$L \longrightarrow \widetilde{\Sigma}_{\geqslant -1}$$

Spectral networks

Lemma: (i) The double cover π is nontrivial on the link ℓ of Σ_{-2}
(ii) The holonomy of L around $\pi^{-1}(\ell)$ is $-\mathrm{id}$





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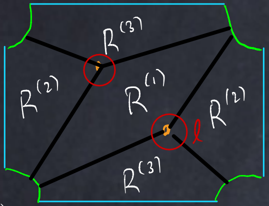
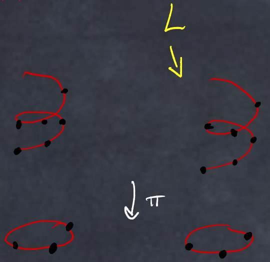
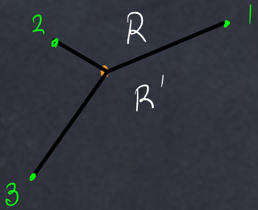




Proof: (i)



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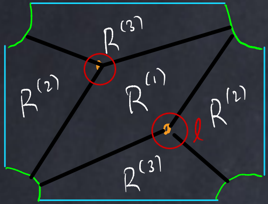
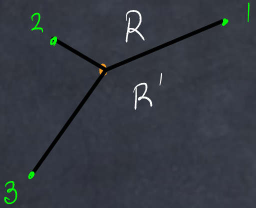
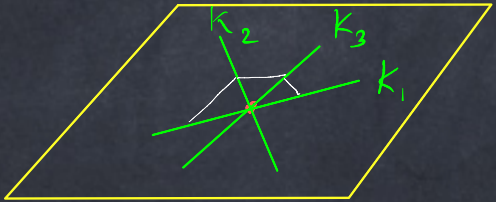




Proof: (ii)

$$K_1 \xrightarrow{id} K_1 \xrightarrow{proj_{K_2}} K_3 \xrightarrow{proj_{K_3}} K_1$$

$$K_3 \xrightarrow{proj_{K_1}} K_2 \xrightarrow{id} K_2 \xrightarrow{id} K_3$$



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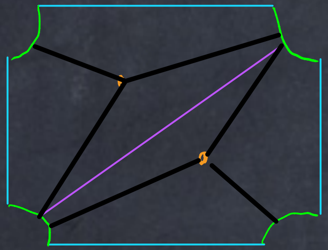
Walls: Σ_{-1}

Branch locus: Σ_{-2}



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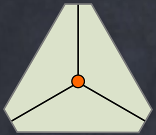
2d spectral networks were introduced by Gaiotto-Moore-Neitzke in their study of supersymmetric 4-dimensional gauge theories. They have motivated many mathematical constructions and conjectures: hyperkähler geometry, enumerative invariants, asymptotic analysis of complex ODE.

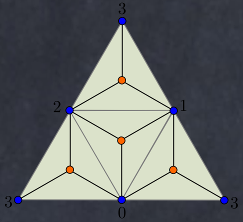
3d Spectral networks

Standard SN on 2-simplex Δ^2

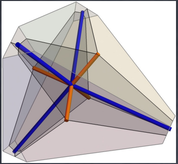


$\Delta = \Delta_0 \amalg \Delta_{-1} \amalg \Delta_{-2b}$

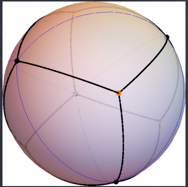
4 copies on $\partial \Delta^3$



Take "cone" 3d SN on Δ^3



Link $\mathcal{L}$ of singularity at center



Holonomies and cross-ratio

Double cover $\pi\colon \widetilde{\mathcal{L}} \longrightarrow \mathcal{L}$ branched at 4 points. $\widetilde{\mathcal{L}} \cong$ 2-torus

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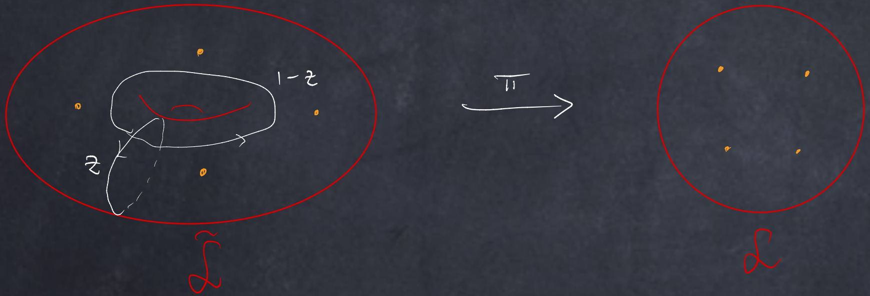
Lemma: Holonomies of $L \longrightarrow \widetilde{\mathcal{L}}$ are z and $1-z$ for some $z \in \mathbb{C}\backslash\{0,1\}$.



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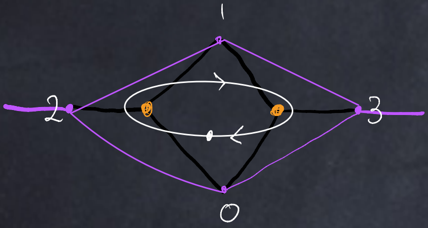

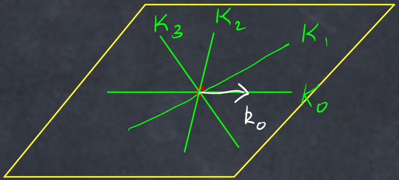

$$k_0 = k_2 + k_1$$
$$k_0 = k_3 + \lambda k_1$$

$$K_0 \xrightarrow{\mathrm{proj}_{K_2}} K_1 \xrightarrow{\mathrm{proj}_{K_3}} K_0$$
$$k_0 \longmapsto \frac{1}{\lambda} k_0$$

$$\frac{1}{\lambda} = \frac{(k_0 \wedge k_2)(k_1 \wedge k_3)}{(k_0 \wedge k_3)(k_1 \wedge k_2)}$$

Abelianization on M^3

$M = \bigcup_{\nu} \triangle_\nu$ ideal triangulation

$$M' = \bigcup_{\nu} \big(\triangle_\nu \backslash B_\epsilon(0) \big)$$

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$\partial \widetilde{M}' = \bigsqcup_\nu (S^1 \times S^1) \quad \cup \quad \partial M \times \{0,1\}$

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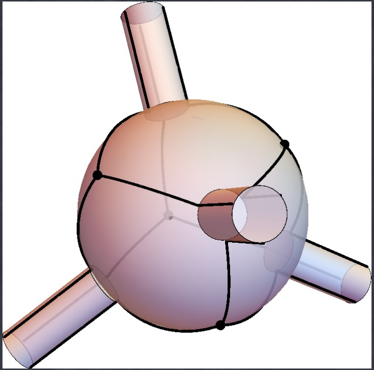

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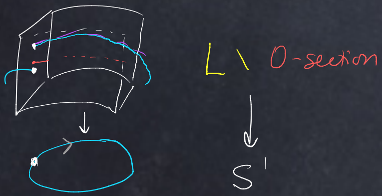
Flat $SL_2\mathbb{C}$-bundle $E \to M$ is abelianized to $\mathbb{C}^{\times}$-bundle $L \to \widetilde{M'}$

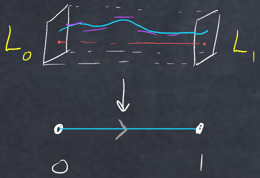
Abelianization: pushforward + unipotent gluing

(2) Invertible field theories

1-dimensional warmup:

$G = \mathbb{C}^\times$	
$\omega(\Theta)$	represents c_1
$\mathscr{F}_{\mathbb{C}^\times}(S^1;\Theta)$	holonomy of Θ around circle
$\mathscr{F}_{\mathbb{C}^\times}([0,1];\Theta)$	parallel transport
$\mathscr{F}_{\mathbb{C}^\times}(\mathrm{pt};\Theta)$	complex line





$P \longrightarrow M$ a principal $\mathrm{SL}_2\mathbb{C}$-bundle with connection Θ: invariants from $c_2 \in H^4(B\mathrm{SL}_2\mathbb{C})$

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Primary invariant: M^4 closed: $\int_M c_2(P) \in \mathbb{Z}$

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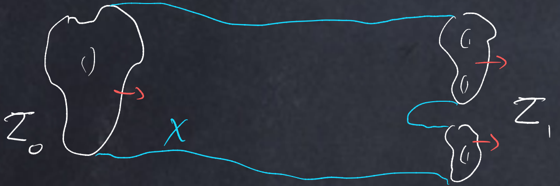
Primary invariant: M^4 closed: $\int_M c_2(P) \in \mathbb{Z}$

Secondary invariant: X^3 closed: $\int_X \check{c}_2(\Theta)$

$$\eta(\Theta) = \frac{1}{8\pi^2}\,\mathrm{trace}\left(\Theta \wedge \Omega - \frac{1}{6}\Theta \wedge [\Theta \wedge \Theta]\right) \in \Omega^3(P;\mathbb{C})$$

If $s\colon X \longrightarrow P$ is a section,

$$\mathscr{F}_{\mathrm{SL}_2\mathbb{C}}(X;\Theta) = \exp\left(2\pi i \int_X s^*\eta(\Theta)\right) \in \mathbb{C}^\times$$



$\mathcal{F}(X) : \mathcal{F}(Z_0) \to \mathcal{F}(Z_1)$

Differential cohomology

$$\begin{tikzcd} \check{c}_2 \in \check{H}^4(B_\nabla G) \arrow[r, dashed] \arrow[d, dashed] & \Omega^4_{\mathrm{cl}}(B_\nabla G) \ni \frac{1}{8\pi^2}\,\mathrm{trace}\,(\Omega\wedge\Omega) \arrow[d] \\ c_2 \in H^4(B_\nabla G;\mathbb{Z}) \arrow[r] & H^4(B_\nabla G;\mathbb{R}) \ni c_2^{\mathbb{R}} \end{tikzcd}$$

$$G = \mathrm{SL}_2\mathbb{C}$$

Abelianization

$$G = \mathrm{SL}_2\mathbb{C}$$

$$T = \mathbb{C}^\times \quad = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\}$$

$$H \quad = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & \mu \\ -\mu^{-1} & 0 \end{pmatrix} \right\}$$

$$\begin{tikzcd} BT \arrow[d, "2:1"', "p"] & \\ BH \arrow[r, "r"] \arrow[d, "q"] & BG \\ B\mu_2 & \end{tikzcd}$$

Abelianization

$$
\begin{tikzcd}
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Introduce new cohomology theory: $H\mathbb{Z} \longrightarrow E \longrightarrow \Sigma^{-2} H\mathbb{Z}/2\mathbb{Z}$

spin structures

Abelianization

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$$\begin{tikzcd} \lambda\ BT \arrow[d, "p"] & (2\lambda = c^2) \\ BH \arrow[r, "r"] \arrow[d, "q"] & BG \\ \alpha\ B\mu_2 & c_2 \end{tikzcd}$$

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Introduce new cohomology theory: $H\mathbb{Z} \longrightarrow E \longrightarrow \Sigma^{-2} H\mathbb{Z}/2\mathbb{Z}$

Proposition: $p_* \lambda = -r^* c_2 + q^* \alpha$ in $E^4(BH)$

Abelianization

Global:

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d, "\pi"'] & B_\nabla \mathbb{C}^\times \arrow[d, "p"] & \\ X \arrow[r] & B_\nabla H \arrow[r, "r"] & B_\nabla \mathrm{SL}_2\mathbb{C} \end{tikzcd}$$

Abelianization

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\end{tikzcd}$$

Stratified:

$$\begin{tikzcd}
\widetilde{X}_{\geqslant -3\mathrm{a}} \arrow[r] \arrow[d, "\pi"'] & B_\nabla \mathbb{C}^\times \arrow[d, "p"] & \\
X_{\geqslant -3\mathrm{a}} \arrow[r] \arrow[d, hook] & B_\nabla H \arrow[r, "r"] & B_\nabla \mathrm{SL}_2\mathbb{C} \\
X \arrow[urr] & &
\end{tikzcd}$$

(3) New construction of dilog via abelian Chern-Simons

$\mathcal{M}_T = (\mathbb{C}^\times)^2$ moduli space of flat $\mathbb{C}^\times$-connections on 2-torus $T = \mathbb{R}^2/\mathbb{Z}^2$ (θ^1, θ^2)

$\widehat{\mathcal{M}}_T = \mathbb{C}^2$ flat connections on trivial (product) $\mathbb{C}^\times$-bundle over T

moduli space of flat $\mathbb{C}^\times$-connections with nonflat trivialization over T/htpy

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- Universal $\mathbb{C}^\times$-connection over $\widehat{\mathcal{M}}_T, \mathcal{M}_T$:

$$\widehat{\mathcal{M}}_T \times T \times \mathbb{C}^\times \underset{\hat{t}_0}{\overset{\hat{\rho}}{\rightleftarrows}} \widehat{\mathcal{M}}_T \times T$$

$$\hat{t}_0^* \hat{\eta} = -u_1\, d\theta^1 - u_2\, d\theta^2 \qquad \in \Omega^1_{\widehat{\mathcal{M}}_T \times T}(\mathbb{C})$$

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- Descend to $\rho\colon \mathcal{Q} \longrightarrow \mathcal{M}_T \times T$ with connection η

- Curvature: $\Omega(\eta) = -\dfrac{d\mu_1}{\mu_1} \wedge d\theta^1 - \dfrac{d\mu_2}{\mu_2} \wedge d\theta^2$

Application of spin Chern-Simons

$$\mathcal{M}'_T = \{(\mu_1, \mu_2) \in \mathcal{M}_T : \mu_1 + \mu_2 = 1\}$$

$$\widehat{\mathcal{M}}'_T = \{(u_1, u_2) \in \widehat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\}$$

$$\begin{tikzcd}
 & (\epsilon')^*\mathcal{Q}' \arrow[dl, shift right, "\hat\rho'"'] & & \epsilon^*\mathcal{Q} \arrow[dl, shift right, "\hat\rho"'] \\
\widehat{\mathcal{M}}'_T \times T \arrow[rr, hook] \arrow[dd, "\epsilon'"] \arrow[ur, red, shift right, "\hat t'_0"'] & & \widehat{\mathcal{M}}_T \times T \arrow[dd, "\epsilon"] \arrow[ur, red, shift right, "\hat t_0"'] & \\
 & \mathcal{Q}' \arrow[dl, "\rho'"'] & & \mathcal{Q} \arrow[dl, "\rho"'] \\
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& \mathcal{Q}' \arrow[dl, "\rho'"'] & & \mathcal{Q} \arrow[dl, "\rho"'] \\
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\end{tikzcd}
\qquad
\begin{tikzcd}
{} \arrow[r, rightsquigarrow, "\mathscr{S}" description] & {}
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\qquad
\begin{tikzcd}
& (e')^*\mathcal{L}' \arrow[dl, shift right] & & e^*\mathcal{L} \arrow[dl, shift right] \\
\widehat{\mathcal{M}}'_T \arrow[ur, shift right, red, "\hat\tau'_0"'] \arrow[rr, hook] \arrow[dd, "e'"] & & \widehat{\mathcal{M}}_T \arrow[ur, shift right, red, "\hat\tau_0"'] \arrow[dd, "e"] & \\
& \mathcal{L}' \arrow[dl] & & \mathcal{L} \arrow[dl] \\
\mathcal{M}'_T \arrow[rr, hook] & & \mathcal{M}_T &
\end{tikzcd}$$

Application of spin Chern-Simons

$$\mathcal{M}'_T = \{(\mu_1, \mu_2) \in \mathcal{M}_T : \mu_1 + \mu_2 = 1\}$$
$$\widehat{\mathcal{M}}'_T = \{(u_1, u_2) \in \widehat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\}$$

$$\begin{tikzcd}
 & (\epsilon')^*\mathcal{Q}' \arrow[dl, "\hat\rho'"'] & & \epsilon^*\mathcal{Q} \arrow[dl, "\hat\rho"'] \\
\widehat{\mathcal{M}}'_T \times T \arrow[ur, shift right=2, "\hat t_0'"'] \arrow[rr, hook] \arrow[dd, "\epsilon'"] & & \widehat{\mathcal{M}}_T \times T \arrow[ur, shift right=2, "\hat t_0"'] \arrow[dd, "\epsilon"] & \\
 & \mathcal{Q}' \arrow[dl, "\rho'"'] & & \mathcal{Q} \arrow[dl, "\rho"'] \\
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Proposition: : $\mathcal{L}' \longrightarrow \mathcal{M}'_T$ is flat with trivial holonomy

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\widehat{\mathcal{M}}'_T \arrow[ur, shift left=2, "(e')^*\tau'"] \arrow[ur, shift right, "\hat{\tau}'_0"'] \arrow[rr, hook] \arrow[dd, "e'"] & & \widehat{\mathcal{M}}_T \arrow[ur, shift right, "\hat{\tau}_0"'] \arrow[dd, "e"] & \\
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$$

Define a function on $\widehat{\mathcal{M}}'_T$ as the ratio of two sections:

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\varphi = \frac{\widehat{\tau}'_0}{(e')^*\tau'} : \widehat{\mathcal{M}}'_T \longrightarrow \mathbb{C}^\times
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$$\frac{d\varphi}{\varphi} = \frac{1}{4\pi\sqrt{-1}}(u_1 du_2 - u_2 du_1)$$

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Define $L\colon \widehat{\mathcal{M}}'_T \to \mathbb{C}/\mathbb{Z}(2)$ by $\varphi = \exp\left(\frac{L}{2\pi\sqrt{-1}}\right)$, so $dL = \frac{u_1 du_2 - u_2 du_1}{2}$

Application of spin Chern-Simons to prove 5-term identity

Theorem: The sum $L(u_1, v_1) + \cdots + L(u_5, v_5)$ is independent of $(u_i, v_i) \in \widehat{\mathcal{M}}'_T$, $i \in \mathbb{Z}/5\mathbb{Z}$, which satisfy

$$v_i = u_{i-1} + u_{i+1} \qquad \text{for all } i$$

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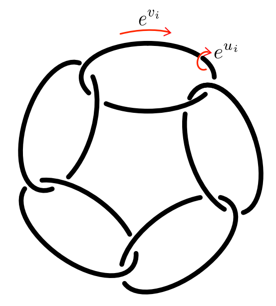
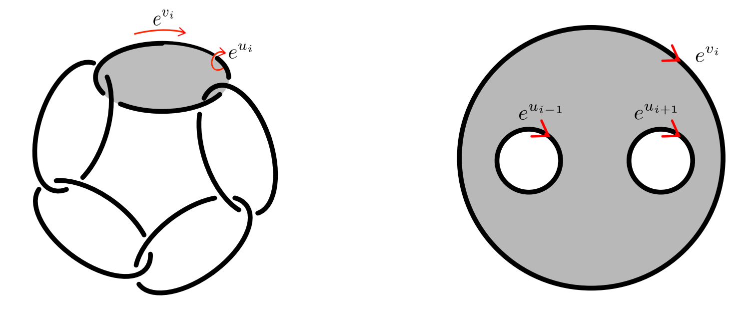
Remark: Write $z_i = e^{u_i}$, so $1 - z_i = z_{i-1} z_{i+1}$. There is a connected complex 2-manifold $\mathcal{M}'_X$ of solutions

$$x \ , \ \frac{1-x}{1-xy} \ , \ \frac{1-y}{1-xy} \ , \ y \ , \ 1-xy$$

parametrized by $x, y \in \mathbb{C}$ satisfying $xy \neq 0$, $x \neq 1$, $y \neq 1$, and $xy \neq 1$.

Proof sketch: Let X be the compact spin 3-manifold with boundary formed from S^3 by removing a tubular neighborhood of the 5-component link





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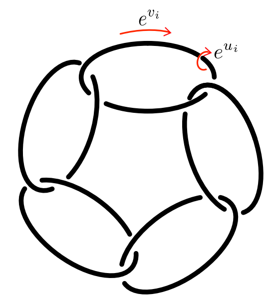

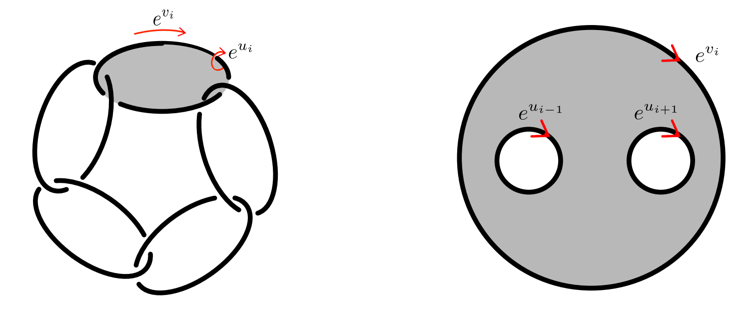

$\partial X \approx T^{\amalg 5}$ and the restriction of a flat connnection on X to the boundary satisfies

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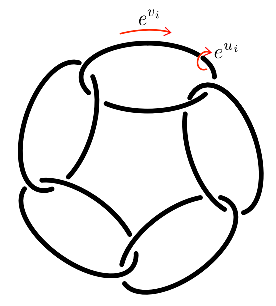

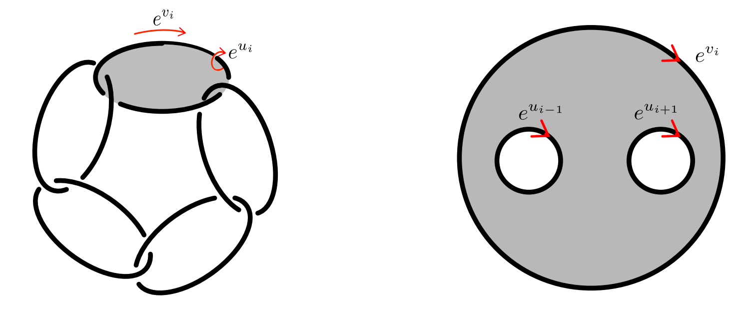

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$\mathcal{M}_X$ moduli space of flat $\mathbb{C}^\times$-connections on X

$\widehat{\mathcal{M}}_X$ moduli space of flat $\mathbb{C}^\times$-connections with nonflat trivialization over X/htpy

$$
\begin{tikzcd}
& (\widehat{r}')^*(\widehat{\mathcal{L}}')^{\boxtimes 5} \arrow[dl] & & (\widehat{\mathcal{L}}')^{\boxtimes 5} \arrow[dl, shift right] \\
\widehat{\mathcal{M}}'_X \arrow[rr, "\widehat{r}'"] \arrow[dd, "e'_X"] & & (\widehat{\mathcal{M}}'_T)^5 \arrow[dd, "(e'_T)^5"] \arrow[ur, shift right, "(\widehat{\tau}_0)^{\boxtimes 5}"'] & \\
& (r')^*(\mathcal{L}')^{\boxtimes 5} \arrow[dl, shift right] & & (\mathcal{L}')^{\boxtimes 5} \arrow[dl, shift right] \\
\mathcal{M}'_X \arrow[rr, "r'"] \arrow[ur, shift right, "\mathscr{S}"'] & & (\mathcal{M}'_T)^5 \arrow[ur, shift right, "(\tau')^{\boxtimes 5}"'] &
\end{tikzcd}
$$

- The section $\mathscr{S}$ is flat (covariant derivative = integral of $\frac{1}{2}\omega \wedge \omega$ vanishes)
- Choose flat section τ' so $\mathscr{S} = (r')^*\big[(\tau')^{\boxtimes 5}\big]$ (unique up to 5th root of unity)
- Pullbacks of $\mathscr{S}$ and $(\widehat{\tau}_0)^{\boxtimes 5}$ to $\widehat{\mathcal{M}}'_X$ agree $\Longrightarrow$ $(\widehat{\tau}_0/\tau')^{\boxtimes 5} = 1$ (exp of 5-term identity) //