DIFFERENTIAL GEOMETRY

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What follows are lecture notes from graduate courses given at the University of Texas at Austin in Fall, 2021 and at Harvard University in Fall, 2024. These notes are a bit rough in many places, so use at your own risk!

The overarching theme is the Felix Klein dictum that places symmetry groups at the center of geometry. I introduce this idea in Lecture 1 in the form of *symmetry types* in affine geometry. Calculus on smooth manifolds is modeled on calculus in real affine spaces, and so too are geometric structures on smooth manifolds. To get to geometric structures on manifolds we develop many ideas that have great interest and application in their own right. The Table of Contents provides a detailed outline of what is contained here. Observe that after preliminaries about flows, Lie derivatives, distributions, and the Frobenius theorem, we introduce curvature in its classical context: curves and surfaces in Euclidean 2- and 3-space. After that we turn to the general theory of connections on principal bundles, eventually focusing on the intrinsic case: linear and affine connections. We tell some basics about curvature in Riemannian geometry, and also introduce complex and Kähler manifolds. Lectures 21–23 contain a proof[1] of the Chern–Gauss–Bonnet theorem, and in the process we expose basics about Chern–Weil and Chern–Simons forms.

I have appended homework exercises to these notes.

I welcome feedback of any kind.

Contents

Date: February 18, 2025.

[1] up to a constant that we invite the reader to fix!

Lecture 1: Parallelism and geometric structures

We begin with the basic geometry of affine space. Affine geometry is the geometry of parallelism, which is expressed by the simply transitive action of a vector space: the action of the vector space on the affine space is called *translation*. We say that the affine space is a *torsor* over the vector space. Smooth manifolds generally do not admit a global parallelism, but rather one can introduce a structure of parallel transport along curves, as we will learn later in the course.

The second topic in this lecture is symmetry groups of vector and affine spaces. A vector space has *linear* symmetries, whereas an affine space has *affine* symmetries. An affine map between affine spaces is characterized by the property that its differential is constant (under translation), and this leads to the definition of an affine symmetry of a fixed affine space. We introduce bases of vector spaces and frames in an affine space (Definition 1.26). The spaces of bases and frames are torsors over the linear and affine symmetry groups.

The final topic is *symmetry types* (Definition 1.37). This is one of the key ideas in the first part of the course. A symmetry type is a Lie group over a general linear group. (We introduce Lie groups systematically soon.) Felix Klein, in his *Erlangen program* (**1.34**), told that a particular type of geometry is characterized by its symmetry group. A symmetry type encodes this idea. A symmetry types leads to both a type of affine geometry—as we explain in this lecture—and a type of geometry on smooth manifolds—which we explain in due course. Thus the orthogonal group as a symmetry type determines both the Euclidean geometry of affine space and Riemannian geometry of smooth manifolds. There are many, many species of geometric structures, some more common than others. The centrality of symmetry groups in geometry—as emphasized by Klein and subsequently by Cartan, Chern, Singer, Bott, . . . —guides our approach throughout the course.

We provide a brief appendix on group extensions for reference.

The notes on Multivariable Analysis and on Differential Topology contain more material about affine spaces. At points in this lecture I use concepts from the theory of smooth manifolds, such as fiber bundles, that you will find explained in the Differential Topology notes.

Affine geometry

Definition 1.1. Let V be a vector space. An *affine space over* V is a set A equipped with a simply transitive action $+\colon A \times V \to A$ of V.

We call V the *tangent space* to A. We do not assume that V is finite dimensional. See Figure 1.

FIGURE 1. An affine space A and its tangent space V

(1.2) *The ground field.* In this class we mostly use real vector spaces, though occasionally we encounter complex vector spaces. Affine geometry may be done over any field. If V is a finite dimensional real vector space, then there is a unique topology on V for which the vector space operations are continuous, and we can transport that topology to A. Then A has a standard smooth structure and may be considered as a smooth manifold. Of course, this is backwards: a smooth manifold is locally modeled on affine space, so we need to know about affine space first.

(1.3) *Torsors.* The simple transitivity of the *translation* action of V on A means given $p, q \in A$ there exists a unique $\xi \in V$ such that $q = p+\xi$. We write $\xi = q-p$. We encounter simply transitive actions often, and we use a standard term for them.

Definition 1.4. Let G be a group. A *(left/right) torsor over* G is a set T equipped with a simply transitive (left/right) G-action. We call T a *(left/right)* G*-torsor.*

So an affine space is a torsor over a vector space. If G is a topological group, then a torsor over G is assumed to be a topological space and the action is assumed continuous. Similarly, if G is a Lie group, then a torsor over G is assumed to be a smooth manifold and the action is assumed smooth. The group G with left/right multiplication is the "trivial" left/right G-torsor.

Suppose T is a right G-torsor. (This discussion works for left G-torsors as well.) A point $t_0 \in T$ induces an isomorphism of right G-torsors

$$\begin{aligned} \theta_{t_0}\colon G &\longrightarrow T \\ g &\longmapsto t_0 g \end{aligned} \tag{1.5}$$

If $t_1 \subset T$ and $t_1 = t_0 h$ for the uniquely determined $h \in G$, then $\theta_{t_1} \circ \theta_{t_0}^{-1}\colon G \to G$ is *left* multiplication by h^{-1}, which is a self-map of the right G-torsor G. One thinks of a choice of $t_0 \in G$ as a trivialization of T, and we have computed the change of trivialization in the diagram

$$\begin{tikzcd} & T & \\ G \arrow[ur, "\theta_{t_0}"] \arrow[rr, dashed] & & G \arrow[ul, "\theta_{t_1}"'] \end{tikzcd} \tag{1.6}$$

(1.7) *Model spaces.* Model geometries play an important role. Fix a dimension $n \in \mathbb{Z}^{\ge 0}$.

Definition 1.8.

(1) The standard n-dimensional vector space is

$$\mathbb{R}^n = \{(\xi^1, \dots, \xi^n) : \xi^i \in \mathbb{R}\} \tag{1.9}$$

equipped with its standard zero vector, vector addition, and scalar multiplication.

(2) The standard n-dimensional affine space is

$$\mathbb{A}^n = \{(x^1, \dots, x^n) : x^i \in \mathbb{R}\} \tag{1.10}$$

equipped with the standard translation action of $\mathbb{R}^n$.

(1.11) *Weighted averages.* Let A be an affine space, $p_0, \dots, p_k \in A$ and $\lambda^0, \dots, \lambda^k \in \mathbb{R}$. Then if $\lambda^0 + \dots + \lambda^k = 1$ the point

$$\lambda^i p_i = \lambda^0 p_0 + \dots \lambda^k p_k \in A \tag{1.12}$$

is well-defined. (Note the summation convention in (1.12).) A *convex* weighted average is one in which $0 \le \lambda^i \le 1$ for all i. A *convex subset* of an affine space is a subset that contains all of its convex weighted averages.

(1.13) *Affine maps and affine subspaces.*

Definition 1.14. Let V, W be vector spaces and A, B affine spaces over V, W, respectively. A map $f\colon A \to B$ is *affine* if there exists a linear map $T\colon V \to W$ such that

$$f(p + \xi) = f(p) + T\xi, \qquad p \in A, \quad \xi \in V. \tag{1.15}$$

In this case we write $T = df$. Alternatively, if V, W are finite dimensional (or equipped with Banach space structures) we can say that a smooth map $f\colon A \to B$ is affine iff its differential $df_p\colon V \to W$ is independent of p. (Recall that in general the differential df of a differentiable map $f\colon A \to B$ is a continuous map $df\colon A \to \mathrm{Hom}(V, W)$.)

Definition 1.16. Let A be an affine space with tangent space V. A subset $A' \subset A$ is an *affine subspace* if there exists a linear subspace $V' \subset V$ such that $q' - p' \in V'$ for all $p', q' \in V'$. We call V' the *tangent space* to A'. Affine subspaces $A', A'' \subset A$ are *parallel* if they have equal tangent spaces. (If one is willing to say that affine subspaces of different dimensions are parallel, then one should require that one tangent space is a subspace of the other.)

The subspace A' is affine over V'. Translation maps affine subspaces to affine subspaces: for all $\xi \in V$, the subset $A' + \xi \subset A$ is an affine subspace with the same tangent space $V' \subset V$. We say that A' and $A' + \xi$ are *parallel*. A linear subspace $V' \subset V$ defines a *foliation* of A by affine subspaces with tangent space V'; see Figure 2.

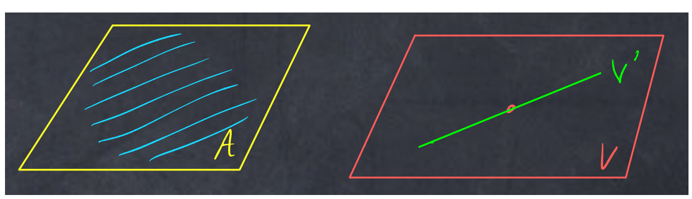

FIGURE 2. The foliation of affine subspaces with tangent space $V' \subset V$

Example 1.17 (geodesic motion)**.** Parallelism leads to the notion of a geodesic motion. Recall that a *motion* in an affine space A over V is a smooth map $\gamma\colon I \to A$ for an open interval $I \subset \mathbb{R}$. The *velocity* $\dot\gamma\colon I \to A$ is constant if and only if γ is the restriction of an affine map $\mathbb{R} \to A$. This holds iff the *acceleration* $\ddot\gamma\colon I \to A$ vanishes. Motions with zero acceleration are *geodesic* motions (or *constant velocity* motions). The range of the motion is a subset of an affine line. We will see later that on a smooth manifold we only need parallelism along curves to define geodesic motion.

Global parallelism

(1.18) *Parallelism in affine space.* As stated earlier, an affine space A has global parallelism. This manifests in various ways. The structure of affine space—the simply transitive (translation) action by a vector space V—is an expression of parallelism. Thus, for example if $A' \in A$ is an affine subspace, and $p \in A$, then there is a unique affine subspace through p parallel to A'. This, of course, is the famous parallel postulate of Euclid.

FIGURE 3. A constant (parallel) vector field on A

The translation action also leads to the notion of a constant or translationally invariant or parallel vector field on an affine space A. In general, a vector field is a function $A \to V$, and a constant vector field is a constant function. So given a vector $\xi \in V$ there is a constant vector field with value ξ. (See Figure 3.) Also, a vector at any point of A extends uniquely to a constant vector field.

(1.19) *Global parallelism on smooth manifolds.*

Definition 1.20. Let X be a smooth manifold. A *(global) parallelism* on X is a vector space V and an isomorphism

$$\begin{tikzcd} X \times V \arrow[rr, "\cong"] \arrow[dr] & & TX \arrow[dl] \\ & X & \end{tikzcd} \tag{1.21}$$

of vector bundles over X.

Thus a parallelism is a structure which identifies each tangent space T_pX with a fixed vector space V. An affine space carries a canonical parallelism in this sense. A smooth manifold is *parallelizable* if a global parallelism exists. Typically global parallelisms do not exist. For example, the Hairy Ball Theorem implies that S^2 is not parallelizable. A classical theorem settles the question of which spheres are parallelizable.

Theorem 1.22 (Kervaire, Bott–Milnor 1958)**.** *Suppose the sphere S^n is parallelizable. Then its dimension satisfies $n \in \{0, 1, 3, 7\}$.*

A *Lie group* is the marriage of a group and a smooth manifold, a ceremony we will perform soon in an upcoming lecture. A Lie group G carries two canonical parallelisms via left and right translation; if G is abelian, then these agree. A torsor over a Lie group also carries a canonical parallelism;

the global parallelism of affine spaces is a special case. The spheres S^0, S^1, S^3 admit Lie group structures, but S^7 does not.

Remark 1.23. Any smooth manifold admits a structure called a *linear connection* or *covariant derivative* which allows us to define parallelism along curves. There are new phenomena of *torsion*, *curvature*, and *holonomy* which do not appear in the global parallelism of affine space. This *intrinsic* parallelism on smooth manifolds has an *extrinsic* generalization: *connections on principal bundles*. Connections are a central focus of this course.

Linear and affine symmetries

Definition 1.24.

(1) Let V be a vector space. A *linear symmetry* of V is an invertible linear map $T\colon V \to V$. The group of linear symmetries of V is denoted $\mathrm{Aut}(V)$.
(2) Let A be an affine space over V. An *affine symmetry* of A is an invertible affine map $f\colon A \to A$. The group of affine symmetries of A is denoted $\mathrm{Aut}(A)$.

If V is finite dimensional (over $\mathbb{R}$), then we can endow $\mathrm{Aut}(V)$ and $\mathrm{Aut}(A)$ with the structure of a Lie group.

The differential d is a group homomorphism that fits into the *group extension*

$$1 \longrightarrow V \longrightarrow \mathrm{Aut}(A) \xrightarrow{\;d\;} \mathrm{Aut}(V) \longrightarrow 1 \tag{1.25}$$

(Recall that the differential of an affine map is constant.) The kernel of d is the normal subgroup of translations. The group extension (1.25) is split by a choice of point $p_0 \in A$; the splitting $\sigma_{p_0}\colon \mathrm{Aut}(V) \to \mathrm{Aut}(A)$ maps a linear symmetry T to the unique affine symmetry that fixes p_0 and has differential T.

Bases and frames

The word choice is a bit fraught: for a vector space we might use 'basis' and 'frame' synonymously.

Definition 1.26.

(1) Let V be an n-dimensional real vector space. A *basis* of V is a linear isomorphism

$$b\colon \mathbb{R}^n \longrightarrow V. \tag{1.27}$$

The set of all bases of V is denoted $\mathcal{B}(V)$.

(2) Let A be an n-dimensional real affine space. A *frame* in A is an affine isomorphism

$$f\colon \mathbb{A}^n \longrightarrow A. \tag{1.28}$$

The set of all frames in A is denoted $\mathcal{B}(A)$.

Remark 1.29.

(1) Automorphisms of $\mathbb{R}^n$ act on the *right* on bases, and automorphism of V act on the *left*. Both actions are simply transitive. The right and left torsor structures are indicated thus:

$$\mathrm{Aut}(V) \circlearrowright \mathcal{B}(V) \circlearrowleft \mathrm{GL}_n\mathbb{R} \tag{1.30}$$

In a similar way we have the right and left torsors

$$\mathrm{Aut}(A) \circlearrowright \mathcal{B}(A) \circlearrowleft \mathrm{Aff}_n\mathbb{R}, \tag{1.31}$$

where $\mathrm{Aff}_n\mathbb{R} = \mathrm{Aff}(\mathbb{A}^n)$ is the group of affine symmetries of $\mathbb{A}^n$.

(2) My convention is that *right* group actions are *structural* and that *left* group actions are *geometric*. Thus the left action in (1.30) is induced from the geometric action of $\mathrm{Aut}(V)$ on V, and in (1.31) from the geometric action of $\mathrm{Aut}(A)$ on A. These are the natural symmetries. The right actions are "internal"; they depend on the choice of frame or basis. You might think of the left action as "active" and the right as "passive". Example: there is a right (internal) action on bases that exchanges the first and second basis elements.

(3) Both $\mathcal{B}(V)$ and $\mathcal{B}(A)$ have natural smooth manifold structures, and the actions in (1.30) and (1.31) are Lie group actions.

(4) Let $0 \in \mathbb{A}^n$ denote the point $(0, \dots, 0)$. The map

$$\begin{aligned} \mathcal{B}(A) &\longrightarrow A \times \mathcal{B}(V) \\ f &\longmapsto f(0),\ df_0 \end{aligned} \tag{1.32}$$

is a diffeomorphism. Furthermore, the diffeomorphism is equivariant for the left V-action and the right $\mathrm{GL}_n\mathbb{R}$-action, and those actions commute. (What is the right $\mathrm{GL}_n\mathbb{R}$-action on $\mathcal{B}(A)$? Its definition depends on the basepoint $0 \in \mathbb{A}^n$.)

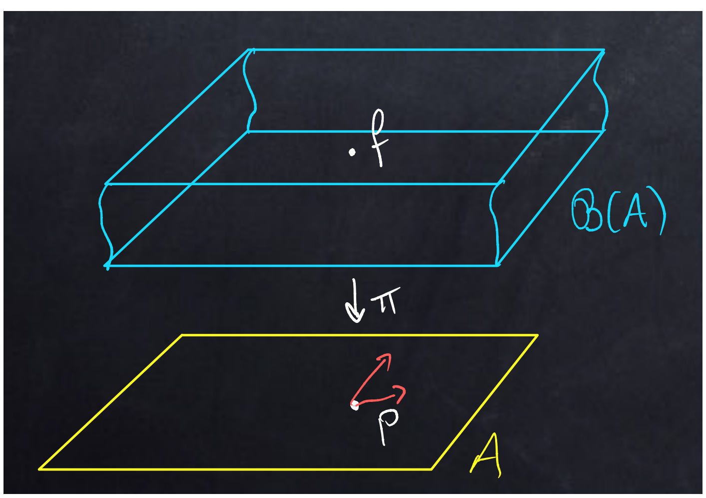

FIGURE 4. The bundle of frames of the affine space A

(5) The projection

$$\begin{aligned} \pi\colon \mathcal{B}(A) &\longrightarrow A \\ f &\longmapsto f(0) \end{aligned} \tag{1.33}$$

is a fiber bundle of a special type: a *principal* $\mathrm{GL}_n\mathbb{R}$-*bundle*. The total space $\mathcal{B}(A)$ carries a right $\mathrm{GL}_n\mathbb{R}$-action which commutes with π, and the fibers of π are right $\mathrm{GL}_n\mathbb{R}$-torsors. We identify the fiber of π over $p \in A$ as the set of bases of $T_pA = V$ by sending f to its differential $df_0\colon \mathbb{R}^n \xrightarrow{\cong} V$.

We observe that π is the quotient map of the right $\mathrm{GL}_n\mathbb{R}$-action on $\mathcal{B}(A)$.

A smooth manifold carries an analog of (1.33)—the *bundle of frames* or *frame bundle*—which plays an important role in our study of differential geometry. However, on a general smooth manifold we do not have the geometric left action of a group. On affine space the left V-action on π encodes global parallelism. On a smooth manifold we will introduce a different notion of parallelism on the total space of the frame bundle.

Geometric structures: symmetry types

To define a geometric structure we (1) specify a symmetry type on the model vector space $\mathbb{R}^n$, (2) extend to the model affine space $\mathbb{A}^n$ imposing translation invariance, and (3) use frames to define a structure on a general n-dimensional affine space. As mentioned in the introduction to this lecture, a symmetry type also defines a notion of geometric structure on smooth manifolds, an idea we develop in due course.

(1.34) *Erlangen program.* Linear geometry is the study of invariants under linear symmetry. Such concepts as basis, spanning subset, linear subspace are all examples of linear notions. In other words, the set of all bases or of linear subspaces (of fixed dimension) in a vector space V is acted on by the group $\mathrm{Aut}(V)$. A similar statement can be made for affine geometry and affine symmetry. In a similar vein, differential topology is the study of C^∞ concepts. By contrast, Euclidean geometry is the study of invariants under the Euclidean group of symmetries, which is a subgroup of the group of affine symmetries. Whereas the notion of a triangle makes sense in both affine and Euclidean geometry, the notion of the perimeter or of the area of a triangle is only defined in Euclidean geometry. On the other hand, a triangle is not a C^∞ concept, so it does not exist in differential topology.

In 1872 Felix Klein introduced a dictum which characterizes different types of geometry by their symmetry. A "small" interpretation of Klein's *Erlangen program* in the case of affine geometry focuses on a single affine space A and tells that affine geometry on A is the study of structures/properties/quantities of/in A which are invariant under the action of $\mathrm{Aut}(A)$ on A. A "large" interpretation says that affine geometry is the study of the category of affine spaces and affine maps between them. The notion of a *symmetry type* generates both of these interpretations.

Remark 1.35. A general principal is that a smaller symmetry group has more invariants, so in this context a smaller symmetry group leads to richer geometry.

(1.36) *Symmetry types.* Following (1.34) we define a linear geometric structure on $\mathbb{R}^n$ by a group acting linearly on $\mathbb{R}^n$.

Definition 1.37. An n-dimensional (linear) *symmetry type* is a pair (G_n, λ_n) in which G_n is a Lie group and $\lambda_n\colon G_n \to \mathrm{GL}_n\mathbb{R}$ is a homomorphism of Lie groups.

Roughly, (G_n, λ_n)-geometry is the study of structures/properties/quantities in $\mathbb{R}^n$ invariant under the G_n-action. A symmetry type induces a canonical symmetry group of the model space $\mathbb{A}^n$ via pullback of group extensions:

(1.38)
$$\begin{tikzcd}
1 \arrow[r] & \mathbb{R}^n \arrow[r, dashed] \arrow[d, equal] & \mathcal{G}_n \arrow[r, dashed] \arrow[d, dashed, "\tilde\lambda_n"] & G_n \arrow[r] \arrow[d, "\lambda_n"] & 1 \\
1 \arrow[r] & \mathbb{R}^n \arrow[r] & \mathrm{Aff}_n\mathbb{R} \arrow[r] & \mathrm{GL}_n\mathbb{R} \arrow[r] & 1
\end{tikzcd}$$

We call the pair $(\mathcal{G}_n, \tilde\lambda_n)$ an *affine symmetry type*. Note that $\mathbb{R}^n \subset \mathcal{G}_n$ is a *normal* subgroup. The existence of a normal subgroup of translations is the statement that the affine symmetry type includes translation invariance. Quotienting by the normal subgroup of translations passes in the other direction from an affine symmetry type to a linear symmetry type. Also, a choice of $p \in \mathbb{A}^n$ splits the group extension (1.38): sitting over G_n is the subgroup of $\mathcal{G}_n$ consisting of symmetries that fix p. We use the canonical choice $p = 0 = (0, \dots, 0)$, as in (1.32).

Remark 1.39. For some symmetry types (G_n, λ_n) there is a geometric structure on $\mathbb{R}^n$ whose group of invariants is $G_n \subset \mathrm{GL}_n\mathbb{R}$. This is true in Example 1.40 below: the orthogonal group $\mathrm{O}_n \subset \mathrm{GL}_n\mathbb{R}$ is the subgroup of matrices that preserves the standard inner product. For any symmetry type (G_n, λ_n), the invariants of $G_n \circlearrowright \mathbb{R}^n$ are important elements of the geometry. They may be tensors of various types or other kinds of geometric objects. But one needn't find a collection that *characterizes* G_n. Rather, the symmetry type defines the geometry completely. Also, we neither assume that λ_n is injective nor that λ_n is surjective. In particular, λ_n may have a nontrivial kernel $K_n \subset G_n$, a normal subgroup of *internal symmetries*. One can imagine an "internal geometric structure" whose symmetry group is K_n, and in affine space such an internal geometric structure exists at each point and is acted on by translations; see Example 1.43 below. Traditionally, Cartan's notion of "G-structure" has λ_n injective, so no internal geometric structures. Notice that this definition excludes Example 1.44.

Example 1.40 (Euclidean geometry)**.** Let $G_n = \mathrm{O}_n$ be the orthogonal group and $\lambda_n\colon \mathrm{O}_n \hookrightarrow \mathrm{GL}_n\mathbb{R}$ be the inclusion. This is the symmetry group of the standard positive definite inner product on $\mathbb{R}^n$, and linear orthogonal geometry is the study of linear invariants, such as the length of a vector, the angle between two vectors, the volume of parallelepipeds of all dimensions, etc. The induced standard affine symmetry group $\mathcal{G}_n$ in (1.38) is the Euclidean group, a subgroup of $\mathrm{Aff}_n\mathbb{R}$. It is the symmetry group of Euclidean geometry in the standard affine space, including the traditional rotations, reflections, and translations of Euclidean geometry.

Example 1.41 (oriented volume geometry)**.** Let $G_n = \mathrm{SL}_n\mathbb{R}$ be the group of invertible $n \times n$ matrices of determinant one, and let $\lambda_n\colon \mathrm{SL}_n\mathbb{R} \hookrightarrow \mathrm{GL}_n\mathbb{R}$ be the inclusion. The invariant in this geometry is the signed volume of an n-dimensional parallelepiped.

Example 1.42 (Lorentz geometry)**.** This is formally similar to Example 1.40, but the positive definite inner product is replaced by a standard nondegenerate symmetric bilinear form on $\mathbb{R}^n$ of signature $(1, n-1)$. In this case $G_n = \mathrm{O}_{1,n-1}$ is the Lorentz group and again λ_n is the inclusion. The induced affine symmetry group is a form of the Poincaré group. The symmetry type one uses to define a general relativistic quantum field theory uses a subgroup of index two in G_n (that preserves a time orientation).

Example 1.43 (internal symmetry)**.** Let K be any Lie group and set $G_n = \mathrm{GL}_n\mathbb{R} \times K$ with λ_n projection onto the first factor. The induced affine symmetry group is $\mathcal{G}_n = \mathrm{Aff}_n\mathbb{R} \times K$. Think of this as geometry on $\mathbb{A}^n$ equipped with some internal structure at each point which transforms under the symmetry group K. For example, if K is cyclic of order two, this is the symmetry group of the product bundle (covering) $\mathbb{A}^n \times \{0,1\} \to \mathbb{A}^n$: the internal symmetry group K permutes the fiber over each point of $\mathbb{A}^n$.

Example 1.44 (spin geometry)**.** The spin group $G_n = \mathrm{Spin}_n$ is a double cover of the identity component $\mathrm{SO}_n \subset \mathrm{O}_n$, and that double cover map followed by the inclusion defines $\lambda_n\colon \mathrm{Spin}_n \to \mathrm{GL}_n\mathbb{R}$. It is difficult to describe a structure on $\mathbb{R}^n$ whose precise symmetry group defines the spin symmetry type.

Remark 1.45 (stable symmetry types)*.* The symmetry types in the previous example can be defined in all dimensions coherently, and implicitly that is what we understand when we say 'Euclidean geometry' or 'Lorentz geometry'. We define a *stable symmetry type* as follows. First, for each $n \in \mathbb{Z}^{\ge 0}$ define an inclusion

$$\begin{aligned} \mathrm{GL}_n\mathbb{R} &\longrightarrow \mathrm{GL}_{n+1}\mathbb{R} \\ A \quad &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \end{aligned} \tag{1.46}$$

A stable symmetry type is a collection $\{(G_n, \lambda_n)\}_{n\in\mathbb{Z}^{\ge 0}}$ of symmetry types, one in each dimension, and inclusions $i_n\colon G_n \hookrightarrow G_{n+1}$ such that the diagram of group homomorphisms

$$\begin{tikzcd} G_n \arrow[r, hook, "i_n"] \arrow[d, "\lambda_n"'] & G_{n+1} \arrow[d, "\lambda_{n+1}"] \\ \mathrm{GL}_n\mathbb{R} \arrow[r, hook] & \mathrm{GL}_{n+1}\mathbb{R} \end{tikzcd} \tag{1.47}$$

commutes and is a pullback diagram. Note that the internal symmetry group $K_n = \ker \lambda_n$ is independent of n up to an isomorphism determined by the data.

From symmetry types to vector and affine geometries

A symmetry type lets us define linear and affine geometric structures of that type on abstract vector and affine spaces. The construction follows a basic theme: torsors control geometric structures. It is the geometric structures on affine spaces that we generalize later to geometric structures on smooth manifolds.

We begin with an important general maneuver on torsors.

(1.48) *The mixing construction.* Let $\lambda\colon G \to \overline{G}$ be a homomorphism of (Lie) groups, and suppose T is a right G-torsor. Define the right $\overline{G}$-torsor $\lambda(T)$ as the *mixing construction*

$$\lambda(T) = T \times_G \overline{G} = T \times \overline{G}/\sim \tag{1.49}$$

where the equivalence relation $\sim$ is

$$(t, \bar{g}) \sim (tg, \lambda(g)^{-1}\bar{g}), \qquad t \in T, \quad g \in G, \quad \bar{g} \in \overline{G}. \tag{1.50}$$

In other words, $\lambda(T)$ is the quotient of $T \times \overline{G}$ by the right G-action $(t, \bar{g}) \xrightarrow{g} (tg, \lambda(g)^{-1}\bar{g})$.

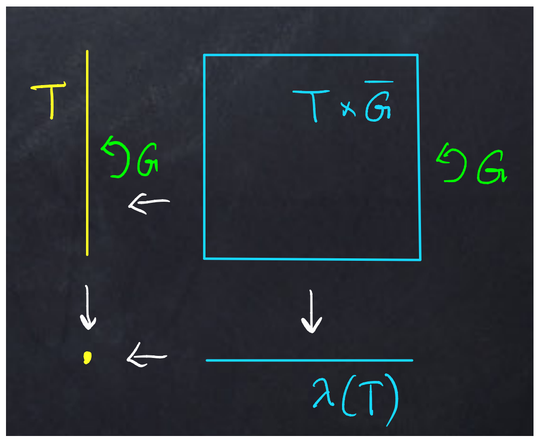

Figure 5. The mixing construction

We invite the reader to visualize the mixing construction using Figure 5. We later encounter this construction fiber-by-fiber in a fiber bundle of torsors, i.e., in a principal bundle.

Think through this construction in terms of trivializations (1.5) of the torsor T. Show that a trivialization of T—a point $t_0 \in T$—identifies $\lambda(T)$ with $\overline{G}$. How does a change of trivialization of T change this identification?

(1.51) *Geometric structure on a vector space.* Given a symmetry type (G_n, λ_n), we specify the data of a geometric structure of type (G_n, λ_n) on an arbitrary n-dimensional vector space V.

Definition 1.52. Fix $n \in \mathbb{Z}^{\ge 0}$ and a symmetry type (G_n, λ_n). Let V be an n-dimensional vector space. A (G_n, λ_n)-*structure* on V is a pair $(\mathcal{B}_G(V), \phi)$ consisting of a right G_n-torsor $\mathcal{B}_G(V)$ and an isomorphism $\phi\colon \mathcal{B}(V) \xrightarrow{\cong} \lambda_n\big(\mathcal{B}_G(V)\big)$ of right $\mathrm{GL}_n\mathbb{R}$-torsors.

In the case of Euclidean structure (Example 1.40) the data $(\mathcal{B}_G(V), \phi)$ is equivalent to a choice of positive definite inner product on V. In that case $\mathcal{B}_G(V)$ can be identified with the space of orthonormal bases.

(1.53) *Geometric structure on an affine space.* Here is the affine extension of Definition 1.52.

Definition 1.54. Fix a symmetry type (G_n, λ_n) and let $(\mathcal{G}_n, \tilde{\lambda}_n)$ be the induced affine symmetry type. Let A be an n-dimensional affine space. A $(\mathcal{G}_n, \tilde{\lambda}_n)$-*structure* on A is a pair $\big(\mathcal{B}_G(A), \tilde{\phi}\big)$ consisting of a right $\mathcal{G}_n$-torsor $\mathcal{B}_G(A)$ and an isomorphism $\tilde{\phi}\colon \mathcal{B}(A) \xrightarrow{\cong} \tilde{\lambda}_n\big(\mathcal{B}_G(A)\big)$ of right $\mathrm{Aff}_n\mathbb{R}$-torsors.

Remark 1.55. Recall Remark 1.29(5) in which we construct the principal bundle of frames of affine space. Now, to a $(\mathcal{G}_n, \tilde{\lambda}_n)$-structure $\big(\mathcal{B}_G(A), \tilde{\phi}\big)$ on an affine space A, let

$$\pi_G\colon \mathcal{B}_G(A) \longrightarrow A \tag{1.56}$$

be the quotient map of the right G_n-action on $\mathcal{B}_G(A)$. (This action uses the splitting described immediately preceding Remark 1.39.) The map $\tilde{\phi}$ associates to each point $f_G \in \mathcal{B}_G(A)$ a frame based at the point $\pi_G(f_G) \in A$. The map π_G is a principal G_n-bundle.

Example 1.57 (Euclidean structure)**.** For the symmetry type in Example 1.40 we say an affine space equipped with an O_n-structure is a *Euclidean space*. (We sometimes abbreviate '(G_n, λ_n)-structure' by 'G_n-structure' if λ_n is unmistakable.) In this case a point of $\mathcal{B}_{\mathrm{O}}(A)$ in the fiber over $p \in A$ is an orthonormal frame of the tangent space $T_pA = V$, and (1.56) is called the *bundle of orthonormal frames*.

Appendix: group extensions

We include this appendix for reference. We do not comment further on topology, but say once and for all that for topological groups all group homomorphisms are assumed continuous, and for Lie groups all group homomorphisms are assumed smooth.

Definition 1.58. A *group extension* is a sequence of group homomorphisms

$$1 \longrightarrow G' \xrightarrow{\;i\;} G \xrightarrow{\;\pi\;} G'' \longrightarrow 1 \tag{1.59}$$

that is exact in the sense that the kernel of any homomorphism equals the image of the preceding homomorphism. We call G' the *kernel* and G'' the *quotient*.

Exactness at G' implies that i is injective. The inclusion i identifies G' with its image, which is a subgroup of G. At G the exactness implies that π factors through an injective map of the quotient group G/G' into G'', and exactness at G'' implies that this is an isomorphism. We use it to identify G'' as this quotient.

There is a category of group extensions with fixed kernel and quotient.

Definition 1.60. Let G', G'' be groups and G^{τ_1}, G^{τ_2} group extensions with kernel G' and quotient G''. A *morphism of group extensions* G^{τ_1}, G^{τ_2} is a group homomorphism $\varphi\colon G^{\tau_1} \to G^{\tau_2}$ which fits into the commutative diagram

$$\begin{tikzcd} & & G^{\tau_1} \arrow[dd, "\varphi"] \arrow[dr] & & \\ 1 \arrow[r] & G' \arrow[ur] \arrow[dr] & & G'' \arrow[r] & 1 \\ & & G^{\tau_2} \arrow[ur] & & \end{tikzcd} \tag{1.61}$$

As usual, φ is an *isomorphism* if there exists a homomorphism $\psi\colon G^{\tau_2} \to G^{\tau_1}$ of group extensions so that the compositions $\psi \circ \varphi$ and $\varphi \circ \psi$ are identity maps, which is simply equivalent to the condition that φ be an isomorphism of groups..

There is a notion of a trivialization of a group extension.

Definition 1.62. A *splitting* of the group extension (1.59) is a homomorphism $j\colon G'' \to G$ such that $\pi \circ j = \mathrm{id}_{G''}$.

Not every group extension is split: for example, the cyclic group of order 4 is a nonsplit extension of the cyclic group of order 2 by the cyclic group of order 2:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \tag{1.63}$$

Definition 1.64. Suppose (1.59) is a group extension and $\rho''\colon \widetilde{G}'' \to G''$ a group homomorphism. Then there is a *pullback* group extension with kernel G' and quotient $\widetilde{G}''$ which fits into the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & G' \arrow[r] \arrow[d, equal] & \widetilde{G} \arrow[r, "\tilde{\pi}"] \arrow[d, "\rho"] & \widetilde{G}'' \arrow[r] \arrow[d, "\rho''"] & 1 \\ 1 \arrow[r] & G' \arrow[r] & G \arrow[r, "\pi"] & G'' \arrow[r] & 1 \end{tikzcd} \tag{1.65}$$

It is defined by setting

$$\widetilde{G} = \bigl\{(g, \tilde{\gamma}'') \in G \times \widetilde{G}'' : \pi(g) = \rho''(\tilde{\gamma}'')\bigr\}. \tag{1.66}$$

Lecture 2: Flows and Lie derivatives

The material here relies on the fundamental theorem in the theory of ordinary differential equations; see Lectures 17 and 18 in the Multivariable Analysis notes. What is not proved there is smooth dependence of the solution on parameters. You can find a proof in several texts, such as in Arnold's books on ordinary differential equations or in *Introduction to Smooth Manifolds* by John Lee. The smooth dependence on initial conditions is used below in the construction of the flow of a vector field. There are further readings posted on Canvas that contain examples and pictures relevant to this lecture.

This is the first of several lectures that develop fundamental background material for our study of geometric structures on smooth manifolds. The topics covered are flows of vector fields and Lie derivatives, introduction to Lie groups, distributions and foliations, and the Frobenius theorem. In this lecture we already encounter several important themes of the course.

A vector field on a smooth manifold gives rise to motion. One has single motions—*integral curves*—for any initial point, and can assemble them together into a *flow* that moves sets of points. *Local* flows always exist; the existence of a *global* flow depends on completeness properties of the manifold and boundedness of the vector field. (We do not delve into these global questions here.) The dichotomy *local vs. global* is one of several important themes in the course.

Another important dichotomy is *intrinsic vs. extrinsic*. Here we use flows to transport intrinsic linear quantities—*tensors*—and then to differentiate them. This *Lie derivative* is an important ingredient in differential calculus on smooth manifolds. We work out the Lie derivative of vector fields, which leads to a *Lie algebra* structure on vector fields. In a subsequent lecture we work out the Lie derivative of differential forms.

Finally, we interpret the Lie derivative of vector fields as a measure of the failure of commutativity of the associated flows. This particular *infinitesimal* manifestation of a *local* phenomenon is another important theme of the course: we will see *curvature* as a geometric manifestation of this general link between commutativity of flows and Lie derivative. (Of course, the passage between local and infinitesimal is the basic idea of differential calculus.)

Notation: The vector space of vector fields on a smooth manifold X (i.e., the space of smooth sections of the tangent bundle $TX \to X$) is denoted $\mathfrak{X}(X)$. For a nonnegative integer $q \in \mathbb{Z}^{\ge 0}$, the vector space of smooth differential q-forms on X is denoted Ω^q_X. In particular, the algebra of smooth functions is denoted Ω^0_X. We often use 'n' as the dimension of a smooth manifold without a proper introduction.

Integral curves and flows

(2.1) *Integral curves.* The basic ordinary differential equation on a manifold is for a motion with prescribed velocity.

Definition 2.2. Let X be a smooth manifold, let ξ be a vector field on X, and suppose $(a,b) \subset \mathbb{R}$ is an open interval. Then a motion $\gamma\colon (a,b) \to X$ is an *integral curve* of ξ if

$$\dot\gamma(t) = \xi_{\gamma(t)}, \qquad t \in (a,b). \tag{2.3}$$

Here and always we use the notation $\dot\gamma = d\gamma/dt$.

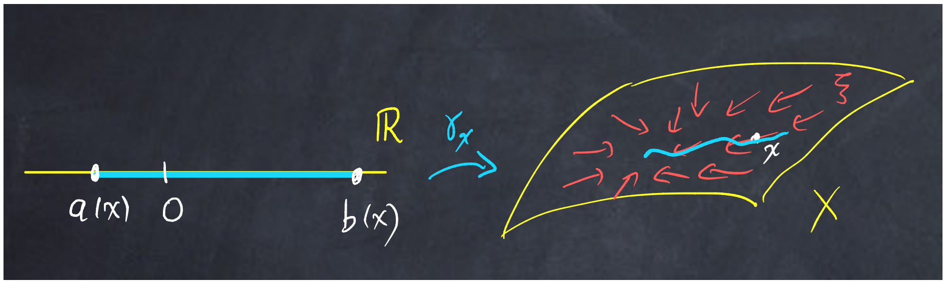

FIGURE 6. An integral curve of a vector field

A globalized form of the fundamental theorem of ODEs is the existence and uniqueness of a maximal integral curve with given initial value; see Figure 6.

Theorem 2.4. *Let X be a smooth manifold, let ξ be a vector field on X, and fix $x \in X$. Then there exists a unique maximal integral curve*

$$\gamma_x\colon \big(a(x), b(x)\big) \longrightarrow X \tag{2.5}$$

such that $\gamma_x(0) = x$. Furthermore, $-\infty \le a(x) < 0 < b(x) \le +\infty$.

The maximality defines the functions a, b. Maximality and uniqueness mean that if $\delta\colon (c,d) \to X$ is any integral curve of ξ satisfying $\delta(0) = x$, then $a(x) \le c < 0 < d \le b(x)$ and $\delta = \gamma_x\big|_{(c,d)}$.

Definition 2.6. The vector field ξ is *complete* if $a(x) = -\infty$ and $b(x) = +\infty$ for all $x \in X$.

In other words, all integral curves of X exist for all time.

Remark 2.7.

(1) If X is compact, then every vector field on X is complete.
(2) The vector field $x^2\,\partial/\partial x$ on $\mathbb{R}$ is not complete. Nor is the vector field $\partial/\partial x$ on $\mathbb{R}^{\ne 0}$.

(2.8) *Global and local flows.* Fix a vector field $\xi \in \mathfrak{X}(X)$. We consider simultaneously all maximal integral curves of ξ and show (well, state) that they assemble into a *flow*.

Definition 2.9. A *global flow* $\varphi\colon \mathbb{R} \times X \to X$ on a smooth manifold X is a smooth function that is also a homomorphism $\mathbb{R} \to \mathrm{Diff}(X)$.

Here $\mathrm{Diff}(X)$ is the group of diffeomorphisms $X \to X$. The diffeomorphism at $t \in \mathbb{R}$ is denoted φ_t, i.e., for $t \in \mathbb{R}$ and $x \in X$ we write $\varphi_t(x) = \varphi(t,x)$. A complete vector field determines a global flow, but a general vector field determines a flow defined on an open subset of $\mathbb{R} \times X$. Set

$$\mathcal{D}_t = \big\{ x \in X : t \in \big(a(x), b(x)\big)\big\}\,, \qquad t \in \mathbb{R}, \tag{2.10}$$

and define

$$\varphi(t,x) = \varphi_t(x) = \gamma_x(t), \qquad \text{if } x \in \mathcal{D}_t. \tag{2.11}$$

The following theorem, which is essentially Theorem 1.48 in Warner's *Foundations of Differentiable Manifolds and Lie Groups*, gives the main properties of $\mathcal{D}_t$ and φ. I defer to that reference for the proof, assuming the fundamental theorem of ODE.

Theorem 2.12.

(1) *$\mathcal{D}_t \subset X$ is open and $\bigcup_{t>0} \mathcal{D}_t = X$.*
(2) *$\varphi_t \colon \mathcal{D}_t \to \mathcal{D}_{-t}$ is a diffeomorphism with inverse φ_{-t}.*
(3) *For $t_1, t_2 \in \mathbb{R}$, we have $\varphi_{t_1}^{-1}(\mathcal{D}_{t_2}) \cap \mathcal{D}_{t_1} \subset \mathcal{D}_{t_1+t_2}$ and $\varphi_{t_2} \circ \varphi_{t_1} = \varphi_{t_1+t_2}$ on their common domain. (The two domains are equal if t_1, t_2 have the same sign.)*
(4) *If $U \subset X$ is an open subset and $x \in U$, then there exist $\epsilon \in \mathbb{R}^{>0}$ and an open subset $V \subset U$ with $x \in V$ such that $\varphi\big((-\epsilon, \epsilon) \times V\big) \subset U$.*
(5) *There exists an open subset $\mathcal{U} \subset \mathbb{R} \times X$ and a smooth function $\widehat{\varphi} \colon \mathcal{U} \to X$ such that $\{0\} \times X \subset \mathcal{U}$ and if $(t, x) \in \mathcal{U}$ then $x \in \mathcal{D}_t$ and $\widehat{\varphi}(t, x) = \varphi_t(x)$.*
(6) *If ξ is complete, then $\mathcal{D}_t = X$ for all $t \in \mathbb{R}$ and a global flow exists.*

We use 'φ' in place of '$\widehat{\varphi}$'. A *local flow* is the restriction of φ to a subset $(-\epsilon, \epsilon) \times V$, as in (4). It is the local flow that we use below to define the Lie derivative. The local flow is unique in the sense that any two local flows agree on their common domain. We say that ξ is the *generator* of the flow φ, or that ξ *generates* φ.

(2.13) *The flow in local coordinates.* We continue with a vector field ξ and the flow φ it generates. Let $(U; x^1, \dots, x^n)$ be a chart, and write $\xi = \xi^i \frac{\partial}{\partial x^i}$ for the vector field in the local coordinates of the chart. As in Theorem 2.12(4) choose $\epsilon > 0$ and $V \subset U$ such that if $t \in (-\epsilon, \epsilon)$ and $x \in V$, then $\varphi(t, x) \in U$. In the formulæ below the arguments t, x are implicitly restricted to $(-\epsilon, \epsilon) \times V$. Write $(t; x) = (t; x^1, \dots, x^n)$ and

$$\varphi(t, x) = \big(\varphi^1(t; x), \dots, \varphi^n(t; x)\big). \tag{2.14}$$

The elementary properties of the flow imply

$$\begin{aligned} \varphi^i(0, x) &= x^i \\ \dot{\varphi}^i(t; x) &= \xi^i_{\varphi(t;x)} \\ \left.\frac{\partial \varphi^i}{\partial x^j}\right|_{t=0} &= \delta^i_j \\ \left.\frac{\partial^2 \varphi^i}{\partial x^k \partial x^j}\right|_{t=0} &= 0. \end{aligned} \tag{2.15}$$

We abbreviate the second equation: $\dot{\varphi}^i = \xi^i$.

Lie derivative

Continue with a smooth manifold X, a vector field $\xi \in \mathfrak{X}(X)$, and a local flow $\varphi \colon \mathcal{U} \to X$ generated by ξ. We use the flow to transport "objects" on X and so define a derivative at $t = 0$

of a curve of transported objects. Without further data, any object intrinsically associated to the tangent bundle so transports.

(2.16) *Tensor fields.* A formal definition: A tensor field is a section of a vector bundle associated to the frame bundle of X. But since we have not yet constructed the frame bundle, we content ourselves with an informal definition: A tensor field is a section of a vector bundle with is a tensor product of tensor powers of the tangent and cotangent bundles. If only the tangent bundle is involved, then the tensor field is *covariant*; if only the cotangent bundle is involved, then the tensor field is *contravariant*. A vector field is a covariant tensor field; a differential form is a contravariant tensor field. If ω is a contravariant tensor field on X, and $\psi\colon X' \to X$ is *any* smooth map, then the pullback $\psi^*\omega$ is a tensor field on X'. If η is a covariant tensor field on X, and $\psi\colon X \to X''$ is a *diffeomorphism*, then the pushforward $\psi_*\eta$ is a tensor field on X''. A tensor field that is mixed—neither covariant nor contravariant—can be pulled back or pushed forward by a diffeomorphism.

(2.17) *Differentiation along a flow.* A modification of the following works for mixed tensor fields.

Definition 2.18. Let X be a smooth manifold, $\xi \in \mathfrak{X}(X)$ a vector field on X, and $\varphi\colon \mathcal{U} \to X$ a local flow it generates. Let T be a tensor field on X that is covariant or contravariant. Then the *Lie derivative* of T with respect to ξ is

$$\mathcal{L}_\xi T = \begin{cases} \left.\dfrac{d}{dt}\right|_{t=0} (\varphi_{-t})_*T, & T \text{ is covariant;} \\ \left.\dfrac{d}{dt}\right|_{t=0} \varphi_t^*T, & T \text{ is contravariant.} \end{cases} \tag{2.19}$$

Intuitively, to compute $\mathcal{L}_\xi T$ at $x \in X$ we stand at X where as time t marches from $-\epsilon$ to ϵ we see a motion in a vector space at x, the fiber of the appropriate vector bundle associated to the frame bundle. It begins with upstream values of T, transported forward to x by the flow, and ends with downstream values of T, transported backward to x. Figure 7 illustrates this in case $T = \eta$ is a vector field. Note that the transport of η by the flow generated by ξ cannot be computed simply from the integral curve γ_x. (Visualize η at a point of X as a germ of a motion, and then transport the motion germs using the flow φ.)

(2.20) *Lie derivative of a function.* A function $f \in \Omega^0_X$ is a contravariant tensor field. (We use the notation 'Ω^q_X' for the vector space of smooth q-forms on the smooth manifold X.) Hence by (2.19),

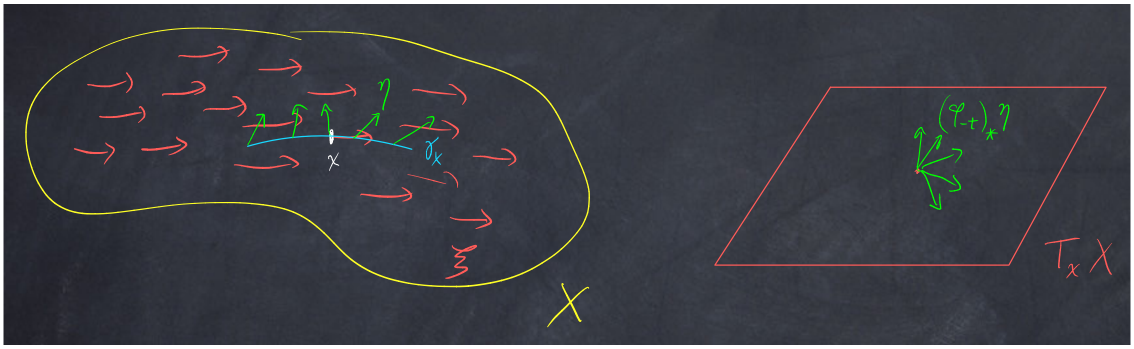

FIGURE 7. The Lie derivative of a vector field

for $x \in X$ we compute

$$\begin{aligned}\mathcal{L}_\xi f(x) &= \frac{d}{dt}\bigg|_{t=0} \varphi_t^* f(x) \\ &= \frac{d}{dt}\bigg|_{t=0} f\big(\varphi_t(x)\big) \\ &= \frac{d}{dt}\bigg|_{t=0} f\big(\gamma_x(t)\big) \\ &= (\xi f)(x) \\ &= df_x(\xi)\end{aligned} \tag{2.21}$$

Therefore, the Lie derivative of a function is the directional derivative: $\mathcal{L}_\xi f = \xi f$.

In a future lecture we compute a formula for the Lie derivative of an arbitrary differential form.

(2.22) *Lie derivative of a vector field.* Compute in a local chart, as in (**2.13**).

Proposition 2.23. *The Lie derivative of a vector field* $\eta = \eta^i \dfrac{\partial}{\partial x^i}$ *with respect to* $\xi = \xi^i \dfrac{\partial}{\partial x^i}$ *is*

$$\mathcal{L}_\xi \eta = \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}. \tag{2.24}$$

Proof. Apply (2.19). First,

$$(\varphi_{-t})_* \eta = \big(\varphi_t^* \eta^i\big) \left((\varphi_{-t})_* \frac{\partial}{\partial x^i} \right). \tag{2.25}$$

Evaluate the first factor in (2.25):

$$\varphi_t^* \eta^i(x) = \eta^i\big(\varphi(t;x)\big). \tag{2.26}$$

For the second factor in (2.25), flow by φ_{-t} the motion germ whose velocity is $\partial/\partial x^i$ at the point $\varphi(t;x)$:

$$\begin{aligned}\Bigl((\varphi_{-t})_*\frac{\partial}{\partial x^i}\Bigr)(x) &= \frac{d}{ds}\Big|_{s=0}\varphi\bigl(-t;\varphi^1(t;x),\dots,\varphi^i(t;x)+s,\dots,\varphi^n(t;x)\bigr)\\ &= \frac{d}{ds}\Big|_{s=0}\Bigl(\varphi^j\bigl(-t;\varphi^1(t;x),\dots,\varphi^i(t;x)+s,\dots,\varphi^n(t;x)\bigr)\Bigr)_j\\ &= \frac{\partial\varphi^j}{\partial x^i}\bigl(-t;\varphi(t;x)\bigr)\frac{\partial}{\partial x^j}.\end{aligned}\tag{2.27}$$

Combine (2.19), (2.25), (2.26), and (2.27) to obtain

$$\begin{aligned}(\mathcal{L}_\xi\eta)(x) &= \frac{d}{dt}\Big|_{t=0}\Bigl\{\eta^i\bigl(\varphi(t;x)\bigr)\frac{\partial\varphi^j}{\partial x^i}\bigl(-t;\varphi(t;x)\bigr)\frac{\partial}{\partial x^j}\Bigr\}\\ &= \frac{\partial\eta^i}{\partial x^k}\dot\varphi^k\frac{\partial\varphi^j}{\partial x^i}\frac{\partial}{\partial x^j}\Big|_{t=0} - \eta^i\frac{\partial\dot\varphi^j}{\partial x^i}\frac{\partial}{\partial x^j}\Big|_{t=0} + \frac{\partial^2\varphi^j}{\partial x^k\partial x^i}\dot\varphi^k\frac{\partial}{\partial x^j}\Big|_{t=0}\\ &= \frac{\partial\eta^j}{\partial x^i}\xi^k\frac{\partial}{\partial x^i} - \eta^i\frac{\partial\xi^j}{\partial x^i}\frac{\partial}{\partial x^j},\end{aligned}\tag{2.28}$$

where we use (2.15) to pass to the last equation. Rearrange the indices to obtain (2.24). □

Lie derivatives and derivations

(2.29) *Derivations.* The following may be generalized to derivations on a module over a ring.

Definition 2.30. Let A be an algebra. A linear map $D\colon A\to A$ is a *derivation* if

$$D(fg) = (Df)g + f(Dg),\qquad f,g\in A.\tag{2.31}$$

If D_1, D_2 are derivations of A, then so is the *commutator* or *Lie bracket*

$$[D_1,D_2] = D_1\circ D_2 - D_2\circ D_1.\tag{2.32}$$

A simple manipulation with (2.32) proves that the bracket endows the vector space of derivations of A with the structure of a *Lie algebra.*

Definition 2.33. A *Lie algebra* $\bigl(L,[-,-]\bigr)$ is a vector space L endowed with a bilinear operation

$$[-,-]\colon L\times L\longrightarrow L\tag{2.34}$$

which for all $D_1,D_2,D_3\in L$ satisfies

$$[D_1,D_2]+[D_2,D_1]=0\tag{2.35}$$

$$\bigl[D_1,[D_2,D_3]\bigr]+\bigl[D_3,[D_1,D_2]\bigr]+\bigl[D_2,[D_3,D_1]\bigr]=0\tag{2.36}$$

The *Jacobi identity* (2.36) can be written in alternative useful forms. For example, it is the statement that the operation $[D_1, -]$ of left bracketing with D_1 is a derivation:

$$\big[D_1, [D_2, D_3]\big] = \big[[D_1, D_2], D_3\big] + \big[D_2, [D_1, D_3]\big]. \tag{2.37}$$

Equivalently, right bracketing with D_1 is a derivation. Another equivalent form of the Jacobi identity is the map $D \mapsto [D, -]$ is a homomorphism of Lie algebras.

(2.38) *Vector fields as derivations.* Let $\xi \in \mathfrak{X}(X)$ be a vector field on a smooth manifold X. Then ξ defines a derivation on functions via directional derivative:

$$\begin{aligned} D_\xi\colon \Omega^0_X &\longrightarrow \Omega^0_X \\ f &\longmapsto \xi f \end{aligned} \tag{2.39}$$

Conversely, every derivation on the algebra Ω^0_X is directional derivative along a vector field.

Proposition 2.40. *Let $D\colon \Omega^0_X \to \Omega^0_X$ be a derivation. Then there exists a vector field $\xi \in \mathfrak{X}(X)$ such that $D = D_\xi$.*

First we prove the following.

Lemma 2.41. *Suppose h is a smooth real-valued function defined on a convex neighborhood U of $0 \in \mathbb{A}^n$, and assume $h(0) = 0$. Then there exist smooth functions $g_i\colon U \to \mathbb{R}$, $i = 1, \dots, n$, such that*

$$h(x) = g_i(x)x^i. \tag{2.42}$$

Furthermore,

$$g_i(0) = \frac{\partial h}{\partial x^i}(0). \tag{2.43}$$

Proof. For any $x \in U$ write

$$h(x) = \int_0^1 dt\, \frac{d}{dt} h(tx) = \int_0^1 dt\, \frac{\partial h}{\partial x^i}(tx)\, x^i, \tag{2.44}$$

so set

$$g_i(x) = \int_0^1 dt\, \frac{\partial h}{\partial x^i}(tx). \tag{2.45}$$

Equation (2.43) follows immediately from (2.45) upon setting $x = 0$. $\square$

Proof of Proposition 2.40. First, use the derivation property to show that $D(1) = 0$ and then D applied to any constant function vanishes. Next, suppose f_1, f_2 are functions that agree in a neighborhood U of a point $x \in X$. Then if ρ is a cutoff function—$\rho(x) = 1$ and ρ vanishes on the complement of U—we have $\rho f_1 = \rho f_2$, and apply D to conclude that $D(f_1)(x) = D(f_2)(x)$. It follows that a derivation of Ω^0_X gives rise to a derivation of Ω^0_U for all open $U \subset X$. Furthermore, a derivation of Ω^0_X is determined by the collection of induced derivations on Ω^0_U as U ranges over an open cover of X.

Cover X by charts $(U; x^1, \dots, x^n)$ such that the image of U is a convex subset of $\mathbb{A}^n$. Define the vector field on U

$$\xi_x = (Dx^i)(x)\frac{\partial}{\partial x^i}, \qquad x \in U. \tag{2.46}$$

We claim that the derivation D induces on Ω^0_U is D_ξ. Namely, fix $x_0 \in U$ and translate the coordinates so that $x^i(x_0) = 0$ for all i. Identify $U \subset X$ with its image in $\mathbb{A}^n$, and so $x_0 \in X$ with $0 \in \mathbb{A}^n$. Fix $f \in \Omega^0_U$. Then Lemma 2.41 produces functions $g_i \in \Omega^0_U$ such that

$$f(x) = f(0) + g_i(x)x^i \tag{2.47}$$

and $g_i(0) = \partial f/\partial x^i(0)$. Then

$$Df(0) = D\big(f - f(0)\big)(0) = D(g_i x^i)(0) = (Dx^i)(0)g_i(0) = Dx^i(0)\frac{\partial f}{\partial x^i}(0) = (\xi f)(0). \tag{2.48}$$

This proves that $D = D_\xi$ as derivations of Ω^0_U, from which it follows that ξ patches to a global vector field on X and $D = D_\xi$ as derivations of Ω^0_X. $\square$

(2.49) *The Lie derivative is the Lie bracket.*

Proposition 2.50. *Let X be a smooth manifold and let $\xi, \eta \in \mathfrak{X}(X)$ be vector fields on X. Then*

$$[D_\xi, D_\eta] = D_{\mathcal{L}_\xi \eta}. \tag{2.51}$$

Proof. Compute in local coordinates using Proposition 2.23. For $f \in \Omega^0_X$ we have

$$\begin{aligned}[D_\xi, D_\eta]f &= \xi(\eta f) - \eta(\xi f) \\ &= \left(\xi^i \frac{\partial}{\partial x^i}\right)\left(\eta^j \frac{\partial f}{\partial x^j}\right) - \left(\eta^i \frac{\partial}{\partial x^i}\right)\left(\xi^j \frac{\partial f}{\partial x^j}\right) \\ &= \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j} \\ &= (\mathcal{L}_\xi \eta) f\end{aligned} \tag{2.52}$$

$\square$

Transport the commutator of derivations of Ω^0_X to a Lie bracket on vector fields, using Proposition 2.40: for $\xi,\eta\in\mathcal{X}(X)$ characterize $[\xi,\eta]\in\mathcal{X}(X)$ by

$$D_{[\xi,\eta]}=\bigl[D_\xi,D_\eta\bigr]. \tag{2.53}$$

Then Proposition 2.50 shows that

$$\mathcal{L}_\xi\eta=[\xi,\eta]. \tag{2.54}$$

Corollary 2.55. $\bigl(\mathcal{X}(X),[-,-]\bigr)$ *is a Lie algebra.*

A geometric interpretation of the Lie derivative

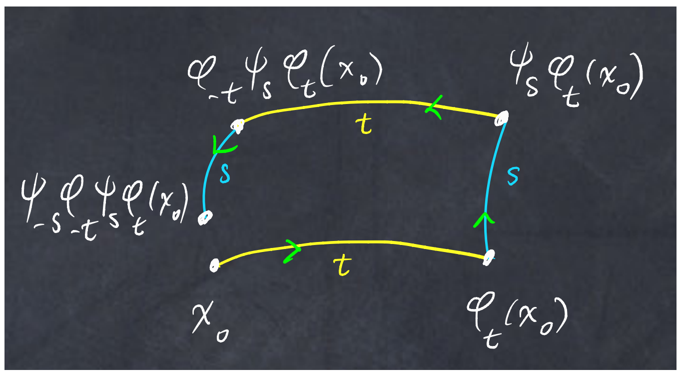

FIGURE 8. The commutator of flows

Let $\xi,\eta\in\mathcal{X}(X)$ be vector fields on a smooth manifold X, and fix $x_0\in X$. We compute the Lie derivative $\mathcal{L}_\xi\eta$ at x_0 in terms of the commutator of the flows that ξ,η generate. Let φ_t,ψ_s be local flows near x_0 generated by ξ,η, and fix local coordinates $x^1,\dots,x^n$ near x_0. For sufficiently small $\epsilon>0$ and $t,s\in(-\epsilon,\epsilon)$, write the function

$$(t,s)\longmapsto\psi_{-s}\varphi_{-t}\psi_s\varphi_t(x_0) \tag{2.56}$$

as

$$(t,s)\longmapsto x^i(t,s). \tag{2.57}$$

Proposition 2.58. *We have*

$$\mathcal{L}_\xi\eta(x_0)=\frac{\partial^2x^i}{\partial t\partial s}\,\frac{\partial}{\partial x^i}\bigg|_{\substack{t=0\\s=0}}. \tag{2.59}$$

Proof. From (2.56) compute

$$\frac{\partial x^i}{\partial s}\frac{\partial}{\partial x^i}\bigg|_{s=0} = -\eta + (\varphi_{-t})_*\eta, \tag{2.60}$$

where the right hand side is evaluated at x_0. Differentiate (2.60) in t at $t = 0$ to derive (2.59). □

Consider the diagonal motion derived from (2.56) as

$$t \longmapsto \psi_{-t}\varphi_{-t}\psi_t\varphi_t(x_0), \tag{2.61}$$

which we write as

$$t \longmapsto \tilde{x}^i(t). \tag{2.62}$$

Corollary 2.63. *We have*

$$\mathcal{L}_\xi\eta(x_0) = \frac{1}{2}\frac{d^2\tilde{x}^i}{dt^2}\frac{\partial}{\partial x^i}\bigg|_{\substack{t=0\\ s=0}}. \tag{2.64}$$

Lie brackets and smooth maps

In general, if $\varphi\colon X' \to X$ is a smooth map, and $\xi' \in \mathfrak{X}(X')$ is a vector field, then there is not a vector field defined on X. For to define a putative pushforward at $x \in X$ we would apply the differential of φ to ξ' at an inverse image point of x. But $\varphi^{-1}(x)$ may not be a singleton: if it is empty, there is no value defined, and if its cardinality is ≥ 2 there may be multiple values defined.

Definition 2.65. Suppose $\varphi\colon X' \to X$ is a surjective map, $\xi' \in \mathfrak{X}(X')$, and $\xi \in \mathfrak{X}(X)$. We say ξ', ξ are *φ-related* if $\varphi_*(\xi'_{x'}) = \xi_{\varphi(x')}$ for all $x' \in X'$.

Proposition 2.66. *If ξ'_1, ξ'_2 are φ-related to ξ_1, ξ_2, respectively, then $[\xi'_1, \xi'_2]$ is φ-related to $[\xi_1, \xi_2]$.*

Proof. ξ'_i and ξ_i are φ-related iff for all $f \in \Omega^0_X$ we have $\varphi^*(\xi_i f) = \xi'_i(\varphi^* f)$. If so, then

$$\begin{aligned}\varphi^*\big([\xi_1,\xi_2]f\big) &= \varphi^*(\xi_1\xi_2 f - \xi_2\xi_1 f)\\ &= \xi'_1\varphi^*(\xi_2 f) - \xi'_2\varphi^*(\xi_1 f)\\ &= \xi'_1\xi'_2\varphi^* f - \xi'_2\xi'_1\varphi^* f\\ &= [\xi'_1,\xi'_2]\varphi^* f.\end{aligned} \tag{2.67}$$

It follows that $[\xi'_1, \xi'_2]$ is φ-related to $[\xi_1, \xi_2]$. □

Lecture 3: Introduction to Lie groups

As we said in (**1.34**), we organize geometric structures (after Klein) in terms of symmetry. In this lecture we introduce symmetry groups in differential geometry: Lie groups. A group is a structure on a set, and discrete groups play an important role in all of mathematics. In geometric situations one often combines the group structure with a geometric structure: topological groups, algebraic groups, group schemes, ... Here we combine with a smooth manifold structure on the underlying set to obtain the notion of a Lie group.

The smooth manifold that underlies a Lie group carries two global parallelisms, and we choose the *left* parallelism as primary. We begin the lecture with some general remarks on manifolds equipped with a global parallelism; see (**1.19**).

We then turn to the definition of a Lie group. The parallel (left invariant) vector fields are closed under Lie bracket and comprise the Lie algebra associated to a Lie group. We give several examples, including the "exceptional" Lie group G_2. (There is a classification of compact simply connected Lie groups that includes three infinite families and five exceptional groups; G_2 is one of the latter.)

The notion of a homomorphism of Lie groups (Definition 3.39) is evident, and by differentiation a homomorphism of Lie groups induces a homomorphism of their Lie algebras. Remark 3.42 on the converse sets up our discussion of distributions and the Frobenius theorem, which we take up in a few lectures. One-parameter groups are a special and important class of homomorphisms. They correspond to elements of the Lie algebra, and often are a useful way to work with such elements. This leads to the exponential map from the Lie algebra to the Lie group.

We conclude the lecture by working out one parameter groups and the exponential map for matrix groups.

In the next lecture we continue our introduction to Lie groups.

Parallelized manifolds

Let X be a smooth manifold equipped with a global parallelism (Definition 1.20). The structure (1.21) of the parallelism identifies each tangent space T_xX with a fixed vector space V. This provides an isomorphism

$$\mathfrak{X}(X) \xrightarrow{\cong} \mathrm{Map}(X, V). \tag{3.1}$$

A vector field $\xi \in \mathfrak{X}(X)$ is *parallel* if it corresponds to a constant function under (3.1). Observe that a vector $\xi_x \in T_xX$ at any $x \in X$ extends to a unique parallel vector field $\xi \in \mathfrak{X}(X)$. Also, given ξ_x there is a distinguished motion germ with initial position x and initial velocity ξ_x defined by an integral curve of the parallel vector field ξ with initial position x. This is a *constant velocity motion* (germ).

Remark 3.2. On a smooth manifold equipped with a global parallelism, the Lie bracket of parallel vector fields is not necessarily parallel. However, for the special left and right parallelisms we define below on a Lie group, the Lie bracket of parallel vector fields is parallel.

The parallelism also induces an isomorphism

$$\Omega^\bullet(X) \xrightarrow{\cong} \mathrm{Map}(X, {\textstyle\bigwedge}^\bullet V^*), \tag{3.3}$$

and the parallel differential forms form a subalgebra

$$\Omega^\bullet_{\parallel}(X) \subset \Omega^\bullet(X). \tag{3.4}$$

As in Remark 3.2, the subspace of parallel differential forms is not necessarily closed under d.

Basic concepts of Lie groups

(3.5) *The definition.* A mathematical marriage imposes compatibility constraints, here smoothness of structure maps.

Definition 3.6. A *Lie group* is a set G endowed with the structures of a group and of a smooth manifold. The multiplication and inversion maps

$$\begin{aligned} m\colon G \times G &\longrightarrow G \\ i\colon G &\longrightarrow G \end{aligned} \tag{3.7}$$

are required to be smooth.

Observe that a Lie group has a distinguished point, the identity element $e \in G$. Therefore, it has a distinguished component, the *identity component* that contains e. Recall too Definition 2.33 of a Lie algebra. We will soon see that attached to a Lie group G is a Lie algebra $\mathrm{Lie}(G)$.

A Lie group can act as symmetries on a smooth manifold.

Definition 3.8. Let G be a Lie group and let X be a smooth manifold. A (left/right) *action* of G on X is a smooth map

$$\begin{aligned} L\colon G \times X &\longrightarrow X \\ R\colon X \times G &\longrightarrow X \end{aligned} \tag{3.9}$$

that satisfies the associativity law required of a group action: for $g_1, g_2 \in G$ and $x \in X$, we have

$$\begin{aligned} L\bigl(g_1, L(g_2, x)\bigr) &= L\bigl(g_1g_2, x\bigr) \\ R\bigl(R(x, g_1), g_2\bigr) &= R\bigl(x, g_1g_2\bigr) \end{aligned} \tag{3.10}$$

for a left action L and a right action R, respectively.

As usual, we write $L(g,x) = g\cdot x = gx$ and $R(x,g) = x\cdot g = xg$.

(3.11) *Parallelisms on a Lie group.* A Lie group G carries canonical left and right actions on itself by left and right multiplication: for $g, h \in G$ and $x \in G$ define

$$\begin{aligned} L_g(x) &= gx \\ R_h(x) &= xh \end{aligned} \tag{3.12}$$

The maps $L_g, R_h \colon G \to G$ are diffeomorphisms. Left multiplication and right multiplication commute, so together define a left action of $G \times G$ on G: the pair $(g,h) \in G \times G$ maps $x \mapsto gxh^{-1}$. The differentials of left and right multiplication each define a global parallelism that identifies each tangent space with T_eG. The *left parallelism* is

$$\begin{aligned} L_g \colon T_eG &\longrightarrow T_gG \\ \xi &\longmapsto (L_g)_*\xi \end{aligned} \tag{3.13}$$

and the *right parallelism* is

$$\begin{aligned} R_h \colon T_eG &\longrightarrow T_hG \\ \xi &\longmapsto (R_h)_*\xi \end{aligned} \tag{3.14}$$

Convention 3.15. We use 'parallel' to mean the *left* parallelism of a Lie group.

Remark 3.16. A parallel vector field ξ is also called 'left-invariant': it is invariant under left translation (3.13). Namely, ξ is parallel iff $(L_g)_*\xi = \xi$ for all $g \in G$. In other words, ξ is L_g-related to itself for all $g \in G$. Similarly, a differential form α is parallel iff $L_g^*\alpha = \alpha$ for all $g \in G$.

Notation 3.17. The subspace of parallel vector fields is denoted $\mathfrak{g} \subset \mathfrak{X}(G)$.

Evaluation at the identity is an isomorphism of vector spaces

$$\mathrm{ev}_e \colon \mathfrak{g} \longrightarrow T_eG \tag{3.18}$$

(3.19) *The Lie algebra of a Lie group.* The space $\mathfrak{g}$ of parallel vector fields has a Lie algebra structure induced from the Lie bracket of vector fields.

Lemma 3.20. *If $\xi_1, \xi_2 \in \mathfrak{g}$ are parallel vector fields, then so too is their Lie bracket $[\xi_1, \xi_2]$.*

Proof. By Remark 3.16 each ξ_i is L_g-related to itself for all $g \in G$, hence by Proposition 2.66 this also holds for $[\xi_1, \xi_2]$. Therefore, $[\xi_1, \xi_2]$ is parallel. □

We use the isomorphism (3.18) to transport the Lie algebra structure on $\mathfrak{g}$, defined by Lie bracket of vector fields, to the vector space T_eG. This is often convenient. For example, if $G = \mathrm{O}_n$ is the group of orthogonal $n \times n$ real matrices, then $T_e\mathrm{O}_n$ is the vector space of skew-symmetric $n \times n$ real matrices. We compute the induced Lie bracket on matrices below in Proposition 3.70.

Examples of Lie groups

Example 3.21 (discrete Lie groups)**.** A finite or countable group is a 0-dimensional Lie group when equipped with the discrete topology. (An uncountable discrete set is not a manifold: it violates second countability.) The associated Lie algebra is the zero vector space equipped with the zero bracket.

Example 3.22 (real vector spaces)**.** A finite dimensional real vector space V is a Lie group. The group law is vector addition and the manifold structure is as usual. A vector space is an *abelian* Lie group; the group law is commutative.

Example 3.23 (circle group)**.** Let $\mathbb{T} \subset \mathbb{C}$ denote the set of complex numbers λ of unit norm: $\lambda\bar{\lambda} = 1$. Then $\mathbb{T}$ is a 1-dimensional Lie group; the group law is multiplication of complex numbers. Regarding $\mathbb{T} \subset \mathbb{C}$, it is natural to identify $\operatorname{Lie}(\mathbb{T}) \cong T_1\mathbb{T} \cong \sqrt{-1}\,\mathbb{R} \subset \mathbb{C}$.

Example 3.24 (general linear group)**.** Let V be a finite dimensional real vector space. Then $\operatorname{End}(V)$ is the vector space of linear maps $A\colon V \to V$, and $\operatorname{Aut}(V) \subset \operatorname{End}(V)$ is the open subset of invertible linear maps $P\colon V \to V$. The group $\operatorname{Aut}(V)$ is a Lie group whose group law is composition of automorphisms. Its dimension is $(\dim V)^2$. The tangent space at the identity is canonically $T_{\mathrm{id}} \operatorname{Aut}(V) = \operatorname{End}(V)$. Note that $\operatorname{End}(V)$ is an associative algebra under composition, a structure we exploit. Quite generally, it is pleasant to compute in Lie groups that are subgroups of associative algebras.

Observe that $\operatorname{Aut}(V)$ has two parallelisms. First, the affine parallelism induced from $\operatorname{End}(V)$ identifies each tangent space with $\operatorname{End}(V)$. Second, as a Lie group $\operatorname{Aut}(V)$ has the left parallelism (3.13). (Of course, it has a third parallelism—right parallelism—but we do not use that.) These two parallelisms do not agree. Namely, if we use the affine parallelism to identify $T_P \operatorname{Aut}(V) \cong \operatorname{End}(V)$ for $P \in \operatorname{Aut}(V)$, then the left parallelism is left composition by P:

$$\begin{aligned} (L_P)_*\colon T_{\mathrm{id}} \operatorname{Aut}(V) \cong \operatorname{End}(V) &\longrightarrow \operatorname{End}(V) \cong T_P(V) \\ A \quad &\longmapsto \quad P \circ A \end{aligned} \tag{3.25}$$

We compute the Lie bracket on $T_e \operatorname{Aut}(V) \cong \operatorname{End}(V)$ in (**3.68**) below.

If $V = \mathbb{R}^n$, then $\operatorname{End}(\mathbb{R}^n) = M_n\mathbb{R}$ is the algebra of $n\times n$ real matrices under matrix multiplication, and $\operatorname{Aut}(\mathbb{R}^n) = \mathrm{GL}_n\mathbb{R}$ is the subgroup of invertible matrices.

Definition 3.26. A Lie group G is a *matrix group* if it is a Lie subgroup $G \subset \operatorname{Aut}(V)$ of a general linear group.

As we spell out in the next lecture, this means that G has a Lie group structure and the inclusion map into the general linear group is a homomorphism of Lie groups (defined below).

Example 3.27 (matrix groups cut out by a bilinear form)**.** Let V be a finite dimensional real vector space and $B\colon V \times V \to \mathbb{R}$ a bilinear form. Define

$$\operatorname{Aut}_B(V) = \{P \in \operatorname{Aut}(V) : B(P\xi_1, P\xi_2) = B(\xi_1, \xi_2) \text{ for all } \xi_1, \xi_2 \in V\}; \tag{3.28}$$

it is a Lie subgroup of $\mathrm{Aut}(V)$. The tangent space at $\mathrm{id} \in \mathrm{Aut}_B(V)$ is computed by differentiating the equation in (3.28) in the variable P:

$$\mathrm{End}_B(V) = \{A \in \mathrm{End}(V) : B(A\xi_1, \xi_2) + B(\xi_1, A\xi_2) = 0 \text{ for all } \xi_1, \xi_2 \in V\}. \tag{3.29}$$

For each pair $\xi_1, \xi_2 \in V$ the equation in (3.28) cuts out a closed set in $\mathrm{Aut}(V)$, and hence $\mathrm{Aut}_B(V)$ is an intersection of closed sets so is a closed subgroup of $\mathrm{Aut}(V)$. This alone is sufficient to prove that $\mathrm{Aut}_B(V)$ is a submanifold of $\mathrm{Aut}(V)$ and that the induced manifold structure is compatible with the multiplication. We explain the indicated theorem in the next lecture. If B is nondegenerate, then for any $A \in \mathrm{End}(V)$ define the *adjoint* operator $A^* \in \mathrm{End}(V)$ via the equation

$$B(\xi_1, A\xi_2) = B(A^*\xi_1, \xi_2), \qquad \xi_1, \xi_2 \in V. \tag{3.30}$$

Then $P \in \mathrm{Aut}(V)$ is in $\mathrm{Aut}_B(V)$ iff $P^*P = \mathrm{id}$. One can prove that this equation defines a submanifold, and from there that $\mathrm{Aut}_B(V)$ is a Lie group.

For $V = \mathbb{R}^n$ this specializes to classical Lie subgroups of $\mathrm{GL}_n\mathbb{R}$. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$.

Example 3.31 (orthogonal groups)**.** Let $p, q \in \{0, 1, \dots, n\}$ satisfy $p + q = n$. Define the bilinear form $B_{p,q} \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$B_{p,q}(e_k, e_\ell) = \begin{cases} +1, & k = \ell \le p; \\ -1, & k = \ell > p; \\ 0, & k \ne \ell. \end{cases} \tag{3.32}$$

Then B is symmetric and nondegenerate. The *orthogonal group* $\mathrm{Aut}_{B_{p,q}}(\mathbb{R}^n)$ is notated $\mathrm{O}_{p,q}$, and in the definite case we write $\mathrm{O}_{n,0} = \mathrm{O}_{0,n} = \mathrm{O}_n$.

An orthogonal matrix P satisfies $\det P = \pm 1$. The orthogonal group is not connected; write $\mathrm{SO}_{p,q} \subset \mathrm{O}_{p,q}$ for the *identity component*, that is, the connected component that contains the identity element. In the definite case, write $\mathrm{SO}_n \subset \mathrm{O}_n$. This subgroup is called the *special orthogonal group*.

Your homework treats nondegenerate skew-symmetric forms.

Example 3.33 (Lie groups from complex vector spaces)**.** A finite dimensional complex vector space U has an underlying finite dimensional real vector space, so is a smooth manifold. Hence, so too is $\mathrm{Aut}(U) \subset \mathrm{End}(U)$, where now $\mathrm{End}(U)$ is the associative algebra of *complex linear* endomorphisms of U. For $U = \mathbb{C}^m$ we write $\mathrm{Aut}(U) = \mathrm{GL}_m\mathbb{C}$; it is the *complex general linear group* of (real) dimension $2m^2$. By contrast with the real case in Example 3.31, there is only a single isomorphism class of nondegenerate symmetric complex bilinear forms on $\mathbb{C}^m$; the corresponding group of symmetries is the *complex orthogonal group* $\mathrm{O}_m\mathbb{C}$. There is a new possibility: *hermitian forms*. Recall that a hermitian form on a complex vector space is a bilinear map[2]

$$h \colon \overline{U} \times U \longrightarrow \mathbb{C}, \tag{3.34}$$

[2] $\overline{U}$ is the complex conjugate vector space to U: the same real vector space with the complex conjugate action of the complex scalars $\mathbb{C}$.

which is equivalent to a bilinear map $U_{\mathbb{R}} \times U_{\mathbb{R}} \to \mathbb{C}$ of the underlying real vector spaces that is complex conjugate linear in the first variable and complex linear in the second variable. Symmetry or skew-symmetry is the equation

$$h(\bar{\xi}_1, \xi_2) = \pm\overline{h(\bar{\xi}_2, \xi_1)}. \tag{3.35}$$

A symmetric hermitian form has a signature, just as in Example 3.31, leading for $U = \mathbb{C}^m$ to the *unitary groups* $\mathrm{U}_{p,q}$, $p + q = m$. In the definite case we write $\mathrm{U}_{m,0} = \mathrm{U}_{0,m} = \mathrm{U}_n$

The next example is not defined by a bilinear form, but rather by a trilinear form.

Example 3.36 (An exceptional Lie group)**.** Let $V = \mathbb{R}^7$ with its standard inner product and standard (orthonormal) basis $e_1, \dots, e_7$. Denote the dual basis of V^* by $e^1, \dots, e^7$, and use the notation $e^{ijk} = e^i \wedge e^j \wedge e^k \in \bigwedge^3 V^*$. Introduce the 3-form $\omega \in \bigwedge^3 V^*$ defined as

$$\omega = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. \tag{3.37}$$

The exceptional (compact) Lie group G_2 is

$$\mathrm{G}_2 = \{P \in \mathrm{SO}_7 : P^*\omega = \omega\}. \tag{3.38}$$

It has dimension 14.

Homomorphisms of Lie groups

The next definition is in keeping with the compatible marriage in Definition 3.6.

Definition 3.39. Let G', G be Lie groups. A function $\psi\colon G' \to G$ is a (*Lie group*) *homomorphism* if it is a smooth map of manifolds and a homomorphism of groups.

Lemma 3.40. *A Lie group homomorphism $\psi\colon G' \to G$ induces a Lie algebra homomorphism $\dot{\psi}\colon \mathfrak{g}' \to \mathfrak{g}$ on parallel vector fields.*

Proof. Let $\xi' \in \mathfrak{g}'$ be a parallel vector field. Then for $g' \in G'$ we have

$$\begin{aligned} \psi_*(\xi'_{g'}) &= \psi_*(L_{g'})_*\xi'_{e'} \\ &= (\psi \circ L_{g'})_*\xi'_{e'} \\ &= (L_{\psi(g')} \circ \psi)_*\xi'_{e'} \\ &= (L_{\psi(g')})_*\xi_e \end{aligned} \tag{3.41}$$

where $\xi_e = \psi_*\xi'_{e'}$. It follows that ξ' pushes forward to a vector field ξ on the image of ψ, and that ξ is parallel on that image. Extend ξ to a parallel vector field on G, i.e., to an element of $\mathfrak{g}$. Proposition 2.66 implies that the map $\dot{\psi}\colon \mathfrak{g}' \to \mathfrak{g}$ is a Lie algebra homomorphism. Finally, observe that under (3.18) the map $\dot{\psi}$ is identified with the differential of ψ at the identity $e' \in G'$. $\square$

Remark 3.42. Lemma 3.40 suggests the following question: Given a Lie algebra homomorphism $\beta\colon \mathfrak{g}' \to \mathfrak{g}$, does there exist a Lie group homomorphism $\psi\colon G' \to G$ such that $\dot{\psi} = \beta$? We address this question by seeking the graph of ψ, which is a submanifold $\Gamma \subset G' \times G$. The equation $\dot{\psi} = \beta$ is encoded by a *distribution* on $G' \times G$, that is, a subbundle of the tangent bundle such that the restriction of this subbundle to Γ is the tangent bundle to Γ. Specifically, identify

$$T_{(g',g)}(G' \times G) \cong \mathfrak{g}' \oplus \mathfrak{g} \tag{3.43}$$

and then the graph of $\dot{\psi}$ determines a subspace of (3.43). The local and global existence and uniqueness of Γ—called an *integral manifold* of the distribution—is the subject of the Frobenius theorem, which we take up in Lecture 5.

(3.44) *One-parameter groups.* Specialize to $G' = \mathbb{R}$. Then the distribution described in Remark 3.42 is 1-dimensional, and the graph we seek is an integral curve of a vector field.

Definition 3.45. A *one-parameter group* in a Lie group G is a homomorphism $\gamma\colon \mathbb{R} \to G$. We say γ is *generated* by $\dot{\gamma}(0) = \xi \in \mathfrak{g}$.

The following is an affirmative answer to the question in Remark 3.42 for $G' = \mathbb{R}$.

Theorem 3.46. *Let $\xi \in \mathfrak{g}$ be a parallel vector field on a Lie group G. Then ξ is complete. Furthermore, the integral curve $\gamma_\xi\colon \mathbb{R} \to G$ of ξ with $\gamma_\xi(0) = e$ is a one-parameter group with generator ξ.*

A one-parameter group is a constant velocity motion: its velocity is parallel.

Remark 3.47. Note that Definition 3.45 and Theorem 3.46 combine to a bijection between the Lie algebra $\mathfrak{g}$ and one-parameter groups in G. Often these motions in G provide a geometric way to work with an element of the Lie algebra of a Lie group, an idea we use often. In particular, a one-parameter group and its translates provide distinguished motions that represent a tangent vector on a Lie group.

Proof. Let $\gamma\colon (-\epsilon, \epsilon) \to G$ be a local integral curve of ξ with $\gamma(0) = e$. Suppose $t_1, t_2 \in (-\epsilon, \epsilon)$ satisfy $t_1 + t_2 \in (-\epsilon, \epsilon)$. Then we claim

$$\gamma(t_1 + t_2) = \gamma(t_1)\gamma(t_2). \tag{3.48}$$

To verify this, fix t_1 and let γ_L, γ_R denote the left and right hand sides of (3.48) as functions of $t = t_2$. Then

$$\dot{\gamma}_L(t) = \dot{\gamma}(t_1 + t) = \xi_{\gamma(t_1+t)} = \xi_{\gamma_L(t)} \tag{3.49}$$

and

$$\dot{\gamma}_R(t) = \gamma(t_1)\dot{\gamma}(t) = (L_{\gamma(t_1)})_* \xi_{\gamma(t)} = \xi_{\gamma(t_1)\gamma(t)} = \xi_{\gamma_R(t)}. \tag{3.50}$$

Furthermore, the initial positions are

$$\gamma_L(0) = \gamma(t_1) = \gamma_R(0). \tag{3.51}$$

Now (3.48) follows from the uniqueness statement in the fundamental theorem of ordinary differential equations.

Next, we prove that $\gamma\colon (-\epsilon,\epsilon) \to G$ extends to an integral curve γ_ξ of ξ with domain $\mathbb{R}$. For any $t \in \mathbb{R}$ choose $N \in \mathbb{Z}^{>0}$ such that $|t/N| < \epsilon$, and define

$$\gamma_\xi(t) = \gamma\left(\frac{t}{N}\right)^N. \tag{3.52}$$

If also $M \in \mathbb{Z}^{>0}$ such that $|t/M| < \epsilon$, then using (3.48) we have

$$\gamma\left(\frac{t}{M}\right)^M = \left[\gamma\left(\frac{t}{NM}\right)^N\right]^M = \left[\gamma\left(\frac{t}{NM}\right)^M\right]^N = \gamma\left(\frac{t}{N}\right)^N. \tag{3.53}$$

This proves that (3.52) is well-defined. It follows from (3.48) that γ_ξ is a homomorphism.

Finally, for any $g \in G$ the map

$$\begin{aligned} \mathbb{R} &\longrightarrow G \\ t &\longmapsto g\,\gamma_\xi(t) \end{aligned} \tag{3.54}$$

is an integral curve of ξ with initial position g. This proves that ξ is a complete vector field. □

Remark 3.55. Notice that (3.54) implies that the flow generated by a parallel (*left*-invariant) vector field is *right* multiplication by the integral curve through $e \in G$. Quite generally, signs and left vs. right are tricky in differential geometry, so it is worth paying attention.

(3.56) *The exponential map.* The integral curves in Theorem 3.46 fit together into a map from the Lie algebra $\mathfrak{g}$ of a Lie group G to the Lie group. Namely, define the *exponential map*

$$\begin{aligned} \exp\colon \mathfrak{g} &\longrightarrow G \\ \xi &\longmapsto \gamma_\xi(1) \end{aligned} \tag{3.57}$$

Proposition 3.58. *The exponential map* (3.57) *is smooth.*

Proof. On $\mathfrak{g} \times G$ define the vector field $\eta_{\xi,g} = 0 \oplus \xi$; see Figure 9. It generates the flow

$$\varphi_t(\xi, g) = \bigl(\xi, g\gamma_\xi(t)\bigr). \tag{3.59}$$

The fundamental theorem of ODE asserts that φ is a smooth function jointly in t, ξ, g. Hence so too is $\exp\colon \xi \mapsto \pi\bigl(\varphi_1(\xi, e)\bigr)$, where $\pi\colon \mathfrak{g} \times G \to G$ is projection. □

Remark 3.60. The device of passing to the universal example—in this case of a parallel vector field—by introducing a parameter space—in this case $\mathfrak{g}$—is a powerful maneuver in many situations.

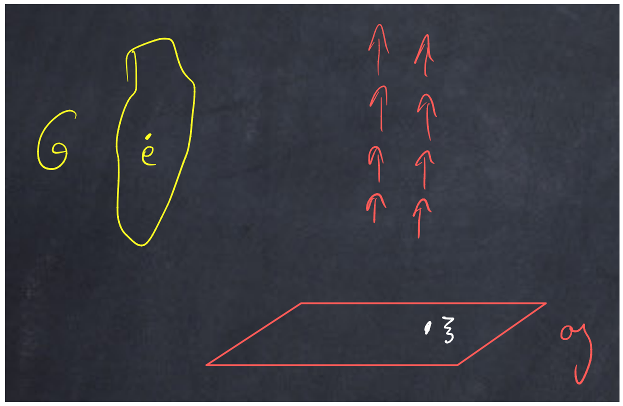

FIGURE 9. The universal parallel vector field on a Lie group

Exponentials and Lie brackets for the general linear group

(3.61) *The exponential.* As in Example 3.24, let V be a finite dimensional real vector space. Suppose $A \in \operatorname{End} V$. Define its exponential by a power series:

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \tag{3.62}$$

A finite truncation is defined algebraically. To define the infinite sum, introduce[3] a norm on V and prove that the sequence of partial sums is Cauchy. By easy manipulations with the partial sums, and then by passage to the limit, you will prove that for $t, t_1, t_2 \in \mathbb{R}$,

$$e^{(t_1+t_2)A} = e^{t_1A} \circ e^{t_2A} \tag{3.63}$$

$$\frac{d}{dt}e^{tA} = A \circ e^{tA} = e^{tA} \circ A. \tag{3.64}$$

The first equation implies that e^A is invertible with inverse e^{-A}. For endomorphisms $A, B \in \operatorname{End}(V)$, if A and B commute then $e^{A+B} = e^A \circ e^B$; if they do not commute, then in general $e^{A+B} \neq e^A \circ e^B$.

In the sequel we drop the '$\circ$' symbol.

(3.65) *The exponential map.* Recall from (3.25) that the parallel vector field ξ_A on the general linear group $\operatorname{Aut}(V)$ with value $A \in \operatorname{End}(V) \cong T_{\mathrm{id}} \operatorname{Aut}(V)$ has value $PA \in \operatorname{End}(V)$ at $P \in \operatorname{Aut}(V)$. By (3.64) the integral curve of ξ_A with initial value $P_0 \in \operatorname{Aut}(A)$ is

$$P_t = P_0e^{tA}. \tag{3.66}$$

In particular, the integral curve with initial value id is $t \mapsto e^{tA}$. This implies the following.

[3]All norms are equivalent, so any works for this argument. You can find details of the arguments in this paragraph in the notes on Multivariable Analysis, and you should work through this paragraph carefully if you have never seen it before. This is an exhortation that applies universally in these notes.

Proposition 3.67. *For the general linear group* $\mathrm{Aut}(V)$*, the exponential map* (3.57) *is the exponential map* (3.62).

(3.68) *The Lie bracket.* Let $A, B \in \mathrm{End}(V)$ generate parallel vector fields ξ_A, ξ_B, and let φ be the flow generated by ξ_A. Then $\mathcal{L}_{\xi_A}\xi_B$ at id is

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} (\varphi_{-t})_*\xi_B(\mathrm{id}) &= \frac{d}{dt}\Big|_{t=0} e^{tA}Be^{-tA} \\ &= AB - BA. \end{aligned} \tag{3.69}$$

In the first line we use that the flow generated by ξ_A is right multiplication by e^{tA}—see (3.66)—and that the value of ξ_B at e^{tA} is $e^{tA}B$—see (3.25). This computation implies the following.

Proposition 3.70. *For the general linear group* $\mathrm{Aut}(V)$*, the Lie bracket on the Lie algebra* $\mathrm{End}(V)$ *is the commutator* $[A, B] = AB - BA$.

Observe that we use (3.18) to identify the Lie algebra with $\mathrm{End}(V)$, and we use the associative algebra structure of $\mathrm{End}(V)$ to define the commutator.

Remark 3.71. Any associative algebra determines a Lie algebra whose Lie bracket is the commutator.

Lecture 4: More on Lie groups

We continue our introduction to Lie groups, and we also develop a few general aspects of the calculus of differential forms.

We begin with a discussion of Lie subgroups of a Lie group. A Lie subgroup need not be closed in the topology of the Lie group. A powerful general theorem asserts that a closed subgroup of a Lie group is itself a Lie group. We state, but do not prove that theorem.

The next topic is the adjoint action of a Lie group on itself and on its Lie algebra. This is essentially conjugation in various forms; it measures the deviation of a Lie group from abelianness. For matrix groups the action is precisely conjugation.

We then make a digression to conclude a topic left over from Lecture 2, namely Lie derivatives of differential forms. In this section and in the remainder of the lecture, we introduce a few formulas (4.39), (4.48), (4.49), (4.69), (4.83) that are so fundamental in Differential Geometry that I recommend you memorize them. The section on Lie derivatives has six commutator formulas (4.30), (4.35), (4.39), (4.42), (4.43), (4.44) that are basic to the calculus of differential forms.

With this in hand, we introduce a tautological vector-valued 1-form on a Lie group: the *Maurer–Cartan form*. It satisfies the *Maurer–Cartan equation* (4.69), the basic structure equation on a Lie group. Variations of this equation will be central in our geometric discussions, for example of curvature. The form (4.81), (4.83) this equation takes on a matrix group will also recur later.

Lie subgroups

(4.1) *One-parameter subgroups on a torus.* A one-parameter subgroup of the vector group $\mathbb{R}^2$ is a homomorphism $\tilde{\gamma}\colon \mathbb{R} \to \mathbb{R}^2$, i.e., a linear map $\tilde{\gamma}(t) = (at, bt)$ for some $a, b \in \mathbb{R}$. Let $G = \mathbb{R}^2/\mathbb{Z}^2$ be the quotient by the subgroup of vectors with integer coefficients. Then G is an abelian Lie group whose underlying manifold is diffeomorphic to a 2-torus. Let $\pi\colon \mathbb{R}^2 \to G$ be the quotient map. Then $\gamma = \pi \circ \tilde{\gamma}$ is a one-parameter subgroup of G. If b/a is rational, then γ is not injective, but factors through an injective Lie group homomorphism $\mathbb{R}/L\mathbb{Z} \to G$ for some $L \in \mathbb{Z}$. The image in G is a closed submanifold diffeomorphic to S^1, and it is a subgroup as well; we summarize this combination as a *closed subgroup.* If b/a is irrational, then γ is an injective immersion with dense image in G; it is not an embedding. See Figure 10 for a crude picture. The image of γ is a subgroup of G, but it is not a submanifold.

(4.2) *Definitions and basic theorems.* The examples in **(4.1)** motivate the following.

Definition 4.3. Let G be a Lie group. A *Lie subgroup* $i\colon G' \to G$ is an injective homomorphism of Lie groups. It is a *closed subgroup* if $i(G') \subset G$ is closed.

The differential of an injective Lie group homomorphism is an injective homomorphism of Lie algebras, so in particular an injective Lie group homomorphism is an injective immersion. We state the following without proof.

Theorem 4.4.

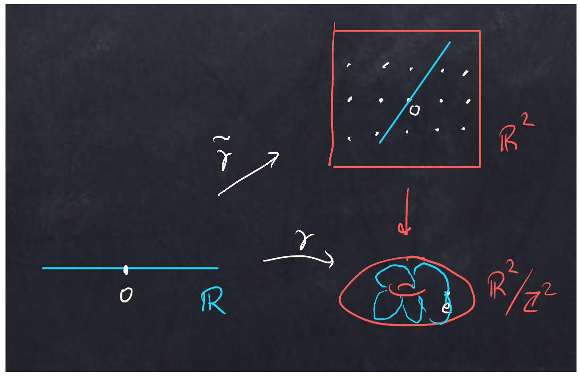

FIGURE 10. The skew line on the torus

(1) *Let $i\colon G' \to G$ be a Lie subgroup. Then $i(G') \subset G$ is closed iff i is an embedding.*
(2) *A closed subgroup of a Lie group is a Lie subgroup.*

The second assertion states that if a subset $G' \subset G$ is a subgroup in the group structure and is closed in the topology, then G' is a submanifold and the group operations on G' are smooth. The proofs of these assertions are not immediate; see Warner.

Remark 4.5. Theorem 4.4(2) is powerful, as pointed out in Example 3.27 for instance.

The adjoint action

(4.6) *The nonlinear action of G by conjugation.* Define the left action (Definition 3.8) of G on G by conjugation:

$$\mathrm{AD}_g(x) = gxg^{-1}, \qquad g, x \in G. \tag{4.7}$$

The *orbit* of $x \in G$ under conjugation is the *conjugacy class* of x, which is a closed submanifold of G. The *stabilizer* of $x \in G$ is the *centralizer* $Z(x) \subset G$, which is a closed subgroup and hence, by Theorem 4.4(2), is a Lie subgroup.

(4.8) *The linear action of G on its Lie algebra.* Let $\xi \in \mathfrak{g}$ be a parallel vector field on G. Then we check that the differential of AD_g maps ξ to a parallel vector field using the canonical motion (germ) with velocity ξ:

$$\mathrm{AD}_g(xe^{t\xi}) = gxe^{t\xi}g^{-1} = (gxg^{-1})(ge^{t\xi}g^{-1}). \tag{4.9}$$

This linear action of G on its Lie algebra of parallel vector fields is denoted

$$\mathrm{Ad}\colon G \longrightarrow \mathrm{Aut}(\mathfrak{g}). \tag{4.10}$$

It is a canonical *linear representation* of G, the *adjoint representation.* Note that $\operatorname{Aut}(\mathfrak{g})$ is a Lie group (Example 3.24), and (4.10) is a Lie group homomorphism. Under the identification (3.18) of $\mathfrak{g}$ with T_eG we have $\operatorname{Ad}_g = d(\operatorname{AD}_g)_e$. Furthermore, linear transformations in the image of the adjoint representation preserve the Lie bracket:

$$\operatorname{Ad}_g[\xi,\eta] = [\operatorname{Ad}_g \xi, \operatorname{Ad}_g \eta], \qquad g \in G, \quad \xi,\eta \in \mathfrak{g}. \tag{4.11}$$

In other words, we can read '$\operatorname{Aut}(\mathfrak{g})$' not just as linear automorphisms, but as automorphisms which preserve the Lie algebra structure. This follows from a computation with the flows generated by ξ,η (see Proposition 2.58 and Remark 3.55), which we evaluate at $e \in G$:

$$\left.\frac{\partial^2}{\partial t \partial s}\right|_{\substack{t=0\\ s=0}} \operatorname{AD}_g(e^{t\xi}e^{s\eta}e^{-t\xi}e^{-s\eta}) = \left.\frac{\partial^2}{\partial t \partial s}\right|_{\substack{t=0\\ s=0}} (ge^{t\xi}g^{-1})(ge^{s\eta}g^{-1})(ge^{-t\xi}g^{-1})(ge^{-s\eta}g^{-1}). \tag{4.12}$$

(4.13) *The infinitesimal linear action of* $\mathfrak{g}$ *on* $\mathfrak{g}$*.* By Lemma 3.40 the Lie group homomorphism (4.10) induces a Lie algebra homomorphism

$$\operatorname{ad}\colon \mathfrak{g} \longrightarrow \operatorname{End}(\mathfrak{g}). \tag{4.14}$$

This is the *infinitesimal adjoint representation* of the Lie algebra $\mathfrak{g}$. Furthermore, ad_ξ, $\xi \in \mathfrak{g}$, acts as a derivation on $\mathfrak{g}$, which follows by differentiating (4.11) in g:

$$\operatorname{ad}_\xi[\eta,\zeta] = [\operatorname{ad}_\xi(\eta),\zeta] + [\eta,\operatorname{ad}_\xi(\zeta)]. \tag{4.15}$$

In fact, we can compute the adjoint action explicitly.

Proposition 4.16. *For* $\xi,\eta \in \mathfrak{g}$ *we have*

$$\operatorname{ad}_\xi(\eta) = [\xi,\eta]. \tag{4.17}$$

Proof. As in (4.12) we compute at $e \in G$ and for $g \in G$ write

$$\operatorname{Ad}_g(\eta)(e) = \left.\frac{d}{ds}\right|_{s=0} ge^{s\eta}g^{-1}. \tag{4.18}$$

Hence

$$\operatorname{ad}_\xi(\eta)(e) = \left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} e^{t\xi}e^{s\eta}e^{-t\xi}. \tag{4.19}$$

Since the flow φ_t generated by ξ is right multiplication by $e^{t\xi}$, the expression on the right hand side of (4.19) is[4] $\left.\frac{d}{dt}\right|_{t=0} (\varphi_{-t})_*(\eta)$, which by (2.19) is $(\mathcal{L}_\xi\eta)(e)$. Conclude with (2.54). ∎

Remark 4.20. Under the identification (4.17), equation (4.15) is a form of the Jacobi identity (2.36) for the Lie bracket.

[4] The value of the parallel vector field η at $\varphi_t(e) = e^{t\xi}$ is $\left.\frac{d}{ds}\right|_{s=0} e^{t\xi}e^{s\eta}$.

(4.21) *Adjoint action for matrix groups.* Suppose G is a matrix group, which means that $G \subset \mathrm{Aut}(V)$ for some finite dimensional real vector space V. In that case the Lie algebra $\mathfrak{g} \subset \mathrm{End}(V)$ is a subset of endomorphisms, after identifying it with the tangent space to G at the identity $e = \mathrm{id}_V$. Then from (4.9) with $x = e$ we deduce that

$$\mathrm{Ad}_P(A) = PAP^{-1}, \qquad P \in G, \quad A \in \mathfrak{g}. \tag{4.22}$$

In other words, the adjoint action is precisely conjugation for matrix groups.

The Lie derivative on differential forms

(4.23) *Derivations on graded algebras.* We begin with the $\mathbb{Z}$-graded variation of (**2.29**).

Definition 4.24. Let $A = \bigoplus_{q\in\mathbb{Z}} A^q$ be a $\mathbb{Z}$-graded algebra. A homogeneous map $D\colon A \to A$ of degree k is s *derivation* if

$$D(\alpha\beta) = (D\alpha)\beta + (-1)^{kq}\alpha(D\beta), \qquad \alpha \in A^q, \quad \beta \in A. \tag{4.25}$$

If D_1, D_2 are derivations of degrees k_1, k_2, respectively, then

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{k_1k_2} D_2 \circ D_1 \tag{4.26}$$

is a derivation of degree $k_1 + k_2$.

Remark 4.27.

1. In (4.25) and (4.26) we use the *Koszul sign rule*, which loosely imposes a sign $(-1)^{pq}$ which commuting objects of degrees p and q in a $\mathbb{Z}$-graded algebra.
2. The commutator (4.26) (with sign) endows the vector space $\mathrm{Der}(A)$ of derivations of A with the structure of a $\mathbb{Z}$*-graded Lie algebra*. As an exercise, write this $\mathbb{Z}$-graded variation of Definition 2.33. You'll need to apply the Koszul sign rule carefully and consistently.

Example 4.28 (Derivations on differential forms)**.** Let X be a smooth manifold and let $A = \Omega^\bullet_X$ be the $\mathbb{Z}$-graded algebra of differential forms. The Cartan differential $d\colon \Omega^\bullet_X \to \Omega^{\bullet+1}_X$ is a derivation of degree $+1$. For any vector field $\xi \in \mathfrak{X}(X)$, the Lie derivative $\mathcal{L}_\xi$ is a derivation of degree 0. To verify this, let φ_t be the flow generated by ξ, differentiate

$$\varphi_t^*(\alpha \wedge \beta) = \varphi_t^*\alpha \wedge \varphi_t^*\beta, \qquad \alpha, \beta \in \Omega^\bullet_X \tag{4.29}$$

in t, and apply (2.19). Furthermore,

$$[d, \mathcal{L}_\xi] = 0 \tag{4.30}$$

follows by differentiating

$$\varphi_t^* d\alpha = d\varphi_t^*\alpha, \qquad \alpha \in \Omega^\bullet_X, \tag{4.31}$$

in t.

(4.32) *Interior product (contraction).* Interior product is an algebraic operation on differential forms: it acts pointwise. There is no differentiation; by contrast, the derivations d and $\mathcal{L}_\xi$ act as first order differential operators. Hence we treat contraction algebraically.

Let V be a vector space.[5] For $\xi \in V$ we give three equivalent characterizations of the derivation

$$\iota_\xi\colon {\textstyle\bigwedge}^\bullet V^* \longrightarrow {\textstyle\bigwedge}^{\bullet-1} V^* \tag{4.33}$$

as follows:

(1) ι_ξ is dual to left exterior multiplication $\epsilon_\xi\colon \bigwedge^\bullet V \to \bigwedge^{\bullet+1} V$.
(2) ι_ξ is a derivation of degree -1 and $\iota_\xi\colon V^* \to \mathbb{R}$ is evaluation on ξ.
(3) If we identify $\alpha \in \bigwedge^q V^*$ with alternating q-linear functionals on V, then

$$(\iota_\xi\alpha)(\xi_1, \dots, \xi_{q-1}) = \alpha(\xi, \xi_1, \dots, \xi_{q-1}), \qquad \xi_1, \dots, \xi_{q-1} \in V. \tag{4.34}$$

This last equation tells that ι_ξ is the operation "plug ξ into the first slot". All three characterizations are useful in practice. It is easy to verify

$$[\iota_\xi, \iota_\eta] = 0, \qquad \xi, \eta \in V. \tag{4.35}$$

For example, this is dual to the (graded) commutativity of exterior multiplication using characterization (1). Take $\xi = \eta$ in (4.35) to deduce

$$(\iota_\xi)^2 = \frac{1}{2}[\iota_\xi, \iota_\xi] = 0. \tag{4.36}$$

Let X be a smooth manifold. Define the interior product $\iota_\xi\colon \Omega^\bullet_X \to \Omega^{\bullet-1}_X$ on differential forms pointwise as interior product on the exterior algebra of the cotangent space.

(4.37) *The Cartan[6] formula.* Now we can compute the Lie derivative on differential forms, a formula you should know so well that you could recite it in your sleep.

Theorem 4.38. *Let $\xi \in \mathfrak{X}(X)$ be a vector field on X. Then*

$$\mathcal{L}_\xi = [d, \iota_\xi] = d\iota_\xi + \iota_\xi d. \tag{4.39}$$

Proof. Both $\mathcal{L}_\xi$ and $[d, \iota_\xi]$ are degree 0 derivations of $\Omega^\bullet_X$, both commute with d, and they agree on functions Ω^0_X; see (4.26), (4.29), and (2.21). Any differential form is locally a sum of terms of the form $f_0\, df_1 \wedge \cdots \wedge df_q$, where $f_0, f_1, \dots, f_q \in \Omega^0_X$. The result follows. $\square$

[5] We needn't assume the ground field is $\mathbb{R}$, nor that V is finite dimensional.

[6] This is Henri Cartan, son of Élie.

Remark 4.40. The derivations of interior product, Lie derivative, and Cartan d endow

$$\mathfrak{X}(X) \oplus \mathfrak{X}(X) \oplus \mathbb{R} \tag{4.41}$$

with the structure of a $\mathbb{Z}$-graded Lie algebra, where the degrees of the summands are $-1, 0, +1$. An element $\eta \in \mathfrak{X}(X)$ in the first summand acts on $\Omega^\bullet_X$ as ι_η. An element $\xi \in \mathfrak{X}(X)$ in the second summand acts on $\Omega^\bullet_X$ as $\mathcal{L}_\xi$. A real number $a \in \mathbb{R}$ acts on $\Omega^\bullet_X$ as a times Cartan d. The bracketing relations are (4.30), (4.35), (4.39), and

$$[d, d] = 0 \tag{4.42}$$

$$[\mathcal{L}_\xi, \iota_\eta] = \iota_{[\xi,\eta]} \tag{4.43}$$

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi,\eta]} \tag{4.44}$$

Remark 4.45. This $\mathbb{Z}$-graded Lie algebra encodes what is needed to compute in first order differential calculus. If one supplements with integration over the fibers of a fiber bundle, and the bracket of that operation with the operations in (4.41), then one has almost all one needs to compute many formulas in calculus. (One needs integration over fiber bundles whose fibers are manifolds with boundary... and then eventually manifolds with corners...).

(4.46) *Two basic formulas.* The following formulas should also inhabit your dreams.

Proposition 4.47. *Let X be a smooth manifold, let $\alpha, \beta \in \Omega^1_X$ be 1-forms, and $\xi, \eta \in \mathfrak{X}(X)$ be vector fields. Then*

$$d\alpha(\xi, \eta) = \xi\alpha(\eta) - \eta\alpha(\xi) - \alpha([\xi, \eta]) \tag{4.48}$$

$$(\alpha \wedge \beta)(\xi, \eta) = \alpha(\xi)\beta(\eta) - \alpha(\eta)\beta(\xi) \tag{4.49}$$

Proof. For (4.48), compute using the commutation relations of (4.41):

$$\begin{aligned} d\alpha(\xi, \eta) &= \iota_\eta \iota_\xi d\alpha \\ &= \iota_\eta(\mathcal{L}_\xi \alpha - d\iota_\xi \alpha) \\ &= \mathcal{L}_\xi \iota_\eta \alpha - \iota_{[\xi,\eta]}\alpha - \iota_\eta d\iota_\xi \alpha \\ &= \xi\alpha(\eta) - \alpha([\xi, \eta]) - \eta\alpha(\xi). \end{aligned} \tag{4.50}$$

The proof of (4.49) is similar:

$$\begin{aligned} (\alpha \wedge \beta)(\xi, \eta) &= \iota_\eta \iota_\xi (\alpha \wedge \beta) \\ &= \iota_\eta(\iota_\xi \alpha \wedge \beta - \alpha \wedge \iota_\xi \beta) \\ &= (\iota_\xi \alpha)(\iota_\eta \beta) - (\iota_\eta \alpha)(\iota_\xi \beta) \\ &= \alpha(\xi)\beta(\eta) - \alpha(\eta)\beta(\xi). \end{aligned} \tag{4.51}$$

$\square$

Differential forms on Lie groups

(4.52) *Parallel forms and Lie algebra cohomology.* Let G be a Lie group. The global parallelism by left translation determines a subalgebra $\Omega^\bullet_{\parallel}(G) \subset \Omega^\bullet(G)$ of parallel differential forms.

Lemma 4.53. *$\Omega^\bullet_{\parallel}(G)$ is closed under d.*

Recall that this is not true for a general global parallelism on a smooth manifold.

Proof. A differential form $\alpha \in \Omega^\bullet_\bullet G$ lies in $\Omega^\bullet_{\parallel}(G)$ iff $L_g^*\alpha = \alpha$ for all $g \in G$. The lemma follows since $L_g^* \circ d = d \circ L_g^*$. ☐

Evaluation at the identity gives an isomorphism of $\mathbb{Z}$-graded algebras

$$\Omega^\bullet_{\parallel}(G) \xrightarrow{\mathrm{ev}_e} {\textstyle\bigwedge}^\bullet \mathfrak{g}^*. \tag{4.54}$$

Use it to transfer d to a differential on $\bigwedge^\bullet \mathfrak{g}^*$. The resulting differential graded algebra, the *Chevalley–Eilenberg complex*, defines the Lie algebra cohomology of $\mathfrak{g}$ and can be generalized to Lie algebras which do not arise as Lie algebras of a Lie group, as well as to Lie algebras over more general fields.

(4.55) *Vector-valued differential forms.* Let X be a smooth manifold and W a real vector space.

Definition 4.56. The $\mathbb{Z}$-graded vector space $\Omega^\bullet_X(W) = \bigwedge^\bullet(X;W)$ of *W-valued differential forms* is the space of smooth sections of the vector bundle over X whose fiber at $x \in X$ is $\bigwedge^\bullet T_x^*X \otimes W$.

For $W = \mathbb{R}^m$ an element of $\Omega^\bullet_X(W)$ is a column vector of differential forms on X. Pointwise exterior multiplication gives $\Omega^\bullet_X(W)$ the structure of a $\mathbb{Z}$-graded module over $\Omega^\bullet_X$. Furthermore, the differential d acts on $\Omega^\bullet_X(W)$ compatibly with the module structure:

$$d(\alpha \wedge \sigma) = d\alpha \wedge \sigma + (-1)^q \alpha \wedge d\sigma, \qquad \alpha \in \Omega^q_X, \quad \sigma \in \Omega^\bullet_X(W). \tag{4.57}$$

In fact, (4.57) can be used to define d on $\Omega^\bullet_X(W)$: first let $\sigma = w$ be a constant function in $\Omega^0_X(W)$, in which case define $d(\alpha w) = (d\alpha)w$, and then extend to $\Omega^\bullet_X(W)$ by the Leibniz rule (4.57). We say that $\Omega^\bullet_X(W)$ is a *differential graded module* over the differential graded algebra $\Omega^\bullet_X$.

Remark 4.58. We will generalize to differential forms valued in a vector bundle $W \to X$. Then we need an additional structure—a *covariant derivative* on $W \to X$—to define a first-order differential operator of degree $+1$ on $\Omega^\bullet_X(W)$. In that case the square is not necessarily zero; it is the *curvature*.

(4.59) *The Maurer–Cartan form.* There is a tautological vector-valued 1-form on a Lie group G.

Definition 4.60. Let G be a Lie group. The *Maurer–Cartan form* $\theta \in \Omega^1_G(\mathfrak{g})$ assigns to $\xi_x \in T_xG$, $x \in G$, the parallel vector field $\xi \in \mathfrak{g}$ whose value at x is ξ_x.

It follows immediately from the definition that θ is parallel:

$$(L_g^*\theta)_x(\xi_x) = \theta_{gx}\bigl((L_g)_*\xi_x\bigr) = \xi = \theta_x(\xi_x), \qquad g,x\in G, \quad \xi_x\in T_xX, \tag{4.61}$$

since the parallel vector field ξ which extends ξ_x has value $(L_g)_*\xi_x$ at $gx\in G$. The tautological nature of θ is brought out by applying the isomorphism (4.54): the image of θ under

$$\Omega^1_{\|}(G;\mathfrak{g}) \xrightarrow{\;\mathrm{ev}_e\;} \mathfrak{g}^*\otimes\mathfrak{g}\cong \mathrm{End}(\mathfrak{g}) \tag{4.62}$$

is the identity endomorphism $\mathrm{id}_{\mathfrak{g}}$.

(4.63) *The wedge-bracket.* For any smooth manifold X, define the *wedge-bracket*

$$[-\wedge -]\colon \Omega^\bullet_X(\mathfrak{g})\times\Omega^\bullet_X(\mathfrak{g})\longrightarrow\Omega^\bullet_X(\mathfrak{g}) \tag{4.64}$$

on $\mathfrak{g}$-valued differential forms by simultaneously executing wedge product and Lie bracket:

$$[\alpha\xi\wedge\beta\eta] = (\alpha\wedge\beta)[\xi,\eta], \qquad \alpha,\beta\in\Omega^\bullet_X, \quad \xi,\eta\in\mathfrak{g}. \tag{4.65}$$

This suffices, since the general element of $\Omega^\bullet_X(\mathfrak{g})$ is locally a sum of forms of this type. The Koszul sign in the wedge product and the skew-symmetry of the Lie bracket combine to yield

$$[\sigma\wedge\tau] = -(-1)^{pq}\,[\tau\wedge\sigma], \qquad \sigma\in\Omega^p_X(\mathfrak{g}), \quad \tau\in\Omega^q_X(\mathfrak{g}). \tag{4.66}$$

Notice in particular that the wedge-bracket is symmetric on $\mathfrak{g}$-valued 1-forms.

(4.67) *The Maurer–Cartan equation.* This is also known as the *structure equation* of a Lie group.[7]

Theorem 4.68. *The Maurer–Cartan form $\theta\in\Omega^1_G(\mathfrak{g})$ satisfies*

$$d\theta + \frac12[\theta\wedge\theta] = 0. \tag{4.69}$$

Proof. Apply Proposition 4.47. Fix parallel vector fields $\xi,\eta\in\mathfrak{g}$. Then (4.48) yields

$$\begin{aligned} d\theta(\xi,\eta) &= \xi\theta(\eta) - \eta\theta(\xi) - \theta([\xi,\eta]) \\ &= -\theta([\xi,\eta]) \\ &= -[\xi,\eta] \end{aligned} \tag{4.70}$$

because the pairing of a parallel 1-form with a parallel vector field is a parallel function, i.e., a constant function. Also, by (4.49) we have

$$\begin{aligned} \frac12[\theta\wedge\theta](\xi,\eta) &= \frac12\bigl([\theta(\xi),\theta(\eta)] - [\theta(\eta),\theta(\xi)]\bigr) \\ &= [\xi,\eta]. \end{aligned} \tag{4.71}$$

Combine (4.70) and (4.71) to obtain (4.69). ∎

[7] This is the fourth formula of this lecture that is worthy of special memorization.

(4.72) *Maurer–Cartan in a basis.* Let $\xi_1, \dots, \xi_m$ be a basis of $\mathfrak{g} = \mathfrak{X}_{\|}(G)$, and let $\theta^1, \dots, \theta^m$ be the dual basis of $\mathfrak{g}^* = \Omega^{\bullet}_{\|}(X)$. The *structure constants* $c^i_{jk} \in \mathbb{R}$ of $\mathfrak{g}$ relative to this basis encode the Lie bracket:

$$[\xi_j, \xi_k] = c^i_{jk}\xi^i, \qquad i \in \{1, \dots, m\}. \tag{4.73}$$

The Maurer–Cartan form (Definition 4.60) is

$$\theta = \theta^i \xi_i, \tag{4.74}$$

and we compute

$$[\theta \wedge \theta] = [\theta^j \xi_j \wedge \theta^k \xi_k] = (\theta^j \wedge \theta^k)[\xi_j, \xi_k] = c^i_{jk}\theta^j \wedge \theta^k \xi_i. \tag{4.75}$$

The Maurer–Cartan equation (4.69) in this basis is a system of m equations:

$$d\theta^i + \frac{1}{2}c^i_{jk}\theta^j \wedge \theta^k = 0, \qquad i \in \{1, \dots, m\}. \tag{4.76}$$

(4.77) *Maurer–Cartan on a matrix group.* Consider the matrix group $G = \mathrm{GL}_n\mathbb{R}$. There is a tautological matrix-valued function

$$g\colon\ \mathrm{GL}_n\mathbb{R} \longrightarrow M_n\mathbb{R}, \tag{4.78}$$

which is simply the inclusion of $\mathrm{GL}_n\mathbb{R}$ into the vector space of $n \times n$ real matrices. It is precisely this function (4.78) which justifies the moniker 'matrix group'. It follows directly from Definition 4.60 and (3.25) that the Maurer–Cartan form on $\mathrm{GL}_n\mathbb{R}$ is

$$\Theta = g^{-1}dg, \tag{4.79}$$

which is an $n \times n$ matrix of 1-forms on $\mathrm{GL}_n\mathbb{R}$. (The capitalized notation 'Θ' is intentional.) The right hand side uses matrix multiplication. The Maurer–Cartan equation in matrix form follows directly by differentiation:

$$d\Theta = -g^{-1}dg\, g^{-1} \wedge dg = -\Theta \wedge \Theta, \tag{4.80}$$

or in the form we will use it

$$d\Theta + \Theta \wedge \Theta = 0. \tag{4.81}$$

Write the matrix Θ of 1-forms in terms of scalar 1-forms

$$\Theta = \left(\Theta^i_j\right), \qquad \Theta^i_j \in \Omega^1_{\mathrm{GL}_n\mathbb{R}}. \tag{4.82}$$

Then (4.81) expands to n^2 equations, another entry for your rapidly expanding mathematical memory bank:

$$d\Theta^i_j + \Theta^i_k \wedge \Theta^k_j = 0, \qquad i,j \in \{1,\dots,n\}. \tag{4.83}$$

We will use these structure equations often.

Lecture 5: The local Frobenius theorem

In this lecture we discuss the local version of the Frobenius theorem and the integrability condition which pertains. In the next lecture we turn to global aspects and applications.

We begin with *line fields*, which to each point of a smooth manifold assign a 1-dimensional subspace of the tangent space. These are closely related to vector fields—an assignment of a vector in the tangent space rather than a line—and in place of a motion that integrates a vector field we have an *integral manifold* whose tangents are the line fields. These always exist. A *distribution* is a higher dimensional analog: a subbundle of the tangent bundle of any rank. If the rank is at least two, then there is an integrability condition for the existence of integral manifolds: the vanishing of the *Frobenius tensor*. This obstruction to integrability is the basis of curvature, which is a central theme of the course.

Next, we turn to local normal forms for a nonzero vector field and then for a finite set of linearly independent vector fields that mutually commute in the sense that their pairwise Lie brackets vanish. We apply this to prove the local Frobenius Theorem 5.41.

Line fields

(5.1) *Recollection.* Let X be a smooth manifold. We recall the notions of an injective immersion and of a submanifold. (Some authors use 'submanifold' for the former; we reserve the term for a subset of a smooth manifold which is locally diffeomorphic to a standard normal form.)

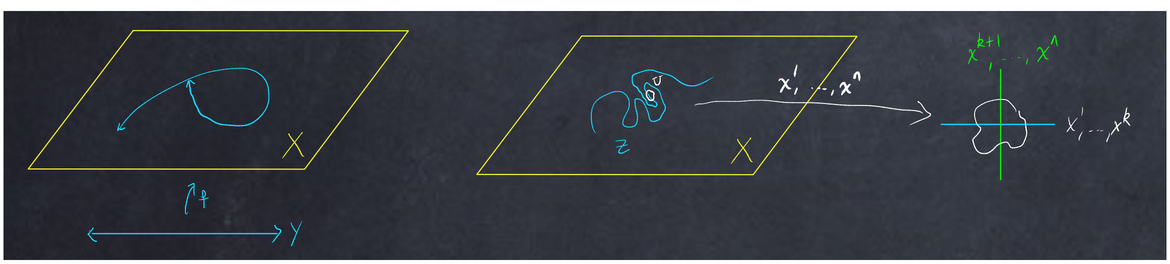

FIGURE 11. An injective immersion which is not an embedding; a submanifold

Definition 5.2.

(1) A smooth injective map $f\colon Y \to X$ is an *injective immersion* if df_y is injective for all $y \in Y$.
(2) A subset $Z \subset X$ is a *submanifold* if every point $z \in Z$ belongs to a chart $(U; x^1, \dots, x^n)$ such that $Z \cap U = \{x^{k+1} = \dots = x^n = 0\}$ for some $0 \le k \le n$.

Theorem 5.3. *The image $f(Y) \subset X$ of an injective immersion $f\colon Y \to X$ is a submanifold iff f factors through a homeomorphism $Y \to f(Y)$ (in which $f(Y)$ has the subspace topology).*

An injective immersion which satisfies this condition is called an *embedding*. The left side of Figure 11 illustrates an injective immersion which is not an embedding: the subspace topology on $f(Y)$ pulls back via f to a topology on Y which is coarser than the given topology. Theorem 5.3 is Theorem 7.1 in the Differential Topology notes, and I defer to that reference for the proof.

(5.4) *Vector fields and line fields.* A vector field $\xi \in \mathfrak{X}(X)$ on a smooth manifold X induces an ordinary differential equation (2.3) whose solutions are called integral curves. Recall that there is local and global existence and uniqueness (Theorem 2.4). An integral curve is a function with domain an interval in $\mathbb{R}$, i.e., a *motion* in X. Now we switch our focus to the *image* of the motion—a subset of X—and its tangent lines. Suppose $L \subset TX$ is a line subbundle of the tangent bundle of X. We seek an injective immersion $f\colon Y \to X$ of a connected 1-manifold Y such that

$$f_*(T_yY) = L_{f(y)}, \qquad y \in Y. \tag{5.5}$$

This is the analog of (2.3) for a line field, and is called an *integral manifold* of L.

Remark 5.6. The existence of a line field on X imposes a topological constraint: the Euler number of X must vanish. For example, S^2 does not admit a line field.

A section of $L \to X$ is a vector field. At least locally we can find a *nonzero* section ξ. Then an integral curve of ξ is an injective immersion that satisfies (5.5). By this device we can parlay the local and global theory of integral curves into theorems about integral manifolds for line fields. But we resist in favor of a general theory for higher rank distributions.

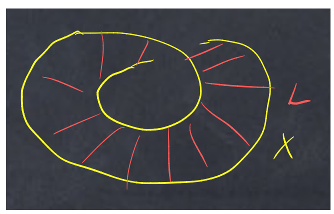

Figure 12. A line field on the Möbius band

Example 5.7 (A line field with no nonzero global section)**.** Figure 12 illustrates a line field on the Möbius band which has the property that every global section has a zero.

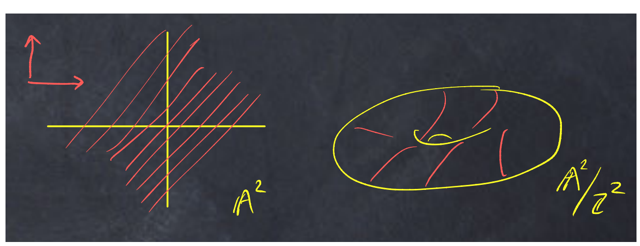

Figure 13. A parallel line field on the 2-torus

Example 5.8 (Line fields on the 2-torus)**.** The standard affine plane $\mathbb{A}^2$ has an action of the vector group $\mathbb{R}^2$ by translation. The countable discrete subgroup $\mathbb{Z}^2 \subset \mathbb{R}^2$ acts freely (and properly discontinuously). The quotient $X = \mathbb{A}^2/\mathbb{Z}^2$ inherits the structure of a smooth manifold; it is diffeomorphic to a 2-torus. The global parallelism of affine space descends to a global parallelism of X. A parallel line field is determined by a 1-dimensional subspace of $\mathbb{R}^2$, for example

$$L^{(a)} = \{(\xi^1, \xi^2) \in \mathbb{R}^2 : \xi^2 = a\xi^1\}, \qquad a \in \mathbb{R}. \tag{5.9}$$

The integral manifolds of the resulting parallel line field on $\mathbb{A}^2$ are affine lines, and there is an unique integral manifold through each point; see Figure 13. Upon descent to the 2-torus X, the picture depends strongly on whether $a \in \mathbb{R}$ is rational or irrational. If a is rational, then $L^{(a)} \cap \mathbb{Z}^2$ is an infinite cyclic group, each integral line in $\mathbb{A}^2$ parallel to $L^{(a)}$ is fixed by translations in this subgroup, and the integral manifolds in $\mathbb{A}^2$ descend to integral manifolds in X, each of which is a submanifold diffeomorphic to a circle. If a is irrational, then translation by $L^{(a)}$ acts freely on each integral manifold in X, and each integral manifold is dense and is the image of an injective immersion, but it is not a submanifold. If $U \subset X$ is the image of a ball in $\mathbb{A}^2$ of radius $< 1/2$, then the intersection of each integral manifold with U has countably many components.

Distributions and the Frobenius tensor

We begin with the definition of a higher dimensional line field.

Definition 5.10. Let X be a smooth manifold.

(1) A *distribution* is a vector subbundle $E \subset TX$ of the tangent bundle.
(2) An injective immersion $f\colon Y \hookrightarrow X$ is an *integral manifold* of E if

$$f_*(T_yY) = E_{f(y)}, \qquad y \in Y. \tag{5.11}$$

(3) A vector field $\xi \in \mathfrak{X}(X)$ *belongs to* E if $\xi_x \in E_x$ for all $x \in X$.
(4) E is *integrable* (or *involutive*) if whenever ξ and η belong to E, then so too does $[\xi, \eta]$.

So a vector field ξ belongs to E iff the map $\xi\colon X \to TX$ factors through the inclusion $E \hookrightarrow TX$:

$$\begin{tikzcd} E \arrow[rr, hook] \arrow[dr] & & TX \arrow[dl] \\ & X \arrow[ul, dashed, bend left] \arrow[ur, bend right, "\xi"'] & \end{tikzcd} \tag{5.12}$$

There are different standard notations for the vector subspace of vector fields that belong to E:

$$\Gamma(E) = \Gamma_X(E) = C^\infty_X(E) = \Omega^0_X(E) = \{\xi \in \mathfrak{X}(X) :\ \xi \text{ belongs to } E\}. \tag{5.13}$$

The following lemma allows us to reexpress Definition 5.10(4) as the vanishing of a tensor.

Lemma 5.14. *The map*

$$\begin{aligned} \Gamma(E) \times \Gamma(E) &\longrightarrow \Gamma(TX/E) \\ \xi \quad , \quad \eta \quad &\longmapsto \quad [\xi, \eta] \pmod{E} \end{aligned} \tag{5.15}$$

is linear over functions.

See **(19.11)** in the Differential Topology notes for the concept of "linear over functions", especially Proposition 19.16 for the relationship to tensors.

Proof. Let $\xi, \eta \in \Gamma(E)$ and $f, g \in \Omega^0_X$. Then

$$[f\xi, g\eta] = fg[\xi,\eta] + f(\xi g)\eta - g(\eta f)\xi. \tag{5.16}$$

Hence

$$[f\xi, g\eta] \pmod{E} = fg\,[\xi,\eta] \pmod{E}, \tag{5.17}$$

as desired. □

Definition 5.18. Let $E \subset TX$ be a distribution on a smooth manifold X. The *Frobenius tensor* of E is

$$\begin{aligned} \phi_E \colon E \times E &\longrightarrow TX/E \\ \xi_x\,, \eta_x &\longmapsto [\xi,\eta]_x \pmod{E}, \end{aligned} \tag{5.19}$$

where $\xi, \eta \in \mathfrak{X}(X)$ are vector fields that extend ξ_x, η_x.

Remark 5.20.

(1) E is integrable iff ϕ_E vanishes.
(2) The Frobenius tensor ϕ_E is a section of

$$\textstyle\bigwedge^2 E^* \otimes TX/E \longrightarrow X. \tag{5.21}$$

The following example gives some insight into the partial differential equations that underlie the geometric condition (5.11) for an integral manifold, and the integrability condition Definition 5.10(4), or equivalently the vanishing of the Frobenius tensor (5.19).

FIGURE 14. A distribution on $\mathbb{A}^2 \times \mathbb{R}$ whose integral manifold is a graph

Example 5.22. Let $P, Q\colon \mathbb{A}^2 \to \mathbb{R}$ be smooth functions, and $\ell_{(x,y)}\colon \mathbb{R}^2 \to \mathbb{R}$ the linear function defined by the matrix $\begin{pmatrix} P(x,y) & Q(x,y) \end{pmatrix}$. Let E be the distribution on $\mathbb{A}^2_{x,y} \times \mathbb{R}_z$ whose value at (x, y, z) is the graph of the linear function $\ell_{(x,y)}$; it is translationally invariant in the z-direction. Then E has a global basis of vector fields

$$\begin{aligned} \xi &= \frac{\partial}{\partial x} + P\frac{\partial}{\partial z} \\ \eta &= \frac{\partial}{\partial y} + Q\frac{\partial}{\partial z} \end{aligned} \tag{5.23}$$

If $f\colon \mathbb{A}^2 \to \mathbb{R}$ is a smooth function, and $\Gamma(f) \subset \mathbb{A}^2 \times \mathbb{R}$ is its graph, then $T_{(x,y,f(x,y))}\Gamma(f)$ has basis

$$\begin{aligned} &\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial z} \\ &\frac{\partial}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial}{\partial z} \end{aligned} \tag{5.24}$$

Hence the condition (5.11) that $\Gamma(f)$ be an integral manifold is the system of PDEs (partial differential equations)

$$\begin{aligned} \frac{\partial f}{\partial x} &= P \\ \frac{\partial f}{\partial y} &= Q \end{aligned} \tag{5.25}$$

These are familiar equations. Any C^2 solution f, much less any C^∞ solution, must have mixed second partials that are equal:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}. \tag{5.26}$$

Combining (5.26) and (5.25) we deduce the integrability condition

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{5.27}$$

This familiar integrability condition for (5.26) is in fact the integrability condition for the distribution E. In fact, for the vector fields (5.23) belonging to E we compute

$$[\xi, \eta] = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\frac{\partial}{\partial z}, \tag{5.28}$$

and so $[\xi, \eta] \pmod{E} = 0$ iff (5.27) holds. It is not too difficult to see that this is true iff $\phi_E = 0$.

Remark 5.29. The integrability condition in Definition 5.10(4) is a geometric form of "mixed partials commute". Recall that the condition $d^2 = 0$ for the Cartan differential on the de Rham complex is another version of "mixed partials commute". The Frobenius tensor measures the failure of this condition for a distribution. This will manifest as *curvature*, as we will see.

Remark 5.30. The distribution E in Example 5.22 is the kernel of the 1-form

$$\theta = P\,dx + Q\,dy - dz. \tag{5.31}$$

We compute

$$d\theta = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy, \tag{5.32}$$

from that we deduce that $\theta\wedge d\theta = 0$ iff (5.27) holds. Quite generally, the condition $\theta\wedge d\theta = 0$ is the integrability condition for a codimension one distribution, and there is a generalization in terms of differential forms for a distribution of any rank.

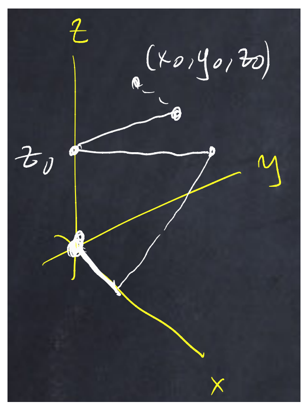

FIGURE 15. A piecewise smooth integral curve of a nonintegrable distribution

Example 5.33 (A nonintegrable distribution)**.** Take the special case of the distribution E in Example 5.22 with $P = 0$ and $Q = x$, so the vector fields (5.23) are

$$\begin{aligned} \xi &= \frac{\partial}{\partial x} \\ \eta &= \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \end{aligned} \tag{5.34}$$

We claim there is a piecewise smooth path in $\mathbb{A}^2\times\mathbb{R}$ that begins at $(0,0,0)$, ends at any given (x_0, y_0, z_0), and has velocity in the distribution E. It is illustrated in Figure 15. Here is an explicit path:

(1) Follow ξ to $(1,0,0)$.
(2) Follow η to $(1, z_0, z_0)$.
(3) Follow $-\xi$ to $(0, z_0, z_0)$.
(4) Follow $\pm\eta$ to $(0, y_0, z_0)$.
(5) Follow $\pm\xi$ to (x_0, y_0, z_0).

If E admitted an integral surface S through $(0,0,0)$, then any piecewise smooth integral curve C of E which contains $(0,0,0)$ would lie in S, since at any smooth $c \in C$ we have $T_cC \subset E = T_cS$. So the fact that we can connect $(0,0,0)$ to *any* (x_0, y_0, z_0) indicates that there is no such integral surface, and is consistent with the fact that E is nonintegrable.

Special coordinate systems

As a preliminary to the local Frobenius theorem, we prove two results that allow us to locally specify coordinate vector fields of a chart. Put differently, we prove a local normal form for a set of linearly independent vector fields of cardinality $\leq n$ under an commutativity condition.

(5.35) *Normal form for a single nonzero vector field.*

Proposition 5.36. *Let X be a smooth manifold of dimension n, $\xi \in \mathfrak{X}(X)$ a vector field, $p \in X$ a point, and suppose $\xi_p \neq 0$. Then there exists a chart $(U; x^1, \dots, x^n)$ about p such that $\frac{\partial}{\partial x^1} = \xi$.*

The chart $(U; x^1, \dots, x^n)$ is illustrated in Figure 16.

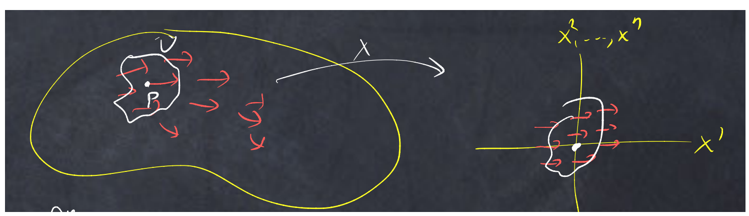

FIGURE 16. Straightening out a single vector field

Proof. Let φ_t be a local flow of ξ near p, and choose a standard chart $(V; y^1, \dots, y^n)$ about p such that the local flow maps a ball about p into V on some time interval $(-\delta, \delta)$ (this is always true); $y^i(p) = 0$, $i = 1, \dots, n$; and $\frac{\partial}{\partial y^1}\Big|_p = \xi_p$. Define

$$\begin{aligned} F \colon U &\longrightarrow X \\ (x^1, \dots, x^n) &\longmapsto \varphi_{x^1}(0, x^2, \dots, x^n) \end{aligned} \tag{5.37}$$

where we use the y-coordinate system on X and $U \subset \mathbb{A}^n$ is an open neighborhood of $(0, \dots, 0) \in \mathbb{A}^n$ on which the map is defined and for which $F(U) \subset V$. Then $dF_{(0,\dots,0)}$ is invertible—in fact, $d(y \circ F)_{(0,\dots,0)} = \mathrm{id}_{\mathbb{R}^n}$ so by the inverse function theorem we can invert F in a neighborhood of the origin. This gives the desired standard chart: in x-coordinates the flow is $\varphi_t \colon (x^1, x^2, \dots, x^n) \to (x^1 + t, x^2, \dots, x^n)$. $\square$

(5.38) *Normal form for a set of commuting vector fields.* The next result generalizes Proposition 5.36 to a set of linearly independent vector fields.

Proposition 5.39. *Now suppose $\xi_1,\dots,\xi_k\in\mathfrak{X}(X)$ with $\{\xi_1\big|_p,\dots,\xi_k\big|_p\}\subset T_pX$ linearly independent. Then there exists a chart $(U;x^1,\dots,x^n)$ about p with $\frac{\partial}{\partial x^i}=\xi_i$, $i=1,\dots,k$, iff $[\xi_i,\xi_j]=0$ for all $1\le i,j\le k$.*

The commutativity of the vector fields—the vanishing of Lie brackets—is a necessary and sufficient condition for the local normal form.

Proof. If the chart exists, then $[\xi_i,\xi_j]$ is a Lie bracket of coordinate vector fields, hence vanishes.[8] Conversely, suppose the pairwise Lie brackets vanish. Let $\varphi_t^{(i)}$, $i=1,\dots,k$, be local flows of the ξ_i, and let $(V;y^1,\dots,y^n)$ be a standard chart such that $\frac{\partial}{\partial y^i}\big|_p=\xi_i\big|_p$. Define

$$\begin{aligned} F\colon U &\longrightarrow X \\ (x^1,\dots,x^n) &\longmapsto \varphi^k_{x^k}\cdots\varphi^{(1)}_{x^1}(0,\dots,0,x^{k+1},\dots,x^n) \end{aligned} \tag{5.40}$$

as in (5.37), for an appropriate domain $U\subset\mathbb{A}^n$. Use the fact that the flows commute (Proposition 2.58) to compute $d(y\circ F)_{(0,\dots,0)}=\mathrm{id}_{\mathbb{R}^n}$ and that, after inverting F, the resulting x-coordinates satisfy $\frac{\partial}{\partial x^i}=\xi_i$. ☐

The local Frobenius theorem

The main theorem in this lecture is a local normal form—a straightening—for an integrable distribution.

Theorem 5.41. *Let X be a smooth manifold of dimension n and $E\subset TX$ a distribution of rank k.*

(1) *E is integrable iff for all $p\in X$ there exists a chart $(U;x^1,\dots,x^n)$ about p such that in U we have*

$$E=\operatorname{span}\left\{\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^k}\right\}. \tag{5.42}$$

(2) *If so, then any connected integral manifold $f\colon Y\to U$ of E has image contained in a submanifold of the form $\{x^m=c^m\}_{m=k+1,\dots,n}$ for some $c^{k+1},\dots,c^n\in\mathbb{R}$.*

Proof. If E has the form (5.42), then since coordinate vector fields commute it follows that E is integrable. Conversely, suppose that E is integrable. Fix $p\in X$ and choose local coordinates $y^1,\dots,y^n$ about p so that

$$E_p=\operatorname{span}\left\{\frac{\partial}{\partial y^1}\bigg|_p,\dots,\frac{\partial}{\partial y^k}\bigg|_p\right\}. \tag{5.43}$$

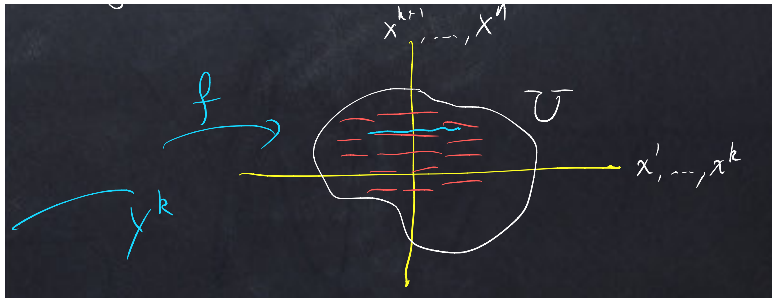

FIGURE 17. The local normal form of an integrable distribution and a connected integral manifold

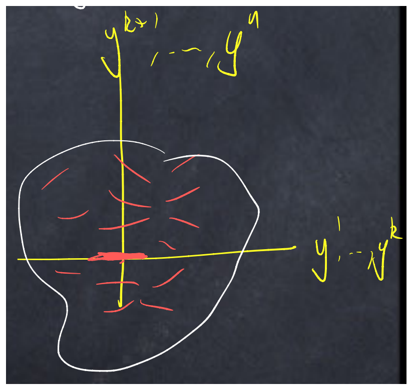

FIGURE 18. A local integrable distribution straightened at a single point

Use these coordinates to transfer to a local problem in $\mathbb{A}^n$; see Figure 18. Let $\pi\colon \mathbb{A}^n \to \mathbb{A}^k$ be affine projection onto the first k coordinates along the last $(n-k)$ coordinates. Choose a neighborhood of p such that $\pi_*\colon \mathbb{R}^n \to \mathbb{R}^k$ restricts to an isomorphism on E. Define vector fields $\xi_1, \dots, \xi_k$ that belong to E and project via π_* to the coordinate vector fields $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^k}$ on $\mathbb{A}^k$. Then for $1 \le i, j \le n$, the Lie bracket $[\xi_i, \xi_j]$ is π-related to $\left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0$, and since by integrability $[\xi_i, \xi_j]$ belongs to E, we conclude that $[\xi_i, \xi_j] = 0$. Now apply Proposition 5.39 to obtain the desired standard chart in (1).

For (2), suppose that $f\colon Y \to U$ is an integral manifold. Then $d(f^*x^m) = f^*(dx^m) = 0$ for $m = k+1, \dots, n$, since E is defined by the simultaneous vanishing of $dx^{k+1}, \dots, dx^n$. It follows that f^*x^m is constant, $m = k+1, \dots, n$, since Y is connected. $\square$

Definition 5.44. Let X be a smooth manifold of dimension n and $E \subset TX$ an integrable distribution of rank k. A chart $(U; x^1, \dots, x^n)$ that satisfies (5.42) is an *E-coordinate system*. For any $c^{k+1}, \dots, c^n \in \mathbb{R}$, the subset $\{x^m = c^m\}_{m=k+1,\dots,n} \subset U$ is a *slice*.

[8] This is another instance of the equality of mixed partials.

Lecture 6: Foliations and the global Frobenius theorem

For a vector field on a manifold there is a local existence and uniqueness theorem for integral curves (with given initial condition), as well as a global existence and uniqueness theorem for maximal integral curves. For a distribution we write the theorems for all initial conditions at once. Theorem 5.41 is the local version: a local normal form for a distribution. In this lecture we turn to the global version, which is a *foliation* on a smooth manifold.

We begin with the definition of a foliation: a partition of a smooth manifold into a union of images of injective immersions. The union is not arbitrary; there is a local model. Hence the study of foliations is a study of global phenomena. We give some examples of foliations, including the Reeb foliation of the solid torus and of the 3-sphere. We encourage the reader to have fun thinking through the geometry of these examples.

The main result (Theorem 6.22) is that an integrable distribution integrates to a foliation. The proof begins with the local Theorem 5.41 and then uses a global equivalence relation to define the global partition of the manifold into leaves. A useful corollary (Theorem 6.25) asserts that a smooth map into a foliated manifold whose image is contained in a single leaf is a smooth map into that leaf.

Foliations

(6.1) *Definition of a foliation.* Let X be a smooth manifold

Definition 6.2. A k-dimensional *foliation* $\mathcal{F}$ on X is a set $f_\alpha\colon Y_\alpha \hookrightarrow X$, $\alpha \in A$, of injective immersions such that (i) $\dim Y_\alpha = k$; (ii) Y_α is connected; (iii) the images $\mathcal{F}_\alpha := f_\alpha(Y_\alpha)$ partition X:

$$X = \bigsqcup_{\alpha\in A} \mathcal{F}_\alpha; \tag{6.3}$$

and (iv) about each $p \in X$ there exists a chart $(U; x^1, \dots, x^n)$ such that $\mathcal{F}_\alpha \cap U$ is a union of slices $\{x^i = c^i\}_{i=k+1,\dots,n}$ for some $c^{k+1}, \dots, c^n \in \mathbb{R}$. Each $\mathcal{F}_\alpha$ is called a *leaf* of the foliation, and a chart in (iv) is an $\mathcal{F}$*-coordinate system*.

Since Y_α is second countable, so too is $f_\alpha^{-1}(U) \subset Y_\alpha$. It follows that $f_\alpha^{-1}(U)$ has countably many components, hence the intersection in (iv) is contained in a union of at most *countably* many slices. An $\mathcal{F}$-coordinate system is depicted in Figure 19.

Remark 6.4. It is convenient to assume that an $\mathcal{F}$-coordinate system $(U; x^1, \dots, x^n)$ is *cubic* in the sense that the image $x(U) \subset \mathbb{A}^n$ is an open cube whose sides are aligned with the axes. In that case each slice is a *connected* subset of U.

(6.5) *Examples.* We give several examples to illustrate some possible behaviors.

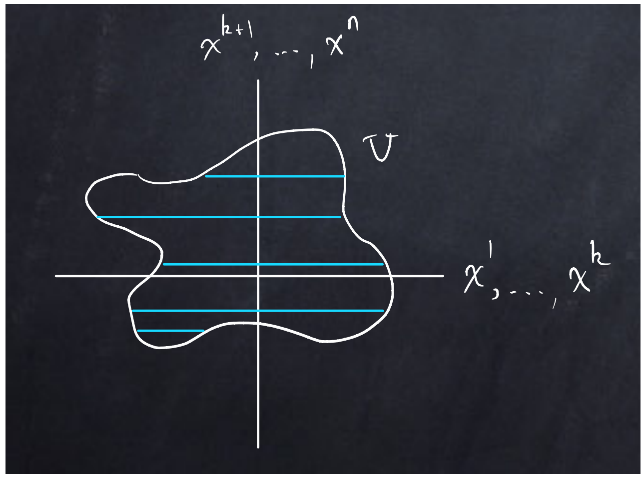

FIGURE 19. An $\mathcal{F}$-coordinate system and slices of a leaf

Example 6.6 (Foliations of the 2-torus)**.** Example 5.8 illustrates parallel 1-dimensional foliations of the 2-torus $\mathbb{A}^2/\mathbb{Z}^2$. For rational $a \in \mathbb{Q}$ in (5.9) the corresponding foliation is by submanifolds diffeomorphic to a circle. In this case the $\mathcal{F}$-coordinate systems in Definition 6.2(iv) may be chosen so that each leaf cuts through the chart in zero or one slices. On the other hand, for irrational $a \in \mathbb{R} \setminus \mathbb{Q}$ the foliation is by images of injective immersions of affine lines whose images are not submanifolds. Each leaf cuts through any $\mathcal{F}$-coordinate system in countably many slices.

Example 6.7 (Fiber bundles)**.** Let $\pi \colon X \to S$ be a fiber bundle (Definition 8.24 of the Differential Topology notes). Then X is foliated by the fibers:

$$\mathcal{F} = \bigsqcup_{s \in S} X_s, \qquad X_s = \pi^{-1}(s). \tag{6.8}$$

Furthermore, each leaf (fiber) is a submanifold of X. Note in this case the *leaf space* S is a smooth manifold. The foliation in Example 6.6 has this form for rational a: the leaf space is diffeomorphic to a circle, and the torus appears as the total space of a circle bundle over the circle. For irrational a the leaf space is not a smooth manifold. (It is not even a Hausdorff topological space.)

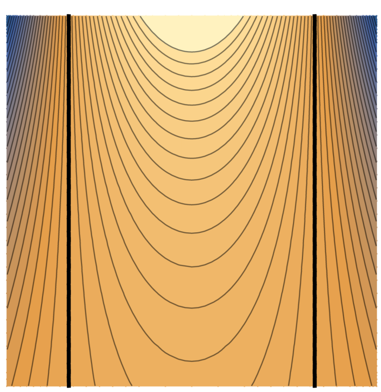

FIGURE 20. Foliation of a surjective submersion

Example 6.9 (Surjective submersions)**.** A surjective submersion $\pi\colon X \to S$ need not be a fiber bundle, and still it defines a foliation of X by submanifolds: the fibers of π. An example is

$$\begin{aligned} \pi\colon \mathbb{A}^2_{x,y} &\longrightarrow \quad \mathbb{R} \\ (x,y) &\longmapsto (1-x^2)e^y \end{aligned} \tag{6.10}$$

Some fibers are illustrated in Figure 20.

Example 6.11 (Reeb foliation)**.** Consider Figure 21, which illustrates a foliation of a closed strip in an affine plane. It is the portion of Figure 20 with $-1 \le x \le 1$. The black lines are $x = \pm 1$. Imagine this x,y-plane as sitting in 3-space and revolve about the vertical blue line $x = 0$. You obtain a foliation of a solid cylinder in which the boundary cylinder is a leaf and each other leaf is diffeomorphic to an affine plane. Each leaf is a submanifold. Now observe that the foliations of the strip and of the solid cylinder are invariant under translation $y \mapsto y + c$ for any $c \in \mathbb{R}$. Restrict to $c \in \mathbb{Z}$ and take the quotient. The quotient of this $\mathbb{Z}$-action on the strip is an annulus, and now the foliation has leaves the two boundary circles together with images of affine lines under injective immersions that are not embeddings. The quotient of the $\mathbb{Z}$-action on the solid cylinder is a solid torus, and now the leaves of the foliation are the boundary torus and the images of affine planes under injective immersions that are not embeddings. This is the *Reeb foliation* of the solid torus.

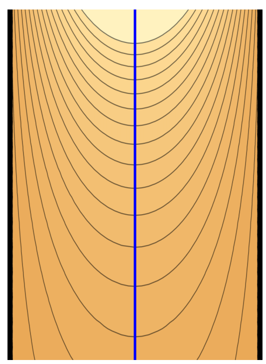

FIGURE 21. The proto-Reeb foliation

We can parlay this into a foliation of S^3 as follows. Present S^3 as the union of two solid tori along a 2-torus by writing

$$S^3 = \{(z^1, z^2) \in \mathbb{C}^2 : |z^1|^2 + |z^2|^2 = 1\}. \tag{6.12}$$

The solid tori are $\{\,|z^1| \le |z^2|\,\}$ and $\{\,|z^1| \ge |z^2|\,\}$ which intersect in the 2-torus $\{\,|z^1| = |z^2| = 1/\sqrt{2}\,\}$. Now transport the Reeb foliation of the solid torus above to these two solid tori. The foliations glue to a foliation of S^3, which can be smoothed. The result is also known as the *Reeb foliation.*

Example 6.13 (Another construction of the Reeb foliation)**.** Define

$$\widetilde{X} = \{(x,y,z) \in \mathbb{A}^3 : z \ge 0\} \ \setminus\ \{(0,0,0)\}. \tag{6.14}$$

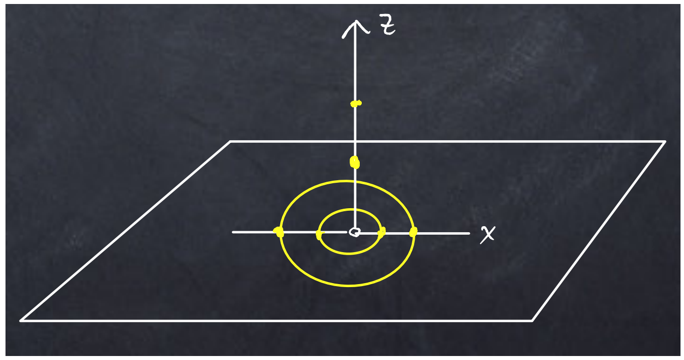

FIGURE 22. The solid torus as a quotient of upper half space minus the origin

Note that $\widetilde{X}$ is a 3-manifold with boundary. Fix $q \in (0,1)$. Set $X = \widetilde{X}/q^{\mathbb{Z}}$ to be the quotient of $\widetilde{X}$ by the action of the infinite cyclic group of homotheties of $\mathbb{A}^3$ centered at $(0,0,0)$ with scale a power of q; see Figure 22. Consider first $\partial\widetilde{X} \approx \mathbb{A}^2 \setminus \{(0,0)\}$. A fundamental domain of the action is the annulus whose boundary circles have radii 1 and q, and under the action those boundary circles are identified. Hence the quotient is a 2-torus. Now the entire quotient $X = \widetilde{X}/q^{\mathbb{Z}}$ is a 3-manifold with boundary a 2-torus, and I leave as a visualization exercise proving that X is diffeomorphic to a solid 3-dimensional torus.[9] Now let $\widetilde{\mathcal{F}}$ be the foliation of $\widetilde{X}$ by horizontal planes $z = c$, $c \in \mathbb{R}^{\ge 0}$. (We remove the origin.) There is a quotient foliation $\mathcal{F}$ of X. The leaf $\{z = 0\}$ of $\widetilde{\mathcal{F}}$ descends to the leaf ∂X of $\mathcal{F}$ diffeomorphic to a 2-torus. The homotheties acts freely on the leaf space of $\widetilde{\mathcal{F}}$ minus the leaf $\{z = 0\}$, so each other leaf of $\mathcal{F}$ is diffeomorphic to an affine plane. The claim is that $\mathcal{F}$ is diffeomorphic to the Reeb foliation of Example 6.11.

(6.15) *Codimension one foliations.* If a connected closed manifold X admits a codimension one foliation $\mathcal{F}$, then it is not difficult to see that the Euler number $\mathrm{Euler}(X)$ must vanish. (It is useful to use the tangent bundle $T\mathcal{F}$, defined shortly.) Thus, for example, even dimensional spheres do not admit a codimensional one foliation. Thurston proved the converse statement.

Theorem 6.16 (Thurston)**.** *A connected closed smooth manifold X admits a codimension one foliation iff* $\mathrm{Euler}(X) = 0$.

The global Frobenius theorem

(6.17) *The tangent bundle to a foliation.* Let X be a smooth manifold and suppose $\mathcal{F}$ is a foliation on X.

Definition 6.18. The *tangent bundle* $T\mathcal{F} \to X$ to $\mathcal{F}$ is

$$\bigsqcup_{p \in X} T_p\mathcal{F}_{\alpha(p)} \to X, \qquad p \in \mathcal{F}_{\alpha(p)}. \tag{6.19}$$

[9]You might first consider the x,z-plane (restricting to $z \ge 0$). You can also consider the rays emanating from the origin in that plane, and then revolution about the nonnegative z-axis. In addition, you can consider the foliation of $\widetilde{X}$ by rays emanating from the origin, and see what happens to that foliation in the quotient.

In (6.19) we write A for the index set of the foliation (the leaf set) and $\alpha\colon X \to A$ for the function that assigns to each $p \in X$ its leaf.

Lemma 6.20. *$T\mathcal{F} \to X$ is a vector bundle. Furthermore, the distribution $T\mathcal{F}$ is integrable.*

Proof. Local triviality and integrability follow from the existence of local $\mathcal{F}$-coordinate systems (Definition 6.2(iv)). ☐

(6.21) *Global Frobenius theorem.* The theorem asserts that every integrable distribution is the tangent bundle to a foliation.

Theorem 6.22. *Let X be a smooth manifold and $E \subset TX$ an integrable distribution.*

(1) *There exists a foliation $\mathcal{F}$ such that $E = T\mathcal{F}$.*
(2) *Each leaf of $\mathcal{F}$ is a maximal connected integral manifold of E.*
(3) *Points $p_0, p_1 \in X$ lie in the same leaf of $\mathcal{F}$ iff there exists a piecewise smooth motion in X from p_0 to p_1 whose velocity vectors lie in E.*

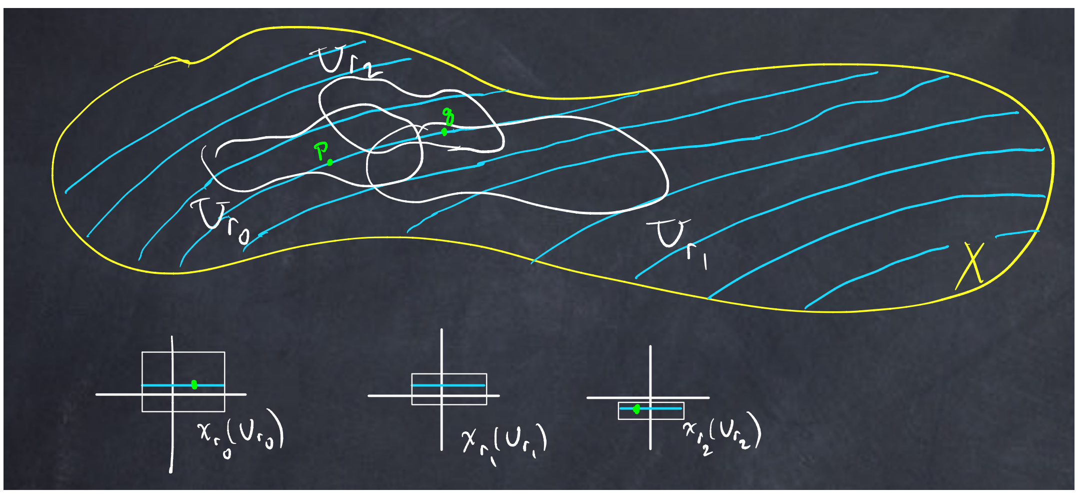

FIGURE 23. A sequence of slices in some $\mathcal{F}_\alpha$

Proof. Apply Theorem 5.41 and second countability of X to cover X by a (finite or countable) sequence $(U_r; x_r^1, \dots, x_r^n)$, $r = 1, 2, \dots$, of E-coordinate systems, which we assume are cubic in the sense of Remark 6.4. Observe that if $S \subset U_r$ is a slice, then for $t \neq r$ the intersection $S \cap U_t$ is empty or is a finite or countable union of slices of U_t. Define an equivalence relation $\sim$ on X that sets $p \in S_r \subset U_r$ equivalent to $q \in S_t \subset U_t$ iff there exist $r_0, \dots, r_\ell \in \mathbb{Z}^{>0}$ and slices $S_{r_i} \subset U_{r_i}$ such that $r_0 = r$, $r_\ell = t$, $S_{r_0} = S_r$, $S_{r_\ell} = S_t$, and $S_{r_{i-1}} \cap S_{r_i} \neq \emptyset$ for all $1 \leq i \leq \ell$. Then we claim that the equivalence classes

$$X = \bigsqcup_{\alpha \in A} \mathcal{F}_\alpha \tag{6.23}$$

of $\sim$ are the leaves of a foliation $\mathcal{F}$. Let Y be an equivalence class. Then each $p \in Y$ lies in a slice $S_r \in U_r$ in one of the chosen E-coordinate systems. If rank $E = k$, then use the first k coordinates in that chart as coordinates on $S_r \subset Y$. In this way we simultaneously topologize Y and construct an atlas. Coordinate charts have C^∞ overlaps since the E-coordinate systems do. The topology on Y is clearly locally Euclidean and path connected, hence is connected. It is also second countable: there are at most countably many sequences $U_{r_0}, \dots, U_{r_\ell}$, and if $p \in S_{r_0} \subset U_{r_0}$, then there are at most countably many sequences $S_{r_0}, \dots, S_{r_\ell}$ in which[10] $S_{r_{i-1}} \cap S_{r_i} \neq \emptyset$, hence in all there are at most countably many slices that appear in Y. The inclusion $Y \hookrightarrow X$ is smooth and is an injective immersion. The E-coordinate systems are $\mathcal{F}$-coordinate systems, and this proves that (6.23) is a foliation. The local form shows $E = T\mathcal{F}$, which completes the proof of (1).

Since $T\mathcal{F} = E$, each leaf $\mathcal{F}_\alpha$ is an integral manifold of E. If $f\colon Y \hookrightarrow X$ is any (connected) integral manifold, then by Theorem 5.41(2) locally its image lies in a slice of an E-coordinate system $(U_r; x_r^1, \dots, x_r^n)$. Any $p, q \in f(Y)$ can be joined by a piecewise smooth path, and along that path we can find a finite sequence of slices $S_{r_0}, \dots, S_{r_\ell}$ as in the definition of $\sim$. In other words, $p \sim q$. This proves $f(Y) \subset \mathcal{F}_\alpha$ for some α, which is the maximality claimed in (2).

If $p_0, p_1 \in X$ lie in the same leaf of $\mathcal{F}$, then from the definition of $\sim$ we construct a smooth motion from p_0 to p_1 which lies in the slices that connect the points. The velocity vectors are in E. Conversely, given such a motion the velocity vectors lie in $E = T\mathcal{F}$, which means that if $p_0 \in \mathcal{F}_\alpha$, then locally the motion lies in $\mathcal{F}_\alpha$, from which it globally lies in $\mathcal{F}_\alpha$. □

(6.24) *Maps into a leaf.* We sometimes encounter a smooth map whose codomain has a foliation and the map factors through the inclusion of a leaf. In that situation we would like to know that the factored map is smooth. This is not true in general for a map that factors through an injective immersion—see Figure 24—but it is true if that injective immersion is the leaf of a foliation.

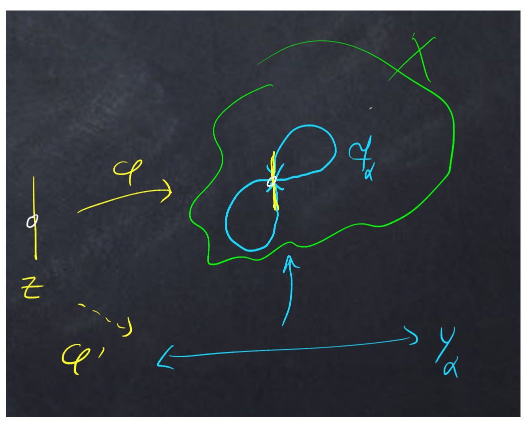

FIGURE 24. A map that factors through an injective immersion but the factored map is not continuous

Theorem 6.25. *Let X be a smooth manifold equipped with a foliation $\mathcal{F}$, and suppose Z is a smooth manifold and $\varphi\colon Z \to X$ is a smooth map. Assume there is a leaf of $\mathcal{F}$, the image $\mathcal{F}_\alpha$ of*

[10]A slice in one chart U_r intersects at most countably many slices in any U_s by the argument following Definition 6.2.

an injective immersion $f_\alpha\colon Y_\alpha \hookrightarrow X$, and a map $\varphi'\colon Z \to Y_\alpha$ such that $\varphi = f_\alpha \circ \varphi'$:

$$
\begin{tikzcd}
Z \arrow[r, "\varphi"] \arrow[dr, dashed, "\varphi'"'] & X \\
& Y_\alpha \arrow[u, hook, "f_\alpha"']
\end{tikzcd}
\tag{6.26}
$$

Then φ' is smooth.

Proof. Fix $z \in Z$ and let $p = \varphi(z)$. Choose an E-coordinate system $(U; x^1, \dots, x^n)$ about p from the sequence in the proof of Theorem 6.22, and choose the slice $S \subset U$ such that $p \in S$. The inverse image $\varphi^{-1}(U) \subset Z$ is open; let $V \subset Z$ be the component of $\varphi^{-1}(U)$ that contains z. Then $\varphi(V) \subset U$ is connected, $p \in \varphi(V)$, hence by the factorization (6.26) we conclude that $\varphi(V)$ lies in the component of $\mathcal{F}_\alpha \cap U$ that contains p. But from the first part of the proof of Theorem 6.22, that component is S. (The intersection $\mathcal{F}_\alpha \cap U$ is a countable union of slices.) Now $x^1, \dots, x^k$ are local coordinates on S (assuming $\operatorname{rank} E = k$), and since φ is smooth, so too is the local representation $(x^1, \dots, x^k) \circ \varphi$ of φ'. $\square$

Lecture 7: Maps into a Lie group

In this lecture we study solutions $f\colon X \to G$ of a first order partial differential equation (PDE) for a map f from a smooth manifold X to a Lie group G. The PDE (or system of PDEs) is specified by a Lie algebra-valued 1-form $\omega \in \Omega^1_X(\mathfrak{g})$; the PDE is the equation $\omega = f^*(\theta)$, where $\theta \in \Omega^1_G(\mathfrak{g})$ is the Maurer–Cartan form. There is an integrability condition—a consequence of "mixed partials commute"—which is the Maurer–Cartan equation for ω. Certainly this is a necessary condition for a solution f, since the Maurer–Cartan equation holds for θ. The existence and uniqueness result is Theorem 7.8.

We give a first application of this theorem internal to the theory of Lie groups. Namely, we prove in Theorem 7.27 that a homomorphism of Lie algebras of Lie groups integrates to a homomorphism of Lie groups, assuming the domain is simply connected. More geometric applications will follow shortly, but many replace the codomain Lie group by a torsor, so in the second part of this lecture we resume the study of torsors that we touched on in Lecture 1.

Main theorem

(7.1) *The differential.* Let G be a Lie group and let $\theta \in \Omega^1_G(\mathfrak{g})$ be its parallel (left-invariant) Maurer–Cartan form (Definition 4.60). Recall that θ satisfies the Maurer–Cartan equation

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0. \tag{7.2}$$

Now suppose X is a smooth manifold and $f\colon X \to G$ is a smooth map.

Lemma 7.3. *The differential df may be identified with $f^*(\theta) \in \Omega^1_X(\mathfrak{g})$.*

Proof. The global parallelism on G is implemented by the Maurer–Cartan form, so the composition identifies the differential

$$T_xX \xrightarrow{df_x} T_{f(x)}G \xrightarrow{\theta_{f(x)}} \mathfrak{g}, \qquad x \in X, \tag{7.4}$$

as a $\mathfrak{g}$-valued 1-form at x, which we claim is $f^*(\theta)_x$. Namely, if $\xi_x \in T_xX$, then

$$f^*(\theta)_x(\xi_x) = \theta_{f(x)}\bigl(df_x(\xi_x)\bigr). \tag{7.5}$$

□

The Maurer–Cartan equation (7.2) implies that $\omega = f^*(\theta)$ satisfies

$$0 = f^*\left(d\theta + \frac{1}{2}[\theta \wedge \theta]\right) = d\omega + \frac{1}{2}[\omega \wedge \omega]. \tag{7.6}$$

Equation (7.6) is a necessary condition on ω for a solution f to the equation $\omega = f^*(\theta)$ to exist.

(7.7) *Existence and uniqueness.* The main theorem tells that we can solve for f with specified differential ω as long as ω satisfies the integrability condition (7.6).

Theorem 7.8. *Let X be a smooth manifold, let G be a Lie group, and let $\theta \in \Omega^1_G(\mathfrak{g})$ be the Maurer–Cartan form on G.*

(1) *Suppose $f_1, f_2\colon X \to G$ are smooth maps, $f_1^*(\theta) = f_2^*(\theta)$, and X is connected. Then there exists $g \in G$ such that*

$$f_2 = L_g \circ f_1, \tag{7.9}$$

where $L_g\colon G \to G$ is left multiplication by g.

(2) *Suppose $\omega \in \Omega^1_X(\mathfrak{g})$ satisfies*

$$d\omega + \frac{1}{2}[\omega \wedge \omega] = 0. \tag{7.10}$$

Then if X is simply connected, there exists $f\colon X \to G$ such that

$$f^*(\theta) = \omega. \tag{7.11}$$

The uniqueness (1) holds for X connected, whereas the existence (2) is for X simply connected. Both hold locally, i.e., on "small" simply connected open subsets of X (that can play the role of 'X' in the theorem). We abbreviate (7.9) to the equation $f_2 = gf_1$.

(7.12) *A vector group.* The simplest case is the Lie group $G = \mathbb{R}$ under addition; the Lie algebra is $\mathfrak{g} = \mathbb{R}$ with zero Lie bracket. The Maurer–Cartan form on $\mathbb{R}_x$ is the parallel 1-form dx. If $f\colon X \to \mathbb{R}$ is a smooth function, then $f^*(dx) = df \in \Omega^1_X$. The uniqueness (1) asserts that if $df_1 = df_2$, then $f_2 = f_1 + c$ for some $c \in \mathbb{R}$, which follows since $d(f_2 - f_1) = 0$, hence is locally constant, hence by connectivity of X is constant. The proof of (1) below is a generalization of this argument. The existence (2) reduces to the claim that if $\omega \in \Omega^1_X$ is a closed 1-form ($d\omega = 0$), and if X is simply connected, then ω is exact. This is a familiar local statement: the Poincaré lemma. The global statement requires more argument, which we give more generally in the proof below.

Use a basis to generalize to $G = V$ for V a finite dimensional real vector space.

(7.13) *Computations with the Maurer–Cartan form.* Recall from (4.77) that for a matrix group G we have $\theta = g^{-1}dg$, where g is the matrix-valued function on G that identifies G as a matrix group. For an arbitrary Lie group we may compute as if we are in a matrix group and then rewrite the result without that assumption. For example, to prove the uniqueness statement Theorem 7.8(1) set $F = f_2 f_1^{-1}\colon X \to G$ and compute

$$\begin{aligned} F^*(\theta) = F^{-1}dF &= (f_1 f_2^{-1})\, d(f_2 f_1^{-1}) \\ &= (f_1 f_2^{-1})(df_2 f_1^{-1} - f_2 f_1^{-1} df_1 f_1^{-1}) \\ &= f_1 f_2^*(\theta) f_1^{-1} - f_1 f_1^*(\theta) f_1^{-1} \\ &= \mathrm{Ad}_{f_1}\big(f_2^*(\theta) - f_1^*(\theta)\big). \end{aligned} \tag{7.14}$$

The equation takes place in $\Omega^1_X(\mathfrak{g})$. In the last expression, the algebraic operator Ad_{f_1} operates pointwise on $\bigl(f_2^*(\theta) - f_1^*(\theta)\bigr) \in \Omega^1_X(\mathfrak{g})$. This last expression makes sense on any Lie group G; it needn't be a matrix group. We claim that the equality of the first and last expressions in (7.14) holds for any Lie group. The manipulations that lead to this equality can be justified in a few ways. If G embeds in a matrix group $\widetilde{G}$, then we interpret (7.14) in the ambient group $\widetilde{G}$. This works for all compact Lie groups, though the proof that G embeds in a matrix group is not easy (Peter-Weyl theorem). A universally applicable argument breaks down the computation into the pullback of θ under multiplication and inversion. Namely, for

$$\begin{aligned} m\colon G\times G &\longrightarrow G \\ i\colon G &\longrightarrow G \end{aligned} \tag{7.15}$$

we have

$$\begin{aligned} (m^*\theta)_{(g_1,g_2)} &= \mathrm{Ad}_{g_2^{-1}}\,\pi_1^*\theta + \pi_2^*\theta \\ (i^*\theta)_g &= \mathrm{Ad}_{g^{-1}}\,\theta \end{aligned} \tag{7.16}$$

where $\pi_i\colon G\times G\to G$, $i=1,2$, are the projections onto the factors. I leave the verification of (7.16) and its application to (7.14) to a homework problem.

(7.17) *The proof.* We return to the main theorem.

Proof of Theorem 7.8. For the uniqueness statement (1) set $F=f_2f_1^{-1}\colon X\to G$. Then $F^*(\theta)=0$ by (7.14) and the hypothesis $f_1^*(\theta)=f_2^*(\theta)$. It follows from Lemma 7.3 that $dF=0$, hence F is locally constant. Since X is connected, F is constant.

For the existence statement (2), we construct the function f, identified with its graph[11] $\Gamma(f)\subset X\times G$, as the maximal integral manifold of a distribution $E\subset T(X\times G)$ that encodes the first order differential equation (7.11). For $(x,g)\in X\times G$ identify

$$T_{(x,g)}(X\times G)\cong T_xX\oplus T_gG\cong T_xX\oplus\mathfrak{g}. \tag{7.18}$$

Now define the distribution $E\subset T(X\times G)$ by

$$E_{(x,g)}=\Gamma(\omega_x),\qquad (x,g)\in X\times G, \tag{7.19}$$

where $\Gamma(\omega_x)\subset T_xX\oplus\mathfrak{g}$ is the graph of the linear function ω_x. Notice that E is invariant under left multiplication in G: the right hand side of (7.19) does not depend on $g\in G$, and the last identification in (7.18) is left-invariant. Alternatively, define $\Theta\in\Omega^1_{X\times G}(\mathfrak{g})$ as

$$\Theta=\pi_1^*\omega-\pi_2^*\theta, \tag{7.20}$$

[11] Recall that a function $X\to G$ is a relation on $X\times G$ that satisfies an extra condition. In other words, a function *is* its graph.

where π_1, π_2 are the projections

$$\begin{tikzcd} X \times G \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & G \\ X \end{tikzcd} \tag{7.21}$$

Then E is the kernel of Θ. Note

$$\begin{aligned} L_g^*\Theta &= L_g^*\pi_1^*\omega - L_g^*\pi_2^*\theta \\ &= \pi_1^*\omega - \pi_2^*L_g^*\theta \\ &= \pi_1^*\omega - \pi_2^*\theta \\ &= \Theta \end{aligned} \tag{7.22}$$

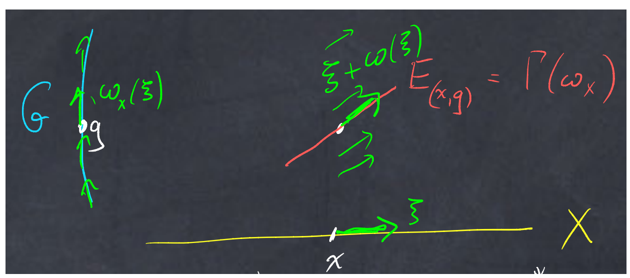

FIGURE 25. A vector field $\xi + \omega(\xi)$ in the distribution E

We claim that E is integrable. A vector field $\xi \in \mathfrak{X}(X)$ gives rise to a vector field $\xi + \omega(\xi)$ on $X \times G$ that belongs to the distribution E, as in Figure 25. This vector field is invariant under left multiplication on G. If $\xi, \eta \in \mathfrak{X}(X)$ we claim that on $X \times G$ we have

$$[\xi, \omega(\eta)] = \xi\omega(\eta). \tag{7.23}$$

The left hand side is the Lie bracket on $X \times G$ of the horizontal vector field ξ and the vertical vector field $\omega(\eta)$. The right hand side is the vertical vector field whose value on $\{x\} \times G$ is the parallel vector field on G with value the directional derivative of the function $\omega(\eta)\colon X \to \mathfrak{g}$ in the direction ξ_x. To prove (7.23), let φ_t be a local flow on X generated by ξ, and as well the induced flow on $X \times G$ that is the identity on the G factor. Then

$$[\xi, \omega(\eta)] = \frac{d}{dt}\bigg|_{t=0} (\varphi_{-t})_*\omega(\eta). \tag{7.24}$$

Since the flow does not move the points of G, the right hand side of (7.24) reduces to the definition of the directional derivative, which proves (7.23). Now compute

$$\begin{aligned}
[\xi+\omega(\xi),\eta+\omega(\eta)] &= [\xi,\eta]+[\xi,\omega(\eta)]-[\eta,\omega(\xi)]+[\omega(\xi),\omega(\eta)]\\
&= [\xi,\eta]+\xi\omega(\eta)-\eta\omega(\xi)+[\omega(\xi),\omega(\eta)]\\
&= [\xi,\eta]+\omega\big([\xi,\eta]\big)+d\omega(\xi,\eta)+[\omega(\xi),\omega(\eta)]\\
&= [\xi,\eta]+\omega\big([\xi,\eta]\big)+d\omega(\xi,\eta)+\frac{1}{2}[\omega\wedge\omega](\xi,\eta)\\
&= [\xi,\eta]+\omega\big([\xi,\eta]\big).
\end{aligned}\tag{7.25}$$

In the second equation we apply (7.23), in the third we apply (4.48), in the penultimate we apply (4.49), and in the last we apply the integrability condition (7.10). This proves that sections of E are closed under Lie bracket, so E is integrable.

An integral manifold Γ_0 of E is locally the graph of the function $f\colon U\to G$ for an open set $U\subset X$, since the differential of the projection $\pi\colon X\times G\to X$ is an isomorphism restricted to E. More precisely, cutting down Γ_0 if necessary we have that $\pi_1\big|_{\Gamma_0}\colon\Gamma_0\to X$ is invertible with inverse $s\colon U\to\Gamma_0\hookrightarrow X\times G$ a local section of π_1. Then $f=\pi_2\circ s$. Compute

$$f^*(\theta)=s^*\pi_2^*\theta=s^*(-\Theta+\pi_1^*\omega)=s^*\pi_1^*(\omega)=\omega\tag{7.26}$$

since the restriction of Θ to $\Gamma_0\subset X\times G$ vanishes. (The tangent distribution E is the kernel of Θ.) In other words, integral manifolds of E satisfy (7.11).

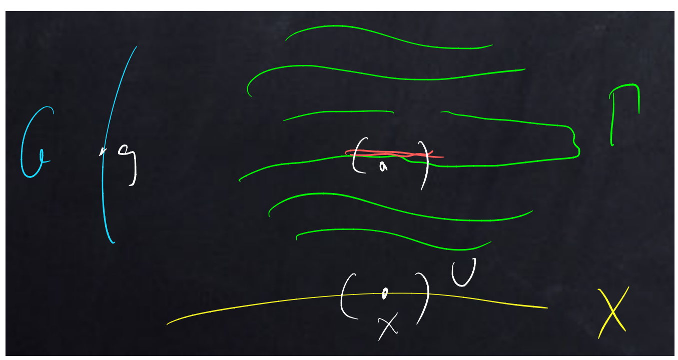

FIGURE 26. A maximal integral manifold of E

Fix $(x_0,g_0)\in X\times G$. The global Frobenius Theorem 6.22 gives a maximal integral manifold $\Gamma\subset X\times G$ through (x_0,g_0), as depicted in Example 1.44. The projection $\pi=\pi_2\big|_{\Gamma}\colon\Gamma\to X$ is a local diffeomorphism, since the differential is an isomorphism, hence for all $x\in X$ in the image $\pi(\Gamma)\subset X$ there is a local inverse $U\to\Gamma$ for some neighborhood $U\subset X$ of x. Suppose that local inverse maps x to $g\in G$. Since E is invariant under left translations in G, if $(x,g')\in\Gamma$ for some $g'\in G$, then composition with left multiplication by $g'g^{-1}$ maps a local inverse with $x\mapsto g$ to a local inverse with $x\mapsto g'$. It follows that U is evenly covered by π. Also, π is an open map since it is a local diffeomorphism, hence $\pi(\Gamma)\subset X$ is open.

We claim the image $\pi(\Gamma) \subset X$ is also closed. Namely, suppose $x_0 \in X$ is a limit point of the image. Then we claim that there exists a neighborhood U of x_0 such that $\pi_1^{-1}(U) \subset X \times G$ is foliated by integral manifolds of E. Certainly there is an integral manifold through every point of $\pi_1^{-1}(x_0)$, but for general distributions we do not have uniformity to conclude that there is a neighborhood U of x_0 to which each integral manifold maps surjectively by π_1. In this case the distribution is invariant under left multiplication on G, and this gives the desired uniformity: the image of any integral manifold under left translation is another integral manifold. Now since x_0 is a limit point of $\pi(\Gamma)$ it follows that one of these integral manifolds Γ' intersects Γ nontrivially. The maximality of Γ then implies that $\Gamma' \subset \Gamma$, and therefore $x_0 \in \pi(\Gamma)$. This proves that $\pi(\Gamma) \subset X$ is closed. Since X is connected, it follows that π is surjective.

Since π evenly covers its image, it is a covering map. And since X is assumed simply connected, it now follows that π is a diffeomorphism. Now define $f = \pi_2 \circ \pi^{-1}$, where $\pi_2 \colon \Gamma \to G$ is the restriction of projection $X \times G \to G$. Then (7.26) shows that f satisfies the differential equation (7.11). This concludes the proof of existence. $\square$

Lie group homomorphisms

Our first application of Theorem 7.8 is internal to the theory of Lie groups; see Remark 3.42.

Theorem 7.27. *Let G be a Lie group and let G' be a simply connected Lie group. Suppose $\dot\psi \colon \mathfrak{g}' \to \mathfrak{g}$ is a homomorphism of their Lie algebras. Then there exists a unique Lie group homomorphism $\psi \colon G' \to G$ whose differential is $\dot\psi$.*

Proof. Let $\theta_G, \theta_{G'}$ denote the Maurer–Cartan 1-forms on G, G', respectively, and let e', e be the respective identity elements. Set $\omega = \dot\psi(\theta_{G'}) \in \Omega^1_{G'}(\mathfrak{g})$. Theorem 7.8 produces a unique smooth map $\psi \colon G' \to G$ such that

$$\begin{aligned} \psi^*(\theta_G) &= \dot\psi(\theta_{G'}) \\ \psi(e') &= e \end{aligned} \tag{7.28}$$

We claim ψ is a homomorphism, which is equivalent to

$$\psi \circ L_{g'} = L_{\psi(g')} \circ \psi, \qquad g' \in G' \tag{7.29}$$

as maps $G' \to G$ for all choices of g'. Observe that both sides Ψ of (7.29) satisfy

$$\begin{aligned} \Psi^*(\theta_G) &= \dot\psi(\theta_{G'}) \\ \Psi(e') &= \psi(g'), \end{aligned} \tag{7.30}$$

and therefore are equal by the uniqueness Theorem 7.8(1). $\square$

Right G-torsors

(7.31) *Recollection.* We repeat Definition 1.4, but now in the smooth category.

Definition 7.32. Let G be a Lie group. A *right G-torsor* is a smooth manifold P equipped with a smooth simply transitive right G-action $P \times G \to P$.

The simple transitivity is equivalent to the invertibility of the map $P \times G \to P \times P$ whose first component is projection and whose section is the action.

Example 7.33 (The trivial right G-torsor)**.** The Lie group acts on itself by right multiplication, and that action is simply transitive. Denote this canonical right G-torsor as G_G.

(7.34) *Transport of structure.* We reprise (1.5) and (1.6). Let P be a right G-torsor. Then any $p \in P$ determines an isomorphism of right G-torsors (cf. (1.5) and (1.6))

$$\begin{aligned} \psi_p \colon G_G &\longrightarrow P \\ g &\longmapsto pg \end{aligned} \tag{7.35}$$

Hence any two right G-torsors are isomorphic, though not canonically so. The overlap between two parametrizations is the composition in the diagram

$$\begin{tikzcd} & P & \\ G \arrow[ur, "\psi_p"] \arrow[rr, dashed] & & G \arrow[ul, "\psi_{ph}"'] \end{tikzcd} \tag{7.36}$$

where $h \in G$. This composition is easily computed to be the left multiplication $L_{h^{-1}} \colon G \to G$. In other words, a right G-torsor is identified with G_G up to a *left* multiplication on G. It follows that the result of transporting a (left) parallel tensor on G to P via ψ_p is independent of $p \in P$.

Remark 7.37. Recall that a smooth structure on a topological manifold is an atlas of C^∞-compatible charts. The C^∞-compatibility means that C^∞ objects and concepts in affine space transport unambiguously to smooth manifolds. The principle here is the same.

(7.38) *Vector fields on P.* Let $\zeta \in \mathfrak{g}$ be a parallel (left-invariant) vector field on G. By transport of structure **(7.34)** it determines a vector field $\widehat{\zeta}$ on P. Using the identification of the Lie algebra $\mathfrak{g}$ as the set of one-parameter subgroups of G (Theorem 3.46), we give a useful formula for the vector field $\widehat{\zeta}$:

$$\widehat{\zeta}_p = \frac{d}{dt}\bigg|_{t=0} pe^{t\zeta}, \qquad p \in P. \tag{7.39}$$

Remark 7.40. Since left and right multiplication on G commute, the parallelism defined by right multiplication is left invariant, so transports to a parallelism on a right G-torsor P. More simply, a simply transitive group action on a manifold gives rise to a parallelism, and the right action of G on P is simply transitive. The vector field $\widehat{\zeta}$ is *not* right parallel on P. Namely, for $p \in P$, $h \in G$,

$$\widehat{\zeta}_{ph} = \frac{d}{dt}\bigg|_{t=0} phe^{t\zeta} = \frac{d}{dt}\bigg|_{t=0} p\left(he^{t\zeta}h^{-1}\right)h = (R_h)_*(\widehat{\mathrm{Ad}_h\,\zeta})_p, \tag{7.41}$$

where $R_h\colon P \to P$ is the action of h and Ad_h is the adjoint action of $h \in G$ on the Lie algebra $\mathfrak{g}$.

Remark 7.42. A right G-torsor is a *left* torsor for the action of its automorphism group $\mathrm{Aut}_G(P)$, which we define in the next lecture, and that action induces a (left) parallelism on P. The vector field $\widehat{\zeta}$ *is* parallel with respect to that parallelism.

(7.43) *The Maurer–Cartan form on P.* The Maurer–Cartan form $\theta \in \Omega^1_G(\mathfrak{g})$ is (left) parallel, so transports to a 1-form $\theta_P \in \Omega^1_P(\mathfrak{g})$ that we call the Maurer–Cartan form on P. The Maurer–Cartan equation transports to the equation

$$d\theta_P + \frac{1}{2}[\theta_P \wedge \theta_P] = 0 \qquad \text{in } \Omega^2_P(\mathfrak{g}). \tag{7.44}$$

Lemma 7.45. *Fix $\zeta \in \mathfrak{g}$ and let $\widehat{\zeta} \in \mathfrak{X}(P)$ denote the corresponding vector field on P. Then*

$$\theta_P(\widehat{\zeta}) = \zeta. \tag{7.46}$$

The right hand side of (7.46) is a constant $\mathfrak{g}$-valued function on P.

Proof. For $p \in P$ we see from (7.35) and (7.39) that $d(\psi_p)(\zeta) = \widehat{\zeta}$. Then $(\theta_P)(\widehat{\zeta}) = \theta(\zeta) = \zeta$. $\square$

Examples of right G-torsors

We have encountered some of these examples earlier.

Example 7.47 (Affine space)**.** Let V be a finite dimensional real vector space, which we regard as a Lie group under vector addition. Then a right V-torsor is an affine space over V (Definition 1.1).

Example 7.48 (Torsors in homogeneous manifolds)**.** Let $\widetilde{G}$ be a Lie group and suppose $\widetilde{G}$ acts transitively on a smooth manifold X via a left action. Fix $x \in X$ and let $G = \mathrm{Stab}_x \subset \widetilde{G}$ be the closed Lie subgroup of elements that fix x. Then for any $x' \in X$ define the right G-torsor

$$P_{x'} = \{\tilde{\gamma} \in \widetilde{G} : \tilde{\gamma}x = x'\}. \tag{7.49}$$

Note that $P_{x'}$ is a left G-coset in $\widetilde{G}$.

As a particular example, let $\widetilde{G} = \mathrm{O}_2$ be the orthogonal group and $X = \{x, x'\}$ a set of cardinality 2. Let $g \in \mathrm{O}_2$ act as the id_X if $\det(g) = +1$ and as the nontrivial permutation if $\det(g) = -1$.

Then $G = \mathrm{Stab}_x = \mathrm{SO}_2 \subset \mathrm{O}_2$, and $P_{x'} = \mathrm{O}_2 \setminus \mathrm{SO}_2$. Hence reflections are a right torsor over rotations.

As another example, let $\widetilde{G} = \mathrm{SO}_3$ act by rotations on the unit $S^2 \subset \mathbb{R}^3$, and let $x = (1,0,0)$ be the north pole. Then $G = \mathrm{Stab}_x \cong \mathrm{SO}_2$ is the group of rotations about the $(0,0,*)$-axis, which can be identified with rotations in the $(*,*,0)$-plane. The rotations that move x to a fixed $x' \in S^2$ are a torsor over SO_2.

Example 7.50 (Bases in a vector space)**.** We conclude with a key example for us. Let V be a finite dimensional real vector space. Recall from Definition 1.26 the right $\mathrm{GL}_n\mathbb{R}$-torsor of bases

$$\mathcal{B}(V) = \{b\colon \mathbb{R}^n \longrightarrow V : b \text{ is an isomorphism}\}\,. \tag{7.51}$$

We use the notation in (4.77) to write the (transported) Maurer–Cartan form on $\mathcal{B}(V)$ as a matrix of scalar 1-forms $\Theta^i_j \in \Omega^1_{\mathcal{B}(V)}$ that satisfy the Maurer–Cartan equation (4.83):

$$d\Theta^i_j + \Theta^i_k \wedge \Theta^k_j = 0. \tag{7.52}$$

(7.53) *Geometric interpretation of Θ^i_j on $\mathcal{B}(V)$.* Let $b \in \mathcal{B}(V)$ be a basis, which we identify with the image $e_1, \dots, e_n$ in V of the standard basis of $\mathbb{R}^n$. Let $e^1, \dots, e^n$ be the dual basis of V^*. A tangent vector $\xi \in T_b\mathcal{B}(V)$ is represented as a motion germ $e_1(t), \dots, e_n(t)$ with initial value the given basis $e_1, \dots, e_n$. Let $\dot e_j = e'_j(0)$ be the initial velocity of the j^{th} basis element.

Proposition 7.54. $(\Theta^i_j)_b(\xi) = \langle e^i, \dot e_j\rangle$.

The brackets on the right hand side denote the duality pairing of V^* and V. In words, Θ^i_j measures the instantaneous rate at which e_j is moving towards e_i, relative to the given basis. This geometric interpretation is important in the sequel.

Proof. Choose $A \in \mathfrak{gl}_n\mathbb{R}$ so that $\xi = \widehat{A}$. Then by Lemma 7.45 we have $(\Theta^i_j)_b(\xi) = A^i_j$. Choose the motion germ with $e_j(t) = b\left[(e^{tA})_j\right]$ is the image of the j^{th} column, regarded as a vector in $\mathbb{R}^n$, and so $\dot e_j = b(A_j)$. Hence the basis b maps the pairing $\langle \hat e^i, A_j\rangle = A^i_j$ of A_j with the standard dual basis element $\hat e^i \in (\mathbb{R}^n)^*$ to the desired pairing $\langle e^i, \dot e_j\rangle$. $\square$

Lecture 8: Curves in a Euclidean plane

We begin the lecture with more discussion of right G-torsors. The group of automorphisms is introduced. It is the transport of left multiplication on the trivial right G-torsor G_G. Then we discuss the important example of frames in affine space as a torsor over the affine group; its metric variation is the space of orthonormal frames in Euclidean space. These bundles of frames in affine geometry provide much intuition for the differential geometry curved manifolds.

In the second half of the lecture we turn to curves in a Euclidean plane. The first instance of curvature—which dates back to Nicole Oresme in the mid 14^{th} century and was developed further by Newton, Leibniz, the Brothers Bernoulli, and others beginning in the latter part of the 17^{th} century—emerges from the restriction of Maurer–Cartan forms to the curve. This is Cartan's *method of moving frames*, which in various forms is basic to our approach in this course.

We conclude with a global theorem about plane curves, the so-called *theorem of turning tangents* or *Hopf umlaursatz*.

More on right G-torsors

(8.1) *The automorphism group.* Let G be a Lie group and suppose P is a right G-torsor.

Definition 8.2. An *automorphism* $\varphi\colon P \to P$ of P is a smooth G-equivariant map, i.e.,

$$\varphi(pg) = \varphi(p)g, \qquad p \in P, \quad g \in G. \tag{8.3}$$

The group of automorphisms is denoted $\mathrm{Aut}_G(P)$.

If $p, p' \in P$, then there is a unique $\varphi \in \mathrm{Aut}_G(P)$ such that $\varphi(p) = p'$: use (8.3) to define $\varphi(pg) = p'g$ for all $g \in G$. Hence P is a left $\mathrm{Aut}_G(P)$-torsor. As in (1.30) and (1.31) we write

$$\mathrm{Aut}_G(P) \circlearrowright P \circlearrowleft G \tag{8.4}$$

Remark 8.5. We remind (Remark 1.29(2)) that right group actions are by *structural* symmetries whereas left group actions are by *geometric* symmetries.

Example 8.6 (Automorphisms of the trivial right G-torsor)**.** Every automorphism of G_G is given by left multiplication:

$$\begin{aligned} G &\longrightarrow \mathrm{Aut}_G(G_G) \\ g &\longmapsto L_g \end{aligned} \tag{8.7}$$

is an isomorphism of groups. ($\varphi \in \mathrm{Aut}_G(G_G)$ is the image of $\varphi(e) \in G$.)

If P is a right G-torsor and $p \in P$, then the map ψ_p in (7.35) determines an isomorphism

$$\begin{aligned} G &\longrightarrow \mathrm{Aut}_G(P) \\ g &\longmapsto \psi_p \circ L_g \circ \psi_p^{-1} \end{aligned} \tag{8.8}$$

of groups, which we use to transport the Lie group structure on G to $\mathrm{Aut}_G(P)$; the resulting Lie group structure is independent of p. The map ψ_p identifies the right and left actions

$$\begin{gathered} G \circlearrowright G \circlearrowleft G \\ \mathrm{Aut}_G(P) \circlearrowright P \circlearrowleft G \end{gathered} \tag{8.9}$$

Remark 8.10. As $p \in P$ changes, the isomorphism (8.8) of the group of automorphisms of a right G-torsor with G changes by conjugation. Therefore, an automorphism of a right G-torsor determines a conjugacy class in G.

Example 8.11 (Homogeneous manifolds)**.** Resuming the setup of Example 7.48, it is straightforward to identify $\mathrm{Aut}_G(P_{x'})$ as a subgroup of $\widetilde{G}$, namely the stabilizer subgroup $\mathrm{Stab}_{x'} \subset \widetilde{G}$. Observe that $\mathrm{Stab}_x, \mathrm{Stab}_{x'} \subset \widetilde{G}$ are conjugate subgroups; any element of the torsor $P_{x'} \subset \widetilde{G}$ gives a conjugacy.

Example 8.12 (Bases in a vector space)**.** Resuming the setup of Example 7.50, identify the automorphism group $\mathrm{Aut}_{\mathrm{GL}_n\mathbb{R}}\bigl(\mathcal{B}(V)\bigr)$ of the torsor of bases with the group $\mathrm{Aut}(V)$ of vector space automorphisms of V. (An automorphism $V \to V$ acts on a basis $\mathbb{R}^n \to V$ by left composition, so commutes with the action of $\mathrm{GL}_n\mathbb{R}$ by right composition.)

(8.13) *Frames in affine space.* Recall from Definition 1.26 that if A is an affine space over a finite dimensional real vector space V of dimension $n \in \mathbb{Z}^{\ge 0}$, then a frame in A is an affine isomorphism $f\colon \mathbb{A}^n \to A$. As in Remark 1.29 the 1-jet of f at $0 \in \mathbb{A}^n$ is a diffeomorphism $\mathcal{B}(A) \to A \times \mathcal{B}(V)$; see (1.32). Let $\pi\colon \mathcal{B}(A) \to A$ be the projection that takes the 0-jet at $0 \in \mathbb{A}^n$; see (1.33). Embed $\mathrm{GL}_n\mathbb{R} \subset \mathrm{Aff}_n$ as the stabilizer of $0 \in \mathbb{A}^n$. The right $\mathrm{GL}_n\mathbb{R}$-action on $\mathcal{B}(A)$ preserves π, and in fact acts simply transitively on the fibers of π, which are identified with $\mathcal{B}(V)$. Hence π is a (product) fiber bundle of right $\mathrm{GL}_n\mathbb{R}$-torsors. This is the structural symmetry. The automorphism group $\mathrm{Aut}_{\mathrm{Aff}_n\mathbb{R}}\bigl(\mathcal{B}(A)\bigr)$ is isomorphic to the Lie group $\mathrm{Aut}(A)$ of affine automorphisms of A. The subgroup $V \subset \mathrm{Aut}(A)$ of translations acts on the left on $\mathcal{B}(A)$ and on A, and the action commutes with the projection π. This is the geometric symmetry that encodes the parallelism of affine space.

The use of frames in differential geometry was pioneered by Élie Cartan. The term *method of moving frames*—or its French equivalent *la méthode du répère mobile*—is often used in this context.

Remark 8.14. We will see shortly fiber bundles of right torsors of frames over curves and surfaces. They are not product bundles, nor is there a global parallelism and consequently no universal left group action as there is on affine space.

(8.15) *Maurer–Cartan forms on the bundle of frames.* Embed $\mathbb{A}^n \hookrightarrow \mathbb{R}^{n+1}$ as the affine plane $\{(\xi^0, \xi^1, \dots, \xi^n) : \xi^0 = 1\}$. This embeds Aff_n as the subgroup of $\mathrm{GL}_{n+1}\mathbb{R}$ of invertible matrices of the block form

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline * & P \end{array}\right) \tag{8.16}$$

in which P is an invertible $n \times n$ matrix. The Maurer–Cartan 1-form can be regarded as having values in the Lie algebra of such matrices, so can be written

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline \theta^1 & \\ \vdots & \Theta^i_j \\ \theta^n & \end{array}\right) \tag{8.17}$$

The Maurer–Cartan equations of $\mathrm{GL}_{n+1}\mathbb{R}$ (see **(4.77)**) imply the Maurer–Cartan equations of Aff_n:

$$\begin{aligned} d\theta^i + \Theta^i_j \wedge \theta^j &= 0 \\ d\Theta^i_j + \Theta^i_k \wedge \Theta^k_j &= 0 \end{aligned} \tag{8.18}$$

(8.19) *Geometric interpretation of Maurer–Cartan.* Let $f \in \mathcal{B}(A)$ be a frame, which we identify as a point $p \in A$ and a basis $e_1, \dots, e_n$ of the tangent space V. A tangent vector ξ in $T_f\mathcal{B}(A)$ is represented as a motion (germ) $\bigl(p(t); e_1(t), \dots, e_n(t)\bigr)$ whose initial condition at $t = 0$ is $(p; e_1, \dots, e_n)$. Let $\dot{p}, \dot{e_1}, \dots, \dot{e_n} \in V$ be the derivatives at $t = 0$. Proposition 7.54 asserts

$$\Theta^i_j(\xi) = \langle e^i, \dot{e}_j \rangle, \tag{8.20}$$

where $e^1, \dots, e^n$ is the basis of V^* dual to $e_1, \dots, e_n$. Similarly, we have the following.

Proposition 8.21. *$\theta^i(\xi) = \langle e^i, \dot{p} \rangle$.*

Proof. From the discussion in **(7.38)** the vector ξ corresponds to an element in the Lie algebra $\mathfrak{aff}_n \cong \mathfrak{gl}_n\mathbb{R} \oplus \mathbb{R}^n$. The form θ^i only detects the $\mathbb{R}^n$ component of ξ, which is an infinitesimal translation $\zeta \in \mathbb{R}^n$. As in the proof of Proposition 7.54, the differential of f maps ζ to $\dot{p}$. The Maurer–Cartan form θ^i on Aff_n evaluates on ζ to $\langle \hat{e}^i, \zeta \rangle$, the duality pairing between $(\mathbb{R}^n)^*$ and $\mathbb{R}^n$, and this is equal to $\langle e^i, \dot{p} \rangle$. ☐

Remark 8.22. We reinterpret Proposition 8.21 in terms of the map $\pi\colon \mathcal{B}(A) \to A$. If ξ is a tangent vector to $\mathcal{B}(A)$ at a frame $(p; e_1, \dots, e_n)$, then $\pi_*\xi$ is a tangent vector to A at p, and its expression in the basis $e_1, \dots, e_n$ has coefficients given by the Maurer–Cartan forms θ^i:

$$\pi_*\xi = \theta^i(\xi)e_i. \tag{8.23}$$

This interpretation of the forms θ^i carries over to smooth manifolds, as we will see.

(8.24) *Orthogonal and Euclidean groups.* If $G \subset \mathrm{GL}_n\mathbb{R}$ is a closed Lie subgroup, then $\mathfrak{g} \subset \mathfrak{gl}_n\mathbb{R}$ is a Lie subalgebra. The Maurer–Cartan form on $\mathrm{GL}_n\mathbb{R}$ restrict to a $\mathfrak{gl}_n\mathbb{R}$-valued 1-form on G whose values lie in this subalgebra $\mathfrak{g} \subset \mathfrak{gl}_n\mathbb{R}$. In other words, the restriction of the Maurer–Cartan form on $\mathrm{GL}_n\mathbb{R}$ to G satisfies some relations.

For the orthogonal group $O_n \subset \mathrm{GL}_n\mathbb{R}$, the Lie algebra $\mathfrak{o}_n \subset \mathfrak{gl}_n\mathbb{R}$ is the space of skew-symmetric matrices, so the relations satisfied are

$$\Theta^i_j + \Theta^j_i = 0, \qquad i, j \in \{1, \dots, n\}. \tag{8.25}$$

These equations, together with (4.83), are the structure equations of O_n:

$$\begin{aligned} \Theta^i_j + \Theta^j_i &= 0 \\ d\Theta^i_j + \Theta^i_k \wedge \Theta^k_j &= 0 \end{aligned} \tag{8.26}$$

For the Euclidean group $\mathrm{Euc}_n \subset \mathrm{Aff}_n$, we also have the relations (8.25), in addition to (8.18), so the Maurer–Cartan forms θ^i, Θ^i_j on Euc_n satisfy the structure equations

$$\begin{aligned} \Theta^i_j + \Theta^j_i &= 0 \\ d\theta^i + \Theta^i_j \wedge \theta^j &= 0 \\ d\Theta^i_j + \Theta^i_k \wedge \Theta^k_j &= 0 \end{aligned} \tag{8.27}$$

These are the basic structure equations of Euclidean geometry, and they transport to Riemannian geometry as well.

Curvature of plane curves

(8.28) *Maurer–Cartan forms on a Euclidean plane.*

Definition 8.29. A *Euclidean space* A is an affine space over an inner product space V. We say A is a *Euclidean plane* if $\dim A = 2$.

The Lie group Euc_2 has dimension 3. It consists of Euclidean transformations of the standard Euclidean plane $\mathbb{E}^2$: translations, rotations, and reflections. The right Euc_2-torsor of orthonormal frames $\mathcal{B}_O(A)$ on a Euclidean plane A, which is a 3-manifold, has a global parallelism given by the Maurer–Cartan 1-forms $\theta^1, \theta^2, \Theta^2_1 = -\Theta^1_2$, which form a basis of parallel 1-forms. They satisfy the Maurer–Cartan equations

$$d\theta^1 - \Theta^2_1 \wedge \theta^2 = 0 \tag{8.30}$$

$$d\theta^2 + \Theta^2_1 \wedge \theta^1 = 0 \tag{8.31}$$

$$d\Theta^2_1 = 0 \tag{8.32}$$

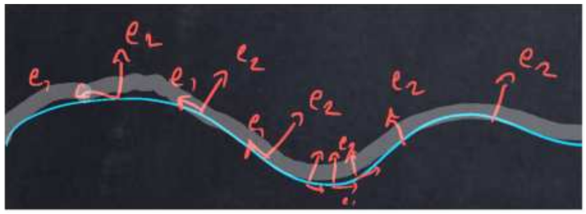

FIGURE 27. A cooriented curve with an adapted moving frame

(8.33) *Restriction to a curve.* Let $C \subset A$ be a 1-dimensional submanifold of a Euclidean plane A. A *coorientation* of $C \subset A$ is an orientation of the normal bundle. (Recall that the normal bundle is defined as the line bundle $TA\big|_C / TC \to C$. Use the metric on V to identify the quotient with the orthogonal complement $TC^\perp \to C$ to the tangent bundle.) We say a frame e_1, e_2 at $p \in C$ is *adapted* to $C \subset A$ if e_2 is the unique cooriented unit normal vector to C; see Figure 27. Let $i\colon C \hookrightarrow A$ be the inclusion. Then an *adapted moving frame* is a lift f of the inclusion

(8.34)
$$\begin{tikzcd} & \mathcal{B}_O(A) \arrow[d, "\pi"] \\ C \arrow[ur, dashed, "f"] \arrow[r, hook, "i"] & A \end{tikzcd}$$

whose image consists of adapted frames. Set $\overline{\theta}^1 = f^*\theta^1$, $\overline{\theta}^2 = f^*\theta^2$, and $\overline{\Theta}^2_1 = f^*\Theta^2_1$.

Lemma 8.35.

(1) $\overline{\theta}^1$ *is a nonzero section of* $T^*C \to C$*, i.e., a global trivialization.*
(2) $\overline{\theta}^2 = 0$.
(3) $\overline{\Theta}^2_1 = \kappa\overline{\theta}^1$ *for some function* $\kappa\colon C \to \mathbb{R}$.

The function κ is the *curvature* of $C \subset A$.

Proof. For (1) and (2) use the geometric interpretation (8.23) of the Maurer–Cartan forms θ^1, θ^2. Namely, if $\xi \in T_pC$ for some $p \in C$, and $f(p) = (p; e_1, e_2)$, then $\xi = \overline{\theta}^1(\xi)e_1 + \overline{\theta}^2(\xi)e_2$. Since ξ is a multiple of e_1, (1) and (2) follow. For (3) we use that any 1-form on C is a multiple of a given nonzero 1-form. □

Remark 8.36. At each $p \in C$ there are two adapted frames; they differ by the sign of e_1. So an adapted lift f in (8.34) is a section of the *orthonormal frame bundle* $\mathcal{B}_O(C) \to C$, which is a principal O_1-bundle over C, i.e., a double cover of C. Hence the diagram in (8.34) expands to

(8.37)
$$\begin{tikzcd} \mathcal{B}_O(C) \arrow[r, hook, "i"] \arrow[d, shift left] & \mathcal{B}_O(A) \arrow[d, "\pi"] \\ C \arrow[u, dashed, shift left, "f"] \arrow[r, hook, "i"] & A \end{tikzcd}$$

(8.38) *The form $\overline{\theta}^1$ and arc length.* The unit vector field e_1 of an adapted moving frame defines a parallelization of the curve C. In particular, it determines an orientation. The dual form $\overline{\theta}^1$ is the "volume" form of the Riemannian metric on C induced from the inclusion $C \subset A$ and the orientation. Namely, if $\xi \in T_pC$ is a tangent vector, then its length is $|\overline{\theta}^1(\xi)|$. The Riemannian measure on C is often denoted '$|ds|$'. This notation is inspired by the situation in which $s\colon C \to \mathbb{R}$ is an arc length parameter on a connected curve: if $p, p' \in C$, then $|s(p') = s(p)|$ is the length of the segment of C between p and p'. If $g\colon C \to \mathbb{R}$ is a function, then its integral with respect to arc length, classically denoted $\int_C g\,|ds|$, is the integral of a differential form on the oriented curve C: $\int_C g\overline{\theta}^1$.

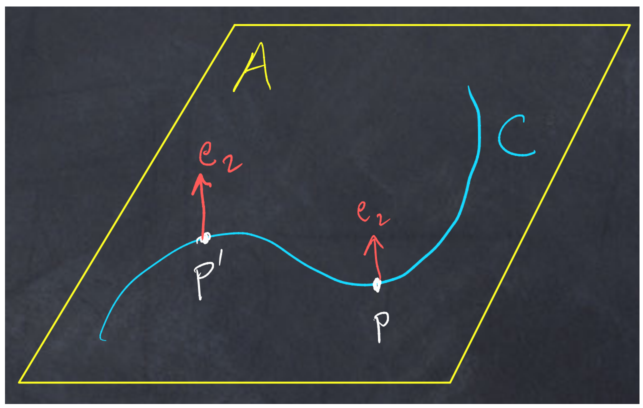

FIGURE 28. A point p of positive curvature and a point p' of negative curvature

(8.39) *Geometric interpretation of curvature.* Recall from **(7.53)** that Θ^2_1 measures the instantaneous rate at which the frame vector e_1 is rotating towards e_2. Evaluate the equation in Lemma 8.35(3) on the unit tangent vector $e_1 \in T_pC$ at some $p \in C$. Then $\kappa(p) = \overline{\Theta}^2_1(e_1)$ is the instantaneous rate at p that C is turning towards its cooriented normal. Observe that if we take the "opposite" moving frame $-e_1, e_2$, then $\overline{\theta}^1$ and $\overline{\Theta}^2_1$ change sign so κ is unchanged. The reader can think this through using Figure 28.

(8.40) *Curvature of a circle.* Let $A = \mathbb{E}^2$ be the standard Euclidean plane, that is, the affine space $\mathbb{A}^2$ over $\mathbb{R}^2$, but the latter endowed with its standard inner product. Use polar coordinates r, θ, which are defined on $A \setminus R$, where R is a ray emanating from the origin. The function r is in fact defined globally on A, but it is not smooth at $r = 0$: the differential dr is well-defined on $A \setminus \{(0,0)\}$. The differential $d\theta$ also exists on $A \setminus \{(0,0)\}$. Define the moving frame and dual moving coframe

$$
\begin{aligned}
e_1 &= \frac{1}{r}\frac{\partial}{\partial\theta} \qquad & \theta^1 &= r\,d\theta \\
e_2 &= \frac{\partial}{\partial r} \qquad & \theta^2 &= dr
\end{aligned}
\tag{8.41}
$$

Now use the structure equations (8.30) and (8.31) to compute $\Theta^2_1 = -\Theta^1_2$. From (8.30),

$$\begin{aligned} 0 &= d\theta^1 + \Theta^1_2 \wedge \theta^2 \\ &= dr \wedge d\theta + \Theta^1_2 \wedge dr \end{aligned} \tag{8.42}$$

which implies

$$\Theta^1_2 = d\theta + f(r)dr \tag{8.43}$$

for some function f. (In fact, we can only conclude this locally in $A \setminus \{(0,0)\}$, but since we prove below that $f = 0$ it is enough to argue locally.) Then (8.31) implies

$$\begin{aligned} 0 &= d\theta^2 + \Theta^2_1 \wedge \theta^1 \\ &= 0 + r\Theta^2_1 \wedge d\theta \end{aligned} \tag{8.44}$$

and combining with (8.43) we conclude $f = 0$. Hence

$$\Theta^2_1 = -d\theta. \tag{8.45}$$

Observe that $d\Theta^2_1 = 0$, as per (8.32).

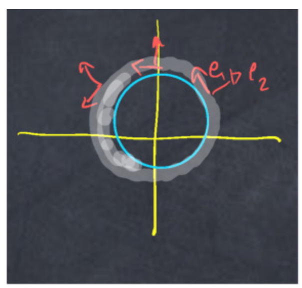

FIGURE 29. An adapted moving frame along a cooriented circle in $\mathbb{E}^2$

Now restrict to the circle $C \subset A \setminus \{(0,0)\}$ of radius $R > 0$ with center $(0,0)$. Note that e_1, e_2 restricts to an adapted moving frame if we coorient $C \subset A$ as indicated in Figure 29. As before, use bars to denote the restrictions of the Maurer–Cartan forms to C with its adapted moving frame. Set $r = R$ and so $dr = 0$ in (8.41) and (8.45) to compute

$$\begin{aligned} \bar{\theta}^1 &= R\, d\theta \\ \bar{\theta}^2 &= 0 \\ \bar{\Theta}^2_1 &= -d\theta \end{aligned} \tag{8.46}$$

Hence from Lemma 8.35(3) we find

$$\kappa = \frac{-1}{R} \tag{8.47}$$

In other words, the circle of radius R has constant curvature $-1/R$. (If we coorient $C \subset A$ by the inward pointing normal, then the curvature is $+1/R$.)

Remark 8.48. Note that curvature of a plane curve has units of reciprocal length. We write

$$|\kappa| = 1/L. \tag{8.49}$$

to indicate the units. This elementary dimensional analysis is useful both for computations and for theoretical understanding.

Total curvature of closed plane curves

We generalize somewhat the notion of a plane curve.

Definition 8.50. Let A be a Euclidean plane with translation group V.
1. A *closed plane curve* is the image of an immersion $S^1 \to A$.
2. A *simple closed plane curve* is the image of an embedding $S^1 \to A$.
3. Let $i\colon S^1 \to A$ be an immersion. A *coorientation* of the resulting closed plane curve is an orientation of the real line bundle $L \to S^1$ whose fiber at $s \in S^1$ is $i_*(T_sS^1)^\perp \subset V$.

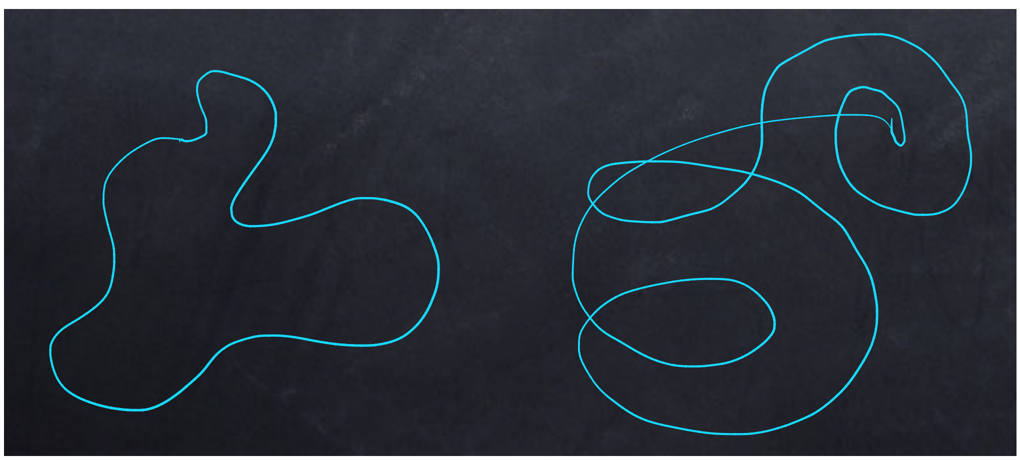

FIGURE 30. Simple and non-simple closed plane curves

Figure 30 depicts a simple closed curve on the left and a closed curve that is not simple on the right. The reader should contemplate the two coorientations of the latter, especially at the double points.

Let $i\colon S^1 \to A$ be an immersion. Since i may not be injective, we define moving frames in terms of the pullback frame bundle, a variation of (8.37):

$$\begin{tikzcd} \mathcal{B}_{\mathrm{O}}(S^1) \arrow[r, hook] \arrow[dr] & i^*\mathcal{B}_{\mathrm{O}}(A) \arrow[d] \\ & S^1 \arrow[ul, dashed, bend left, "f"] \end{tikzcd} \tag{8.51}$$

Here $\mathcal{B}_{\mathrm{O}}(S^1) \to S^1$ is the orthonormal frame bundle of the circle, which is a product double cover map. There are two moving frames f, the two global sections of this double cover. A unit norm tangent vector $e_1 \in T_sS^1$ determines an adapted orthonormal frame of A at the image point $i(s)$; this determination is the top arrow in (8.51).

Remark 8.52. As geometers we like to know what every point of a space means geometrically, and the same for every map. We recommend the reader linger over the previous paragraph taking the previous sentence to heart.

The pullback Maurer–Cartan forms on C obey the conclusions of Lemma 8.35. In particular, curvature is defined as a function $\kappa\colon S^1 \to \mathbb{R}$. As before, it is independent of the section f, though it does depend on the coorientation.

Definition 8.53. The *total curvature* of a closed plane curve is

$$\int_{S^1} \overline{\Theta}{}^2_1. \tag{8.54}$$

Here we assume a global moving frame, which orients S^1 and so defines the integral. For the opposite frame, both the orientation and $\overline{\Theta}{}^2_1$ change sign, so the total curvature does not change.

Theorem 8.55. *The total curvature of a simple closed plane curve is* $\pm 2\pi$*.*

The reader can check this explicitly for a circle using (8.46).

Proof. Since A is simply connected, the embedding $i\colon S^1 \to A$ extends to a continuous map $F\colon D^2 \to A$, which we can perturb to be smooth. (Here $D^2 \subset \mathbb{A}^2$ is the unit closed disk with boundary S^1.) We can perturb further so that the differential at the origin $0 \in D^2$ is an isomorphism. Then using the inverse function theorem we can further perturb so that for some $\delta > 0$ the ball $B_\delta(0) \subset \mathbb{A}^2$ maps isometrically to A. Consider the pullback of the orthonormal frame bundle to A, restricted to the disk minus the origin:

$$F^*\mathcal{B}_{\mathrm{O}}(A) \longrightarrow D^2 \setminus \{0\}. \tag{8.56}$$

Choose a section f that (i) restricts to an adapted frame of the simple closed curve at the boundary S^1, and (ii) restricts near $0 \in D^2$ to the framing (8.41). The pullback of $d\Theta^2_1$ vanishes—as follows from (8.32)—so by Stokes' theorem the total curvature at the boundary equals the total curvature of a small circle about the origin, which is $\pm 2\pi$. $\square$

We now sketch the classical formulation of Definition 8.53. For a closed plane curve $S^1 \to \mathbb{A}^2$ with global moving frame, define a *local* real-valued function θ on connected proper subsets of S^1, up to shifts by multiples of 2π, by setting

$$e_1(s) = \big(\cos\theta(s), \sin\theta(s)\big), \qquad s \in S^1. \tag{8.57}$$

Then one can see $d\theta = \kappa\theta^1$ is globally defined on S^1. Then one can integrate to a global periodic function $\theta\colon \mathbb{R} \to \mathbb{R}$ on the total space of the universal cover $\mathbb{R} \to S^1$. The total curvature is the jump in θ over a period, the total "turning" of a tangent vector around the closed curve.

Lecture 9: Surfaces in a Euclidean 3-space

After we finish up a few loose items about planar curves, mainly the "fundamental theorem" about prescribed curvature, we turn our attention to surfaces. Curves and surfaces are the focus of a typical first undergraduate course on differential geometry. Here we use more high powered techniques than are typically available in such a course to get to some of the main theorems quickly. This illustrates the geometric power inherent in our techniques. We also encounter maneuvers, equations, and ideas that develop experience and intuition for the more abstract study of parallelism and curvature to follow.

Prescribed curvature of plane curves

(9.1) *Variation of Theorem 7.8 for right G-torsors.* The proof of the following proceeds by identifying the right G-torsor with G using a basepoint; the conclusion is independent of the basepoint.

Theorem 9.2. *Let X be a smooth manifold, let G be a Lie group, let P be a right G-torsor, and let $\theta_P \in \Omega^1_P(\mathfrak{g})$ be the Maurer–Cartan form on P.*

(1) *Suppose $f_1, f_2 \colon X \to P$ are smooth maps, $f_1^*(\theta_P) = f_2^*(\theta_P)$, and X is connected. Then there exists $\varphi \in \mathrm{Aut}(P)$ such that*

$$f_2 = \varphi \circ f_1. \tag{9.3}$$

(2) *Suppose $\omega \in \Omega^1_X(\mathfrak{g})$ satisfies*

$$d\omega + \frac{1}{2}[\omega \wedge \omega] = 0. \tag{9.4}$$

Then if X is simply connected, there exists $f \colon X \to P$ such that

$$f^*(\theta_P) = \omega. \tag{9.5}$$

(9.6) *Plane curves with prescribed curvature.* A curve $C \subset A$ in a Euclidean plane inherits both a Riemannian metric and a curvature function. Now we start with an abstract 1-manifold equipped with this data and seek an immersion that induces it.

Theorem 9.7. *Let C be a simply connected Riemannian 1-manifold, let $\kappa \colon C \to \mathbb{R}$ be a smooth function, and suppose A is a Euclidean plane. Then there exists an isometric immersion $i \colon C \to A$ such that κ is the induced curvature function for some coorientation of C. Furthermore, i is unique up to left composition with a Euclidean transformation of A.*

If κ is a constant function, then the image of i is a circle, so if C is a sufficiently long open interval, then the map i is not injective. Note that the normal bundle to an immersion is defined as a vector bundle over the domain—it is the quotient of the pullback of the tangent bundle to the codomain by the tangent bundle to the domain—and so the notion of coorientation also makes sense.

Proof. Let $\mathfrak{o}(C)$ be the 2-element set of orientations of C. For $\mathfrak{o} \in \mathfrak{o}(C)$ there corresponds a 1-form $\sigma_{\mathfrak{o}} \in \Omega^1_C$ of unit length $|\sigma_{\mathfrak{o}}| = 1$. Apply Theorem 9.2(2) to construct $f\colon C \to \mathcal{B}_O(A)$ that satisfies

$$\begin{aligned} f^*\theta^1 &= \sigma_{\mathfrak{o}} \\ f^*\theta^2 &= 0 \\ f^*\Theta^2_1 &= \kappa\sigma_{\mathfrak{o}}, \end{aligned} \tag{9.8}$$

where $\theta^1, \theta^2, \Theta^2_1$ are the Maurer–Cartan forms; see **(8.28)**. Note that the Maurer–Cartan equation (9.4) is automatically satisfied since 2-forms on C are identically zero. Let $\pi\colon \mathcal{B}_O(A) \to A$ be projection, and define $i = \pi \circ f\colon C \to A$. Then by Lemma 8.35 the immersion i is an immersion with curvature κ relative to the coorientation given by e_2 (which is defined on C by f).

Theorem 9.2(2) proves uniqueness up to Euclidean transformations of the solution to (9.8), but those equations depend on a choice of orientation. So we make a more elaborate argument. Let $\mathcal{S} = \bigl\{(\mathfrak{o}, f) \in \mathfrak{o}(C) \times \operatorname{Map}\bigl(C, \mathcal{B}_O(A)\bigr)\bigr\}$ be the space of orientations $\mathfrak{o}$ and solutions f to (9.8). Let $\overline{\mathcal{S}}$ be the set of immersions i that are isometric and have curvature κ relative to some coorientation. Let $p\colon \mathcal{S} \to \overline{\mathcal{S}}$ take $(\mathfrak{o}, f)$ to $\pi \circ f$. Now let $\rho \in \mathrm{O}_2 \subset \mathrm{Euc}_2$ be the Euclidean transformation which in the standard x,y-plane is reflection in the y-axis. The *right* action R_ρ of ρ on $\mathcal{B}_O(A)$ satisfies

$$\begin{aligned} R^*_\rho(\theta^1) &= -\theta^1 \\ R^*_\rho(\theta^2) &= \theta^2 \\ R^*_\rho(\Theta^2_1) &= -\Theta^2_2 \end{aligned} \tag{9.9}$$

It follows that R_ρ acts on $\mathcal{S}$ by sending $(\mathfrak{o}, f)$ to $(-\mathfrak{o}, R_\rho \circ f)$. The isometry group $\operatorname{Aut}(A)$ also acts, now on the left: $\varphi \in \operatorname{Aut}(A)$ sends $(\mathfrak{o}, f)$ to $(\mathfrak{o}, \varphi \circ f)$. These two actions commute and generate a left action on $\mathcal{S}$ of a two component direct product Lie group G whose identity component is $\operatorname{Aut}(A)$, and this action is simply transitive. Noting that R_ρ fixes the composition $\pi \circ f$, we see that the G-action on $\mathcal{S}$ descends to an $\operatorname{Aut}(A)$-action on $\overline{\mathcal{S}}$ that is also simply transitive.[12] That action composes i with $\varphi \in \operatorname{Aut}(A)$. ∎

(9.10) *Alternative discussion of plane curvature.* This is an expansion of Remark 8.36. In **(8.33)** we fixed an adapted moving frame along a plane curve $C \subset A$; see Figure 27. Another option is to consider all adapted moving frames at once. At each $p \in C$ we already have e_2 fixed as the

[12] This action is the *quotient* of G by the order two subgroup generated by R_ρ; the map of $\operatorname{Aut}(A) \subset G$ into the quotient is an isomorphism.

cooriented unit normal, so there are two choices for e_1: the two unit norm vectors in T_pC. We have

$$\begin{tikzcd} \mathcal{B}_O(C) \arrow[r, hook] \arrow[d, "\pi"] & \mathcal{B}_O(A) \\ C \end{tikzcd} \tag{9.11}$$

where $\mathcal{B}_O(C) = \{(p; e_1, e_2) \in \mathcal{B}_O(A) : p \in C,\ e_2 \text{ is cooriented}\}$. Then π is a double cover. Observe that the diffeomorphism $R_\rho\colon \mathcal{B}_O(A) \to \mathcal{B}_O(A)$ restricts to the deck transformation of π. We can repeat the analysis of **(8.33)** using the restrictions of $\theta^1, \theta^2, \Theta^2_1$ to $\mathcal{B}_O(C)$. The argument produces a function $\hat{\kappa}\colon \mathcal{B}_O(C) \to \mathbb{R}$, and we must prove it descends under π to a function $\kappa\colon C \to \mathbb{R}$. That follows from the equation in Lemma 8.35(3) and (9.9). A similar argument occurs at the end of this lecture in the proof of Proposition 9.52.

Remark 9.12. Observe that $\pi\colon \mathcal{B}_O(C) \to C$ is *intrinsically* defined from the Riemannian 1-manifold C. Namely, it is canonically the *bundle of orthonormal frames*, i.e., pairs (p, e_1) in which $p \in C$ and $e_1 \in T_pC$ is a unit norm vector. However the restriction of Θ^2_1 to $\mathcal{B}_O(C)$ is *not* intrinsic; it depends on the embedding $C \hookrightarrow A$. (The existence of unit speed parametrizations proves that any two Riemannian 1-manifolds are locally isometric.)

Surfaces in Euclidean 3-space

(9.13) *Euclidean 3-space.* Let A be a 3-dimensional Euclidean space: an affine space over a 3-dimensional real inner product space V. Let

$$\pi\colon \mathcal{B}_O(A) \longrightarrow A \tag{9.14}$$

be the bundle of orthonormal frames; see (1.33) and **(8.13)**. The 6-manifold $\mathcal{B}_O(A)$ is a right Euc_3-torsor, so it is parallelized by the 6 Maurer–Cartan 1-forms $\theta^1, \theta^2, \theta^3, \Theta^2_1, \Theta^3_1, \Theta^3_2$ (recall the skew-symmetry $\Theta^2_1 + \Theta^1_2 = 0$, etc.) They satisfy the 6 Maurer–Cartan equations

$$d\theta^1 + \Theta^1_2 \wedge \theta^2 + \Theta^1_3 \wedge \theta^3 = 0 \tag{9.15}$$
$$d\theta^2 + \Theta^2_1 \wedge \theta^1 + \Theta^2_3 \wedge \theta^3 = 0 \tag{9.16}$$
$$d\theta^3 + \Theta^3_1 \wedge \theta^1 + \Theta^3_2 \wedge \theta^2 = 0 \tag{9.17}$$
$$d\Theta^2_1 + \Theta^2_3 \wedge \Theta^3_1 = 0 \tag{9.18}$$
$$d\Theta^3_1 + \Theta^3_2 \wedge \Theta^2_1 = 0 \tag{9.19}$$
$$d\Theta^3_2 + \Theta^3_1 \wedge \Theta^1_2 = 0 \tag{9.20}$$

Important to our arguments are the geometric interpretations of these Maurer–Cartan forms in **(8.19)**.

(9.21) *Pullback of a fiber bundle.* In the basic material on fiber bundles in Lecture 9 of the Differential Topology notes there is no material on pullbacks, so we provide a brief description here. (We already used a pullback in (8.51).) Let $\pi\colon E \to M$ be a fiber bundle and suppose $f\colon N \to M$ is a smooth map of smooth manifolds. In the *pullback diagram*

$$\begin{tikzcd} f^*E \arrow[r, dashed, "\tilde f"] \arrow[d, dashed, "\pi'"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd} \tag{9.22}$$

f^*E is defined as the submanifold of $N \times E$ consisting of pairs (n, e) such that $f(n) = \pi(e)$. One uses the fact that π is a submersion to prove that f^*E is a smooth manifold, that π' is a submersion, and more strongly that pull backs of local trivializations of π give local trivializations of π', which is therefore a fiber bundle. Note that the fiber of π' over $n \in N$ is diffeomorphic to the fiber of π over $f(n)$ via the map $\tilde f$.

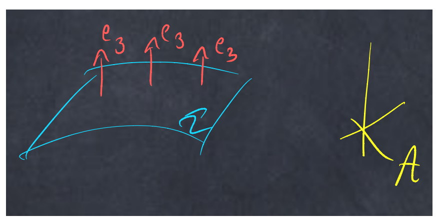

FIGURE 31. A cooriented surface in Euclidean 3-space

(9.23) *Bundle of frames of an embedded surface.* Let $\Sigma \subset A$ be a 2-dimensional submanifold, and suppose $\Sigma \subset A$ is cooriented. Let ν be the cooriented unit normal vector field along Σ; see Figure 31. As in (9.11) define the bundle of adapted orthonormal frames

$$\mathcal{B}_O(\Sigma) = \{(p; e_1, e_2, e_3) \in \mathcal{B}_O(A) : p \in \Sigma,\ e_3 = \nu\}. \tag{9.24}$$

We have a diagram

$$\begin{tikzcd} \mathcal{B}_O(\Sigma) \arrow[r, hook, "i"] \arrow[d, "\pi"] & \mathcal{B}_O(A) \\ \Sigma & \end{tikzcd} \tag{9.25}$$

Lemma 9.26. *i is an embedding of a submanifold, and π is a fiber bundle.*

Proof. Let $S(V) \subset V$ be the unit sphere. Define $\mathcal{E} \subset S(V) \times \mathcal{B}_{\mathrm{O}}(V)$ as the codimension 2 submanifold of pairs $\big(\xi, (e_1, e_2, e_3)\big)$ such that $e_3 = \xi$. (To see it is a submanifold, show that

$$\begin{aligned} S(V) \times \mathcal{B}_{\mathrm{O}}(V) &\longrightarrow \mathbb{R}^2 \\ \xi, (e_1, e_2, e_3) &\longmapsto \langle e_1, \xi\rangle,\ \langle e_2, \xi\rangle \end{aligned} \tag{9.27}$$

is transverse to $(0,0)$. Then $\mathcal{E}$ is the inverse image of $(0,0)$.) Next, we claim that the restriction $\mathcal{E} \to S(V)$ of the projection $S(V) \times \mathcal{B}_{\mathrm{O}}(V) \to S(V)$ is a fiber bundle. To exhibit a local trivialization, fix $\xi_0 \in S(V)$ and suppose $\xi \in S(V)$ satisfies $\langle \xi, \xi_0\rangle > 0$. Construct a diffeomorphism $\mathcal{E}_{\xi_0} \to \mathcal{E}_{\xi}$ that maps a frame e_1, e_2, ξ_0 to its image e_1', e_2', ξ under a Gram–Schmidt process: let e_1' be the normalized orthogonal projection of e_1 to the 2-dimensional subspace $\mathbb{R} \cdot \xi^{\perp} \subset V$, and then let e_2' be the normalized orthogonal projection of e_2 onto the line of vectors orthogonal to ξ and e_1'. Finally use the cooriented normal vector field ν on Σ to form the pullback

$$\begin{tikzcd} \mathcal{B}_{\mathrm{O}}(\Sigma) \arrow[r, dashed] \arrow[d, dashed, "\pi"] & \mathcal{E} \arrow[d] \\ \Sigma \arrow[r, "\nu"] & S(V) \end{tikzcd} \tag{9.28}$$

This proves that π is a fiber bundle, and in particular gives a smooth manifold structure on $\mathcal{B}_{\mathrm{O}}(\Sigma)$.

A closely related construction proves that i is an embedding. Namely, in the diagram

$$\begin{tikzcd} \mathcal{B}_{\mathrm{O}}(\Sigma) \arrow[r, dashed] \arrow[d, dashed, "\pi"] & \mathcal{B}_{\mathrm{O}}(A)\big|_{\Sigma} \arrow[r, dashed] \arrow[d, dashed] & \mathcal{B}_{\mathrm{O}}(A) \arrow[d] \\ \Sigma \arrow[r, equal] & \Sigma \arrow[r, hook] & A \end{tikzcd} \tag{9.29}$$

the right hand square is a pullback and the upper left arrow is the submanifold cut out by the functions $\langle e_1, \nu\rangle$ and $\langle e_2, \nu\rangle$, as in (9.27). The composition of the top arrows is i. $\square$

Remark 9.30. The map $\nu\colon \Sigma \to S(V)$ is the *Gauss map*. We sometimes denote it as e_3.

Remark 9.31. As in Remark 9.12, the fiber bundle $\pi\colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$ is intrinsically associated to the Riemannian 2-manifold Σ. The fiber over $p \in \Sigma$ consists of orthonormal frames of $T_p\Sigma$.

(9.32) *Principal bundles.* The fibers of $\pi\colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$ are right O_2-torsors. This inspires the following generalization.

Definition 9.33. Let X be a smooth manifold, and let G be a Lie group. A *principal G-bundle* over X is a fiber bundle $\pi\colon P \to X$ whose fibers are right G-torsors.

Local trivializations are required to preserve the right G-torsor structure. In other words, a principal bundle is locally isomorphic to the trivial principal bundle $\mathrm{pr}_2\colon X \times G \to G$ in which the fibers are the trivial right G-torsor G_G. The local triviality shows that the G-action on each fiber fits to a smooth G-action on P, and π is a quotient map for the G-action. The fiber bundle $\pi\colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$ is a principal O_2-bundle, the *bundle of orthonormal frames* of Σ. We will study the geometry of principal bundles in future lectures.

(9.34) *Vertical and horizontal distributions.* Let $\pi\colon E \to X$ be a fiber bundle. (The discussion generalizes to submersions.) The differential $d\pi = \pi_*$ is part of a short exact sequence

$$0 \longrightarrow \ker \pi_* \longrightarrow TE \xrightarrow{\pi_*} \pi^*TX \longrightarrow 0 \tag{9.35}$$

of vector bundles over the total space E of π. The subbundle $\ker \pi_* \subset TE$ is an integrable distribution whose associated foliation is the decomposition of E by the fibers of π. We sometimes write $T(E/X)$ for $\ker \pi_*$. A tangent vector to E that lies in $T(E/X) = \ker \pi_*$ is termed *vertical*, as depicted by the vector η in Figure 32.

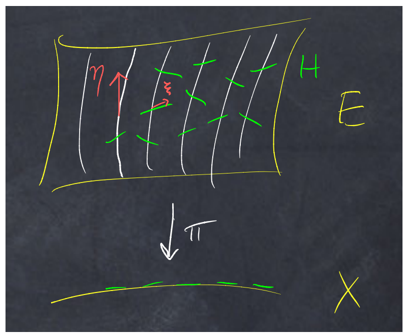

FIGURE 32. A horizontal distribution on the total space of a fiber bundle

Definition 9.36. A *horizontal distribution* $H \subset TE$ is a complement to the vertical distribution $\ker \pi_* \subset TE$, i.e., $H \oplus \ker \pi_* = TE$.

Figure 32 depicts a horizontal distribution. A horizontal distribution is additional data not derived from the data of the fiber bundle $\pi\colon E \to X$. A tangent vector to E that lies in H is termed *horizontal*, as depicted by the vector ξ in Figure 32. The data of a horizontal distribution is equivalent to the data of a *splitting* σ of (9.35):

$$\begin{tikzcd} 0 \arrow[r] & \ker \pi_* \arrow[r] & TE \arrow[r, shift left, "\pi_*"] & \pi^*TX \arrow[l, shift left, "\sigma"] \arrow[r] & 0 \end{tikzcd} \tag{9.37}$$

Recall that a splitting is a vector bundle map $\sigma\colon \pi^*TX \to TE$ that satisfies the affine equation $\pi_* \circ \sigma = \mathrm{id}_{\pi^*TX}$ in $\operatorname{End} \pi^*TX$. Local splittings can be constructed from a local trivialization; they blend to a global splitting via a partition of unity. (For details, see **(16.16)** in the Differential Topology notes.) The distribution H is the image of σ. The map σ induces for each $e \in E$ an isomorphism $T_{\pi(e)}X \to H_e$; it is called *horizontal lift*.

(9.38) *The tautological vertical vector field on the frame bundle of a surface.* The Lie algebra $\mathfrak{o}_2$ of the orthogonal group O_2 is the real line of 2×2 skew-symmetric matrices. Set

$$\zeta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{9.39}$$

It generates the 1-parameter group of rotations

$$e^{t\zeta} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \tag{9.40}$$

Recall $\mathcal{B}_{\mathrm{O}}(\Sigma)$ in (9.24); it is the total space of the orthonormal frame bundle $\mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$ in (9.25). The right action of the one-parameter group (9.40) produces a flow on $\mathcal{B}_{\mathrm{O}}(\Sigma)$. Let $\widehat{\zeta}$ be the vertical vector field that generates the flow; see Figure 33.

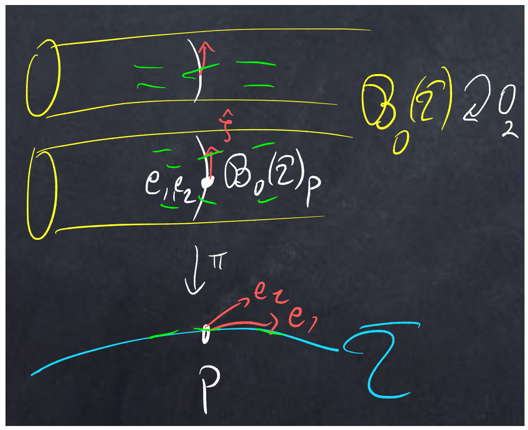

FIGURE 33. The bundle of orthonormal frames $\pi \colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$

(9.41) *Maurer–Cartan forms on* $\mathcal{B}_{\mathrm{O}}(\Sigma)$. As in **(9.10)**, restrict the Maurer–Cartan forms θ^i, Θ^i_j on $\mathcal{B}_{\mathrm{O}}(A)$ to $\mathcal{B}_{\mathrm{O}}(\Sigma)$ using the inclusion i in (9.25), and then analyze the restricted forms using the Maurer–Cartan equations (9.15)–(9.20). Hence set

$$\begin{aligned} \overline{\theta}^i &= i^*\theta^i \\ \overline{\Theta}^i_j &= i^*\Theta^i_j \end{aligned} \tag{9.42}$$

Recall the geometric interpretation (8.23) of θ^i. Namely, if $(p; e_1, e_2, e_3) \in \mathcal{B}_{\mathrm{O}}(\Sigma)$ and we are given a motion germ at this point that represents a tangent vector ξ, and $\pi \colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$ is the projection that forgets the orthonormal frame, then

$$\pi_*\xi = \overline{\theta}^i(\xi)e_i. \tag{9.43}$$

Since $\pi_*\xi \in T_p\Sigma$ is a tangent vector to Σ, and e_1, e_2 is a basis of $T_p\Sigma$, we deduce that $\overline{\theta}^3 = 0$ (which we reiterate as Proposition 9.52(2) below). Keep in mind that $\pi \colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$ is *intrinsic*[13] (Remark 9.31), and for that reason alone e_3 cannot enter (9.43).

[13] The intrinsic/extrinsic dichotomy is fundamental. ‘Intrinsic’ to Σ means ‘does not depend on an embedding $\Sigma \hookrightarrow A$ into Euclidean space’.

(9.44) *The horizontal distribution on the frame bundle.* The next geometric structure we deduce from the Maurer–Cartan forms $\overline{\theta}^i, \overline{\Theta}^i_j$ is a horizontal distribution on the principal O_2-bundle $\pi\colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$. In Lecture 10 we interpret it as a "parallelism" on Σ. However, except in special cases, it is *not* a *global* parallelism in the sense of Definition 1.20. It is a new kind of parallelism that we will study in great generality in a few weeks.

As a preliminary, let $\overline{\Theta} = \bigl(\overline{\Theta}^i_j\bigr)_{1\le i,j\le 2}$ be the 2×2 skew-symmetric matrix of 1-forms on $\mathcal{B}_{\mathrm{O}}(\Sigma)$.

Lemma 9.45.

(1) $\overline{\Theta}^2_1(\widehat{\zeta}) = 1$.
(2) *For* $g \in \mathrm{O}_2$ *we have* $R_g^*\overline{\Theta} = g^{-1}\overline{\Theta}g$.

Proof. For (1), use the geometric interpretation (8.20) of $\overline{\Theta}^2_1$ and the fact that $\widehat{\zeta}$ is generated by the skew-symmetric matrix (9.39) whose $(2,1)$-entry is 1. Or at $f \in \mathcal{B}_{\mathrm{O}}(\Sigma)$ compute

$$(\overline{\Theta}^2_1)_p(\widehat{\zeta}) = \frac{d}{dt}\bigg|_{t=0} \overline{\Theta}^2_1(f\cdot e^{t\zeta}) = \frac{d}{dt}\bigg|_{t=0} \overline{\Theta}^2_1\left(f\cdot\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\right) = \frac{d}{dt}\bigg|_{t=0}\sin t = 1. \tag{9.46}$$

For (2) write an orthonormal frame e_1, e_2 as the row vector $e = \bigl(e_1\ e_2\bigr)$. If $\bigl(p(t); e(t)\bigr)$ is a motion germ on $\mathcal{B}_{\mathrm{O}}(\Sigma)$ that represents a tangent vector ξ, then a rewriting of (8.20) is

$$\dot e = e\overline{\Theta}(\xi). \tag{9.47}$$

Hence

$$\dot e g = e\overline{\Theta}(\xi)g = (eg)(g^{-1}\overline{\Theta}(\xi)g). \tag{9.48}$$

But also

$$\dot e g = (eg)\overline{\Theta}\bigl((R_g)_*\xi\bigr) = (eg)(R_g^*\overline{\Theta})(\xi). \tag{9.49}$$

Combine these expressions for $\dot e g$ to deduce (2). ∎

Proposition 9.50. *The kernel* $H = \ker\overline{\Theta}^2_1 \subset T\bigl(\mathcal{B}_{\mathrm{O}}(\Sigma)\bigr)$ *is an* O_2*-invariant 2-dimensional distribution on the 3-manifold* $\mathcal{B}_{\mathrm{O}}(\Sigma)$.

Proof. Lemma 9.45(1) implies that $H = \ker\overline{\Theta}^2_1$ is 2-dimensional at each point of $\mathcal{B}_{\mathrm{O}}(\Sigma)$, and furthermore H is transverse to the vertical tangent line $\ker\pi_*$. Thus H is a horizontal distribution. If a tangent vector $\xi \in H$, i.e., $\overline{\Theta}^2_1(\xi) = 0$, then for $g \in \mathrm{O}_2$ we have from Lemma 9.45(2) that $\overline{\Theta}^2_1\bigl((R_g)_*\xi\bigr) = 0$, so $(R_g)_*(\xi) \in H$ as well. This proves that H is O_2-invariant. ∎

(9.51) *Computation of* $d\overline{\Theta}{}^2_1$. The analog of Lemma 8.35 for surfaces is the following.

Proposition 9.52.
(1) $\overline{\theta}{}^1, \overline{\theta}{}^2, \overline{\Theta}{}^2_1$ *are a global basis of 1-forms on* $\mathcal{B}_O(\Sigma)$, *so define a global parallelism.*
(2) $\overline{\theta}{}^3 = 0$.
(3) *There exists* $\widetilde{K} \colon \Sigma \to \mathbb{R}$ *such that*

$$d\overline{\Theta}{}^2_1 = (\pi^*\widetilde{K})\,\overline{\theta}{}^1 \wedge \overline{\theta}{}^2. \tag{9.53}$$

By analogy with Lemma 8.35(3) we call $\widetilde{K}$ the *curvature* of Σ. As we will see in the next lecture, there are several curvatures of a surface in Euclidean 3-space. This particular curvature is the *Gauss* curvature, and importantly it is intrinsic: it only depends on the Riemannian structure induced on Σ and not on the embedding $i \colon \Sigma \hookrightarrow A$. We will see that it can be computed from a local section of $\mathcal{B}_O(\Sigma) \to \Sigma$, i.e., from a single local moving frame.

Proof. The horizontal distribution H gives a direct sum decomposition

$$T\big(\mathcal{B}_O(\Sigma)\big) \cong H \oplus \ker \pi_*, \tag{9.54}$$

of vector bundles over $\mathcal{B}_O(\Sigma)$, and π_* restricts to an isomorphism $H \xrightarrow{\cong} \pi^*T\Sigma$. The 1-form $\overline{\Theta}{}^2_1$ is nonzero[14] on $\ker \pi_*$ and vanishes on H. At a point $f = (p; e_1, e_2) \in \mathcal{B}_O(\Sigma)$, the horizontal lifts of $e_1, e_2 \in T_p\Sigma$ comprise a basis of $T_f\mathcal{B}_O(\Sigma)$, and by (9.43) the forms $\overline{\theta}{}^1, \overline{\theta}{}^2$ comprise the dual basis. This proves (1). The proof of (2) is indicated following (9.43).

The global basis $\overline{\theta}{}^1, \overline{\theta}{}^2, \overline{\Theta}{}^2_1$ of 1-forms on $\mathcal{B}_O(\Sigma)$ induces a global basis of 2-forms, and so there exist functions $A, B, C \colon \mathcal{B}_O(\Sigma) \to \mathbb{R}$ such that

$$d\overline{\Theta}{}^2_1 = A\,\overline{\theta}{}^1 \wedge \overline{\theta}{}^2 + B\,\overline{\theta}{}^1 \wedge \overline{\Theta}{}^2_1 + C\,\overline{\theta}{}^2 \wedge \overline{\Theta}{}^2_1. \tag{9.55}$$

The restriction of the structure equations (9.15) and (9.16) to $\mathcal{B}_O(\Sigma)$ give

$$\begin{aligned} d\overline{\theta}{}^1 + \overline{\Theta}{}^1_2 \wedge \overline{\theta}{}^2 &= 0 \\ d\overline{\theta}{}^2 + \overline{\Theta}{}^2_1 \wedge \overline{\theta}{}^1 &= 0 \end{aligned} \tag{9.56}$$

Apply d to each equation (when in doubt, differentiate!), and substitute (9.56) to conclude

$$\begin{aligned} d\overline{\Theta}{}^2_1 \wedge \overline{\theta}{}^2 &= 0 \\ d\overline{\Theta}{}^2_1 \wedge \overline{\theta}{}^1 &= 0 \end{aligned} \tag{9.57}$$

Plug (9.55) into these equations to deduce $B = C = 0$. Thus

$$d\overline{\Theta}{}^2_1 = A\,\overline{\theta}{}^1 \wedge \overline{\theta}{}^2. \tag{9.58}$$

[14] More is true: the fibers of π are right O_2-torsors, and the restriction of $\overline{\Theta}$ to a fiber is the Maurer–Cartan form.

We claim that A descends to the base of $\pi\colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$. In other words, there exists a function $\widetilde{K}\colon \Sigma \to \mathbb{R}$ so that $A = \pi^*\widetilde{K}$ and hence (9.53) holds. As a first step, differentiate (9.58):

$$0 = dA \wedge \overline{\theta}^1 \wedge \overline{\theta}^2. \tag{9.59}$$

Contract with the vertical vector field $\widehat{\zeta}$:

$$0 = \iota_{\widehat{\zeta}}(dA \wedge \overline{\theta}^1 \wedge \overline{\theta}^2) = (\iota_{\widehat{\zeta}} dA)\, \overline{\theta}^1 \wedge \overline{\theta}^2, \tag{9.60}$$

from which $0 = \iota_{\widehat{\zeta}} dA = \widehat{\zeta} A$. In other words, A is locally constant on each fiber of π. Now the fibers of π are O_2-torsors, so they have two components; see Figure 33. Use the reflection $\rho \in \mathrm{O}_2$ that appears in (9.11) to argue that A is constant on each fiber. Namely, $\rho \in \mathrm{O}_2$ induces a diffeomorphism $R_\rho\colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \mathcal{B}_{\mathrm{O}}(\Sigma)$ that exchanges the components of each fiber. Its effect on the 1-forms $\overline{\theta}^1, \overline{\theta}^2, \overline{\Theta}_1^2$ is as in (9.9). Apply R_ρ^* to (9.58) to deduce $R_\rho^* A = A$, which implies the desired descent. $\square$

We continue our study of surfaces in the next lecture. In particular, we will use the heretofore unemployed structure equations (9.17)–(9.20). We will also discuss something you may have already divined: $d\overline{\Theta}_1^2$ is the obstruction to integrability of the horizontal distribution H.

Lecture 10: Extrinsic geometry of a surface in a Euclidean 3-space

In this lecture we continue to mine the Maurer–Cartan forms and Maurer–Cartan equations on the right Euc_3-torsor of orthonormal bases in a Euclidean 3-space A to explore the local differential geometry of an embedded surface $\Sigma \subset A$.

Recall Proposition 9.52(3) in which we introduced a function $\widetilde{K}\colon \Sigma \to \mathbb{R}$ that encodes the differential of the Maurer–Cartan form $\overline{\Theta}_1^2$ with tangential indices. Our first task here is to explore the Maurer–Cartan forms $\overline{\Theta}_1^3, \overline{\Theta}_2^3$ with one normal and one tangent index. This quickly leads to the *second fundamental form* (Definition 10.11) and the associated *shape operator* (Definition 10.16). They control the second-order geometry, which was known as far back as Euler and the French geometers who followed. We identify the function $\widetilde{K}$ with the negative of the *Gauss curvature*. The differentials $d\overline{\Theta}_1^3, d\overline{\Theta}_2^3$ lead to the Codazzi–Mainardi equations for the second fundamental form, but we do not explore those in any depth.

In the second part of the lecture we explore the implications of the global parallelism of A for the embedded surface $\Sigma \subset A$. We are immediately led (through consideration of the Maurer–Cartan forms) to the horizontal distribution on $\mathcal{B}_O(\Sigma) \to \Sigma$ introduced in **(9.44)**. First we show that this distribution is integrable iff the Gauss curvature vanishes. Then we use it to define *parallel transport* of orthonormal frames along curves in Σ. We only have the space/time to hint at beautiful intuition-building ideas in classical differential geometry (Remark 10.44).

In the next lecture we prove that the Gauss curvature, the horizontal distribution on $\mathcal{B}_O(\Sigma) \to \Sigma$, and so the induced parallel transport are all *intrinsic* to Σ; that is, they are determined by the Riemannian metric induced from the embedding $\Sigma \hookrightarrow A$.

The second fundamental form

We continue with the setup and notation of Lecture 9. So A is a Euclidean space of dimension three with inner product space V of translations, and $\Sigma \subset A$ is a cooriented embedded surface in A. Let $\nu\colon \Sigma \to V$ be the oriented unit normal vector field. The manifold $\mathcal{B}_O(\Sigma)$ parametrizes adapted orthonormal frames of A along Σ, and there is an embedding $\mathcal{B}_O(\Sigma) \hookrightarrow \mathcal{B}_O(A)$. Recall $\dim \mathcal{B}_O(\Sigma) = 3$ and $\dim \mathcal{B}_O(A) = 6$.

(10.1) *The forms $\overline{\Theta}_i^3$.* Recall from Proposition 9.52(1) that $\overline{\theta}^1, \overline{\theta}^2, \overline{\Theta}_1^2$ form a global basis of 1-forms on $\mathcal{B}_O(\Sigma)$. This defines a parallelization of $\mathcal{B}_O(\Sigma)$.

Proposition 10.2. *There exist functions*

$$h_{ij}\colon \mathcal{B}_O(\Sigma) \longrightarrow \mathbb{R}, \qquad i,j \in \{1,2\}, \tag{10.3}$$

such that $\overline{\Theta}_i^3 = h_{ij}\overline{\theta}^j$. Furthermore $h_{ij} = h_{ji}$.

Throughout the statement and proof, we have $i,j \in \{1,2\}$.

Proof. Write

$$\overline{\Theta}{}^3_i = h_{ij}\overline{\theta}{}^j + f_i\overline{\Theta}{}^2_1 \tag{10.4}$$

for some functions $h_{ij}, f_i \colon \mathcal{B}_O(\Sigma) \to \mathbb{R}$. Apply $\iota_{\widehat{\zeta}}$. Using the geometric interpretations (8.20) and (8.23) of the Maurer–Cartan forms on the orthonormal frame bundle, we see that $\iota_{\widehat{\zeta}}(\overline{\Theta}{}^3_i) = 0$, since $\zeta \in \mathfrak{o}_2 \subset \mathfrak{o}_3$ has no $(3,1)$-entry; see (9.39). By the same token, $\iota_{\widehat{\zeta}}(\overline{\Theta}{}^2_1) = 1$, which is Lemma 9.45(1). Also, $\iota_{\widehat{\zeta}}\overline{\theta}{}^j = 0$, since $\pi_*\widehat{\zeta} = 0$. Plug into (10.4) to prove the first assertion.

For the symmetry, restrict (9.17) from $\mathcal{B}_O(A)$ to $\mathcal{B}_O(\Sigma)$ and use Proposition 9.52(2):

$$\begin{aligned} 0 &= \overline{\Theta}{}^3_1 \wedge \overline{\theta}{}^1 + \overline{\Theta}{}^3_2 \wedge \overline{\theta}{}^2 \\ &= h_{1j}\overline{\theta}{}^j \wedge \overline{\theta}{}^1 + h_{2j}\overline{\theta}{}^j \wedge \overline{\theta}{}^2 \\ &= (h_{21} - h_{12})\,\overline{\theta}{}^1 \wedge \overline{\theta}{}^2, \end{aligned} \tag{10.5}$$

from which $h_{21} = h_{12}$, as desired. $\square$

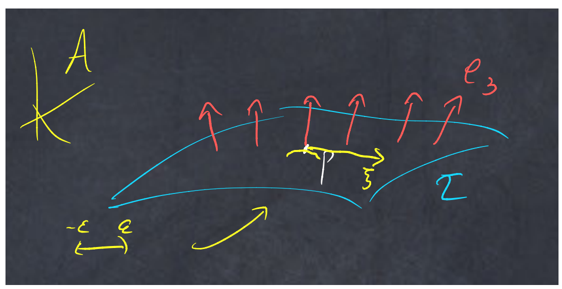

FIGURE 34. Differentiating the cooriented unit normal

(10.6) *Geometric interpretation.* Fix $p \in \Sigma$ and $\xi \in T_p\Sigma$. Let $t \mapsto p(t)$ be a motion (germ) that represents ξ; see Figure 34. Then $t \mapsto \nu(t)$ is a motion in V whose initial velocity is $\dot\nu \in V = T_p\Sigma \oplus T_p\Sigma^\perp$. Since $\langle \nu(t), \nu(t)\rangle = 1$ is constant, $0 = \langle \nu,\nu\rangle^{\cdot} = 2\langle \dot\nu, \nu\rangle$, which implies $\dot\nu \in T_p\Sigma$. Write $D_\xi \nu = \dot\nu$ for the directional derivative of ν in the direction $\xi \in T_p\Sigma \subset V$ at p. Then by (8.20) we have

$$D_\xi\nu = \overline{\Theta}{}^i_3(\xi)e_i = -\sum_{i=1}^2 \overline{\Theta}{}^3_i(\xi)e_i = -\sum_{i=1}^2 h_{ij}\overline{\theta}{}^j(\xi)e_i. \tag{10.7}$$

(Note the fact that $\overline{\Theta}{}^3_3 = 0$ implies $\langle D_\xi\nu, \nu\rangle = 0$, which we (re-)derived a few lines ago.) In particular,

$$D_{e_j}\nu = -\sum_i h_{ij}e_i, \tag{10.8}$$

or equivalently

$$h_{ij} = -\langle D_{e_j}\nu, e_i\rangle = -\langle D_{e_i}\nu, e_j\rangle, \tag{10.9}$$

where we use the symmetry $h_{ij} = h_{ji}$. This tells that h_{ij} defines a symmetric bilinear form that computes the differential of the normal vector field ν.

(10.10) *The first and second fundamental forms.*

Definition 10.11. Let $\Sigma \subset A$ be a cooriented 2-dimensional submanifold of a Euclidean 3-space. Fix $p \in \Sigma$.

(1) The *first fundamental form*

$$\begin{aligned} \mathrm{I}_p\colon T_p\Sigma \times T_p\Sigma &\longrightarrow \mathbb{R} \\ \xi \quad , \quad \eta &\longmapsto \langle \xi, \eta\rangle_p \end{aligned} \tag{10.12}$$

is the induced Riemannian metric on Σ.

(2) The *second fundamental form* is

$$\begin{aligned} \mathrm{II}_p\colon T_p\Sigma \times T_p\Sigma &\longrightarrow \mathbb{R} \\ \xi \quad , \quad \eta &\longmapsto -\langle D_\xi\nu, \eta\rangle_p \end{aligned} \tag{10.13}$$

where ν is the cooriented unit normal vector field along Σ, and D_ξ is the directional derivative computed using the global parallelism in the affine space A.

Both I and II are symmetric bilinear forms; the form I is nondegenerate, in fact positive definite.

Remark 10.14. The second fundamental form on Σ is a section of the vector bundle $\operatorname{Sym} T^*\Sigma \to \Sigma$ whose fiber at p is the linear space $\operatorname{Sym}^2 T_p^*\Sigma$ of symmetric bilinear forms on $T_p\Sigma$. The pullback of $T\Sigma \to \Sigma$ to the total space $\mathcal{B}_{\mathrm{O}}(\Sigma)$ of the frame bundle is trivialized by the frames, and so the second fundamental form pulls back to a vector-valued function, with values in the linear space $\operatorname{Sym}^2(\mathbb{R}^2)^*$ of symmetric bilinear forms on $\mathbb{R}^2$. This pullback $\pi^*\,\mathrm{II}$ is represented by the functions $h_{ij}\colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \mathbb{R}$. This untwisting by pullback to the frame bundle is a recurrent theme.

(10.15) *The shape operator.* Given a nondegenerate bilinear form on a vector space, any other bilinear form can be obtained by applying an endomorphism to one argument.

Definition 10.16. Let $\Sigma \subset A$ be a cooriented 2-dimensional submanifold of a Euclidean 3-space. Fix $p \in \Sigma$. The *shape operator* $S_p\colon T_p\Sigma \to T_p\Sigma$ is characterized by

$$\begin{aligned} \mathrm{II}_p\big(\xi, \eta\big) &= \mathrm{I}\big(\xi, S_p(\eta)\big), \qquad \xi, \eta \in T_p\Sigma \\ &= \langle\, \xi, S_p(\eta)\,\rangle \end{aligned} \tag{10.17}$$

Since I and II are both symmetric, the shape operator S is self-adjoint.

Remark 10.18. The pullback of S to the frame bundle is the $\mathrm{End}(\mathbb{R}^2)$-valued function

$$\pi^*S = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \tag{10.19}$$

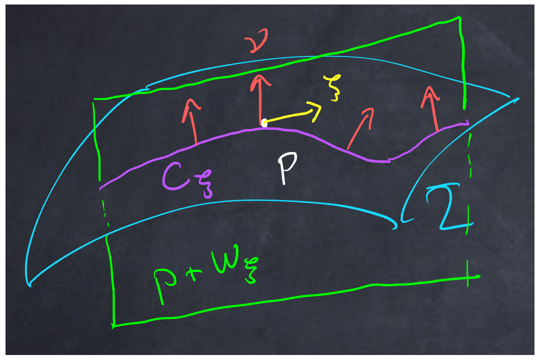

FIGURE 35. A normal section of a surface in Euclidean space

(10.20) *Geometric interpretation on the diagonal.* If $\xi \in T_p\Sigma$ is a nonzero vector, then ν, ξ span a 2-dimensional subspace $W_\xi \subset V$ that intersects $T_p\Sigma$ in the span of ξ. The affine subspace $p+W_\xi \subset A$, which is a Euclidean plane, intersects Σ near p in a plane curve C_ξ with cooriented unit normal ν; see Figure 35. Then the curvature of the plane curve $C_\xi \subset p+W_\xi$ at p is Lemma 8.35(3), which we rewrite as

$$\kappa(p;\xi) = \frac{-\langle D_\xi \nu, \xi\rangle}{\langle \xi, \xi\rangle} = \frac{\mathrm{II}_p(\xi,\xi)}{\mathrm{I}_p(\xi,\xi)}. \tag{10.21}$$

The intersection $\Sigma \cap (p + W_\xi)$ is called a *normal section* of Σ. The curvatures of normal sections were studied by Euler in 1760.

(10.22) *Geometry of the shape operator.* Since S_p is self-adjoint, it is diagonalizable. Let the eigenvalues be $\lambda_1 \ge \lambda_2$. The normal curvature (10.21) is a function on the real projective line $\mathbb{P}T_p\Sigma$ with maximum value λ_1 and minimum value λ_2. There are several possibilities.

If $\lambda_1 = \lambda_2$, i.e., if $S_p = \lambda\,\mathrm{id}_{T_p\Sigma}$ for some $\lambda \in \mathbb{R}$, then the normal curvature is constant and we say that p is an *umbilic point.* If $\lambda = 0$ then Σ is *flat* at p and is approximated to second order by its affine tangent plane. If $\lambda \neq 0$ then Σ is approximated to second order by a sphere of radius $1/\lambda$ passing through p with the same affine tangent plane as Σ.

If $\lambda_1 \neq \lambda_2$, then there is a decomposition $T_p\Sigma = L_1 \oplus L_2$ into an orthogonal sum of lines, where $S_p = \lambda_i$ on L_i. The L_i are called the *principal directions* at p and the λ_i the *principal curvatures.* It follows from (10.21) that the normal curvature on a line L that makes an angle θ with L_1 is

$$\lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta, \tag{10.23}$$

a result known as *Euler's formula*.

If $\lambda_1 \geq 0 \geq \lambda_2$ and $\lambda_1 \neq \lambda_2$, then there are two points of $\mathbb{P}T_p\Sigma$ at which the normal curvature vanishes. These are called *asymptotic directions*.

In every case define the *mean curvature* and *Gauss curvature* as

$$\begin{aligned} H_p &= \frac{\lambda_1+\lambda_2}{2} &&= \frac{1}{2}\operatorname{trace}(S_p) \\ K_p &= \lambda_1\lambda_2 &&= \det(S_p). \end{aligned} \tag{10.24}$$

Observe that the mean curvature has units $1/L$ of reciprocal length and the Gauss curvature has units $1/L^2$ of reciprocal area.

(10.25) *Second order approximation.* Introduce an orthonormal system of affine coordinates x, y, z on A centered at p such that the x- and y-axes lie in the affine tangent space to Σ at p and align with the lines L_1, L_2; the z-axis is the normal affine line at p. (At an umbilic point we are free to choose the x- and y-axes as arbitrary orthogonal lines in the affine tangent space.) Then the osculating quadric surface Q_p is cut out by the equation

$$z = \frac{\lambda_1}{2}\,x^2 + \frac{\lambda_2}{2}\,y^2. \tag{10.26}$$

The quadric is nondegenerate if both λ_1 and λ_2 are nonzero, in which case Q_p well-approximates the surface near p in a sense that can be made precise (Taylor's theorem).

(10.27) *The Gauss equation.* Recall the function $\widetilde{K}\colon \Sigma \to \mathbb{R}$ defined in Proposition 9.52(3).

Proposition 10.28. *$\widetilde{K} = -K$ is minus the Gauss curvature.*

Proof. Apply the 4[th] structure equation (9.18), restricted to $\mathcal{B}_O(\Sigma)$. So

$$\begin{aligned} 0 &= d\overline{\Theta}^2_1 + \overline{\Theta}^2_3 \wedge \overline{\Theta}^3_1 \\ &= (\pi^*\widetilde{K})\,\overline{\theta}^1 \wedge \overline{\theta}^2 \;-\; (h_{21}\overline{\theta}^1 + h_{22}\overline{\theta}^2) \wedge (h_{11}\overline{\theta}^1 + h_{12}\overline{\theta}^2) \\ &= \Big[(\pi^*\widetilde{K}) + (h_{11}h_{22} - h_{12}h_{21})\Big]\,\overline{\theta}^1 \wedge \overline{\theta}^2 \\ &= \Big[\pi^*\widetilde{K} + \pi^*K\Big]\,\overline{\theta}^1 \wedge \overline{\theta}^2 \end{aligned} \tag{10.29}$$

where at the last step we use (10.24) and (10.19). Conclude by observing $\overline{\theta}^1 \wedge \overline{\theta}^2$ is nowhere zero. $\square$

(10.30) *The Codazzi–Mainardi equations.* The final two structure equations (9.19) and (9.20), restricted to $\mathcal{B}_O(\Sigma)$. have leading terms (in terms of derivative count) the differentials $d\overline{\Theta}^3_1$ and $d\overline{\Theta}^3_2$. Thus they involve derivatives of the second fundamental form. To get a feel for what they look like, let $U \subset \Sigma$ be an open subset on which we have an orthonormal moving frame e_1, e_2, i.e., a section σ of $\mathcal{B}_O(U) \to U$. Then $\sigma^*\overline{\theta}^1, \sigma^*\overline{\theta}^2$ is the dual moving coframe, which we simply denote θ^1, θ^2. The 1-form $\sigma^*\overline{\Theta}^2_1$ in this coframe is

$$\sigma^*\overline{\Theta}^2_1 = p\theta^1 + q\theta^2 \tag{10.31}$$

for some functions $p, q\colon U \to \mathbb{R}$. Simply write h_{ij} for σ^*h_{ij}; this is the second fundamental form in the given moving frame. Then the Codazzi–Mainardi equations are

$$\begin{aligned} e_1 \cdot h_{12} - e_2 \cdot h_{11} + p(h_{11} - h_{22}) + 2qh_{12} &= 0 \\ -e_2 \cdot h_{12} + e_2 \cdot h_{22} \qquad + \qquad q(h_{11} + h_{22}) &= 0 \end{aligned} \tag{10.32}$$

In these equations the vector fields e_i act by directional derivative on the functions h_{ij}.

The induced parallelism along curves on Σ

The affine space A has a global parallelism encoded by the action of the vector space V. Unless $\Sigma \subset A$ is an affine subspace, Σ is not preserved by this action, hence there is no induced global parallelism on Σ. We will re-express the global parallelism on A in infinitesimal terms, and in that way we will see that the global parallelism on A restricts to the horizontal distribution **(9.44)** on the total space $\mathcal{B}_O(\Sigma)$ of the orthonormal frame bundle. This restriction determines a parallelism along curves in Σ. In the next lecture, we will prove that this parallelism along curves is *intrinsic* to the Riemannian 2-manifold Σ, as is the Gauss curvature.

(10.33) *Parallelism on A.* We usually say **(1.18)** that a vector $\xi \in T_pA = V$ at any point $p \in A$ induces a global parallel vector field on A. Equally, an orthonormal frame e_1, e_2, e_3 at $p \in A$ induces a global parallel frame on A, i.e., a section σ of the orthonormal frame bundle $\pi\colon \mathcal{B}_O(A) \to A$. Recalling that $\mathcal{B}_O(A) \approx A \times \mathcal{B}_O(V)$ is a global product, we see that the image of σ is horizontal, i.e., is of the form $A \times \{f\}$ for some $f \in \mathcal{B}_O(V)$, namely $f = (e_1, e_2, e_3)$. In other words, the global parallelism is encoded in the product structure of the total space $\mathcal{B}_O(A)$ of the orthonormal frame bundle. Equivalently, it is encoded by an integrable horizontal distribution H_A on $\pi\colon \mathcal{B}_O(A) \to A$, namely the distribution

$$H_A = \bigcap_{i,j} \ker \Theta^i_j. \tag{10.34}$$

This distribution has codimension 3: there are 3 linearly independent 1-forms that cut it out. Note that σ is parallel iff $\sigma^*\Theta^i_j = 0$ iff in the global moving frame $f = (e_1, e_2, e_3)$, the vector e_j does not turn instantaneously towards e_i.

(10.35) *Restriction to* Σ. The 1-forms Θ^i_j on $\mathcal{B}_{\mathrm{O}}(A)$ restrict to 1-forms $\overline{\Theta}^i_j$ on $\mathcal{B}_{\mathrm{O}}(\Sigma)$. The manifold $\mathcal{B}_{\mathrm{O}}(\Sigma)$ has dimension 3, and the fibers of the projection $\pi\colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$ have dimension 1, so a horizontal distribution has codimension 1: it is cut out by a single 1-form that is nonvanishing on vertical vectors. (Vertical vectors comprise the kernel of the differential π_*.) Proposition 10.2 shows that $\overline{\Theta}^3_1, \overline{\Theta}^3_2$ are linear combinations of $\overline{\theta}^1, \overline{\theta}^2$, hence they vanish identically on vertical vectors. Among the forms in (10.34) that leaves $\overline{\Theta}^2_1$, which is nonvanishing on vertical vectors (Lemma 9.45(1)). Thus the horizontal distribution

$$H = \ker \overline{\Theta}^2_1, \tag{10.36}$$

already defined in Proposition 9.50, is the analog of (10.34) on Σ. A *horizontal curve* in H is a curve of adapted frames on Σ, which via the inclusion $\mathcal{B}_{\mathrm{O}}(\Sigma) \hookrightarrow \mathcal{B}_{\mathrm{O}}(A)$ is a curve of orthonormal frames in A. The Maurer–Cartan form Θ^2_1 restricts to zero on that curve. This means that along the curve there is no rotation in the $(1,2)$-plane, as measured using the global parallelism of A. The "No $(1,2)$-Rotation Rule" characterizes the induced parallelism on Σ.

The distribution H is not integrable in general. More precisely, we have the following.

Proposition 10.37. *$H \subset T\bigl(\mathcal{B}_{\mathrm{O}}(\Sigma)\bigr)$ is integrable iff $K = 0$.*

Proof. The codimension one distribution H is cut out by the global nonzero 1-form $\overline{\Theta}^2_1$. Quite generally, a global nonzero 1-form Θ on a smooth manifold X defines a distribution $H = \ker\Theta \subset TX$ and a trivialization $\Theta\colon TX/H \to \underline{\mathbb{R}}$ of the quotient bundle $TX/H \to X$. (Here $\underline{\mathbb{R}} \to X$ is the product vector bundle with constant fiber $\mathbb{R}$.) The Frobenius tensor of H is then the 2-form $-d\Theta$ restricted to $\bigwedge^2 H \to X$. (Recall Definition 5.18 and use (4.48).) For the horizontal distribution $H = \ker\overline{\Theta}^2_1$ on $\mathcal{B}_{\mathrm{O}}(\Sigma)$, by (9.53) and Proposition 10.28 we have $d\overline{\Theta}^2_1 = -(\pi^*K)\,\overline{\theta}^1 \wedge \overline{\theta}^2$. Observe that $\overline{\theta}^1 \wedge \overline{\theta}^2$ is a global basis of $\bigwedge^2 H^* \to \mathcal{B}_{\mathrm{O}}(\Sigma)$, so the Frobenius tensor vanishes iff π^*K vanishes iff K vanishes. $\square$

Remark 10.38. The horizontal distribution $H \subset T\bigl(\mathcal{B}_{\mathrm{O}}(\Sigma)\bigr)$ is invariant under the O_2-action. Quite generally, an invariant horizontal distribution on the total space of a principal bundle is a *connection*, so here we see that H is a connection on the principal O_2-bundle $\mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$ of orthonormal frames. The general study of connections on principal bundles will occupy us shortly.

(10.39) *Parallel transport along curves.* We begin more generally.

Definition 10.40. Let Y be a smooth manifold and $\varphi\colon Y \to \Sigma$ a smooth map. A *horizontal lift* of φ is a map $\widetilde{\varphi}\colon Y \to \mathcal{B}_{\mathrm{O}}(\Sigma)$ that satisfies $\pi \circ \widetilde{\varphi} = \varphi$ and $\widetilde{\varphi}^*(\overline{\Theta}^2_1) = 0$.

Equivalently, $d\widetilde{\varphi}_y(T_yY) \subset H_{\widetilde{\varphi}(y)}$ for all $y \in Y$.

Specialize to $Y = [0,T]$ for some $T > 0$, so φ is a motion on Σ. Let $p = \varphi(0)$ and fix an orthonormal frame f of $T_p\Sigma$.

Proposition 10.41. *There is a unique horizontal lift $\widetilde{\varphi}\colon [0,T] \to \mathcal{B}_{\mathrm{O}}(\Sigma)$ such that $\widetilde{\varphi}(0) = f$.*

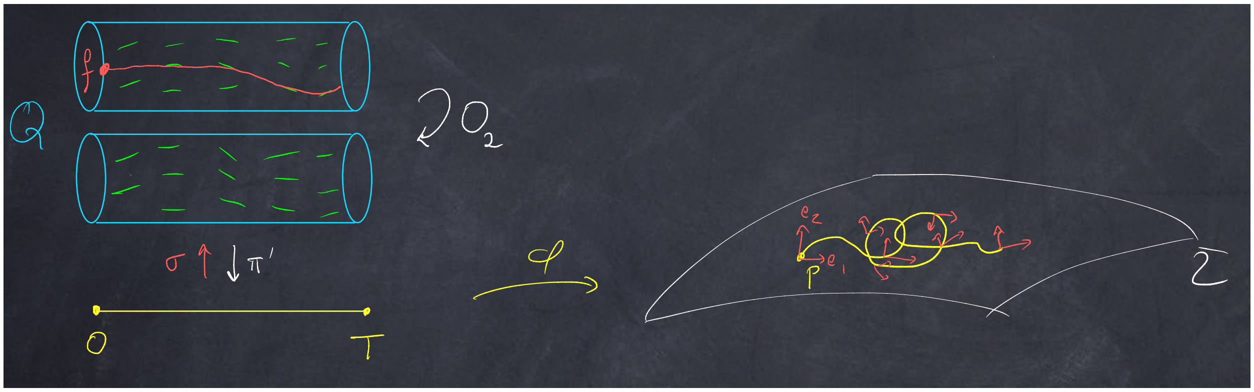

FIGURE 36. Parallel transport

Proof. Construct the pullback principal O_2-bundle π' in

$$\begin{tikzcd} Q \arrow[r, dashed, "\widehat{\varphi}"] \arrow[d, dashed, "\pi'"'] & \mathcal{B}_{\mathrm{O}}(\Sigma) \arrow[d, "\pi"] \\ {[0,T]} \arrow[r, "\varphi"] & \Sigma \end{tikzcd} \tag{10.42}$$

The pullback 1-form $\widehat{\varphi}^*(\overline{\Theta}^2_1)$ is nonzero, since it is so on nonzero vertical vectors, and its kernel $H' = \ker \widehat{\varphi}^*(\overline{\Theta}^2_1)$ is a horizontal distribution on π'. (It can be defined directly as the pullback of H.) Let $\xi \in \mathfrak{X}(Q)$ be the horizontal lift of $\partial/\partial t$ on $[0,T]$. We claim that an integral curve $\sigma\colon [0,T_0) \to Q$ of ξ with initial condition $\sigma(0) = f$ extends to the entire interval $[0,T]$. This follows from compactness of Q: for a sequence $t_n \to T_0$ in $[0,T]$, a subsequence of $\sigma(t_n)$ in Q converges to some $q_0 \in Q$ and then the integral curve $\sigma'\colon (T_0 - \delta, T_0 + \delta) \to Q$ with $\sigma'(T_0) = q_0$ can be glued to σ to extend the domain of the latter to $[0,T]$. Finally, set $\widetilde{\varphi} = \widehat{\varphi} \circ \sigma$. $\square$

The composition $\widetilde{\varphi} = \widehat{\varphi} \circ \sigma\colon [0,T] \to \mathcal{B}_{\mathrm{O}}(\Sigma)$ is a horizontal moving frame along φ. We say $\widetilde{\varphi}(T)$ is the *parallel transport* of $\widetilde{\varphi}(0)$ *along the parametrized curve* φ.

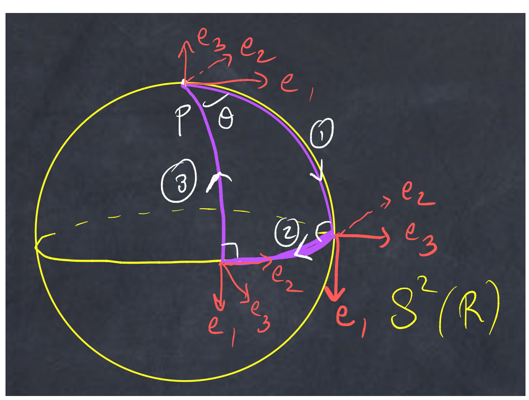

FIGURE 37. Parallel transport around a spherical triangle

Example 10.43. To get a feel for the induced parallelism on a surface in a Euclidean 3-space, consider the 2-sphere $\Sigma = S^2(R)$ centered at the origin in standard Euclidean 3-space $A = \mathbb{E}^3$ with coordinates x, y, z. Coorient Σ so that the exterior of the ball is the "positive side" of Σ in $\mathbb{E}^3$. Let $p = (0,0,1)$ be the north pole, and fix the frame $e_1 = \partial/\partial x$, $e_2 = \partial/\partial y$ at p. In Figure 37 the x-axis points to the right, the y-axis points into the page, and the z-axis points up. The first side 1 begins at p and follows the great circle $y = 0$ down to the equator $z = 0$. Use the "No $(1,2)$-Rotation Rule" in **(10.35)** to deduce the parallel transport. The first frame vector e_1 remains tangent to 1—we will see in the next lecture that this follows from the fact that the great circle segment 1 is a geodesic—and so at the other endpoint the parallel transported frame is $e_1 = -\partial/\partial z$, $e_2 = \partial/\partial y$. (The key fact is that for any parametrization of this great circle segment, the velocity vector is a multiple of the normal e_3.) Take this frame as the initial value for parallel transport along the great circle segment 2, which sweeps out an arc of length θR on the equator. The "No $(1,2)$-Rotation Rule" implies that at the other endpoint the parallel transported frame is $e_1 = -\partial/\partial z$, $e_2 = (\sin\theta)\,\partial/\partial x + (\cos\theta)\,\partial/\partial y$. Now parallel transport back to p along the great circle segment 3. The "No $(1,2)$-Rotation Rule" implies that at p we arrive at the parallel transported frame $e_1 = (\cos\theta)\,\partial/\partial x - (\sin\theta)\,\partial/\partial y$, $e_2 = (\sin\theta)\,\partial/\partial x + (\cos\theta)\,\partial/\partial y$. So the terminal frame is obtained from the initial frame by rotation through angle θ.

As a preliminary to Remark 10.44, we compute the Gauss curvature of $\Sigma = S^2(R) \subset \mathbb{E}^3$ using the geometric interpretation **(10.20)** of the shape operator. Namely, a normal section at any point of Σ in any direction is a circle of radius R, and from (8.47) its curvature is $\kappa = -1/R$. Therefore, from (10.24) the Gauss curvature of $S^2(R)$ is the constant $1/R^2$.

Remark 10.44. There is much to be learned from extended contemplation of this example.

(1) In the right O_2-torsor $P = \mathcal{B}_{\mathrm{O}}(T_p\Sigma)$ of orthonormal frames at the north pole, there is a simply transitive right action by O_2 and a simply transitive left action by the group $\operatorname{Aut} P$ of right O_2-torsor automorphisms. The terminal frame is obtained from the initial frame by a unique element of either group, and both are rotations. Be sure you know exactly which rotations, and be sure you see the distinction.
(2) Picture the parallel transported moving frame around the spherical triangle as a lift to $\mathcal{B}_{\mathrm{O}}(\Sigma)$ that begins and ends at the fiber over p. The lift is a path that is not closed, except for special values of θ. (Which?) That lifted path is horizontal. Compare to the discussion in Example 5.33.
(3) The area enclosed by the spherical triangle is $A = \theta R^2$.
(4) The Gauss curvature of Σ is the constant function $K = 1/R^2$, so $A = \theta/K$.
(5) The sum of the angles at the vertices of the spherical triangles is $\pi + \theta$, so the excess over the Euclidean plane sum π is $\theta = KA$.
(6) How do (3)–(5) generalize for arbitrary spherical triangles? Identify θ as the rotation obtained by parallel transport around the triangle. Experiment with examples!

Lecture 11: Intrinsic geometry of a surface in a Euclidean 3-space

One important dichotomy in geometry is *intrinsic* vs. *extrinsic*. We have been studying a surface Σ embedded as a submanifold of a 3-dimensional Euclidean space A. The extrinsic geometry uses this embedding to derive structures and theorems about Σ, and the results depend—at least apparently depend—on the embedding. In this lecture we turn to the intrinsic geometry of Σ as an abstract 2-manifold equipped with a Riemannian metric. (If we start with this abstract situation, and we want to study the extrinsic geometry of an embedding, we require that the embedding be *isometric*, i.e., that it preserve inner products.)

The main theorem is that the parallelism along curves of $\Sigma \subset A$ inherited from the global parallelism of A is in fact intrinsic; it can be derived directly from the Riemannian structure on Σ. Recall that the induced parallelism is characterized by the "No $(1,2)$-Rotation Rule", and this Rule uses the global parallelism of A to compute instantaneous rotation. The intrinsicality of this parallelism has Gauss' *Theorema Egregium* as an immediate corollary: The Gauss curvature is intrinsic. (By contrast, the mean curvature depends on the embedding.)

The proof of the main theorem, and its generalization to arbitrary dimensions, is at its heart computational. The parallelism is encoded by a (horizontal) distribution H on the 3-manifold $\mathcal{B}_{\mathrm{O}}(\Sigma)$ of orthonormal frames, and that distribution is produced by a formula, not by a "synthetic" construction. (For an embedded surface $\Sigma \in A$, the distribution H is constructed directly from the global product structure on $\mathcal{B}_{\mathrm{O}}(A)$; see **(10.33)** and **(10.35)**.) This special distribution is the *Levi=Civita connection* on the orthonormal frame bundle $\pi\colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$. It is a fundamental structure in Riemannian geometry, yet remains somewhat mysterious in that one way or another it is only obtained via formulas.

Historically, curvature predates the more primitive notion of parallel transport along curves; from a modern point of view we derive curvature from infinitesimal parallel transport as encoded in a horizontal distribution. The principal curvatures **(10.22)** of an embedded surface $\Sigma \subset A$ were studied by Euler in the 1760s. The product was proved to be intrinsic by Gauss in 1827. He does not discuss parallelism explicitly, though he does treat geodesics. Riemann generalized Gauss curvature to arbitrary dimensions in 1854. Parallel transport along curves was introduced by Levi–Civita in 1917, based on earlier work of Christoffel, and there is contemporaneous work by Schouten and Weyl. The approach using horizontal distributions is due to Ehresmann in 1950.

We illustrate the computational power of moving frames in Example 11.28, which only scratches the surface of the classical theory. In Theorem 11.34 we state the fundamental theorem of surface theory, which roughly states that a surface in Euclidean 3-space is determined up to Euclidean transformations by its induced metric and second fundamental form. That theorem is proved by constructing a map into the right Euc_3-torsor of orthonormal frames in Euclidean space. We conclude with the definition of *geodesic curvature* of a curve on a Riemannian 2-manifold. This leads to the definition of a *geodesic curve* and of a *geodesic motion*.

Intrinsic parallelism from a metric

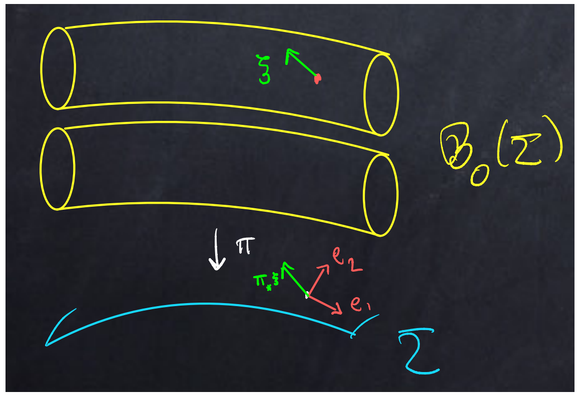

FIGURE 38. The tautological 1-forms θ^1, θ^2

(11.1) *The intrinsic setup.* Let Σ be a Riemannian 2-manifold, and let $\pi\colon \mathcal{B}_O(\Sigma) \to \Sigma$ be its principal O_2-bundle of orthonormal frames. (There is a homework problem to construct this principal bundle for any Riemannian manifold.) A point $f = (p; e_1, e_2) \in \mathcal{B}_O(\Sigma)$ is a point $p \in \Sigma$ and an orthonormal frame e_1, e_2 of $T_p\Sigma$. The 3-manifold $\mathcal{B}_O(\Sigma)$ has a global nonzero vertical vector field $\widehat{\zeta}$ defined in **(9.38)**. It also carries global linearly independent 1-forms θ^1, θ^2 defined as in Remark 8.22. Namely, if $\xi \in T_f\mathcal{B}_O(\Sigma)$ is a tangent vector at $f = (p; e_1, e_2)$, then the equation

$$\pi_*\xi = \theta^i(\xi)e_i \tag{11.2}$$

in $T_p\Sigma$ defines the forms θ^1, θ^2. See Figure 38 for an illustration.

(11.3) *Main theorem and its corollaries.* The horizontal distribution that defines the parallelism we seek is the kernel of a 1-form.

Theorem 11.4. *There exists a unique 1-form ψ on $\mathcal{B}_O(\Sigma)$ that satisfies*

$$d\theta^1 - \psi \wedge \theta^2 = 0 \tag{11.5}$$

$$d\theta^2 + \psi \wedge \theta^1 = 0 \tag{11.6}$$

$$\iota_{\widehat{\zeta}}\psi = \psi(\widehat{\zeta}) = 1 \tag{11.7}$$

Observe that these equations are affine in ψ.

Corollary 11.8. *If $\Sigma \subset A$ is isometrically embedded in a Euclidean 3-space, then $\psi = \overline{\Theta}{}^2_1$.*

Recall from **(9.41)** that $\overline{\Theta}{}^2_1$ is the restriction of the Maurer–Cartan form Θ^2_1 on $\mathcal{B}_O(A)$ to $\mathcal{B}_O(\Sigma)$.

Proof. For $\psi = \overline{\Theta}{}^2_1$, equation (11.7) is satisfied by Lemma 9.45(1); equations (11.5) and (11.6) reduce to (9.56). $\square$

Corollary 11.9 (Gauss). *The Gauss curvature K is intrinsic.*

Proof. Combine (9.53) and Proposition 10.28. $\square$

Corollary 11.10. *If $\Sigma \subset A$ is isometrically embedded in a Euclidean 3-space, then the horizontal distribution $H = \ker \overline{\Theta}_1^2$ is intrinsic.*

Proof. Namely, it is the horizontal distribution $\ker \psi$, and ψ is intrinsic. $\square$

Thus there is an intrinsic parallelism along curves in a Riemannian 2-manifold; see (**10.39**). The horizontal distribution—the Levi–Civita connection—is depicted in Figure 39.

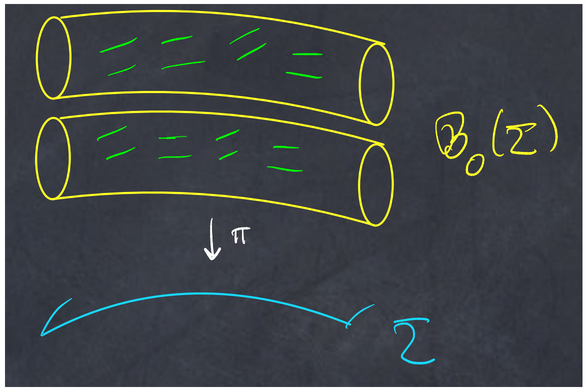

FIGURE 39. The orthonormal frame bundle $\pi\colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$ of the Riemannian 2-manifold Σ and its horizontal distribution: the Levi–Civita connection

(11.11) *A preliminary lemma.* Introduce the matrix notation

$$e = \begin{pmatrix} e_1 \ e_2 \end{pmatrix} \qquad \theta = \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} \qquad \Theta = \begin{pmatrix} 0 & -\psi \\ \psi & 0 \end{pmatrix} \tag{11.12}$$

In this notation, the right hand side of (11.2) is a matrix product:

$$\pi_*\xi = e\,\theta(\xi). \tag{11.13}$$

Lemma 11.14. *For $g \in \mathrm{O}_2$ we have*

$$R_g^*\theta = g^{-1}\theta, \tag{11.15}$$

where $R_g\colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \mathcal{B}_{\mathrm{O}}(\Sigma)$ is the right *action of g.*

Proof. For $f = (p; e) \in \mathcal{B}_{\mathrm{O}}(\Sigma)$ and $\xi \in T_f\mathcal{B}_{\mathrm{O}}(\Sigma)$ we have

$$e\theta(\xi) = \pi_*\bigl((R_g)_*\xi\bigr) = eg\,\theta\bigl((R_g)_*\xi\bigr) = eg\,(R_g^*\theta)(\xi), \tag{11.16}$$

where at the first stage we use $\pi \circ R_g = \pi$, which implies $\pi_*\bigl((R_g)_*\xi\bigr) = \pi_*(\xi)$. The lemma follows by comparing the first and last expressions in (11.16). $\square$

(11.17) *Proof of main theorem.* The proof proceeds in three steps: (1) prove global uniqueness, (2) prove local existence, (3) deduce global existence. Step (3) follows immediately from (1) and (2): uniqueness implies that patching of local solutions leads to a well-defined global solution.

Proof of Theorem 11.4. First we prove uniqueness. The difference δ of two solutions ψ satisfies the linear equations

$$\begin{aligned} \delta\wedge\theta^2 &= 0\\ \delta\wedge\theta^1 &= 0\\ \iota_{\hat{\varsigma}}\delta &= 0 \end{aligned} \tag{11.18}$$

The last equation implies that δ is a linear function on the quotient bundle $T\mathcal{B}_{\mathrm{O}}(\Sigma)/\ker\pi_*\cong\pi^*T\Sigma$ over $\mathcal{B}_{\mathrm{O}}(\Sigma)$. It follows from (11.2) that θ^1,θ^2 is a global basis of these linear functions, and now the first two equations in (11.18) imply $\delta=0$.

For local existence, let $U\subset\Sigma$ be an open subset equipped with a moving frame, i.e., a section σ of $\mathcal{B}_{\mathrm{O}}(U)\to U$. Use σ to construct a local trivialization

$$\begin{aligned} U\times\mathrm{O}_2 &\longrightarrow \mathcal{B}_{\mathrm{O}}(U)\\ p\ ,\ g &\longmapsto \sigma(p)g \end{aligned} \tag{11.19}$$

Let $\underline{\theta}=\begin{pmatrix}\underline{\theta}^1\\ \underline{\theta}^2\end{pmatrix}=\sigma^*(\theta)$ be the moving coframe. We claim there exists a unique $\underline{\psi}\in\Omega^1_U$ such that

$$\begin{aligned} d\underline{\theta}^1-\underline{\psi}\wedge\underline{\theta}^2 &= 0\\ d\underline{\theta}^2+\underline{\psi}\wedge\underline{\theta}^1 &= 0 \end{aligned} \tag{11.20}$$

To see this write

$$\begin{aligned} d\underline{\theta}^1 &= a\,\underline{\theta}^1\wedge\underline{\theta}^2\\ d\underline{\theta}^2 &= b\,\underline{\theta}^1\wedge\underline{\theta}^2 \end{aligned} \tag{11.21}$$

then $\underline{\psi}=-a\,\underline{\theta}^1-b\,\underline{\theta}^2$ is the unique solution to (11.20). Set $\underline{\Theta}=\begin{pmatrix}0&-\underline{\psi}\\ \underline{\psi}&0\end{pmatrix}$.

Under the identification (11.19) the matrix-valued 1-form θ pulls back to $g^{-1}\underline{\theta}$; this follows from Lemma 11.14. Inspired by Lemma 9.45(2), define

$$\Theta=g^{-1}\underline{\Theta}g+g^{-1}dg, \tag{11.22}$$

which under (11.19) is a 1-form on $\mathcal{B}_{\mathrm{O}}(U)$. We claim Θ satisfies (11.5)–(11.7). Recall the matrix notation (11.12), and the corresponding underlined versions. Equation (11.20) in matrix form is $d\underline{\theta}+\underline{\Theta}\wedge\underline{\theta}=0$. Compute using the local trivialization (11.19):

$$\begin{aligned} d\theta\ +\ \Theta\wedge\theta &= d(g^{-1}\underline{\theta})\ +\ (g^{-1}\underline{\Theta}g\ +\ g^{-1}dg)\wedge(g^{-1}\underline{\theta})\\ &= -g^{-1}dg\wedge g^{-1}\underline{\theta}\ +\ g^{-1}d\underline{\theta}\ +\ g^{-1}\underline{\Theta}\wedge\underline{\theta}\ +\ g^{-1}dg\wedge g^{-1}\underline{\theta}\\ &= 0 \end{aligned} \tag{11.23}$$

This proves (11.5)–(11.6).

The vertical vector field $\widehat{\zeta}$ on $\mathcal{B}_{\mathrm{O}}(U)$ corresponds under (11.19) to the vector field on $U \times \mathrm{O}_2$ that projects to the parallel vector field on O_2 defined by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{o}_2$. Thus

$$\iota_{\widehat{\zeta}}\Theta = \iota_{\widehat{\zeta}}(g^{-1}dg) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{11.24}$$

from which $\iota_{\widehat{\zeta}}\psi = 1$. This proves (11.7).

Global existence follows, as explained at the beginning of **(11.17)**. This completes the proof of Theorem 11.4. ∎

Remark 11.25. Theorem 11.4 asserts the existence of a canonical *Levi–Civita connection* on the orthonormal frame bundle $\mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$ of a Riemannian 2-manifold. Later we will prove the analog for a general Riemannian manifold. The Levi–Civita connection is the horizontal distribution $\ker \psi \subset T\mathcal{B}_{\mathrm{O}}(\Sigma)$, which is O_2-invariant.

Local computations

Let Σ be a Riemannian 2-manifold, and suppose $U \subset \Sigma$ is an open subset. A(n orthonormal) *moving frame* on U is a section of the orthonormal frame bundle $\mathcal{B}_{\mathrm{O}}(U) \to U$. So at each $p \in U$ it is a choice of orthonormal frame $e_1(p), e_2(p)$ of $T_pU = T_p\Sigma$. There is a dual coframe of 1-forms $\theta^1, \theta^2 \in \Omega^1_U$. The argument in (11.20)–(11.21) proves that there is a unique Θ^2_1 such that

$$\begin{aligned} d\theta^1 - \Theta^2_1 \wedge \theta^2 &= 0 \\ d\theta^2 + \Theta^2_1 \wedge \theta^1 &= 0 \end{aligned} \tag{11.26}$$

If $\sigma\colon U \to \mathcal{B}_{\mathrm{O}}(U)$ is the section that defines the moving frame e_1, e_2, then these forms $\theta^1, \theta^2, \Theta^2_1 \in \Omega^1_U$ are the pullbacks via σ of the corresponding forms on the total space $\mathcal{B}_{\mathrm{O}}(U)$ of the orthonormal frame bundle. But here, for convenience, we drop the 'σ^*'. The Gauss curvature $K \in \Omega^0_U$ satisfies

$$d\Theta^2_1 = -K\theta^1 \wedge \theta^2. \tag{11.27}$$

The "method of moving frames" is: (1) choose a moving frame e_1, e_2 convenient for your problem, (2) solve for Θ^2_1 that satisfies (11.26), and (3) compute the Gauss curvature using (11.27). (This is for an intrinsic problem on a Riemannian 2-manifold. A similar method works for surfaces embedded in a Euclidean 3-space, and then one can also compute the second fundamental form.)

Example 11.28. *Geodesic polar coordinates* on a Riemannian 2-manifold are local coordinates r, ϕ with respect to which the Riemannian metric is

$$dr^2 + G(r,\phi)d\phi^2 \tag{11.29}$$

for some function G. An example is usual polar coordinates on a Euclidean plane, in which case $G = r^2$. As in that case, the coordinate functions are defined for $r \in (0, C)$ for some $C \in \mathbb{R}^{>0}$ and

$\phi \in (0, 2\pi)$. The differential form $d\phi$ extends over the ray $\phi = 0$, so the computation that follows is valid in a punctured disk of radius R. Define the orthonormal coframing θ^1, θ^2 as follows:

$$\begin{aligned} \theta^1 &= dr & d\theta^1 &= 0 \\ \theta^2 &= \sqrt{G}d\phi & d\theta^2 &= \frac{\partial\sqrt{G}}{\partial r} dr \wedge d\phi \\ & & &= \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial r}\theta^1 \wedge \theta^2 \end{aligned} \tag{11.30}$$

We easily compute from (11.26) that

$$\Theta^2_1 = (\sqrt{G})_r \, d\phi, \tag{11.31}$$

where we use the subscript notation for the partial derivative. Then

$$d\Theta^2_1 = (\sqrt{G})_{rr} \, dr \wedge d\phi, \tag{11.32}$$

so from (11.27) we find

$$K = -\frac{(\sqrt{G})_{rr}}{\sqrt{G}}. \tag{11.33}$$

This is a very useful classical formula for the curvature. For a Euclidean plane, $G = r^2$ and so $K = 0$ as it must. For a sphere of radius R, suitably spherical coordinates provide geodesic polar coordinates, and then $G = R^2 \sin^2(r/R)$. It follows that the Gauss curvature is the constant $1/R^2$. For the hyperbolic plane, $G = R^2 \sinh^2(r/R)$ and now the Gauss curvature is $-1/R^2$. One can show that, locally, any constant Gauss curvature metric has this form.

Fundamental theorem of surface theory

There is an analog of Theorem 9.7 for surfaces embedded in a Euclidean 3-space. It too is a corollary of Theorem 9.2, and so prescribes six Maurer–Cartan forms subject to the six Maurer–Cartan equations. We did not develop these equations in their most intrinsic global form, so the following statement is a bit schematic. The proof, which we do not give here, is similar to that of Theorem 9.7.

Theorem 11.34 (rough statement)**.** *Let A be a Euclidean 3-space, and suppose Σ is a 2-dimensional Riemannian manifold. Let* II *denote a symmetric bilinear form on $T\Sigma$. Suppose the metric and* II *satisfy the Gauss and Codazzi-Mainardi equations.*

(1) *If Σ is simply connected, there exists an isometric immersion $f\colon \Sigma \to A$ such that the associated second fundamental form is* II.

(2) *If Σ is connected, and $f_1, f_2\colon \Sigma \to A$ are isometric immersions that induce the second fundamental form* II*, then there exists a Euclidean transformation $\varphi \in \mathrm{Aut}_{\mathrm{O}}(A)$ such that $f_2 = \varphi \circ f_1$.*

See **(10.27)** and **(10.30)** for the Gauss and Codazzi–Mainardi equations.

Remark 11.35. Theorem 11.4 is used in the proof. Namely, the ψ computed intrinsically is the form $\overline{\Theta}_1^2$ we must specify to set up the system of equations for a map $\mathcal{B}_{\mathrm{O}}(\Sigma) \to \mathcal{B}_{\mathrm{O}}(A)$. The forms $\overline{\theta}^1, \overline{\theta}^2$ are intrinsic as well (11.2). We set $\overline{\theta}^3 = 0$ in light of Proposition 9.52(2). The forms $\overline{\Theta}_1^3, \overline{\Theta}_2^3$ are defined from II; see Proposition 10.2, (10.8), and (10.13). The Maurer–Cartan equation (9.4) in Theorem 9.2 is satisfied in view of the Gauss and Codazzi–Mainardi equations.

Curves on surfaces

Let Σ be a Riemannian 2-manifold and $C \subset \Sigma$ a cooriented 1-dimensional submanifold. In Lecture 8 we studied the special case in which Σ is a 2-dimensional Euclidean space, and in particular we defined the curvature $\kappa\colon C \to \mathbb{R}$; see Lemma 8.35(3). Now we generalize to an arbitrary Riemannian 2-manifold Σ, following the version in **(9.10)**.

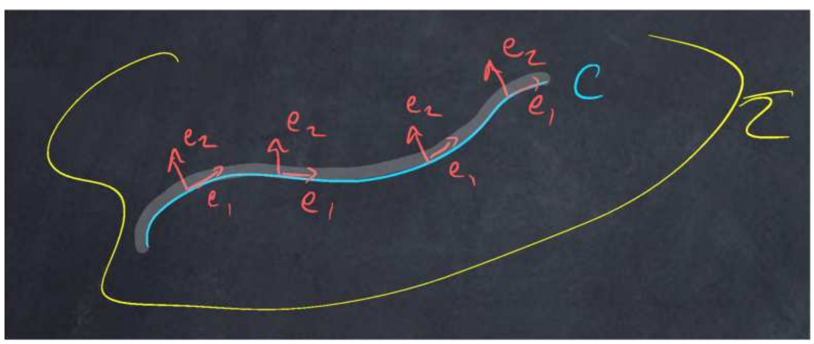

FIGURE 40. An adapted moving frame along C

(11.36) *Adapted frames.* Let $\mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$ be the principal O_2-bundle of orthonormal frames of Σ, and $\mathcal{B}_{\mathrm{O}}(\Sigma)\big|_C \to C$ its restriction to a principal O_2-bundle over C. There is a sub principal O_1-bundle

$$\mathcal{B}_{\mathrm{O}}(C) = \Big\{ (p; e_1, e_2) \in \mathcal{B}_{\mathrm{O}}(\Sigma)\big|_C \colon e_2 \text{ is the unit cooriented normal vector at } p \Big\} \longrightarrow C. \tag{11.37}$$

The bundle $\mathcal{B}_{\mathrm{O}}(C) \to C$ is naturally isomorphic to the principal O_1-bundle of orthonormal frames of C. This is a double cover, which can also be identified with the orientation double cover of C.

(11.38) *Geodesic curvature.* It is customary to denote the 1-form ψ in Theorem 11.4 as Θ_1^2. Restrict $\theta^1, \theta^2, \Theta_1^2$ along the inclusion $\mathcal{B}_{\mathrm{O}}(C) \hookrightarrow \mathcal{B}_{\mathrm{O}}(\Sigma)$ to obtain 1-forms $\overline{\theta}^1, \overline{\theta}^2, \overline{\Theta}_1^2$. Let $\rho \in \mathrm{O}_1$ be the non-identity element. Then $R_\rho \colon \mathcal{B}_{\mathrm{O}}(C) \to \mathcal{B}_{\mathrm{O}}(C)$ is the non-identity deck transformation of the double cover $\mathcal{B}_{\mathrm{O}}(C) \to C$. The following is Lemma 8.35 in this situation.

Lemma 11.39.

(1) $\overline{\theta}^1$ *is a nowhere zero 1-form on* $\mathcal{B}_{\mathrm{O}}(C)$ *that satisfies* $R_\rho^*\overline{\theta}^1 = -\overline{\theta}^1$.
(2) $\overline{\theta}^2 = 0$.
(3) $\overline{\Theta}_1^2$ *satisfies* $R_\rho^*\overline{\Theta}_1^2 = -\overline{\Theta}_1^2$, *and* $\overline{\Theta}_1^2 = (\pi^*\kappa_C)\overline{\theta}^1$ *for some function* $\kappa_C \colon C \to \mathbb{R}$.

The κ_C is the *geodesic curvature* of $C \subset \Sigma$. It reverses sign if we reverse the coorientation. The proof involves no new ideas, but note that $\overline{\Theta}_1^2 = \hat{\kappa}\overline{\theta}^1$ for some $\hat{\kappa} \colon \mathcal{B}_{\mathrm{O}}(C) \to \mathbb{R}$, since $\overline{\theta}^1$ is nowhere zero. To descend $\hat{\kappa}$ to C, use the fact that the deck transformation R_ρ reverses both $\overline{\theta}^1$ and $\overline{\Theta}_1^2$.

(11.40) *Geodesics.* Curves with vanishing geodesic curvature are special.

Definition 11.41. Let $C \subset \Sigma$ be a 1-dimensional submanifold of the 2-dimensional Riemannian manifold Σ. If $\kappa_C = 0$, then we say $C \subset \Sigma$ is a *geodesic curve*.

Fix a choice of adapted moving frame e_1, e_2 along a subset of C, which is equivalent to choosing an orientation of C. This defines a map

$$f \colon C \longrightarrow \mathcal{B}_{\mathrm{O}}(C) \longrightarrow \mathcal{B}_{\mathrm{O}}(\Sigma). \tag{11.42}$$

It follows that C is a geodesic iff f is an integral curve in the horizontal distribution $H = \ker \Theta_1^2$ on $\mathcal{B}_{\mathrm{O}}(\Sigma)$. (It is not an integral manifold since $\dim C = 1$ and an integral manifold has dimension 2. The condition is $f_*(T_pC) \subset H_{f(p)}$ for all $p \in C$.)

Remark 11.43. The parallelism along curves determines a differentiation on vector fields along C. That derivative of the unit tangent vector e_1 is the acceleration of a unit speed parametrization, and is easily determined to be $\kappa_C e_2$. So a geodesic has no (intrinsic) acceleration in a unit speed parametrization. In general, the acceleration is the geodesic curvature times the normal vector.

Lecture 12: Horizontal distributions on fiber bundles

In this lecture we discuss the geometry of general fiber bundles. These occur throughout differential geometry as well as more generally in topology. In the next lecture we take up Steenrod's approach, which puts the emphasis on symmetry groups. The fundamental object is a principal bundle, and other fiber bundles are associated to it: the geometry of the principal bundle simultaneously controls the geometry of all associated fiber bundles. The theory of fiber bundles, following Élie Cartan, was exposited by Charles Ehresmann in *Les connexions infinitésimales dans un espace fibré différentiable*. A horizontal distribution on a fiber bundle is called an *Ehresmann connection*.

We begin with basic definitions and examples. Fiber bundles encode mathematical objects parametrized by a smooth manifold. There is an important dichotomy between *intrinsic* objects and *extrinsic* objects. The examples we give are intrinsic, associated to the first-order geometry of the tangent bundle. We then turn to general constructions: pullback and fiber product.

The main topic is parallelism in the context of general fiber bundles. Infinitesimal parallelism is given by a horizontal distribution. This defines a notion of horizontal/flat/parallel sections, but even locally these may not exist: the Frobenius tensor measure the obstruction. Along curves there are parallel sections, but in general their domain of definition may not be uniform in the initial data, and so in general there is no *parallel transport*. (We first encountered parallel transport in **(10.39)**.)

Reminder on fiber bundles

For convenience, we repeat some definitions from differential topology.

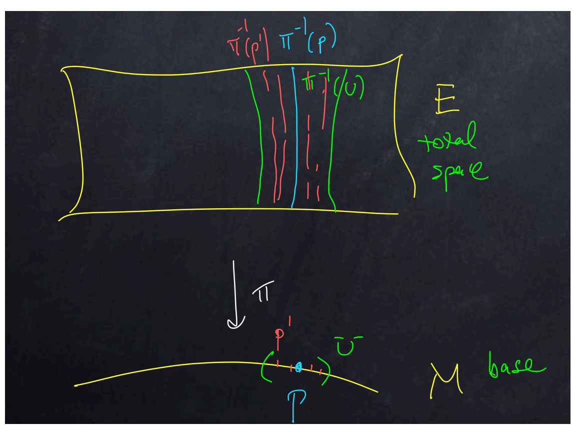

FIGURE 41. A fiber bundle with local trivialization about p

(12.1) *Definitions.*

Definition 12.2. Let $\pi\colon E \to X$ be a surjective smooth map of smooth manifolds.

(1) We say π is a *fiber bundle* if for all $x \in X$ there exists an open neighborhood $U \subset X$ about x and a diffeomorphism $\varphi\colon U \times \pi^{-1}(x) \to \pi^{-1}(U)$ such that the diagram

$$\begin{tikzcd} U \times \pi^{-1}(x) \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd} \tag{12.3}$$

commutes.

(2) Let F be a nonempty smooth manifold. We say π is a *fiber bundle with fiber F* if for all $x \in X$ there exists an open neighborhood $U \subset X$ about x and a diffeomorphism $\varphi\colon U \times F \to \pi^{-1}(U)$ such that the diagram

$$\begin{tikzcd} U \times F \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}_1"'] & & \pi^{-1}(U) \arrow[dl, "\pi"] \\ & U & \end{tikzcd} \tag{12.4}$$

commutes.

Remark 12.5.

(1) The domain E of π is called the *total space* and the codomain X is called the *base* of the fiber bundle π.

(2) A product bundle $\mathrm{pr}_1\colon X \times F \to X$ is called *trivial*, though the term *product bundle* is more informative. We denote the product bundle as $\underline{F} \to X$. Heuristically, a fiber bundle over X is a function from X to some "space" of smooth manifolds. With this heuristic, a trivial bundle is a constant function, and the existence of *local trivializations* (12.3) means a fiber bundle is—heuristically—a locally constant function.

(3) A fiber bundle is a surjective submersion, but not every surjective submersion is a fiber bundle.

(4) The fibers of π might carry extra structure, such as a vector space structure, a Lie group structure, a Lie algebra structure, a right G-torsor structure, etc. We ask that such structures be smooth over the base, which is formalized by requiring local trivializations (12.3) to preserve such structures fiberwise.

(5) Concepts/constructions with single manifolds typically extend over a base, that is, to concepts/constructions for fiber bundles. Hence a smooth map of fiber bundles is a smooth map of the total spaces that commutes with the projections over a common base:

$$\begin{tikzcd} E' \arrow[rr, "\varphi"] \arrow[dr, "\pi'"'] & & E'' \arrow[dl, "\pi''"] \\ & X & \end{tikzcd} \tag{12.6}$$

The local trivializations (12.3) and (12.4) are examples of smooth maps of fiber bundles.

Examples of fiber bundles

Let X be a smooth manifold. The following fiber bundles are *intrinsic*, which loosely means they are naturally constructed from the tangential geometry.[15]

Example 12.7. The tangent bundle $TX \to X$ is a fiber bundle of real vector spaces, i.e., a real vector bundle.

Example 12.8. The (fiberwise) projectivization $\mathbb{P}(TX) \to X$ is a fiber bundle whose fibers are diffeomorphic to real projective spaces. But we do not allow arbitrary diffeomorphisms of real projective spaces in the local trivializations, only projective transformations. In other words, projective geometry makes sense in the fibers of $\mathbb{P}(TX) \to X$. The restriction on maps used to glue together a fiber bundle from a trivial bundle are imposed by realizing the fiber bundle as an associated bundle to a principal bundle, an idea we develop in the next lecture.

Example 12.9. Variations of Example 12.8 include the Grassmannian bundle $\mathrm{Gr}_k(TX) \to X$, whose fiber over $x \in X$ is the Grassmannian of k-dimensional subspaces of the tangent space T_xX; flag bundles such as $\mathrm{F}\ell_{2,4}(TX) \to X$, whose fiber at x is the flag manifold of sequences of subspaces $0 \subset W_2 \subset W_4 \subset T_xX$ of the indicated dimension; the Stiefel bundle $\mathrm{St}_k(TX) \to X$, whose fiber at x is the Stiefel manifold of injections $\mathbb{R}^k \to T_xX$; etc.

Example 12.10. The fiber bundle $\mathrm{End}(TX) \to X$ is a bundle of associative algebras; the fiber at $x \in X$ is the algebra of endomorphisms of T_xX. It is a bundle of Lie algebras via the commutator (Remark 3.71).

Example 12.11. The fiber bundle $\mathrm{Met}(TX) \to X$ has fiber at x the space of inner products on T_xX. The fiber $\mathrm{Met}(TX)_x$ is a convex subset of the vector space $\mathrm{Sym}^2(T_x^*X)$ of symmetric bilinear forms $T_xX \times T_xX \to \mathbb{R}$, so the fiber bundle $\mathrm{Met}(TX) \to X$ is a convex subbundle of the vector bundle $\mathrm{Sym}^2(T^*X) \to X$. Sections of $\mathrm{Met}(X) \to X$ are Riemannian metrics on X.

Example 12.12. There is a fiber bundle $AX \to X$ whose fiber at $x \in X$ is the trivial affine space $A_xX = T_xX$ over the vector space T_xX. So $AX \to X$ is an *affine bundle* over the vector bundle $TX \to X$.

Operations on fiber bundles

(12.13) *Pullback.* Pullback is also called *base change.* Consider the diagram

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd} \tag{12.14}$$

in which π is a fiber bundle and f is a smooth map of smooth manifolds.

[15] A more precise definition, at least for first-order tangential geometry, is that an intrinsic fiber bundle is a fiber bundle associated to the principal $\mathrm{GL}_n\mathbb{R}$-bundle of frames. (We spell out the underlying definitions in the next lecture.)

Definition 12.15. The *pullback bundle* π' in

(12.16)
$$\begin{tikzcd} E' \arrow[r, dashed, "\tilde{f}"] \arrow[d, dashed, "\pi'"'] & E \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

is defined by setting

(12.17)
$$E' = \bigl\{(x', e) \in X' \times E : f(x') = \pi(e)\bigr\}$$

and letting $\pi', \tilde{f}$ be the restrictions of the projections of $X' \times E$ onto its factors.

The notation $f^*E \to X'$ or $f^{-1}E \to X'$ is often used for the pullback. The fiber of π' at $x' \in X'$ is the naturally the fiber of π at $f(x')$. In other words, if we heuristically view the bundle as a function to a "space" of manifolds, then pullback of bundles is the usual pullback of functions.

Remark 12.18. We leave the reader to prove that E' is a smooth manifold. Hint: Prove that the map $f \times \pi \colon X' \times E \to X \times X$ is transverse to the diagonal $\Delta \subset X \times X$.

Proposition 12.19. *π' is a fiber bundle.*

Proof. The pullback of a local trivialization of π is a local trivialization of π'. In more detail, let $x' \in X'$, choose a local trivialization φ of π in some neighborhood U of $f(x')$, and pullback:

(12.20)
$$\begin{tikzcd} \underline{(\pi')^{-1}(x')} \arrow[rr, dashed, "\varphi'"] \arrow[rrrr, dashed, bend left] \arrow[dr, dashed, "\pi'"'] & & (\pi')^{-1}\bigl(f^{-1}(U)\bigr) \arrow[dl, dashed] \arrow[rrrr, dashed, bend left] & & \underline{\pi^{-1}\bigl(f(x')\bigr)} \arrow[rr, "\varphi"] \arrow[dr] & & \pi^{-1}(U) \arrow[dl] \\ & f^{-1}(U) \arrow[rrrr, "f"] & & & & U & \end{tikzcd}$$

The key observation is that the pullback of a product bundle is a product bundle. $\square$

(12.21) *Fiber product.* The following is a special case of pullback.

Definition 12.22. Suppose $\pi' \colon E' \to X$ and $\pi'' \colon E'' \to X$ are fiber bundles over the same base X. The *fiber product* of π' and π'' is the composition $q = \pi'' \circ p' = \pi' \circ p''$ in the pullback

(12.23)
$$\begin{tikzcd} E \arrow[r, dashed, "p''"] \arrow[d, dashed, "p'"'] \arrow[dr, dashed, "q"] & E' \arrow[d, "\pi'"] \\ E'' \arrow[r, "\pi''"] & X \end{tikzcd}$$

The fiber of q at $x \in X$ is $E'_x \times E''_x$. The fiber product of fiber bundles is a parametrized version of the Cartesian product of smooth manifolds. The fiber product is often written as $q\colon E' \times_X E'' \to X$.

Proposition 12.24. *The map $q\colon E \to X$ is a fiber bundle.*

Proof. Pullback local trivializations as in (12.20). The key point is that if $U \subset X$ is an open set on which there are simultaneous local trivializations of π' and π'' with constant fibers F' and F'', respectively, then, the local model over $U' = f^{-1}(U)$ is

(12.25)
$$\begin{tikzcd} & \underline{F' \times F''} \arrow[dl] \arrow[dr] \arrow[dd] & \\ \underline{F''} \arrow[dr] & & \underline{F'} \arrow[dl] \\ & U' & \end{tikzcd}$$

We leave the reader to fill out (12.25) to a diagram analogous to (12.20). ☐

Example 12.26. Building on Example 12.12, the fiber of $AX \times_X AX \to X$ over $x \in X$ is an ordered pair of points in T_xX. There is a map of fiber bundles

(12.27)
$$\begin{tikzcd} AX \times_X AX \arrow[rr, "-"] \arrow[dr] & & TX \arrow[dl] \\ & X & \end{tikzcd}$$

that computes the displacement vector between two points in the affine tangent space.

(12.28) *Existence of global sections.* Recall that a section of a fiber bundle $\pi\colon E \to X$ is a right inverse of π, i.e., a smooth map $s\colon X \to E$ such that $\pi \circ s = \mathrm{id}_X$. We speak of local sections defined on open subsets of X, in which case a section of π is termed a 'global section'. A fiber bundle need not admit a global section: the fiber bundle $\pi\colon E \to S^2$ whose total space is TS^2 with the zero section removed has no global section. The following general positive result is often useful. Recall that an *affine bundle* is a fiber bundle of affine spaces.

Proposition 12.29. *Let X be a smooth manifold, let $\pi\colon E \to X$ be an affine bundle, and suppose $\pi'\colon E' \to X$ is a subbundle of fiberwise convex subsets. Then π' admits a global section.*

The subbundle hypothesis means that there is an injective map $\varphi\colon E' \to E$ such that $\pi \circ \varphi = \pi'$.

Proof. Locally a fiber bundle is a product, hence locally it admits sections. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X such that each U_α is equipped with a section $s'_\alpha\colon U_\alpha \to E'\big|_{U_\alpha}$ of $\pi'\big|_{(\pi')^{-1}(U_\alpha)}$. Let $\{\rho_\alpha\}_{\alpha \in A}$ be a partition of unity subordinate to the cover $\{U_\alpha\}_{\alpha \in A}$. Then $\sum_\alpha \rho_\alpha s'_\alpha$ is a global section of π'. ☐

Remark 12.30.

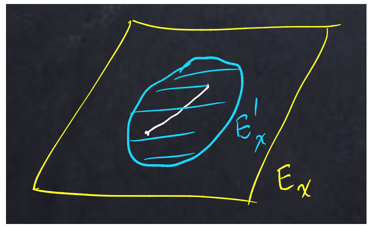

FIGURE 42. A convex subbundle of an affine bundle

(1) Convexity implies that the weighted average $\sum_\alpha \rho_\alpha s'_\alpha$ lies in E'.
(2) As a special case of Proposition 12.29, an affine bundle admits a global section.

Example 12.31. The fiber bundle $\mathrm{Met}(TX) \to X$ in Example 12.11 admits global sections. In other words, a smooth manifold carries Riemannian metrics.

Horizontal splittings (Ehresmann connections)

(12.32) *The relative tangent bundle.* Let $\pi\colon E \to X$ be a fiber bundle. The differential of π is surjective, so its kernel is a subbundle of $TE \to E$.

Definition 12.33. The kernel $\ker d\pi = \ker \pi_* = T\pi = T(E/X) \to X$ is called the *relative tangent bundle* to π.

Note that for $e \in E$ we have

$$T(E/X)_e = T_e(E_e) \subset T_eE, \tag{12.34}$$

where $E_e = \pi^{-1}\bigl(\pi(e)\bigr)$ is the fiber of π through e.

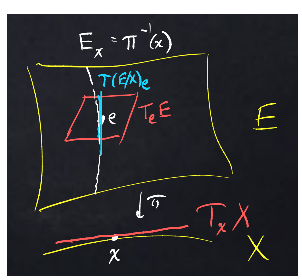

FIGURE 43. The relative tangent space is the tangent space to the fiber

The relative tangent bundle is the sub in a short exact sequence of vector bundles over E:

$$0 \longrightarrow T(E/X) \longrightarrow TE \xrightarrow{\pi_*} \pi^*TX \longrightarrow 0 \tag{12.35}$$

(12.36) *Splittings of a short exact sequence.* Let V, V', V'' be a vector spaces and

$$0 \longrightarrow V' \xrightarrow{\ i\ } V \xrightarrow{\ j\ } V'' \longrightarrow 0 \tag{12.37}$$

a short exact sequence of linear maps.

Definition 12.38. A *splitting* of (12.37) is a right inverse $s\colon V'' \to V$ to j.

In other words,

$$j \circ s = \mathrm{id}_{V''}\,. \tag{12.39}$$

A splitting is equivalent to a left inverse to i, and a splitting is equivalent to an expression of V as a direct sum: $V = s(V'') \oplus i(V')$. Splittings exist since linear subspaces have linear complements. Observe that (12.39) is an affine equation in s: the left hand side is linear and the right hand side is constant. From this we deduce that splittings of (12.37) form an affine space over $\mathrm{Hom}(V'', V')$.

Passing to smooth families of short exact sequences of vector spaces, the existence of splittings is an immediate corollary of Proposition 12.29.

Proposition 12.40. *Let X be a smooth manifold; let $\pi\colon E \to X$, $\pi'\colon E' \to X$, $\pi''\colon E'' \to X$ be vector bundles over X; and suppose*

$$0 \longrightarrow E' \xrightarrow{\ i\ } E \xrightarrow{\ j\ } E'' \longrightarrow 0 \tag{12.41}$$

is a short exact sequence of linear maps. Then there exists a splitting $s\colon E'' \to E$.

(12.42) *Horizontal distributions.* As depicted in Figure 43, we call $T(E/X) \to X$ the *vertical* tangent bundle.

Definition 12.43. Let $\pi\colon E \to X$ be a fiber bundle. A *horizontal distribution* on π is a splitting of (12.35).

A splitting is a linear map σ in

$$\begin{tikzcd} 0 \arrow[r] & T(E/X) \arrow[r] & TE \arrow[r, shift left, "\pi_*"] & \pi^*TX \arrow[l, shift left, "\sigma"] \arrow[r] & 0 \end{tikzcd} \tag{12.44}$$

such that $\pi_* \circ \sigma = \mathrm{id}_{\pi^*TX}$. The image of σ is a distribution $H \subset TE$, which we depict in Figure 44 as horizontal. The restriction

$$\pi_*\big|_H\colon H \longrightarrow \pi^*TX \tag{12.45}$$

is an isomorphism of vector bundles over E with inverse σ. The map σ takes a tangent vector to X to its *horizontal lift* in E.

Proposition 12.46. *Let $\pi\colon E \to X$ be a fiber bundle. Then π admits a horizontal distribution.*

Proof. This is a direct corollary of Proposition 12.40. $\square$

Remark 12.47. If $\pi\colon E \to X$ is a fiber bundle with *discrete* fibers—a covering map—then there is a unique horizontal distribution $H = TE$ on π.

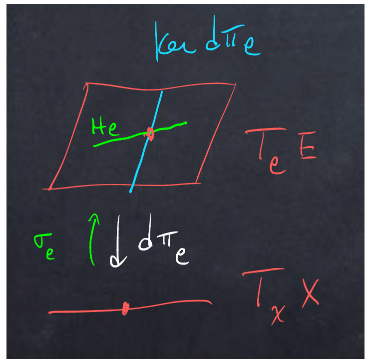

FIGURE 44. The splitting σ and horizontal distribution H

(12.48) *Pullback of a horizontal distribution.* Horizontal distributions play well with base change.

Proposition 12.49. *Let*

(12.50)
$$\begin{tikzcd} E' \arrow[r, dashed, "\tilde{f}"] \arrow[d, dashed, "\pi'"'] & E \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

be a pullback diagram of fiber bundles, and suppose $H \subset TE$ is a horizontal distribution. Then $H' = (d\tilde{f})^{-1}(H) \subset TE'$ is a horizontal distribution on π'.

Proof. Since (12.50) is a pullback, it follows that the right square in the commutative diagram

(12.51)
$$\begin{tikzcd} 0 \arrow[r] & T(E'/X) \arrow[r] \arrow[d, "\tilde{f}_*"] & TE' \arrow[r, "\pi'_*"] \arrow[d, "\tilde{f}_*"] & \pi'^*TX' \arrow[r] \arrow[d, "f_*"] & 0 \\ 0 \arrow[r] & T(E/X) \arrow[r] & TE \arrow[r, "\pi_*"] & \pi^*TX \arrow[r] & 0 \end{tikzcd}$$

is also a pullback. (We leave the proof to a homework exercise.) Then given $\xi' \in \pi'^*TX'$, there exists a unique $\eta' \in TE'$ such that

(12.52)
$$\begin{aligned} \pi'_*\eta' &= \xi' \\ \tilde{f}_*\eta' &= \sigma(f_*\xi') \end{aligned}$$

These equations produce a right inverse to π'_* in (12.51) with image $(d\tilde{f})^{-1}(H)$. □

(12.53) *Flat sections.* A horizontal distribution is a first order differential equation for a section.

Definition 12.54. Let $\pi\colon E\to X$ be a fiber bundle, $H\subset TE$ a horizontal distribution, $f\colon X'\to X$ a smooth map, and

$$\begin{tikzcd} E' \arrow[r, dashed, "\tilde f"] \arrow[d, dashed, "\pi'"'] & E \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd} \tag{12.55}$$

the pullback of f and π.

(1) A section $s\colon X\to E$ of π is *horizontal/flat/parallel* if it is an integral manifold of H.
(2) A *horizontal lift* of f is a flat section of π' relative to the pullback horizontal distribution H'.

In particular, if s is flat then $s_*(T_xX)=H_{s(x)}$ for all $x\in X$.

Recall from Theorem 5.41 that the Frobenius tensor is a local obstruction to the existence of integral manifolds, hence a local obstruction to a flat section. In this case the Frobenius tensor of H on E is a skew-symmetric bilinear map

$$\phi_H\colon H\times H\longrightarrow T(E/X). \tag{12.56}$$

It maps two horizontal vectors to a vertical vector. Alternatively, it is a map

$$\phi_H\colon \pi^*(TX\times TX)\longrightarrow T(E/X). \tag{12.57}$$

In the case of connections on principal bundles, this map descends to the base X, but in general there is no such descent.

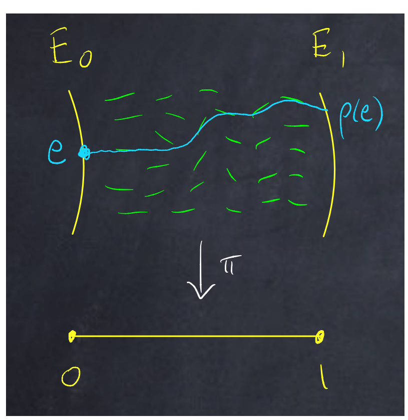

FIGURE 45. Parallel transport using a horizontal distribution

(12.58) *The one-dimensional case; parallel transport.* If $\dim X = 1$ the Frobenius tensor vanishes, and so local flat sections exist. Furthermore, for each $e \in E_x$ there is a unique local section s defined in some maximal open neighborhood $J_{\max}(e) \subset X$ of $x = \pi(e)$ such that $s(x) = e$. If $X = [0,1]$ is a closed interval, then we would like to use flat sections to define a *parallel transport* map (Figure 45)

$$\rho_H \colon E_0 \longrightarrow E_1. \tag{12.59}$$

Namely, for $e \in E_0$ we would let s_e denote the flat section with $s_e(0) = e$ and define $\rho_H(e) = s_e(1)$. That works if $J_{\max}(e) = [0,1]$ for all $e \in E_0$, in which case ρ_H is defined and is a diffeomorphism. If we begin with a fiber bundle over any X, and we equip it with a horizontal distribution, then in a favorable case we can use (12.59) to define parallelism along curves in this fiber bundle, as in **(10.39)**. However, it is possible that $\bigcap_{e\in E_0} J_{\max}(e) = \{0\}$, in which case parallel transport $E_0 \to E_t$ is not defined for *any* $t > 0$.

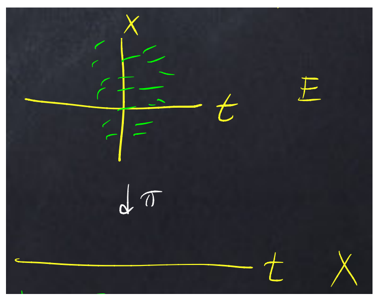

FIGURE 46. A horizontal distribution on $E = \underline{\mathbb{R}} \to \mathbb{R}$

Example 12.60. Let $E = \underline{\mathbb{R}} = \mathbb{R}_t \times \mathbb{R}_x \xrightarrow{\pi} \mathbb{R}_t$ be the product fiber bundle over $\mathbb{R}$. For any horizontal distribution H, the space $H_{t,x}$ is the graph of a linear function $\mathbb{R} \to \mathbb{R}$ that is multiplication by $F(t,x)$ for some smooth $F\colon \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$; see Figure 46. Then a section $s\colon \mathbb{R}_t \to \mathbb{R}_x$ is flat iff s satisfies the ordinary differential equation

$$\frac{ds}{dt}(t) = F\bigl(t, s(t)\bigr). \tag{12.61}$$

For $x_0 \in \mathbb{R}$, the maximal domain of definition $J_{\max}(x_0)$ of a solution with $s(0) = x_0$ is an open interval in $\mathbb{R}$ that contains 0. For

$$F(t,x) = x^2 \tag{12.62}$$

we do indeed find $\bigcap_{x_0\in\mathbb{R}} J_{\max}(x_0) = \{0\}$.

Remark 12.63. The failure in Example 12.60 is due to a lack of uniformity of the horizontal distribution in each fiber. If the fiber bundle $\pi\colon E \to X$ is *proper*, then uniformity is achieved and parallel transport exists. Another favorable situation is that a Lie group G acts on the total space E and π is a quotient map for the action. In that case we can achieve uniformity by demanding that the horizontal distribution be G-invariant. This is precisely a connection on a principal bundle. Such a connection induces a horizontal distribution on every associated fiber bundle, and the parallel transport on the principal bundle induces parallel transport on associated fiber bundles. We study the geometry of principal bundles and their connections in the next few lectures.

Lecture 13: Principal bundles and associated bundles

The general Definition 12.2(2) of a fiber bundle with fiber a smooth manifold F allows arbitrary diffeomorphisms of F for local trivializations. In Remark 12.5(4) we pointed out that the fibers of a fiber bundle often carry a geometric structure that should be preserved by local trivializations. Following Klein's philosophy **(1.34)** this means that the model fiber is not merely a smooth manifold F, but rather the model is a smooth action $G \circlearrowright F$ of a Lie group G on a smooth manifold F. This leads us to the study of principal G-bundles, which we take up in this lecture. From a principal G-bundle $\pi\colon P \to X$ we construct an *associated* fiber bundle with fiber F by *mixing* the right G-action on P with the left G-action on F. Fiber bundles associated to a principal bundle have a richer theory than of general fiber bundles. This is particularly true for horizontal distributions, which we take up in the next lecture. The notion of a fiber bundle associated to a principal bundle is due to Ehresmann. Steenrod's book *The Topology of Fiber Bundles* is a classic.

We have emphasized throughout the role of torsors over a Lie group: affine spaces (Definition 1.1), bases or frames (Definition 1.26), and bundles of frames in the geometry of curves and surfaces (Lectures 8–10). Recall too the convention (Remark 1.29(2)) that *right* group actions are *structural*, and so we use right actions for principal bundles. Some generalities on right torsors over Lie groups appear in **(7.31)**–**(8.24)** *passim*; they are worth a review before proceeding further.

The total space P of a principal G-bundle $\pi\colon P \to X$ carries a free right G-action. Given a smooth manifold P with a free right G-action, the quotient map $\pi\colon P \to P/G$ may be a principal G-bundle. If so, certainly the quotient P/G must carry a smooth manifold structure. In Theorem 13.9 we construct one assuming that the G-action is *proper*, in which case π *is* a principal bundle.

We give many examples of principal bundles and associated fiber bundles.

We conclude with a discussion of change of structure group. A particular case of this concerns symmetry types, as introduced in Definition 1.37. There we used the mixing construction to define a geometric structure on an affine space (Definition 1.54). Now, in Definition 13.52, we define a geometric structure on a smooth manifold.

Principal bundles

(13.1) *Definition.* For convenience, we repeat Definition 9.33.

Definition 13.2. Let X be a smooth manifold, and let G be a Lie group. A *principal G-bundle* over X is a fiber bundle $\pi\colon P \to X$ whose fibers are right G-torsors.

In terms of Definition 12.2, a principal bundle is a fiber bundle with fiber $F = G_G$, the model right G-torsor which is G with the G-action of right multiplication. Sometimes the notations

$$G \longrightarrow P \xrightarrow{\ \pi\ } X \qquad \text{or} \qquad \begin{array}{ccc} G & \longrightarrow & P \\ & & \big\downarrow{\scriptstyle \pi} \\ & & X \end{array} \tag{13.3}$$

are used. They make the *structure group* of the principal bundle explicit, and we use these notations when that is desirable.

Proposition 13.4. *Let G be a Lie group, and suppose P is a smooth manifold equipped with a free G-action. Assume the quotient map $\pi\colon P\to X$ is a submersion onto a smooth manifold X. Then π is a principal G-bundle iff π admits local sections.*

Proof. A local trivialization (12.3) of any fiber bundle gives rise to local sections, which implies the forward direction. Conversely, if $U\subset X$ is open and $s\colon U\to\pi^{-1}(U)$ is a section of $\pi\big|_{\pi^{-1}(U)}$, then

$$\begin{aligned} U\times G &\longrightarrow \pi^{-1}(U)\\ x\ ,\ g &\longmapsto s(x)g \end{aligned}\tag{13.5}$$

is G-equivariant local trivialization, i.e., (13.5) is an isomorphism of principal G-bundles. ☐

(13.6) *Proper free Lie group actions.* We begin with a definition.

Definition 13.7. A smooth right action of a Lie group G on a smooth manifold P is *proper* if

$$\begin{aligned} \alpha\colon P\times G &\longrightarrow P\times P\\ p\ ,\ g &\longmapsto p\ ,\ pg \end{aligned}\tag{13.8}$$

is a proper map.

Recall that a map is proper if the inverse image of every compact set is compact. A proper map is closed[16]—the image of every closed set is closed—and in particular has closed image.

Theorem 13.9. *Suppose a Lie group G acts smoothly, freely, and properly on a smooth manifold P. Then there is a smooth manifold structure on the quotient P/G such that the quotient map $\pi\colon P\to P/G$ is a submersion.*

It then follows from Proposition 13.4 that π is a principal G-bundle: the construction in the proof produces local sections of π. We leave the reader to show that the smooth manifold structure in Theorem 13.9 is unique. More details for the theorem can be found in John Lee, *Introduction to Smooth Manifolds*, Chapter 7.

Proof. First, topologize P/G by the quotient topology. Since P is second countable, so is the quotient P/G. The space P/G is Hausdorff iff the diagonal $\Delta\subset P/G\times P/G$ is closed, and by the definition of the quotient topology that is true iff

$$(\pi\times\pi)^{-1}(\Delta)\subset P\times P\tag{13.10}$$

is closed. This subset is the image of the map α in (13.8), which is closed since α is proper.

We construct an atlas on P/G, and in the process prove that P/G is locally Euclidean. Since the action is free, P is foliated by the orbits, each of which is diffeomorphic to G. We claim that about each $p_0\in P$ there exists a *special chart* $(U;x^1,\dots,x^n,y^1,\dots,y^m)$ such that (1) the image

[16]This is true for a proper map of manifolds, and in general for a proper map of topological spaces if the codomain is locally compact Hausdorff.

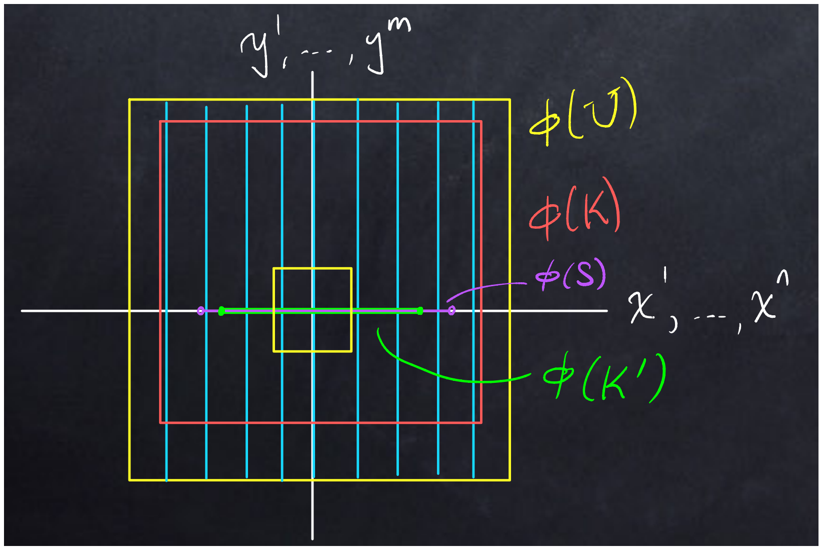

FIGURE 47. The image of the coordinate chart (U, ϕ) in the proof that special charts exist

of U in affine space is an open cube about the origin and the image of p_0 is the origin; and (2) the intersection of each G-orbit with U is either empty or is a G-*slice* $\{x^i = c^i\}$ for some constants $c^i \in \mathbb{R}$, $i = 1, \dots, n$. To prove the claim, fix p_0. By[17] Theorem 5.41 we can find cubic coordinate system (Remark 6.4) about p_0 such that each G-slice lies in an orbit, but it may be that a given orbit intersects U in multiple G-slices. We now show that U can be cut down so this does not happen; see Figure 47. First, the map

$$\begin{aligned} \alpha_{p_0}\colon G &\longrightarrow P \\ g &\longmapsto p_0 \cdot g \end{aligned} \tag{13.11}$$

is proper: for $K \subset P$ compact, the subset[18] $\alpha_{p_0}^{-1}(K) = \mathrm{pr}_2\big(\alpha^{-1}(\{p_0\} \times K)\big) \subset G$ is compact. Let $K \subset U$ be a compact subset whose image under the chart contains an open cubic set that includes the origin. Then $\alpha_{p_0}^{-1}(K)$ is compact, so has finitely many components. This means that the orbit of p_0 intersects K in finitely many G-slices. Hence we can cut down U so that the intersection of the orbit of p_0 with U is the G-slice of U through p_0. In this cut down coordinate system, define $S := \{y^j = 0\}$ and let $K' \subset S$ be a compact subset that contains p_0 in its interior. Then by properness of α, the subset

$$\mathrm{pr}_1\big(\alpha^{-1}(K' \times K') \setminus (K' \times \{e\})\big) \subset S \tag{13.12}$$

is compact and does not include p_0. Therefore, an open neighborhood of $p_0 \in S$ is not contained in (13.12), and every G-orbit that intersects that neighborhood only does so once: the subset (13.12) consists of points of K' whose G-orbit has multiple intersections with K'. Now it is clear that a further restriction of the given chart is special, i.e., it satisfies the desired conditions (1) and (2).

[17] The tangents to the orbits form an integrable distribution.

[18] As always, we use the notation $\mathrm{pr}_i\colon X_1 \times X_2 \to X_i$, $i = 1, 2$, for the projections off a Cartesian product.

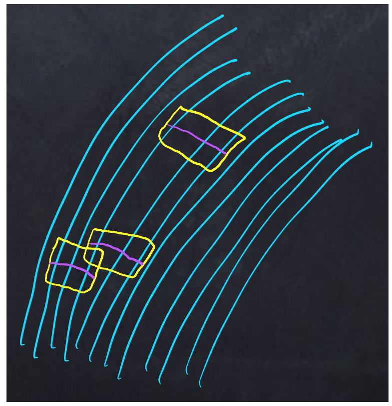

FIGURE 48. Special charts (U,ϕ) with the slice $\phi(S)$ in purple

For each special chart $(U;x,y)$, the function $x=(x^1,\dots,x^n)$ induces a homeomorphism of $\pi(U)$ with an open subset of $\mathbb{A}^n$. This proves that P/G is locally Euclidean. If two special charts intersect in P, then the induced overlap function between the x-coordinates is smooth, since the entire overlap function is smooth. If the images under π of two special charts intersect, then use G-action to move one of them so that they intersect in P; see Figure 48. The overlap function of the moved charts is smooth, and since the G-action is smooth so too is the overlap in P/G for the original charts. $\square$

The subset S constructed in the proof is called a *slice* of the G-action; it is the image of a local section of the quotient map $\pi\colon P\to P/G$. The key hypothesis is the properness of the group action, which guarantees that a compact transversal to the orbits intersects each orbit finitely often.

Examples of principal bundles

(13.13) *Frame bundle.* We have already encountered frame bundles several times: Remark 1.29(5), **(1.53)**, **(9.10)**, and **(9.23)**. Let X be an n-dimensional smooth manifold. Define

$$\mathcal{B}(X)\subset \underbrace{TX\times_X\cdots\times_X TX}_{n\text{ times}} \tag{13.14}$$

as the open submanifold consisting of linearly independent ordered n-tuples $(\xi_1,\dots,\xi_n)$ of tangent vectors at some $x\in X$. Let

$$\pi\colon \mathcal{B}(X)\longrightarrow X \tag{13.15}$$

be the restriction of projection off the n-fold fiber product. The fiber $\pi^{-1}(x)=\mathcal{B}(X)_x$ over $x\in X$ is the right $GL_n\mathbb{R}$-torsor of bases $\mathbb{R}^n\xrightarrow{\cong}T_xX$ of T_xX. Let $(U;x^1,\dots,x^n)$ be a chart on X. Then the moving frame $\partial/\partial x^1,\dots,\partial/\partial x^n$ is a local section of π, and so induces a local trivialization (13.5). Since the $GL_n\mathbb{R}$-action on $U\times GL_n\mathbb{R}$ is smooth, so too is the $GL_n\mathbb{R}$-action on $\pi^{-1}(U)$. It follows from Proposition 13.4 that (13.15) is a principal $GL_n\mathbb{R}$-bundle.

Example 13.16 (free action whose quotient map is not a principal bundle)**.** Let $G = \mathbb{R}$, set $P = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, fix $a \in \mathbb{R} \setminus \mathbb{Q}$, and let G act on P by $(x, y) \cdot t = (xt, ayt)$ for $t \in G$ and $(x, y) \in P$. This action is free but not proper; the quotient is not Hausdorff, much less a manifold. See (**4.1**) and Example 5.8 for related discussions.

Example 13.17 (Galois covers)**.** A covering map $\pi\colon P \to X$ is a principal bundle if it is *Galois/normal/regular*. (All of these terms are used.) In this case G is the group of deck transformations. Given a smooth manifold X and a basepoint $x \in X$, there is a canonical *universal covering* map $\pi\colon \widetilde{X} \to X$, and it is a principal $\pi_1(X, x)$-bundle. (The total space $\widetilde{X}$ consists of homotopy classes of smooth motions $[0, 1] \to X$ that begin at x.)

Example 13.18 (homogeneous manifolds)**.** Let G be a Lie group and $H \subset G$ a *closed* Lie subgroup. Then the action of H on G by right multiplication is proper, hence by Theorem 13.9 and Proposition 13.4 the quotient map $\pi\colon G \to G/H$ is a principal H-bundle. As in Example 7.48, a variation is to start with a smooth manifold X equipped with a transitive left G-action. We say that X is a *homogeneous manifold*. Fix $x \in X$ and let $H \subset G$ be the stabilizer subgroup of x. Then

$$\begin{aligned} \pi\colon G &\longrightarrow X \\ g &\longmapsto gx \end{aligned} \tag{13.19}$$

is a principal H-bundle (and π descends to a diffeomorphism $G/H \to X$).

We give some special cases of Example 13.18.

Example 13.20 (Hopf fibration)**.** Let $X = \mathbb{CP}^1 = \mathbb{P}(\mathbb{C}^2)$ be the complex projective line. The Lie group SU_2 of 2×2 unitary matrices of unit determinant acts transitively. The stabilizer of the line $\mathbb{C}\cdot\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ is the subgroup $\left\{\left(\begin{smallmatrix}\lambda & \\ & \lambda^{-1}\end{smallmatrix}\right) : \lambda \in \mathbb{T}\right\}$, which we identify with the group $\mathbb{T} \subset \mathbb{C}$ of unit norm complex numbers. The principal bundle $\pi\colon \mathrm{SU}_2 \to \mathbb{CP}^1$ is an example of a *Hopf fibration*. There are diffeomorphisms $\mathrm{SU}_2 \approx S^3$, $\mathbb{CP}^1 \approx S^2$, and $\mathbb{T} \approx S^1$, so the total space, base, and fiber are all spheres. There are real, quaternionic, and octonionic versions of this construction, and together we have Hopf fibrations

$$\begin{aligned} S^0 &\longrightarrow S^1 \longrightarrow S^1 \\ S^1 &\longrightarrow S^3 \longrightarrow S^2 \\ S^3 &\longrightarrow S^7 \longrightarrow S^4 \\ S^7 &\longrightarrow S^{15} \longrightarrow S^8 \end{aligned} \tag{13.21}$$

However, the last of these is *not* a principal bundle since S^7 does not carry a Lie group structure. For the three principal bundles in (13.21), it is better to write the groups as $\mathrm{O}_1, \mathrm{U}_1, \mathrm{Sp}_1$, respectively.

Example 13.22 (spheres as homogeneous manifolds)**.** For any $n \in \mathbb{Z}^{>0}$ the orthogonal group O_{n+1} acts transitively on the n-sphere S^n, and this gives rise to a principal bundle

$$\mathrm{O}_n \longrightarrow \mathrm{O}_{n+1} \longrightarrow S^n \tag{13.23}$$

There are similar real and quaternionic constructions with the unitary and symplectic groups:

$$\begin{aligned} \mathrm{U}_n &\longrightarrow \mathrm{U}_{n+1} \longrightarrow S^{2n+1} \\ \mathrm{Sp}_n &\longrightarrow \mathrm{Sp}_{n+1} \longrightarrow S^{4n+3} \end{aligned} \tag{13.24}$$

These principal fiber bundles are useful when studying the topology of classical Lie groups.

Example 13.25 (principal bundles over projective space)**.** Let V be a finite dimensional vector space over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Then the map

$$\pi\colon V \setminus \{0\} \longrightarrow \mathbb{P}(V), \tag{13.26}$$

that assigns to a nonzero vector the line it spans, is a principal $\mathbb{F}^\times$-bundle. If we endow V with a (hermitian) inner product, then we can restrict to unit norm vectors, in which case we obtain a principal O_1-, U_1-, or Sp_1-bundle in case $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, respectively:

$$\begin{aligned} S^0 &\longrightarrow S^n \longrightarrow \mathbb{RP}^n \\ S^1 &\longrightarrow S^{2n+1} \longrightarrow \mathbb{CP}^n \\ S^3 &\longrightarrow S^{4n+3} \longrightarrow \mathbb{HP}^n \end{aligned} \tag{13.27}$$

Example 13.28 (Orthonormal frame bundle of a sphere)**.** Fix $n \in \mathbb{Z}^{\ge 0}$. Let V be an $(n+1)$-dimensional real inner product space, let $S(V) \subset V$ be the sphere of unit norm vectors, and let $\pi\colon \mathcal{B}_O\big(S(V)\big) \to S(V)$ be the principal O_n-bundle of orthonormal frames. The orthogonal group $\mathrm{O}(V)$ acts on $\mathcal{B}_O\big(S(V)\big)$ on the left. In fact, $\mathrm{O}(V)$ also acts compatibly on $S(V)$, so we could say that $\mathrm{O}(V)$ acts on π. We claim the action on $\mathcal{B}_O\big(S(V)\big)$ is simply transitive. More precisely, the map

$$\varphi\colon \mathcal{B}_{\mathrm{O}}(V) \longrightarrow \mathcal{B}_O\big(S(V)\big) \tag{13.29}$$

that takes an orthonormal frame $e_0, e_1, \dots, e_n$ of V to the point $e_0 \in S(V)$ and the orthonormal frame $e_1, \dots, e_n$ of $T_{e_0}S(V)$ is an isomorphism of left $\mathrm{O}(V)$-torsors. This identifies the orthonormal frame bundle of the *standard* sphere $S^n = S(\mathbb{R}^{n+1})$ with the orthogonal group O_{n+1}: an orthogonal matrix maps to the point on S^n given by the first column and the basis of the tangent space given by the subsequent columns.

Associated bundles

(13.30) *Base change.* This is also known as pullback. It is a special case of base change for fiber bundles **(12.13)**. We simply observe that the pullback

$$\begin{tikzcd} P' \arrow[r, dashed, "\tilde f"] \arrow[d, dashed, "\pi'"'] & P \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd} \tag{13.31}$$

of a principal G-bundle is a principal G-bundle.

(13.32) *Associated fiber bundles.* This is also called the *mixing construction.* Let G be a Lie group and F a smooth manifold equipped with a left G-action. Then to any principal G-bundle $\pi\colon P \to X$ we associate a fiber bundle $F_P \to X$ with fiber F as follows. The Cartesian product $P \times F$ carries a right G-action

$$(p, f)\cdot g = (pg, g^{-1}f), \qquad p \in P, \quad f \in F, \quad g \in G. \tag{13.33}$$

Definition 13.34. Let $P \times_G F = F_P$ be the quotient of $P \times F$ by the G-action (13.33). The descent $\pi_F\colon F_P \to X$ of $\mathrm{pr}_1\colon P \times F \to P$ is the *associated fiber bundle* with fiber F.

A local section $s\colon U \to \pi^{-1}(U)$ of π induces a local trivialization

$$\begin{aligned} U \times F &\longrightarrow F_P\big|_U \\ x\ ,\ f &\longmapsto [s(x), f] \end{aligned} \tag{13.35}$$

of $\pi_F\colon F_P \to X$, where $[p, f] \in F_P$ is the equivalence class of $(p, f) \in P \times F$ under the G-action (13.33).

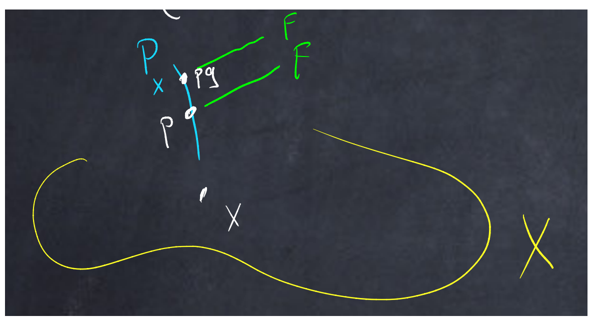

FIGURE 49. The associated bundle $F_P \to X$ descends from $P \times F \to P$

Remark 13.36.

(1) The construction by descent means there is a canonical global product structure on the pullback of $\pi_F\colon F_P \to X$ via π:

$$\begin{tikzcd} P \times F \arrow[r, dashed] \arrow[d, dashed, "\mathrm{pr}_1"'] & F_P \arrow[d, "\pi_F"] \\ P \arrow[r, "\pi"] & X \end{tikzcd} \tag{13.37}$$

(2) A section of π_F is a function $\psi\colon P \to F$ that satisfies the G-equivariance condition

$$\psi(pg) = g^{-1}\psi(p), \qquad p \in P, \quad g \in G. \tag{13.38}$$

This reduces the study of twisted F-valued functions on X to the study of equivariant F-valued functions on P. We can apply our known calculus techniques to the latter.

(3) If F has extra structure—examples: vector space, associative algebra, Lie algebra, Lie group, action of a Lie group, affine space, ...—and if the G-action on F preserves that structure, then $F_P \to X$ is a smooth family of manifolds with that structure.

(4) A particular case is when $F = \mathbb{E}$ is a vector space and G acts via a linear representation $\rho\colon G \to \operatorname{Aut}(\mathbb{E})$. The associated bundle $\mathbb{E}_P = E \to X$ is a vector bundle. Then (13.38) is written as

$$\psi(pg) = \rho(g)^{-1}\psi(p). \tag{13.39}$$

(5) Given F with its G-action, the associated bundle construction is a functor from the category[19] of principal G-bundles to the category of fiber bundles with fiber F.

Example 13.40 (the tangent bundle)**.** Let X be a smooth n-dimensional manifold and $\pi\colon \mathcal{B}(X) \to X$ its principal $GL_n\mathbb{R}$-bundle of frames. The vector bundle associated to the standard representation $GL_n\mathbb{R} \to \operatorname{Aut}(\mathbb{R}^n)$ is canonically isomorphic to the tangent bundle via the isomorphism

$$\varphi\colon \mathcal{B}(X) \times_{GL_n\mathbb{R}} \mathbb{R}^n \longrightarrow TX \tag{13.41}$$

that maps a pair $[b, v]$ of an isomorphism $b\colon \mathbb{R}^n \to T_xX$ and $v \in \mathbb{R}^n$ to the tangent vector $b(v) \in T_xX$. Observe that for $g \in GL_n\mathbb{R}$, we have $(b \circ g)(g^{-1}v) = b(v)$, which proves that (13.41) is a well-defined map on the associated bundle.

Example 13.42. The fiber bundles in Example 12.7–Example 12.11 are all associated to the frame bundle $\pi\colon \mathcal{B}(X) \to X$. We have already treated the tangent bundle. The projectivization $\mathbb{P}(TX) \to X$ is associated to the action $GL_n\mathbb{R} \to PGL_n\mathbb{R} \to \operatorname{Aut}(\mathbb{RP}^{n-1})$ by projective transformations. The Grassmannian, Stiefel, and flag bundles are associated to the corresponding Grassmannian, Stiefel, and flag manifolds of $\mathbb{R}^n$. Similarly, the fiber bundle of metrics $\operatorname{Met}(TX) \to X$ is associated to the action of $GL_n\mathbb{R}$ on $\operatorname{Met}(\mathbb{R}^n)$. The bundle of associative algebras $\operatorname{End}(TX) \to X$ is associated to the action of $GL_n\mathbb{R}$ on $M_n\mathbb{R}$ by conjugation: $g \cdot A = gAg^{-1}$ for $g \in GL_n\mathbb{R}$ and $A \in M_n\mathbb{R}$.

Change of structure group

(13.43) *Associated principal bundles.* Let G', G be Lie groups and $\rho\colon G' \to G$ a Lie group homomorphism. This induces a functor from the category of principal G'-bundles to the category of principal G-bundles that associates to a principal G'-bundle $\pi'\colon P' \to X$ the bundle $\pi\colon P \to X$ associated to the left action of G' on G via ρ: namely, $g' \cdot g = \rho(g')g$ for $g' \in G'$ and $g \in G$. Since the left G'-action commutes with right multiplication by G, this left G'-action on G is by maps of right G-torsors, hence the associated bundle $\pi\colon P \to X$ is a fiber bundle of right G-torsors, i.e., a principal G-bundle. We denote the map $\pi\colon P \to X$ as $\rho(\pi')\colon \rho(P') \to X$.

Example 13.44. Let X be a Riemannian n-manifold and $\mathcal{B}_O(X) \to X$ the principal O_n-bundle of orthonormal frames. Let $\rho\colon O_n \hookrightarrow GL_n\mathbb{R}$ be the inclusion. Then $\rho\big(\mathcal{B}_O(X)\big) \to X$ is canonically the $GL_n\mathbb{R}$-bundle of frames $\mathcal{B}(X) \to X$. We leave the reader to construct an explicit identification.

[19]There are different ways to flesh this out. We can use bundles over a fixed base or allow base change. We must also specify the morphisms.

(13.45) *The affine bundle of frames.* Let X be a smooth manifold. At each $x \in X$ there is a linear space T_xX, the tangent space to X at x. Let A_xX be the trivial affine space over T_xX, that is, the tangent vector space regarded as an affine space. The union of A_xX over $x \in X$ is the total space of the affine bundle $AX \to X$; see Example 12.12. An *affine frame* at $x \in X$ is an affine isomorphism $\mathbb{A}^n \to A_xX$. Affine frames form a right torsor for $\mathrm{Aff}_n\mathbb{R}$. The union of affine frames form a principal $\mathrm{Aff}_n\mathbb{R}$-bundle $\mathcal{A}(X) \to X$. The homomorphisms $\mathrm{GL}_n\mathbb{R} \rightleftarrows \mathrm{Aff}_n\mathbb{R}$ induce passage between the frame bundle and the affine frame bundle via the associated bundle construction. The affine bundle $AX \to X$ is associated to $\mathcal{A}(X) \to X$ via the affine action of $\mathrm{Aff}_n\mathbb{R}$ on $\mathbb{A}^n$.

(13.46) *Lift of structure group.* This is also called *reduction* of structure group, especially in case the homomorphism ρ is injective.

Definition 13.47. Let $\rho\colon G' \to G$ be a homomorphism of Lie groups, and suppose $\pi\colon P \to X$ is a principal G-bundle. A *lift* (π', θ) *of* π *along* ρ is a principal G'-bundle $\pi'\colon P' \to X$ and an isomorphism

$$\begin{tikzcd} P \arrow[rr, "\theta"] \arrow[dr, "\pi"'] & & \rho(P') \arrow[dl, "\rho(\pi')"] \\ & X & \end{tikzcd} \tag{13.48}$$

of principal G-bundles.

While the associated bundle construction **(13.43)** is a functor, a lift may or may not exist and may or may not be unique (up to isomorphism).

Example 13.49. Consider the principal $\mathrm{GL}_2\mathbb{R}$-bundle of frames $\pi\colon \mathcal{B}(S^2) \to S^2$, and let $\rho\colon \{e\} \hookrightarrow \mathrm{GL}_2\mathbb{R}$ be the inclusion. A reduction of π along ρ is equivalent to a global section of π, which does not exist by the Hairy Ball Theorem.

Proposition 13.50. *Suppose $G' \subset G$ is a subgroup of a Lie group G, and let $\pi\colon P \to X$ be a principal G-bundle. Then a reduction of π to G' is equivalent to a section of the associated bundle $P/G' \to X$.*

The fiber bundle $P/G' \to X$ is associated to π via the left G-action on the homogeneous manifold G/G'. We leave the proof of Proposition 13.50 to a homework exercise.

(13.51) *Geometric structures.* Recall Definition 1.37 in which symmetry types are defined.

Definition 13.52. Let (G_n, λ_n) be an n-dimensional symmetry type, and suppose X is a smooth n-dimensional manifold with frame bundle $\pi\colon \mathcal{B}(X) \to X$. A (G_n, λ_n)-*structure on* X is a reduction of π along $\rho_n\colon G_n \to \mathrm{GL}_n\mathbb{R}$.

For example, a $(\mathrm{O}_n, \lambda_n)$-structure (Example 1.40) on a smooth n-manifold is equivalent to a Riemannian structure. In this case the fiber bundle $\mathcal{B}(X)/\mathrm{O}_n \to X$ is canonically $\mathrm{Met}(X) \to X$, which appeared in Example 12.11. According to Proposition 13.50 a $(\mathrm{O}_n, \lambda_n)$-structure is a section of this bundle, i.e., is a Riemannian metric.

Lecture 14: Connections on principal bundles

In this lecture we take up special horizontal distributions on principal bundles, namely those invariant under the gauge group.[20] The group invariance allows us to descend a connection to the base, where it is a section of an affine bundle, and thus prove existence. The horizontal distribution is the kernel of a Lie algebra valued 1-form, which lets us apply exterior calculus. In particular, the Frobenius tensor—which up to a sign is the *curvature* of the connection—has an easy formula in terms of this 1-form. It is convenient to switch between the horizontal distribution and the connection 1-form, depending on the context. We conclude with parallel transport along curves, a construction that fails in general for horizontal distributions on fiber bundles. (See Example 12.60.)

The relative tangent bundle

(14.1) *Global parallelism of vertical tangents.* Let G be a Lie group and suppose $\pi\colon P\to X$ is a principal G-bundle over a smooth manifold X. As is true for any fiber bundle (12.35), there is a short exact sequence

$$0\longrightarrow T(P/X)\longrightarrow TP\xrightarrow{\;\pi_*\;}\pi^*TX\longrightarrow 0 \tag{14.2}$$

of vector bundles over the total space P; the kernel of π_* is the *relative* (or *vertical*) tangent bundle. The free right G-action on P induces an injective Lie group homomorphism

$$\begin{aligned}\mathfrak{g}&\longrightarrow \mathcal{X}(P)\\ \zeta&\longmapsto \widehat{\zeta}\end{aligned} \tag{14.3}$$

that maps $\zeta\in\mathfrak{g}$ to the initial velocity of the motion $t\mapsto pe^{t\zeta}$ at each $p\in P$. The motion lies in the fiber $P_{\pi(p)}$, and so $\widehat{\zeta}$ is a *vertical* vector field—a section of $T(P/X)\to P$. Since π is a quotient map of the G-action, it follows that composition of (14.3) with evaluation at p is an isomorphism $\mathfrak{g}\to T_p(P/X)$. In other words, there is an isomorphism of vector bundles

$$\underline{\mathfrak{g}}\xrightarrow{\;\cong\;}T(P/X) \tag{14.4}$$

where $\underline{\mathfrak{g}}\to P$ is the constant vector bundle with fiber $\mathfrak{g}$. This defines a parallelism on the relative tangent bundle: an isomorphism of the relative tangent bundle with a constant vector bundle.

(14.5) *Right G-action and vertical vector fields.* The Lie group G acts on P on the right, and via the differential it acts on the relative tangent bundle. The isomorphism (14.4) is G-equivariant if we use the (right) adjoint action on the Lie algebra, as we now prove.

[20] If G is a Lie group and $\pi\colon P\to X$ a principal G-bundle, we say that G is the *gauge group* of π.

Lemma 14.6. *For $p \in P$ and $g \in G$ the diagram*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\cong"] \arrow[d, "\mathrm{Ad}_{g^{-1}}"'] & T_p(P/X) \arrow[d, "(R_g)_*"] \\ \mathfrak{g} \arrow[r, "\cong"] & T_{pg}(P/X) \end{tikzcd} \tag{14.7}$$

commutes.

Proof. A vector $\zeta \in \mathfrak{g}$ is the initial velocity of the motion $t \mapsto e^{t\zeta}$ on G. Map it around (14.7):

$$\begin{tikzcd} e^{t\zeta} \arrow[rr, mapsto] \arrow[d, mapsto, "\mathrm{Ad}_{g^{-1}}"'] & & pe^{t\zeta} \arrow[d, mapsto, "(R_g)_*"] \\ g^{-1}e^{t\zeta}g^{-1} \arrow[r, mapsto] & (pg)(g^{-1}e^{t\zeta}g) = pe^{t\zeta}g \end{tikzcd} \tag{14.8}$$

Now apply $\frac{d}{dt}\big|_{t=0}$ to verify that (14.7) commutes. $\square$

Corollary 14.9. *Let $\pi\colon P \to X$ be a principal G-bundle. The linear space of right-invariant vertical vector fields on P is isomorphic to the space of sections of the adjoint bundle $\mathfrak{g}_P \to X$.*

Proof. A section of the adjoint bundle pulls back to a function $\zeta\colon P \to \mathfrak{g}$ that satisfies

$$\zeta(pg) = \mathrm{Ad}_{g^{-1}}\,\zeta(p) \tag{14.10}$$

for $p \in P$, $g \in G$. $\square$

(14.11) *Descent conditions.* The adjoint representation descends the constant vector bundle $\underline{\mathfrak{g}} \to P$ to the adjoint bundle $\mathfrak{g}_P \to X$. We will need to descend functions and differential forms on the total space P, both scalar- and vector-valued, to the base X, so we pause to discuss descent.

Pullback of differential forms from the base of a principal bundle to the total space is an injection. The following result characterizes the image inside all differential forms on the total space. These are *descent conditions*. Recall (Remark 13.36(4)) that if $\rho\colon G \to \mathrm{Aut}(\mathbb{E})$ is a linear representation of G on a vector space $\mathbb{E}$, then to each principal bundle $\pi\colon P \to X$ is associated a vector bundle $\mathbb{E}_P \to X$. For each $q \in \mathbb{Z}^{\ge 0}$ the vector space of sections of $\bigwedge^q T^*X \otimes \mathbb{E}_P \to X$ is denoted $\Omega^q_X(\mathbb{E}_P)$; it is the space of differential forms with values in the vector bundle $\mathbb{E}_P \to X$.

Lemma 14.12. *Let G be a Lie group and suppose $\pi\colon P \to X$ is a principal G-bundle.*

(1) *A form $\omega \in \Omega^\bullet_P$ descends to a form in $\Omega^\bullet_X$ iff*

$$\begin{aligned} \iota_{\hat\zeta}\omega_p &= 0 && \text{for all } \zeta \in \mathfrak{g},\ p \in P, \\ R_g^*\omega &= \omega && \text{for all } g \in G. \end{aligned} \tag{14.13}$$

(2) *Suppose* $\rho\colon G \to \mathrm{Aut}(\mathbb{E})$ *is a linear representation. Then* $\omega \in \Omega^\bullet_P(\mathbb{E})$ *descends to a form in* $\Omega^\bullet_X(\mathbb{E}_P)$ *iff*

$$\begin{aligned} \iota_{\hat\zeta}\omega_p &= 0 && \text{for all } \zeta \in \mathfrak{g},\ p \in P, \\ R_g^*\omega &= \rho(g)^{-1}\omega && \text{for all } g \in G. \end{aligned} \tag{14.14}$$

We leave the proof as a homework exercise.

Connections

(14.15) *Definition.* A connection on a principal bundle is a special horizontal distribution on the total space.

Definition 14.16. Let G be a Lie group, and suppose $\pi\colon P \to X$ is a principal G-bundle. A *connection* on π is a G-invariant horizontal distribution on P.

Recall that a horizontal distribution is a splitting of (14.2). While such splittings exist, it is not immediately clear that G-invariant splittings exist. We sometimes use the term 'G-connection' without making explicit the underlying principal G-bundle. A G-connection is depicted in Figure 50.

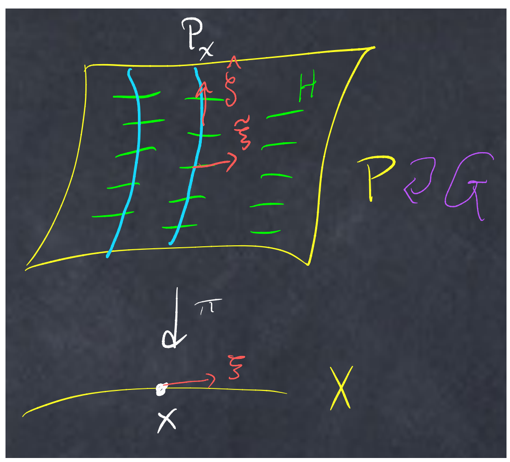

FIGURE 50. A connection on $\pi\colon P \to X$ and the induced horizontal lift $\xi \mapsto \widetilde{\xi}$

(14.17) *Horizontal lift.* A splitting of (14.2) is a map σ in

$$0 \longrightarrow T(P/X) \longrightarrow TP \underset{\sigma}{\overset{\pi_*}{\rightleftarrows}} \pi^*TX \longrightarrow 0 \tag{14.18}$$

such that $\pi \circ \sigma = \mathrm{id}_{\pi^*TX}$. The image $H \subset TP$ of σ is a horizontal distribution. The map σ takes a tangent vector $\xi \in T_xX$ to its *horizontal lift* $\widetilde{\xi} \in H_p \subset T_pP$ for any $p \in P_x$. The G-invariance condition is

$$(R_g)_*(H_p) = H_{pg} \qquad p \in P, \quad g \in G. \tag{14.19}$$

The horizontal lift $\widetilde{\xi}$ is a G-invariant section of $H\big|_{P_x} \to P_x$, the restriction of H to the fiber P_x.

(14.20) *Connections as sections of an affine bundle.* Splittings σ in (14.18) form sections of an affine bundle over P, and it is an affine bundle over the vector bundle

$$\mathrm{Hom}\big(\pi^*TX, T(P/X)\big) \cong \mathrm{Hom}(\pi^*TX, \mathfrak{g}) \longrightarrow P. \tag{14.21}$$

By Lemma 14.6, G-invariant sections of (14.21) descend to sections of the vector bundle

$$\mathrm{Hom}(TX, \mathfrak{g}_P) \longrightarrow X. \tag{14.22}$$

Therefore, the affine space of *G-invariant* splittings of (14.18) is isomorphic to the space of sections of an affine bundle $A_\pi \to X$ over (14.22). The following is then an immediate consequence of (7.52).

Proposition 14.23. *A principal bundle admits a connection. Connections on $\pi\colon P \to X$ are points of an (infinite dimensional) affine space $\mathcal{A}_\pi$ over $\Omega^1_X(\mathfrak{g}_P)$.*

For the second statement, note that the space of sections of $\mathrm{Hom}(TX, \mathfrak{g}_P) \to X$ is $\Omega^1_X(\mathfrak{g}_P)$.

Remark 14.24. G-invariance descends a connection—a horizontal distribution on the total space P—to data on the base—a section of an affine bundle.

Remark 14.25. If the gauge group G is discrete, so π is a Galois covering map, then π admits a unique connection. If G has positive dimension, then the connection is not unique.

(14.26) *The connection 1-form.* A splitting of (14.18) is equivalent to a linear map $TP \to T(P/X)$ that splits the inclusion:

$$\begin{tikzcd} 0 \arrow[r] & T(P/X) \arrow[r, shift left] & TP \arrow[l, shift left] \arrow[r, shift left, "\pi_*"] \arrow[dl, "\Theta"] & \pi^*TX \arrow[l, shift left, "\sigma"] \arrow[r] & 0 \\ & \underline{\mathfrak{g}} \arrow[u, "(14.4)"] & & & \end{tikzcd} \tag{14.27}$$

The 1-form $\Theta \in \Omega^1_P(\mathfrak{g})$ has kernel the horizontal distribution H. We now characterize the subspace of $\Omega^1_P(\mathfrak{g})$ consisting of *connection 1-forms*. (We typically use 'connection' for 'connection 1-form'.)

Lemma 14.28.

(1) *A 1-form $\Theta \in \Omega^1_P(\mathfrak{g})$ defines a splitting of (14.2) iff*

$$i_x^*\Theta = \theta_{P_x} \qquad \text{for all } x \in X, \tag{14.29}$$

where $i_x\colon P_x \hookrightarrow P$ is the inclusion and $\theta_{P_x} \in \Omega^1_{P_x}(\mathfrak{g})$ is the Maurer–Cartan form on the right G-torsor P_x.

(2) *The horizontal distribution* $\ker\Theta$ *is* G*-invariant iff*

$$R_g^*\Theta = \mathrm{Ad}_{g^{-1}}\Theta \qquad \text{for all } g\in G. \tag{14.30}$$

Proof. For (1) observe that Θ is a splitting iff $\Theta(\widehat{\zeta}) = \zeta$ for all $\zeta\in\mathfrak{g}$ iff (14.29) holds, this last implication by Lemma 7.45. Assume $\ker\Theta$ is G-invariant. Equation (14.30) is an equality of 1-forms, so it suffices to check it on horizontal and vertical vectors separately, since

$$TP = H\oplus T(P/X), \tag{14.31}$$

The tangent vector $\widetilde{\xi}$ is horizontal iff $\Theta(\widetilde{\xi}) = 0$. The left hand side of (14.30) on $\widetilde{\xi}$ is

$$(R_g^*\Theta)(\widetilde{\xi}) = \Theta\bigl((R_g)_*\widetilde{\xi}\bigr) = 0, \tag{14.32}$$

since $(R_g)_*\widetilde{\xi}$ is horizontal by the G-invariance of H. The right hand side is

$$(\mathrm{Ad}_{g^{-1}}\Theta)(\widetilde{\xi}) = \mathrm{Ad}_{g^{-1}}\bigl(\Theta(\widetilde{\xi})\bigr) = 0. \tag{14.33}$$

To check (14.30) on vertical vectors, observe that $R_g\circ i_x = i_x\circ R_g$, where the last '$R_g$' acts on the right G-torsor P_x. Apply i_x^* to (14.30) to reduce it on vertical vectors to the equation $R_g^*\theta_{P_x} = \mathrm{Ad}_{g^{-1}}\,\theta_{P_x}$ on P_x. This is the transformation law of the Maurer–Cartan form on P_x, as follows from the first equation in (7.16) by setting $g_1 = e$ and transporting to right G-torsors. $\square$

(14.34) *The space of connections.* We already observed in Proposition 14.23 that the space $\mathcal{A}_\pi$ of connections on a principal bundle $\pi\colon P\to X$ is an affine space over $\Omega^1_X(\mathfrak{g}_P)$. We can also derive this from Lemma 14.28. Namely, $\mathcal{A}_\pi\subset\Omega^1_P(\mathfrak{g})$ is cut out by a continuum of affine linear equations:

$$\begin{aligned} i_x^*\Theta &= \theta_{P_x}, &\qquad x&\in X,\\ R_g^*\Theta &= \mathrm{Ad}_{g^{-1}}\Theta, &\qquad g&\in G. \end{aligned} \tag{14.35}$$

The first equation is affine linear in Θ; the second is linear. The linear translate of the affine subspace defined by (14.35) is cut out by the following linear equations for $\Delta\in\Omega^1_P(\mathfrak{g})$:

$$\begin{aligned} i_x^*\Delta &= 0, &\qquad x&\in X,\\ R_g^*\Delta &= \mathrm{Ad}_{g^{-1}}\Delta, &\qquad g&\in G. \end{aligned} \tag{14.36}$$

By Lemma 14.12(2) the space of solutions to (14.36) is isomorphic to $\Omega^1_X(\mathfrak{g}_P)$.

Curvature

Definition 14.37. Let G be a Lie group, and suppose $\pi\colon P \to X$ is a principal G-bundle with connection a G-invariant horizontal distribution $H \subset TP$. The *curvature* of the connection is minus the Frobenius tensor ϕ_H.

Recall that the Frobenius tensor is a skew-symmetric bilinear form

$$\phi_H\colon H \times H \longrightarrow T(P/X), \tag{14.38}$$

and in view of (14.4) and the isomorphism $\pi_*\colon H \to \pi^*TX$ it can be regarded as a skew-symmetric bilinear form

$$\phi_H\colon \pi^*TX \times \pi^*TX \longrightarrow \underline{\mathfrak{g}}. \tag{14.39}$$

Then G-invariance and Lemma 14.6 imply that $-\phi_H$ descends to a 2-form $\Omega \in \Omega^2_X(\mathfrak{g}_P)$ on X with values in the adjoint bundle. This descent is also called the *curvature*.

Remark 14.40. The pullback $\pi^*\Omega$ of the curvature to the total space P has values in the vector space $\mathfrak{g}$. This lift agrees with $-\phi_H$ on $H \times H$, and it vanishes when contracted with a vertical vector. We conflate Ω and $\pi^*\Omega$, meaning that we also use the notation 'Ω' for '$\pi^*\Omega$'. The "untwisting" of the pullback is typical, and it allows us to do calculus with vector-valued differential forms rather than vector bundle-valued differential forms. We use this idea often.

(14.41) *The curvature formula.* For computational purposes it is important to express the curvature as an explicit first derivative of the connection 1-form.

Proposition 14.42. *The pullback of the curvature to P is the $\mathfrak{g}$-valued 2-form*

$$\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta]. \tag{14.43}$$

Proof. We check that (1) for $\zeta \in \mathfrak{g}$, the contraction of the right hand side of (14.43) with $\widehat{\zeta}$ vanishes, and (2) the restriction of the right hand side to $H \times H$ is $-\phi_H$. These properties characterize the pullback of curvature to P.

Compute

$$\begin{aligned} \iota_{\widehat{\zeta}}\left(d\Theta + \frac{1}{2}[\Theta \wedge \Theta]\right) &= \quad \iota_{\widehat{\zeta}}d\Theta + [\iota_{\widehat{\zeta}}\Theta, \Theta] \\ &= -d\iota_{\widehat{\zeta}}\Theta + \mathcal{L}_{\widehat{\zeta}}\Theta + [\iota_{\widehat{\zeta}}\Theta, \Theta] \\ &= -d(\zeta) + \mathcal{L}_{\widehat{\zeta}}\Theta + [\zeta, \Theta] \\ &= \quad \mathcal{L}_{\widehat{\zeta}}\Theta + [\zeta, \Theta] \end{aligned} \tag{14.44}$$

At the last stage we use $\iota_{\hat{\zeta}}\Theta = \Theta(\hat{\zeta}) = \zeta$; see (14.29). To compute the Lie derivative $\mathcal{L}_{\hat{\zeta}}\Theta$ observe that the flow generated by $\hat{\zeta}$ is the right action $R_{e^{t\zeta}}$ on P. Hence

$$\begin{aligned} \mathcal{L}_{\hat{\zeta}}\Theta &= \frac{d}{dt}\Big|_{t=0} R^*_{e^{t\zeta}}\Theta \\ &= \frac{d}{dt}\Big|_{t=0} \operatorname{Ad}^*_{e^{-t\zeta}}\Theta \\ &= -[\zeta,\Theta], \end{aligned} \tag{14.45}$$

where we use (14.30) to pass to the second equation. Combine (14.44) and (14.45) to prove the first claim (1).

Let $\xi_x, \eta_x \in T_xX$ be tangent vectors at some $x \in X$, extend to vector fields ξ, η in a neighborhood of x, and let $\tilde{\xi}, \tilde{\eta}$ be their horizontal lifts to vector fields on P. Then $\iota_{\tilde{\xi}}\Theta = \iota_{\tilde{\eta}}\Theta = 0$. Hence

$$\begin{aligned} \iota_{\tilde{\eta}}\iota_{\tilde{\xi}}\left(d\Theta + \frac{1}{2}[\Theta\wedge\Theta]\right) &= d\Theta(\tilde{\xi},\tilde{\eta}) \\ &= -\Theta\big([\tilde{\xi},\tilde{\eta}]\big) \\ &= -\phi_H(\tilde{\xi},\tilde{\eta}) \end{aligned} \tag{14.46}$$

The last expression is tensorial in $\tilde{\xi}, \tilde{\eta}$; the value at x of its descent to the base depends only on ξ_x, η_x. This proves (2). $\square$

Remark 14.47. The curvature is a map

$$\Omega\colon \mathcal{A}_\pi \longrightarrow \Omega^2_X(\mathfrak{g}) \tag{14.48}$$

from an affine space to a linear space. According to (14.43) it is an affine quadratic map.

(14.49) *Descent and non-descent.* Apply Lemma 14.12 to conclude the following.

(i) The connection form $\Theta \in \Omega^1_P(\mathfrak{g})$ does *not* descend to X since $\iota_{\hat{\zeta}}\Theta = \zeta \neq 0$ if ζ is nonzero.

(ii) The difference $\Theta_1 - \Theta_0$ of two connections descends to an element of $\Omega^1_X(\mathfrak{g}_P)$; see (14.36).

(iii) The curvature 2-form in $\Omega^2_P(\mathfrak{g})$ descends to an element of $\Omega^2_X(\mathfrak{g}_P)$. This follows from Proposition 14.42. We can also check the descent conditions (14.14) directly on the right hand side of (14.43). The first check is (14.44) and (14.45). The second is

$$\begin{aligned} R^*_g\left(d\Theta + \frac{1}{2}[\Theta\wedge\Theta]\right) &= dR^*_g\Theta + \frac{1}{2}[R^*_g\Theta\wedge R^*_g\Theta] \\ &= d\operatorname{Ad}_{g^{-1}}\Theta + \frac{1}{2}[\operatorname{Ad}_{g^{-1}}\Theta\wedge\operatorname{Ad}_{g^{-1}}\Theta] \\ &= \operatorname{Ad}_{g^{-1}}\left(d\Theta + \frac{1}{2}[\Theta\wedge\Theta]\right) \end{aligned} \tag{14.50}$$

Flat connections and parallel transport

(14.51) *Flat connections and integral manifolds of H.* Let $\pi\colon P \to X$ be a principal G-bundle for some Lie group G, and suppose Θ is a connection with horizontal invariant distribution $H = \ker\Theta$. The Frobenius tensor, hence the curvature Ω, is the obstruction to the existence of integral manifolds.

Definition 14.52. A connection Θ is *flat* if its curvature Ω vanishes.

Assume, then, that Θ is flat, so the horizontal distribution H is integrable. Theorem 6.22 implies that the total space P is foliated by integral manifolds $f_\alpha\colon Y_\alpha \hookrightarrow P$ where Y_α is connected. Denote the image of f_α as $\mathcal{F}_\alpha \subset P$.

Lemma 14.53. *Assume that X is connected. Then the composition $\pi_\alpha = \pi \circ f_\alpha\colon Y_\alpha \to X$ is a Galois covering map.*

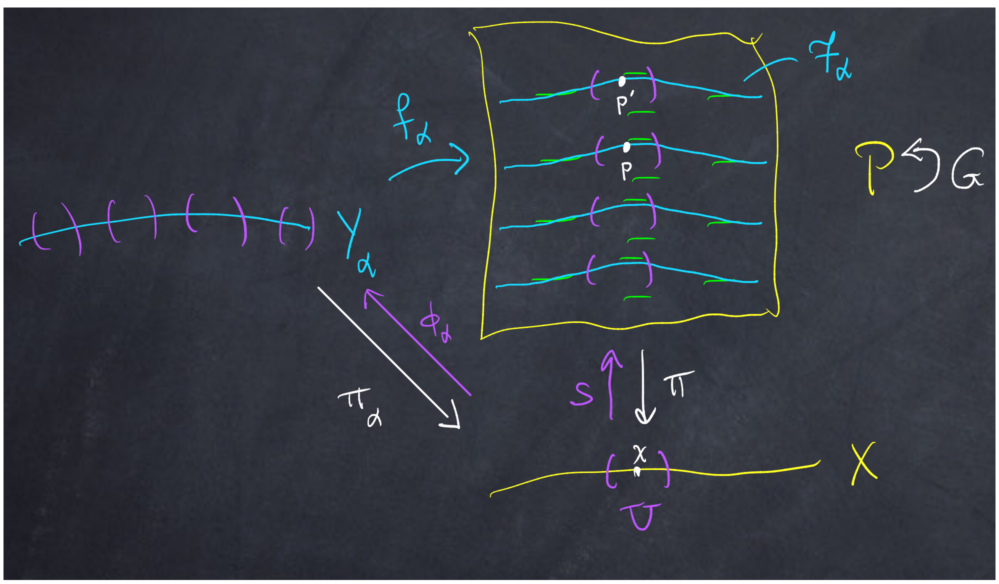

FIGURE 51. The proof of Lemma 14.53

Proof. If $y \subset Y_\alpha$, then $df_\alpha(T_yY_\alpha) = H_{f_\alpha(y)}$, and since $d\pi$ is an isomorphism when restricted to the horizontal distribution, it follows from the inverse function theorem that π_α is a local diffeomorphism. Let $\phi_\alpha\colon U \to Y_\alpha$ be a local inverse to π_α on an open subset $U \subset X$. Then $s := f_\alpha \circ \phi_\alpha$ is a local section of π. Fix $x \in U$ set $p = s(x) \in P_x$. If also $p' \in P_x \cap \mathcal{F}_\alpha$, then $p' = pg$ for some $g \in G$ and $R_g \circ s\colon U \to \mathcal{F}_\alpha$ is also a local section of π. Furthermore, $R_g(\mathcal{F}_\alpha) = \mathcal{F}_\alpha$ since there is a unique maximal integral manifold through any point of P. By Theorem 6.25, $R_g \circ f_\alpha\colon Y_\alpha \to P$ factors through a smooth map $\varphi\colon Y_\alpha \to Y_\alpha$ and $R_g \circ s$ factors through a smooth map $U \to Y_\alpha$ which is a local inverse to π_α. It follows that U is evenly covered by π_α. Since π_α is a local diffeomorphism, it is an open map. Its image is connected, since Y_α is connected. Suppose $x \in X$ is a limit point of the image of π_α. The argument given above implies we can find a neighborhood $U' \subset X$ of x and a local flat section $s\colon U' \to \pi^{-1}(U')$. Its right translates $R_g \circ s$, $g \in G$, foliate $\pi^{-1}(U')$ by flat sections. The image of at least one of these sections s' intersects $\mathcal{F}_\alpha$, since x is a limit point of $\pi(\mathcal{F}_\alpha) \subset X$. Since f_α is a *maximal* integral manifold, its image must include the image of s'; if not, we can paste a right translate of $s(U')$ to $\mathcal{F}_\alpha$ to construct a larger integral manifold. This proves that x is in the image of π_α, which is therefore closed. Since X is assumed connected, it follows that π_α is surjective. Therefore, π_α is a covering map. The existence of φ above shows that automorphisms of π_α act transitively on each fiber, i.e., π_α is a *Galois* covering map. □

Remark 14.54. The argument for surjectivity uses strongly the G-invariance to give uniformity of the domain of flat sections about x. See Example 12.60 for the dire consequences when such uniformity is lacking.

(14.55) *Flat connections and flat sections.*

Definition 14.56. Let $\pi\colon P \to X$ be a principal bundle with connection Θ. A section $s\colon X \to P$ is *flat/horizontal/parallel* if $s^*\Theta = 0$.

Equivalently, s is an integral manifold of the horizontal distribution $H = \ker\Theta$; see Figure 52. For example, in the proof of Lemma 14.53, the local section $s = f_\alpha \circ g$ is flat.

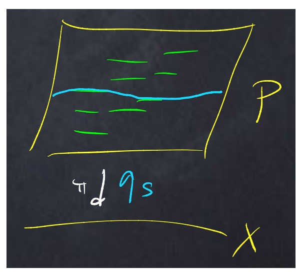

FIGURE 52. A flat section of π

Proposition 14.57. *Let $\pi\colon P \to X$ be a principal bundle with connection Θ. Then Θ is flat iff π admits local flat sections.*

Proof. The forward argument was just given. If $s\colon U \to P\big|_U$ is a flat section of $\pi\big|_{\pi^{-1}(U)}$, then

$$\Omega = (\pi \circ s)^*\Omega = s^*\left(d\Theta + \frac{1}{2}[\Theta \wedge \Theta]\right) = ds^*\Theta + \frac{1}{2}[s^*\Theta \wedge s^*\Theta] = 0. \qquad (14.58)$$ □

Example 14.59. Consider the map

$$\begin{aligned} \pi\colon \mathbb{T} &\longrightarrow \mathbb{T} \\ \lambda &\longmapsto \lambda^2 \end{aligned} \qquad (14.60)$$

It is the nontrivial double cover of the circle, so as in Remark 14.25 it admits a unique connection. This connection is flat, but there is no global (flat) section of π, only local (flat) sections.

(14.61) *Connections over* $[0,1]$. Let $\pi'\colon P' \to [0,1]$ be a principal G-bundle with connection. Since the curvature is a 2-form on a 1-manifold, it necessarily vanishes: the connection is flat. For $q \in P'_0$ let $f_q\colon Y_q \hookrightarrow P'$ be the leaf through q in the foliation on P' determined by the rank one horizontal distribution. Then by Lemma 14.53 the composition $\pi' \circ f_q\colon Y_q \to [0,1]$ is a covering map, and since $[0,1]$ is simply connected it follows that $\pi' \circ f_q$ is a diffeomorphism. Let $\phi_q\colon [0,1] \to Y_q$ be its inverse. Then $s_q := f_q \circ \phi_q$ is a flat section of π' with $s_q(0) = q$.

Definition 14.62. The *parallel transport* map is

$$\begin{aligned} \tau\colon P'_0 &\longrightarrow P'_1 \\ q &\longmapsto s_q(1) \end{aligned} \tag{14.63}$$

The G-invariance of the horizontal distribution implies $s_{qg} = R_g \circ s_q$ for all $g \in G$. It follows that τ is an isomorphism of right G-torsors.

(14.64) *Alternative approach.* To construct parallel transport (14.63), we can argue as follows. First, we prove that any principal bundle $P' \to [0,1]$ admits a global section s. (In fact, any fiber bundle over a contractible base is trivializable.) We want to find $g\colon [0,1] \to G$ such that sg is a *flat* section, i.e., $(sg)^*(\Theta) = 0$, and $g(0) = e$. Compute

$$(sg)^*(\Theta) = \mathrm{Ad}_{g^{-1}}\, s^*(\Theta) + g^*\theta, \tag{14.65}$$

where θ is the Maurer–Cartan form on G. Now apply Theorem 7.8(2).

For a matrix group (14.65) reduces to an ordinary differential equation for a map $g\colon [0,1] \to G$:

$$\frac{dg}{dt} = g(t)A(t) \tag{14.66}$$

for a given function $A\colon [0,1] \to \mathfrak{g}$. If $A(t)$ is a constant function of t, then the solution is the exponential of the matrix A. For a nonconstant function the solution is sometimes called the "path-ordered exponential".

(14.67) *Parallel transport along curves.* Let $\pi\colon P \to X$ be a principal G-bundle with connection, and suppose $\gamma\colon [0,1] \to X$ is a smooth motion. Define the pullback bundle

$$\begin{tikzcd} P' \arrow[r, dashed] \arrow[d, dashed, "\pi'"'] & P \arrow[d, "\pi"] \\ {[0,1]} \arrow[r, "\gamma"] & X \end{tikzcd} \tag{14.68}$$

with its pullback connection. Identify $P'_0 \cong P_{\gamma(0)}$ and $P'_1 \cong P_{\gamma(1)}$. The parallel transport (14.63) is then an isomorphism of right G-torsors

$$\tau_\gamma\colon P_{\gamma(0)} \longrightarrow P_{\gamma(1)} \tag{14.69}$$

Remark 14.70. A connection does not in general give a global parallelism. Proposition 14.57 shows that the curvature is a local obstruction. There is a global obstruction as well—holonomy—which we take up in the next lecture. Parallel transport *along curves* (14.69) is the parallelism that any connection determines.

Proposition 14.71. *Let $\phi\colon [0,1] \to [0,1]$ be a monotonic increasing diffeomorphism. Then the parallel transport along $\gamma \circ \phi$ equals parallel transport along γ.*

In other words, parallel transport is reparametrization invariant.

Proof. Briefly, (14.63) is constructed from integral manifolds of a line field, and the line field is independent of parametrization. $\square$

Remark 14.72.

(1) Define $\gamma^{\mathrm{rev}}(t) = \gamma(1-t)$. Then

$$\tau_{\gamma^{\mathrm{rev}}} = \tau_\gamma^{-1}. \tag{14.73}$$

(2) By composing parallel transport along smooth motions, we can define parallel transport along piecewise smooth motions.

(3) If γ_1, γ_2 are motions with $\gamma_1(1) = \gamma_2(0)$, then there is a composite piecewise smooth motion, and parallel transport composes:

$$\tau_{\gamma_2 \circ \gamma_1} = \tau_{\gamma_2} \circ \tau_{\gamma_1}. \tag{14.74}$$

Parallelism along curves in associated fiber bundles

Let G be a Lie group, and let F be a smooth manifold endowed with a left G-action. In **(13.32)** we constructed a functor[21]

$$\begin{Bmatrix} \text{principal} \\ G\text{-bundles} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{fiber bundles} \\ \text{with fiber } F \end{Bmatrix} \tag{14.75}$$

(14.76) *The induced horizontal distribution.* Now we construct a functor

$$\begin{Bmatrix} \text{principal} \\ G\text{-bundles} \\ \text{with} \\ \text{connection} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{fiber bundles} \\ \text{with fiber } F \\ \text{and horizontal} \\ \text{distribution} \end{Bmatrix} \tag{14.77}$$

Namely, if $\pi\colon P \to X$ is a principal G-bundle with connection given by a horizontal distribution $H \subset TP$, then $H \oplus 0 \subset TP \oplus TF$ is a horizontal distribution on the total space of the product bundle $\underline{F} \to P$. By G-invariance it descends to a horizontal distribution $H_F \subset TF_P$ on the total space $F_P = P \times_G F$ of the associated fiber bundle $\pi_F\colon F_P \to X$

As a generalization[22] of Definition 14.56 we have the following.

Definition 14.78. A section $s\colon X \to F_P$ of π_F is *flat/horizontal/parallel* if $ds_x(T_xX) = (H_F)_{s(x)}$ for all $x \in X$.

[21]We did not spell out what happens on morphisms, only on objects.
[22]in case $F = G$ with G acting by left multiplication

(14.79) *Parallel transport in an associated fiber bundle.* Transport parallel transport in the principal bundle to parallel transport in the associated fiber bundle.

Proposition 14.80. *Let G be a Lie group, let $\pi'\colon P' \to [0,1]$ be a principal G-bundle with connection, and suppose F is a smooth manifold with a left G-action. Assume $t \mapsto p'_t$ is a flat section of π'. Then $t \mapsto [p'_t, f]$ is a flat section of $\pi_F\colon F_{P'} \to X$ for any $f \in F$.*

Recall that $F_{P'}$ is the quotient of $P' \times F$ by a free G-action; the equivalence class of $(p', f) \in P' \times F$ under this action is denoted $[p', f]$. The equivalence relation is

$$[p'g, f] = [p', gf], \qquad p' \in P', \quad g \in G, \quad f \in F. \tag{14.81}$$

Proof. The map $t \mapsto (p'_t, f)$ is a flat section of the product bundle $\mathrm{pr}_1\colon P' \times F \to P'$ with horizontal distribution $H \oplus 0 \subset TP \oplus TF$. Since H_F is descended from this horizontal distribution, the descended section is also flat. ☐

As in **(14.67)** we define parallel transport along curves in associated bundles by applying Proposition 14.80 to the pullback equipped with its pullback horizontal distribution; see **(12.48)**. We use that the pullback of the associated horizontal distribution is the associated horizontal distribution of the pullback. Details are left to the industrious reader.

Lecture 15: Holonomy

Having constructed parallel transport of a connection on a principal bundle, we now specialize to parallel transport around a loop. This is an automorphism of the fiber called *holonomy* or *monodromy*. The group we obtain from all loops is the holonomy group. It is an important invariant of a connection.

We begin with a few definitions of holonomy, both as an automorphism of a fiber of a principal bundle and as an element, or conjugacy class, in the structure group. The precise concept depends on which basepoints are chosen, if any. For abelian structure groups the holonomy is the exponential of an integral of the connection form. For a general group the holonomy is computed from the solution of an ODE; see (14.66).

There are several flavors of holonomy group. We define only the basic holonomy group and the restricted holonomy group, which turns out to be the identity component of the former. Theorem 15.21 tells basic structural properties of these holonomy groups, but we do not prove it in this version of the notes.

We do prove some basic relations between holonomy and curvature. For abelian structure groups, the holonomy around a bounding loop is the exponential of an integral of the curvature. This leads to one variation of the Gauss–Bonnet theorem on surfaces in **(15.36)**. The holonomy bundle is an important construction I believe was introduced by Ambrose–Singer. We construct it and prove the Ambrose–Singer characterization of the Lie algebra of the holonomy group in terms of curvature.

Finally, we turn to flat sections of associated fiber bundles and relate them to fixed points of the action of the holonomy group on the model fiber.

The holonomy group and the restricted holonomy group

(15.1) *Recollection of parallel transport.* Let G be a Lie group, X a smooth manifold, and $\pi\colon P \to G$ a principal G-bundle with connection Θ. To a piecewise smooth motion $\gamma\colon [0,1] \to X$ we constructed in **(14.67)** a parallel transport map

$$\tau_\gamma\colon P_{\gamma(0)} \longrightarrow P_{\gamma(1)} \tag{15.2}$$

It is invariant under reparametrization, inverts when the path is reversed, and composes when paths are concatenated; see Remark 14.72.

(15.3) *Definition of holonomy.* Fix $x \in X$. Let $\Omega_x X$ be the space of piecewise smooth loops $\gamma\colon [0,1] \to X$ at x, i.e., $\gamma(0) = \gamma(1) = x$. Topologize $\Omega_x X$ with the compact-open topology. Let $\Omega_x^0 X \subset \Omega_x X$ denote the subspace of contractible loops, i.e., loops that represent the identity element in $\pi_1(X, x)$. For $\gamma \in \Omega_x X$, the parallel transport

$$\tau_\gamma\colon P_x \longrightarrow P_x \tag{15.4}$$

defined in (15.2) is an automorphism of P_x as a right G-torsor, called the *holonomy* of Θ around γ. A point $p \in P_x$ induces an isomorphism $G_G \xrightarrow{\cong} P_x$ as right G-torsors, which in turn induces

$$G \xrightarrow{\cong} \mathrm{Aut}_G(P_x), \tag{15.5}$$

an isomorphism of Lie groups; see **(8.1)**, especially (8.8). Changing the point $p \in P_x$ conjugates (15.5) by an element of G, so the holonomy in $\mathrm{Aut}_G(P_x)$ determines a conjugacy class in G.

We elaborate on this last assertion. For $p \in P_x$ the holonomy $h_\gamma(p) \in G$ about γ is computed by lifting γ to a horizontal motion $\tilde{\gamma}\colon [0,1] \to P$ with $\tilde{\gamma}(0) = p$, noting that $\tilde{\gamma}(1) \in P_x$, and then defining $h_\gamma(p)$ as the unique group element such that

$$\tilde{\gamma}(1) = p\, h_\gamma(p). \tag{15.6}$$

If we shift $p \in P_x$ to $R_g(p) = pg \in P_x$ for some $g \in G$, then the new horizontal lift of γ is $R_g \circ \tilde{\gamma}$, since the connection (horizontal distribution) is G-invariant. Since

$$(R_g \circ \tilde{\gamma})(1) = p\, h_\gamma(p) g = pg\, g^{-1} h_\gamma(p) g, \tag{15.7}$$

the holonomy transforms as

$$h_\gamma(pg) = g^{-1} h_\gamma(p) g. \tag{15.8}$$

This equivariance descends h_γ to an element of $\mathrm{Aut}_G(P_x)$, which brings us back to our initial definition of holonomy. Note that if G is abelian, then $h_\gamma(p)$ is independent of $p \in P_x$.

We can also shift the basepoint x. Suppose also $x' \in X$, and we are given a piecewise smooth motion γ from x to x' and a piecewise smooth motion γ' from x' to x. The piecewise smooth loop $\gamma' \circ \gamma$ at x and the loop $\gamma \circ \gamma'$ at x' trace out the same loop motion with different starting points. It follows from Remark 14.72(3) that

$$\tau_{\gamma' \circ \gamma} = \tau_\gamma^{-1} \circ (\tau_{\gamma \circ \gamma'}) \circ \tau_\gamma \tag{15.9}$$

Now $\tau_\gamma \colon P_x \to P_{x'}$ is an isomorphism of right G-torsors, so in that sense $\tau_{\gamma' \circ \gamma}$ is conjugate to $\tau_{\gamma \circ \gamma'}$. If we choose a basepoint $p \in P_x$ and let $p' \in P_{x'}$ be its parallel transport along the horizontal lift of γ, then the elements of G that correspond to $\tau_{\gamma' \circ \gamma}$ and $\tau_{\gamma \circ \gamma'}$ are equal. Combining with (15.8) we see that the holonomy is well-defined as a conjugacy class in G independent of any basepoint on P or on X.

(15.10) *Abelian structure groups.* Suppose $G = A$ is an abelian Lie group with Lie algebra $\mathfrak{a}$. Since conjugacy classes in A are singletons, it follows from the previous section that the holonomy of a connection around a loop γ in X is a well-defined element $h_\gamma \in A$.

Let $\gamma\colon [0,1]\to X$ be a piecewise smooth loop with a horizontal lift $\tilde\gamma\colon [0,1]\to P$, and suppose the holonomy $a\in A$ lies in the identity component of A. (Recall that $a\in A$ satisfies $\tilde\gamma(1)=\tilde\gamma(0)a$.) Then there exists $\zeta\in\mathfrak{a}$ such that $a=e^\zeta$. Define

$$\begin{aligned}\delta\colon [0,1]&\longrightarrow P\\ t&\longmapsto \tilde\gamma(t)e^{-t\zeta}\end{aligned}\tag{15.11}$$

Then $\delta(1)=\delta(0)$, so δ is a lift of γ that is a loop. We claim

$$\delta^*\Theta=-\zeta\,dt,\tag{15.12}$$

where $\Theta\in\Omega^1_P(\mathfrak{a})$ is the connection 1-form. This follows from the general transformation formula for connections[23] and the fact that $\tilde\gamma^*(\Theta)=0$, since $\tilde\gamma$ is a horizontal lift. Then the holonomy is

$$h_\gamma=a=\exp(\zeta)=\exp\left(-\int_{[0,1]}\delta^*\Theta\right).\tag{15.13}$$

We claim that (15.13) holds for *any* lift δ of γ to a loop. For any other lift is $\delta'=\delta g$ for a function $g\colon [0,1]\to A$ that satisfies $g(1)=g(0)$. By the transformation law, $(\delta')^*\Theta=\delta^*\Theta+g^*\theta_A$, where θ_A is the Maurer–Cartan form on A.

Lemma 15.14. *Let A be an abelian Lie group and suppose $g\colon [0,1]\to A$ is a loop in A. Then*

$$\exp\left(-\int_{[0,1]}g^*\theta_A\right)=e.\tag{15.15}$$

Proof. The statement is that θ_A is a closed $\mathfrak{a}$-valued 1-form and all of its periods lie in the lattice $\exp^{-1}(e)\subset\mathfrak{a}$. First, $d\theta_A=0$ by the Maurer–Cartan equation. This means the integral in (15.15), as a function of the loop g, factors to a function $H_1A\to A$. Furthermore, the image of the exponential map $\exp\colon\mathfrak{a}\to A$ is contained in the identity component of A, since $\mathfrak{a}$ is connected. It suffices to consider a connected abelian Lie group, since all components of a Lie group are diffeomorphic. Since A is abelian, exp is a homomorphism of Lie groups. Its differential is the identity map, so it is a local diffeomorphism, hence exp is an open map. It also has closed image: if $\exp(\zeta_n)\to a$ for a sequence $\{\zeta_n\}\subset\mathfrak{a}$ and $a\in A$, then $a^{-1}\exp(\zeta_n)\to e$. Since exp is a local diffeomorphism at e, for sufficiently large n there exists $\eta_n\in\mathfrak{a}$ (in a neighborhood of $0\in\mathfrak{a}$) such that $\exp(\eta_n)=a^{-1}\exp(\zeta_n)$. Then $a=\exp(\zeta_n-\eta_n)$ is in the image of exp. It follows that exp is surjective, since we are now assuming that A is connected. Now $\exp^{-1}(e)\subset\mathfrak{a}$ is a discrete subgroup, hence is a finitely generated free subgroup[24] $\Lambda\subset\mathfrak{a}$. Therefore, $A\cong\mathfrak{a}/\Lambda$ is isomorphic to the Cartesian product of a torus and a vector space. In particular, there is an isomorphism $H_1A\cong\Lambda$, so it suffices to check (15.15) on loops that correspond to elements of Λ. For $\lambda\in\Lambda$ consider the path $\xi(t)=t\lambda$ in $\mathfrak{a}$; its exponential is a loop in A. For the path $t\mapsto\exp\bigl(\xi(t)\bigr)$ the integral in (15.15) is λ, from which (15.15) follows. □

[23] I am giving this as a homework problem this week; it was already used in (14.65).

[24] We do not prove this here. In a basis, this is $\mathbb{Z}^r\subset\mathbb{R}^N$ for some $r\le N=\dim A$.

Example 15.16. Let $A = \mathbb{T}$ be the circle group, and suppose $\pi\colon P \to X$ is a principal $\mathbb{T}$-bundle with connection $\Theta \in \sqrt{-1}\Omega^1_P$. (The Lie algebra of $\mathbb{T} \subset \mathbb{C}$ is naturally $\sqrt{-1}\,\mathbb{R} \subset \mathbb{C}$: differentiate the 1-parameter group $t \mapsto e^{ixt}$ at $t = 0$.) Let $\gamma\colon S^1 \to X$ be a loop. The argument around (15.11) proves that the pullback $\gamma^*P \to S^1$ admits a global section s. Write $\alpha = s^*\Theta$. Then (15.13) asserts that the holonomy of Θ around γ is

$$h_\gamma = \exp\left(-\int_{S^1} \alpha\right). \tag{15.17}$$

Note the minus sign in (15.17): the log holonomy is *minus* the integral of the connection form.

(15.18) *Holonomy groups.* Here are two flavors of holonomy groups. (There are more.)

Definition 15.19. Let G be a Lie group and let Θ be a connection on a principal G-bundle $\pi\colon P \to X$.

(1) The *holonomy group* at $x \in X$ is the subgroup $\mathrm{Hol}_x(\Theta) \subset \mathrm{Aut}_G(P_x)$ which consists of elements τ_γ for $\gamma \in \Omega_x X$. The *holonomy group* at $p \in P_x$ is the subgroup $\mathrm{Hol}_p(\Theta) \subset G$ that corresponds to $\mathrm{Hol}_x(\Theta)$ under the isomorphism (15.5) induced by the choice of $p \in P_x$.
(2) The *restricted holonomy group* at $x \in X$ is the subgroup $\mathrm{Hol}^0_x(\Theta) \subset \mathrm{Aut}_G(P_x)$ which consists of elements τ_γ for $\gamma \in \Omega^0_x X$. The *restricted holonomy group* at $p \in P_x$ is the subgroup $\mathrm{Hol}^0_p(\Theta) \subset G$ that corresponds to $\mathrm{Hol}^0_x(\Theta)$ under the isomorphism (15.5) induced by the choice of $p \in P_x$.

There is an inclusion of groups

$$\mathrm{Hol}^0_p(\Theta) \subset \mathrm{Hol}_p(\Theta) \subset G \tag{15.20}$$

in which G is a Lie group. In fact, the subgroups are also Lie groups.

Theorem 15.21.

(1) $\mathrm{Hol}^0_p(\Theta)$ *is a connected Lie subgroup of* G.
(2) $\mathrm{Hol}_p(\Theta)$ *is a Lie subgroup of* G *with identity component* $\mathrm{Hol}^0_p(\Theta)$.
(3) *There is a surjective homomorphism of (discrete) groups*

$$\pi_1(X, x) \longrightarrow \pi_0 \mathrm{Hol}_p(\Theta) \cong \mathrm{Hol}_p(\Theta)/\mathrm{Hol}^0_p(\Theta). \tag{15.22}$$

The identity component of any Lie group is open and closed in the Lie group, so $\mathrm{Hol}^0_p(\Theta) \subset \mathrm{Hol}_p(\Theta)$ is a closed subgroup. However there is no claim that either $\mathrm{Hol}^0_p(\Theta)$ or $\mathrm{Hol}_p(\Theta)$ is closed in G. In fact, that is not true in general, as we illustrate below in examples. We do not provide a proof of Theorem 15.21 in this version of these notes.

For the Levi–Civita connection of a Riemannian manifold (to be defined soon), we have the following important result.

Theorem 15.23 (Borel–Lichnerowicz)**.** *Let* X *be a Riemannian manifold with orthonormal frame bundle* $\pi\colon \mathcal{B}_O(X) \to X$. *Let* Θ *be the Levi–Civita connection on* π. *Then* $\mathrm{Hol}^0_b(\Theta) \subset \mathrm{SO}_n$ *is a* closed *Lie subgroup for any* $b \in \mathcal{B}_O(X)$.

Examples

Example 15.24 (Klein bottle)**.** The affine torus $\mathbb{A}^2/\mathbb{Z}^2$ has a global parallelism inherited from the global parallelism of $\mathbb{A}^2$ since the translation action of $\mathbb{Z}^2$ preserves the parallelism. Alternatively, $\mathbb{A}^2/\mathbb{Z}^2$ is a right $\mathbb{R}^2/\mathbb{Z}^2$-torsor, and as such it has a global parallelism. The Klein bottle K is the quotient of $\mathbb{A}^2/\mathbb{Z}^2$ by the free involution $(x, y) \mapsto (x + 1/2, -y)$. For any $x \in K$ we have $\mathrm{Hol}^0_x = \{e\}$ and $\mathrm{Hol}_x(\Theta)$ is cyclic of order 2: the holonomy about the noncontractible loop $t \mapsto [t, 0]$, $0 \le t \le 1/2$ in K is the reflection $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ on $\mathbb{R}^2$ (in some basis), and it has order 2.

Example 15.25 (a nonclosed Riemannian holonomy group)**.** Let X be the Riemannian manifold that is the quotient of $\mathbb{E}^1 \times \mathbb{E}^2$ by the free action of the infinite cyclic group generated by the transformation

$$(t, y) \longmapsto \bigl(t + 1, R_\theta(y)\bigr), \qquad t \in \mathbb{E}^1, \quad y \in \mathbb{E}^2, \tag{15.26}$$

where R_θ is rotation through angle θ. Then for any b in the frame bundle of X, the restricted holonomy group is trivial whereas the holonomy group $\mathrm{Hol}_b(\Theta)$ is the subgroup of $\mathrm{SO}_2 \subset \mathrm{SO}_3$ generated by the rotation R_θ. This subgroup is not closed if θ is an irrational multiple of π. This does not contradict Theorem 15.23, which only asserts that the *restricted* holonomy group is a closed subgroup of the special orthogonal group.

Example 15.27 (a nonclosed restricted holonomy group)**.** Let $\pi \colon P \to \mathbb{A}^2$ be a principal $\mathbb{R}$-bundle, which is a synonym for a principal affine line bundle—a fiber bundle with fiber $\mathbb{A}^1$. Let Θ be a connection on π whose curvature $\Omega \in \Omega^2(\mathbb{A}^2)$ is compactly supported and satisfies $\int_{\mathbb{A}^2} \Omega = 1$. We prove below in **(15.35)** that $\mathrm{Hol}^0_p(\Theta) = \mathrm{Hol}_p(\Theta) = \mathbb{R}$ for any $p \in P$. (By general principles, since $\mathbb{A}^2$ is simply connected the restricted holonomy group equals the holonomy group.) Now let $a \in \mathbb{T}$ be a complex number of unit norm that is not a root of unity. Define

$$\begin{aligned} \rho \colon \mathbb{R} &\longrightarrow \mathbb{T}^2 \\ t &\longmapsto (e^{it}, ae^{it}) \end{aligned} \tag{15.28}$$

The (restricted) holonomy group of the associated principal $\mathbb{T}^2$-bundle $\rho(P) \to \mathbb{A}^2$ with connection $\rho(\Theta)$ is $\mathrm{Hol}^0\bigl(\rho(\Theta)\bigr) = \rho\bigl(\mathrm{Hol}^0(\Theta)\bigr) = \rho(\mathbb{R}) \subset \mathbb{T}^2$ is not a closed subgroup.

Holonomy and curvature; Gauss-Bonnet for surfaces

We prove some theorems that illustrate the relationship between holonomy—an integrated form of a connection—and curvature.

(15.29) *Abelian structure groups.* Observe that if A is an abelian Lie group with Lie algebra $\mathfrak{a}$, then the adjoint action of A on $\mathfrak{a}$ is trivial, so the adjoint bundle of a principal A-bundle has constant fiber $\mathfrak{a}$. Hence the curvature is an $\mathfrak{a}$-valued 2-form on the base.

We work with a connection over the closed 2-disk $D^2 \subset \mathbb{A}^2$, which one may think is the pullback via a smooth map of D^2 into another manifold with connection. The first theorem is for the structure group $\mathbb{T}$.

Theorem 15.30. *Let $\pi\colon P \to D^2$ be a principal $\mathbb{T}$-bundle and suppose Θ is a connection on π with curvature $\Omega \in \sqrt{-1}\,\Omega^2_{D^2}$. Endow D^2 with the standard orientation inherited from $\mathbb{A}^2$, and give ∂D^2 the boundary orientation. Then the holonomy of Θ around the boundary is*

$$\tau_{\partial D^2} = \exp\left(-\int_{D^2}\Omega\right). \tag{15.31}$$

For an abelian structure group the holonomy is independent of the basepoint; see (**15.10**).

Proof. Let $s\colon D^2 \to P$ be a section of π.[25] Set $\alpha = s^*\Theta \in \sqrt{-1}\Omega^1_{D^2}$, and then the curvature is $\Omega = d\alpha$. From Example 15.16 we see that the holonomy around the boundary is

$$\tau_{\partial D^2} = \exp\left(-\int_{\partial D^2}\alpha\right) = \exp\left(-\int_{D^2}\Omega\right), \tag{15.32}$$

where we use Stokes' theorem. $\square$

There is a generalization of Theorem 15.30 for arbitrary abelian structure groups.

Theorem 15.33. *Let A be an abelian Lie group with Lie algebra $\mathfrak{a}$, and let $\pi\colon P \to D^2$ be a principal A-bundle with connection Θ and curvature $\Omega \in \Omega^2_{D^2}(\mathfrak{a})$. Endow D^2 with the standard orientation inherited from $\mathbb{A}^2$, and give ∂D^2 the boundary orientation. Then the holonomy of Θ around the boundary is*

$$\tau_{\partial D^2} = \exp\left(-\int_{D^2}\Omega\right), \tag{15.34}$$

where $\exp\colon \mathfrak{a} \to A$ *is the exponential map of the Lie group A.*

The proof, which we leave to the reader, uses the discussion in (**15.10**).

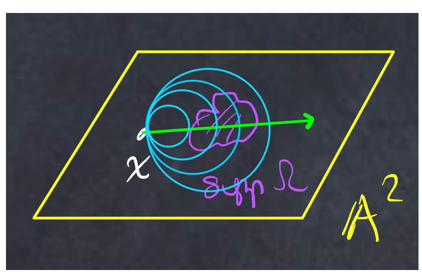

FIGURE 53. Computing holonomy via curvature

[25]A fiber bundle over a contractible base admits a section. In this case we can pick an arbitrary point in the fiber over the center of D^2 and use parallel transport along radii to extend to a section.

(15.35) *An application.* In Example 15.27 we claimed that $\mathrm{Hol}_p(\Theta) = \mathbb{R}$. We sketch a proof using Theorem 15.33. Fix a basepoint $x \in \mathbb{A}^2 \setminus \mathrm{supp}(\Omega)$. As in Figure 53, fix an open ray S emanating from x and passing through $\mathrm{supp}(\Omega)$ and consider the family of closed disks that centers on that ray and containing x in the boundary. Their disjoint union E can be given the structure of a 3-manifold with boundary that is the total space of a fiber bundle $\pi\colon E \to S$. There is a smooth map $f\colon E \to \mathbb{A}^2$. The function[26] $\pi_* f^*(\Omega)\colon S \to \mathbb{R}$ is smooth and has image $[0,1] \subset \mathbb{R}$. Apply Theorem 15.33 to conclude that $\mathrm{Hol}_p(\Theta)$ contains $[0,1] \subset \mathbb{R}$. Since $\mathrm{Hol}_p(\Theta)$ is a subgroup of $\mathbb{R}$, it must equal $\mathbb{R}$.

(15.36) *A Gauss–Bonnet theorem on surfaces.* We resume discussions from Lecture 9–Lecture 11. Let X be a Riemannian 2-manifold. Theorem 11.4 proves the existence of a distinguished connection Θ on the principal O_2-bundle of frames $\mathcal{B}_{\mathrm{O}}(X) \to X$. Let $D \subset X$ be a closed region diffeomorphic to D^2, and fix an orientation of D, so a reduction of the frame bundle to a principal SO_2-bundle $\mathcal{B}_{\mathrm{SO}}(X) \to X$ with restricted connection Θ. Recall from (11.27) that the curvature of Θ is $-K\omega$, where $K\colon X \to \mathbb{R}$ is the Gauss curvature and $\omega \subset \Omega^2_D$ is the area form determined by the metric and orientation. Then Theorem 15.33 implies that the holonomy around the oriented boundary of D is rotation through angle

$$\theta_{\partial D} = \exp\left(\int_D K\,\omega\right). \tag{15.37}$$

The reader should re-examine Example 10.43 and Remark 10.44 in light of this formula.

(15.38) *Another relation between holonomy and curvature.* Let G be a Lie group and suppose $\pi\colon P \to X$ is a principal G-bundle with connection Θ. Fix $x \in X$ and vectors $\xi_x, \eta_x \in T_xX$. Extend the vectors to local vector fields ξ, η that commute: $[\xi,\eta] = 0$. For example, choose a chart $(U; x^1,\dots,x^n)$ about x, write ξ_x,η_x in terms of the frame $\partial/\partial x^1,\dots,\partial/\partial x^n$ at x, and then extend to ξ,η on U which have constant coefficients in the moving frame of the coordinate vector fields. Let $\widetilde{\xi},\widetilde{\eta}$ denote the horizontal lifts of ξ,η to $\pi^{-1}(U)$. Fix $p \in P_x$, and let φ,ψ be local flows near p with generators $\widetilde{\xi},\widetilde{\eta}$, respectively. The vector fields $\widetilde{\xi},\widetilde{\eta}$ are π-related to ξ,η in the sense of Definition 2.65, so the flows φ,ψ project onto the *commuting* flows of ξ,η. For sufficiently small $|t|,|s|$ define $h(t,s) \in G$ by

$$\psi_{-s}\varphi_{-t}\psi_s\varphi_t(p) = ph(t,s). \tag{15.39}$$

The left hand side lies in P_x since the projected local flows on X commute. Note that $h(t,s)$ is the holonomy around the image of the rectangle in the (t,s)-plane with vertices $(0,0)$, $(t,0)$, (t,s), $(0,s)$.

[26] Here $\pi_*\colon \Omega^2(E) \to \Omega^0(S)$ is integration along the fibers of π, an operation we have not discussed in these notes. We orient the relative tangent bundle using the orientation of $\mathbb{A}^2$ and the fact that *df* restricted to the relative tangent bundle is an isomorphism at each point of E.

Theorem 15.40. *The curvature at p is*

$$\Omega_p(\widetilde{\xi}_p, \widetilde{\eta}_p) = -\frac{\partial^2}{\partial s \partial t}\bigg|_{t=s=0} h(t,s). \tag{15.41}$$

Proof. By a computation similar to (14.44) we have

$$\Omega_p(\widetilde{\xi}_p, \widetilde{\eta}_p) = -\Theta\big([\widetilde{\xi}, \widetilde{\eta}]_p\big). \tag{15.42}$$

Apply Proposition 2.58 to compute the Lie bracket as a second derivative of the left hand side of (15.39). Then (15.41) follows by differentiating the right hand side. □

The holonomy bundle and the Ambrose–Singer theorem

Let G be a Lie group and $\pi\colon P \to X$ a principal G-bundle with connection Θ. For $p \in P$ define

$$P(p) = \{q \in P : \text{there exists a piecewise smooth horizontal motion} \\ \gamma\colon [0,1] \to P \text{ such that } \gamma(0) = p, \gamma(1) = q\} \tag{15.43}$$

and let $H = \mathrm{Hol}_p(\Theta)$. Suppose X is connected.

Theorem 15.44. *There exists a smooth manifold structure on $P(p)$ and a free right H-action on $P(p)$ such that the restriction of π to $P(p) \subset P$ is a principal H-bundle $\pi'\colon P(p) \to X$. There is a canonical connection Θ' on π' that induces Θ via the inclusion $H \hookrightarrow G$. Furthermore, this reduced connection has the same holonomy group: $\mathrm{Hol}_p(\Theta') = H = \mathrm{Hol}_p(\Theta)$.*

Remark 15.45. By G-invariance of the connection, $P(pg) = R_g\big(P(p)\big)$ for all $g \in G$. Right translation induces isomorphisms of the holonomy bundles $\pi'\colon P(p) \to X$ for different choices of $p \in P$. If Θ is flat, then $P(p)$ is a leaf of the foliation of P generated by the connection; see **(14.51)**.

The proof of Theorem 15.44 follows fairly quickly from the following.

Lemma 15.46. *Let $\pi\colon P \to X$ be a principal G-bundle, and suppose $H \subset G$ is a Lie subgroup. Assume $Q \subset P$ is a subset such that*

(i) *$\pi\big|_Q\colon Q \to X$ is surjective;*
(ii) *$Q \cdot H = Q$;*
(iii) *if $q, q' \in Q$ satisfy $\pi(q) = \pi(q')$, then there exists $h \in H$ such that $q' = qh$; and*
(iv) *for all $x \in X$ there exists a local cross section of π with image in Q.*

Then $\pi\big|_Q\colon Q \to X$ is a reduction of π to a principal bundle with structure group H.

Proof. Let σ be a local cross section of π with domain $U \subset X$ that satisfies the condition in (iv). As in (13.5) use σ to construct an isomorphism of principal G-bundles $\varphi\colon U \times G \to \pi^{-1}(U)$; the condition in (iv) implies that the restriction

$$\varphi\big|_{U \times H}\colon U \times H \longrightarrow \big(\pi\big|_Q\big)^{-1}(U) \tag{15.47}$$

is a bijection. Cover X by such local trivializations and use the isomorphisms (15.47) construct a smooth manifold structure on Q. The reader should carefully think through the argument that the overlaps are smooth; Theorem 6.25 is used in the proof. The maps (15.47) then provide local trivializations of the quotient map $\pi\big|_Q$ of the free H-action on Q, so $\pi\big|_Q$ is a principal H-bundle. $\square$

Proof of Theorem 15.44. First, $H = \mathrm{Hol}_p(\Theta) \subset G$ is a Lie subgroup by Theorem 15.21(2). It is straightforward to prove that $P(p) \subset P$ satisfies (i)–(iii) of Lemma 15.46. To verify (iv), fix $x \in X$ and choose a chart about x that identifies a neighborhood of $x \in X$ with a neighborhood of $0 \in \mathbb{A}^n$. Choose $q \in P(p)_x$ and use parallel transport to lift radial rays emanating from 0 to horizontal paths in P that emanate from q. The paths lie in $P(p) \subset P$ by definition, so in this way we construct a local section of $\pi\big|_Q$. Now Lemma 15.46 implies that $\pi\big|_{P(p)}$ is a principal H-bundle. It remains to observe that $\ker(\Theta)\big|_Q \subset TQ$, where $\ker(\Theta) \subset TP$ is the G-invariant horizontal distribution that defines the connection Θ. For this, note that the differential of the local section just constructed maps T_xX onto $\ker(\Theta)_q \subset T_qQ$. The H-invariance of $\ker(\Theta)\big|_Q$ follows from the G-invariance of $\ker(\Theta)$. $\square$

Continuing with the setup of Theorem 15.44, let $\Omega \in \Omega^2_P(\mathfrak{g})$ be the curvature of the connection Θ.

Theorem 15.48 (Ambrose–Singer)**.** *The Lie algebra of* $\mathrm{Hol}_p(\Theta)$ *is the subspace* $\mathfrak{g}' \subset \mathfrak{g}$ *spanned by*

$$\Big\{\Omega_{p'}(\widetilde{\xi},\widetilde{\eta}) : p' \in P(p), \quad \widetilde{\xi},\widetilde{\eta} \in \ker(\Theta)_{p'}\Big\} \tag{15.49}$$

Proof. As before, write $H = \mathrm{Hol}_p(\Theta)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. Then $\mathfrak{g}' \subset \mathfrak{h}$ since the curvature Ω equals the curvature of the connection Θ restricted to the holonomy bundle $P(p) \to X$ with structure group H. By (14.4), $\mathfrak{g}' \subset \mathfrak{h}$ determines a subbundle $V' \subset T\big(P(p)/X\big)$. Define the distribution

$$E = \ker(\Theta) \oplus V' \subset T\big(P(p)\big). \tag{15.50}$$

We claim that E is integrable. To verify this, fix $\zeta, \zeta_1, \zeta_2 \in \mathfrak{g}'$ and use hats to denote the induced sections of $V' \to P(p)$. Also, fix local vector fields ξ, ξ_1, ξ_2 on X and use tildes to denote their horizontal lifts to $\ker(\Theta)$. Then

$$\begin{aligned}
&[\widehat{\zeta}_1', \widehat{\zeta}_2'] \text{ lies in } V' \text{ since } \mathfrak{g}' \text{ is a Lie algebra,}\\
&[\widehat{\zeta}', \widetilde{\xi}] = \frac{d}{dt}\Big|_{t=0} \big(R_{e^{t\zeta'}}\big)_* \widetilde{\xi} \text{ lies in } \ker(\Theta), \text{ and}\\
&[\widetilde{\xi}_1, \widetilde{\xi}_2] = -\Omega(\widetilde{\xi}_1, \widetilde{\xi}_2) \text{ lies in } V' \text{ by the definition of } \mathfrak{g}'.
\end{aligned} \tag{15.51}$$

Let $P' \subset P(p)$ be the leaf of the induced foliation that passes through p. For $p' \in P(p)$, let γ be a horizontal curve joining p to p'. Then the tangents to γ lie in $\ker(\Theta) \subset E$, so $\gamma' \subset P'$. In particular, $p' \subset P'$. It follows that $P' = P(p)$. Therefore, $V' = T\big(P(p)/X\big)$ and so $\mathfrak{g}' = \mathfrak{h}$. $\square$

Flat sections of associated fiber bundles

Let G be a Lie group, F a smooth manifold equipped with a left G-action, and $\pi\colon P \to X$ a principal G-bundle with connection Θ. The associated fiber bundle $\pi_F\colon F_P \to X$ is defined in **(13.32)**, and the horizontal distribution on π_F associated to Θ is defined in **(14.76)**.

(15.52) *Horizontal sections.* Recall that a section ψ of π_F lifts to an equivariant function

$$\psi\colon P \longrightarrow F, \qquad \psi(pg) = g^{-1}\psi(p), \qquad p \in P, \quad g \in G, \tag{15.53}$$

and conversely such equivariant functions descend to sections of π_F.

Proposition 15.54. *If a section ψ is parallel, then for all $p \in P$ the point $\psi(p) \in F$ is fixed by the holonomy subgroup $\mathrm{Hol}_p(\Theta) \subset G$.*

Recall that we use 'flat', 'horizontal', and 'parallel' interchangeably in this context.

Proof. Fix $p \in P$ and let $t \mapsto p_t$ be a horizontal motion in P with $p_0 = p$. Then by Proposition 14.80 the motion $t \mapsto [p_t, f]$ is horizontal in F_P for any $f \in F$. Now suppose $\pi(p_1) = \pi(p_0)$ and write $p_1 = p_0h$ for $h \in G$; then $h \in \mathrm{Hol}_p(\Theta)$. Let $\psi\colon P \to F$ satisfy the equivariance property (15.53). Set $f = \psi(p_0)$. If ψ is parallel, then

$$[p_t, \psi(p_t)] = [p_t, f], \qquad t \in [0, 1], \tag{15.55}$$

so in particular

$$[p_0, f] = [p_0, \psi(p_0)] = [p_1, \psi(p_1)] = [p_0h, f] = [p_0, h^{-1}f] \tag{15.56}$$

from which we conclude that h fixes f. ☐

Proposition 15.57. *Assume now that X is connected and for some $p \in P$ there is a $\mathrm{Hol}_p(\Theta)$ fixed point $f \in F$. Then there exists a unique parallel section ψ of π_F such that $\psi(p) = f$.*

Proof. Let $\pi'\colon P(p) \to X$ be the holonomy bundle through p; see (15.43). The fiber bundle $\pi_F\colon F_P \to X$ is canonically isomorphic to the fiber bundle $\pi'_F\colon F_{P'} \to X$ associated to π' via the restricted action of $\mathrm{Hol}_p(\Theta) \subset G$ on F. Define $\psi'\colon P(p) \to F$ as the constant function with value f; it is a parallel section of π'_F. Furthermore, every parallel section of π'_F lifts to a constant function on $P(p)$, so ψ' is the unique parallel section with $\psi'(p) = f$. The extension ψ of ψ' to an equivariant function $P \to F$ descends to the same section of π'_F. ☐

Corollary 15.58. *Parallel sections of π_F correspond to constant functions $P(p) \to F$ on the holonomy bundle, which in turn correspond to $\mathrm{Hol}_p(\Theta)$ fixed points in F.*

Example 15.59. Let X be a Riemannian manifold with orthogonal frame bundle $\pi\colon \mathcal{B}_{\mathrm{O}}(X) \to X$. The Riemannian metric is a section of $T^*X \otimes T^*X \to X$. (In fact, it descends to a section of the quotient bundle $\mathrm{Sym}^2 T^*X \to X$.) This is the vector bundle associated to π via the representation of O_n on $(\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*$. The standard metric on $\mathbb{R}^n$ is a fixed point of that action, and the associated *parallel* section of $T^*X \otimes T^*X \to X$ is the Riemannian metric.

(15.60) *Parallel tensor fields and geometric structures.* Let (G_n, λ_n) be an n-dimensional symmetry type. Recall Definition 13.52 of a (G_n, λ_n)-structure on a smooth manifold X. The G_n-invariant tensors in tensor powers of $\mathbb{R}^n$ are the linear invariants of (G_n, λ_n)-geometry, and they induce parallel tensor fields on any smooth manifold with a (G_n, λ_n)-structure and a linear (G_n, λ_n)-connection. The (G_n, λ_n)-structure is a principal G_n-bundle $\pi\colon P \to X$ with an isomorphism of the frame bundle and the associated bundle $\lambda_n(\pi) \to X$. A linear (G_n, λ_n)-connection is a connection on π.

For example, the O_n-invariant standard metric and volume form on $\mathbb{R}^n$ induce a parallel metric and parallel volume form on any Riemannian manifold with orthogonal connection. Similarly, any symplectic manifold with symplectic connection has a parallel orientation[27] and parallel volume form, together with the parallel symplectic form and its parallel powers. A complex manifold has a complex structure on its tangent bundle that is parallel for any compatible connection.

The invariants of (G_n, λ_n)-geometry need not be linear. Let $F = \mathrm{Map}\bigl(\mathrm{Gr}_k(\mathbb{R}^n), \mathrm{Gr}_{n-k}(\mathbb{R}^n)\bigr)$ denote the space[28] of smooth assignments of an $(n-k)$-plane in $\mathbb{R}^n$ to a k-plane in $\mathbb{R}^n$. The general linear group $\mathrm{GL}_n\mathbb{R}$ acts of F as does any Lie group G_n equipped with a homomorphism $\lambda_n\colon G_n \to \mathrm{GL}_n\mathbb{R}$. There is an O_n fixed point, namely the map $W \mapsto W^{\perp}$. Therefore, on any Riemannian manifold with an orthogonal connection, there is a parallel map that takes a k-dimensional subspace of a tangent space to its orthogonal complement. However, there is no $\mathrm{GL}_n\mathbb{R}$ fixed point in F, which means there is no such *parallel* assignment on a general smooth manifold.

Example 15.61. As opposed to **(15.60)**, this example is *extrinsic*, not intrinsic.

Consider the $\mathbb{C}^\times$-action on the complex projective line $\mathbb{CP}^1$ by linear fractional (projective) transformations. Namely, recall that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2\mathbb{C}$ acts on $[z^0, z^1] \in \mathbb{CP}^1$ by

$$[z^0, z^1] \longmapsto [az^0 + bz^1, cz^0 + dz^1], \tag{15.62}$$

and furthermore the action drops to the projective general linear group $\mathrm{PSL}_2\mathbb{C}$. Let $\lambda \in \mathbb{C}^\times$ act by

$$[z^0, z^1] \longmapsto [\lambda z^0, \lambda^{-1} z^1]. \tag{15.63}$$

The action is by a composition of a homothety and a rotation, both of which fix the poles $n = [1, 0]$ and $s = [0, 1]$ of $\mathbb{CP}^1$. Now a principal $\mathbb{C}^\times$-bundle $\pi\colon P \to X$ has an associated complex line bundle[29] $L \to X$. The fiber bundle associated to π with fiber $\mathbb{CP}^1$ may be identified with

$$\mathbb{P}(L \oplus \underline{\mathbb{C}}) \longrightarrow X, \tag{15.64}$$

the projectivization of the rank two bundle obtained by stabilizing $L \to X$ by the trivial line bundle with fiber $\mathbb{C}$. The north and south poles induce sections of (15.64) that correspond to the lines L and $\underline{\mathbb{C}}$. These sections are parallel with respect to any connection on π.

[27] An orientation is parallel under any linear connection!

[28] For this illustration we can simply consider F as a set and not contemplate the associated fiber bundle with fiber F. If you like, you can topologize F with the compact-open topology, and there are various ways to make (completions) of F into a smooth infinite dimensional manifold, but none of that is necessary for the point of this example.

[29] In fact, there is an equivalence of categories between principal $\mathbb{C}^\times$-bundles and complex line bundles, which I invite you to spell out in detail. (There are different possible interpretations.)

Lecture 16: Homogeneous manifolds; linear connections

This lecture treats two distinct topics: (1) homogeneous manifolds and (2) connections on the frame bundle.

Homogeneous manifolds are characterized as carrying a transitive action of a Lie group G. This is a geometric action—G acts on the *left*—and it can be put to good use to study the geometry. G-torsors are particular examples, but in general there is a nontrivial stabilizer group at each point. This stabilizer group H is treated as a structure group for the geometry, and we show that the *intrinsic* geometry is controlled by a principal H-bundle. A *reductive structure* yields a G-invariant connection on this principal bundle, which is often called a *homogeneous connection*. The curvature of a homogeneous connection has a simple formula (Proposition 16.19).

Since homogeneous connections are intrinsic, they provide a transition from the general study of connections on principal bundles in the last few lectures to the intrinsic situation: connections on the principal bundle of frames. These *linear connections* give a smooth manifold the structure of parallelism along curves, which in future lectures is used to define geodesic motions and directional (covariant) derivatives of tensor fields. We also introduce linear connections compatible with a geometric structure (Definition 16.45). For a general linear connection we define *torsion*, which is a translational analog of curvature. In this lecture we only give formulas; in a future lecture we discuss the geometric interpretation.

Reductive homogeneous manifolds

(16.1) *Homogeneous manifolds.* Let G be a Lie group and suppose X is a smooth manifold equipped with a transitive left action of G. We say that X is a *homogeneous manifold*. Recall that the G-action determines a linear map

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathfrak{X}(X) \\ \xi &\longmapsto \widehat{\xi} \end{aligned} \tag{16.2}$$

which is an *anti*homomorphism of Lie algebras. Fix $x \in X$ and let $H = \mathrm{Stab}_x \subset G$ be the stabilizer subgroup. Then $H \subset G$ is a closed Lie subgroup, and the G-action induces a diffeomorphism

$$\begin{aligned} G/H &\longrightarrow X \\ gH &\longmapsto gx \end{aligned} \tag{16.3}$$

In fact, (16.3) is an isomorphism of G-manifolds.

Remark 16.4. The manifold G has a basepoint: the identity element $e \in G$. Its image in G/H is a basepoint in the homogeneous manifold: the coset $[H] \in G/H$. The guiding principle for homogeneous manifolds is that H-invariant geometry at the basepoint extends to G-invariant geometry on the manifold.

Observe that any complement $\mathfrak{p} \subset \mathfrak{g}$ to the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ determines an isomorphism

$$\mathfrak{p} \hookrightarrow \mathfrak{g} \xrightarrow{(16.2)} \mathfrak{X}(X) \xrightarrow{\mathrm{ev}_x} T_xX \tag{16.5}$$

when composed with evaluation at x. The adjoint representation of G on its Lie algebra $\mathfrak{g}$ restricts to an action of H, and the H-action preserves $\mathfrak{h} \subset \mathfrak{g}$, where it restricts to the adjoint action of H.

Definition 16.6. A *reductive structure* on G/H is an H-invariant complement $\mathfrak{p} \subset \mathfrak{g}$ to $\mathfrak{h} \subset \mathfrak{g}$. We say that G/H is *reductive* if it admits a reductive structure.

In other terms, a reductive structure is an H-invariant splitting of the short exact sequence

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0, \tag{16.7}$$

or equivalently an H-invariant direct sum decomposition

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}. \tag{16.8}$$

Infinitesimally, $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ is a module over $\mathfrak{h}$ with module structure the Lie bracket in $\mathfrak{g}$:

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &\subset \mathfrak{h} \\ [\mathfrak{h}, \mathfrak{p}] &\subset \mathfrak{p} \end{aligned} \tag{16.9}$$

Remark 16.10. If also $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$, then we say that G/H is a *symmetric space*.

Remark 16.11. If H is a *compact* Lie group, then any homogeneous manifold G/H is reductive.

Example 16.12 (spheres)**.** A sphere is a homogeneous manifold, potentially in many different ways. In the following chart we indicate the sphere X, the Lie group G that acts transitively, and the stabilizer group H.

(16.13)

X	G	H
S^n	O_{n+1}	O_n
S^{2m+1}	U_{m+1}	U_m
S^{4k+3}	Sp_{k+1}	Sp_k
S^6	G_2	SU_3
S^7	Spin_7	G_2
S^{15}	Spin_9	Spin_7

Here $n, m, k \in \mathbb{Z}^{\geq 0}$. Each example has H compact, so H-invariant complements $\mathfrak{p} \subset \mathfrak{g}$ to $\mathfrak{h} \subset \mathfrak{g}$ exist. Note that S^7 is a homogeneous manifold in 4 different ways in this chart. Also, each Lie group G acts by isometries if X is endowed with the round metric, so these are Riemannian homogeneous manifolds. Another noteworthy observation is that S^6 carries a G_2-invariant almost complex structure, since SU_3 preserves a complex structure on $\mathbb{R}^6$. (It might be called an "almost Calabi-Yau structure".) A famous open question: Does S^6 admit a complex structure?

Example 16.14 (a non-reductive example)**.** Let $G = \mathrm{Aff}_1\mathbb{R}$ be the group of symmetries of the standard affine line $\mathbb{A}^1$, and let X be the one dimensional real vector space of translationally-invariant vector fields on $\mathbb{A}^1$. Then G acts transitively on X with stabilizer group H the translation subgroup $\mathbb{R} \subset \mathrm{Aff}_1\mathbb{R}$. There is no H-invariant complement to the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of infinitesimal translations. To see this, write $\mathfrak{g} = \mathfrak{aff}_1\mathbb{R} = \mathfrak{gl}_1\mathbb{R} \oplus \mathfrak{h}$, where $\mathfrak{gl}_1\mathbb{R}$ is the Lie algebra of infinitesimal affine transformations that fix $0 \in \mathbb{A}^1$. The complement of the line $[\mathfrak{h}]$ in the real projective line $\mathbb{P}\mathfrak{g} = \mathbb{P}(\mathfrak{aff}_1\mathbb{R})$ is an affine line[30] A. The action of H on this projective line fixes the point $[\mathfrak{h}]$ and acts by translation on A. So there is no H fixed point other than $[\mathfrak{h}]$, hence no H-invariant complement to $\mathfrak{h} \subset \mathfrak{g}$. The matrix representation

$$(16.15)\quad G = \left\{\begin{pmatrix} \lambda & a \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{R}^{\neq 0},\ a \in \mathbb{R}\right\} \qquad H = \left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}\right\} \qquad \mathbb{A}^1 = \left\{\begin{pmatrix} t \\ 1 \end{pmatrix} : t \in \mathbb{R}\right\}$$

is useful for making computations. Then

$$(16.16)\qquad \mathfrak{g} = \left\{\begin{pmatrix} \dot\lambda & \dot a \\ 0 & 0 \end{pmatrix} : \dot\lambda, \dot a \in \mathbb{R}\right\},$$

the projective space $\mathbb{P}\mathfrak{g}$ can be identified with $\big\{[\dot\lambda, \dot a]\big\}$, and then $A = \{[1, \dot a] : \dot a \in \mathbb{R}\}$.

Homogeneous connections

(16.17) *The canonical connection on a reductive homogeneous manifold.* Let G/H be a reductive homogeneous manifold with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Then $\mathfrak{p} \subset \mathfrak{g}$ defines a left-invariant distribution $D_{\mathfrak{p}}$ on the Lie group G. Since the adjoint action by H preserves $\mathfrak{p}$, we have $(R_h)_*D_{\mathfrak{p}} = D_{\mathfrak{p}}$ for $h \in H$ acting by right multiplication. Furthermore, the distribution $D_{\mathfrak{p}}$ is horizontal for the principal H-bundle

$$(16.18)\qquad \pi\colon G \longrightarrow G/H,$$

the quotient of G under right multiplication by H. (It is true at $e \in G$, since $\mathfrak{p}$ is a complement to $\mathfrak{h}$, and by G-invariance it is true everywhere.) Hence $D_{\mathfrak{p}}$ is a connection on the principal bundle π.

Proposition 16.19. *The curvature tensor of the connection $D_{\mathfrak{p}}$ is*

$$(16.20)\qquad \Omega(\xi, \eta) = -[\xi, \eta]_{\mathfrak{h}}, \qquad \xi, \eta \in \mathfrak{p}.$$

Here we write $\zeta = \zeta_{\mathfrak{p}} + \zeta_{\mathfrak{h}}$ relative to the direct sum decomposition (16.8). Also, the curvature is (left) G-invariant, since the connection $D_{\mathfrak{p}}$ is, so we identify it with an $\mathfrak{h}$-valued 2-form on the horizontal space $(D_{\mathfrak{p}})_e = \mathfrak{p}$ at the identity $e \in G$.

Proof. The vectors $\xi, \eta \subset \mathfrak{p}$ induce vector fields on G that belong to the distribution $D_{\mathfrak{p}}$. Now (16.20) follows since the curvature is minus the Frobenius tensor (Definition 14.37). □

[30]This is the standard decomposition of a projective line as an affine line union a point ("at infinity").

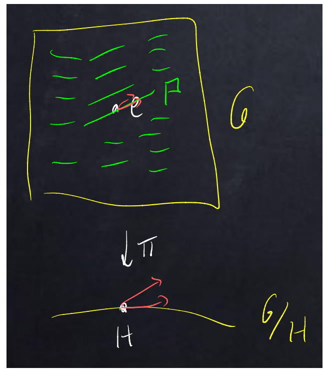

FIGURE 54. The principal H-bundle (16.18) with its connection

(16.21) *The canonical bundle is* intrinsic. Recall that a symmetry type is a pair (H,λ) of a Lie group H and a homomorphism $\lambda\colon H\to GL_n\mathbb{R}=\mathrm{Aut}(\mathbb{R}^n)$. A reductive homogeneous space produces a variation: a Lie group H and a homomorphism $\lambda\colon H\to\mathrm{Aut}(\mathfrak{p})$. Recall from (16.5) that $\mathfrak{p}$ is isomorphic to the tangent space to G/H at any point. The frame bundle $\varpi\colon\mathcal{B}(G/H)\to G/H$ consists of a point $x\in G/H$ and an isomorphism $\mathbb{R}^n\to T_x(G/H)$. Modify the construction to have an isomorphism $\mathfrak{p}\to T_x(G/H)$ instead, and call the modified frame bundle $\varpi'\colon\mathcal{B}(G/H)'\to G/H$; its structure group is $\mathrm{Aut}(\mathfrak{p})$. A basis $\mathbb{R}^n\xrightarrow{\cong}\mathfrak{p}$ produces an identification of the frame bundles.

We claim there is a canonical isomorphism of the modified frame bundle ϖ' with the principal bundle $\lambda(\pi)$ associated to (16.18) via the homomorphism $\lambda\colon H\to\mathrm{Aut}(\mathfrak{p})$. Namely, this is the map of principal $\mathrm{Aut}(\mathfrak{p})$-bundles

$$\begin{tikzcd} G\times_H\mathrm{Aut}(\mathfrak{p}) \arrow[rr] \arrow[dr, "\rho(\pi)"'] & & \mathcal{B}'(G/H) \arrow[dl, "\varpi'"] \\ & G/H & \end{tikzcd} \tag{16.22}$$

that maps $[g,\phi]\subset G\times_H\mathrm{Aut}(\mathfrak{p})$ to the isomorphism (twisted frame)

$$\begin{aligned} \mathfrak{p} &\longrightarrow T_{gH}(G/H) \\ \xi &\longmapsto \widehat{\mathrm{Ad}_g\,\phi(\xi)}\big|_{gH} \end{aligned} \tag{16.23}$$

The hat denotes the vector field defined by the G-action; see (16.2).

Remark 16.24. Somewhat easier to parse at first is that the tangent bundle is associated to π in (16.18) via the action of H on $\mathfrak{p}$. This is the canonical isomorphism

$$\begin{aligned} G\times_H\mathfrak{p} &\longrightarrow T(G/H) \\ [g,\xi] &\longmapsto \widehat{\mathrm{Ad}_g\,\xi}\big|_{gH} \end{aligned} \tag{16.25}$$

In other terms, a section of the associated vector bundle $G \times_H \mathfrak{p} \to G/H$ is an H-equivariant function $\psi\colon G \to \mathfrak{p}$; the equivariance condition is $\psi(gh) = \mathrm{Ad}_{h^{-1}}\,\psi(g)$, $g \in G$, $h \in H$. If we regard $\psi(g) \in \mathfrak{p}$ as an element of the distribution $D_{\mathfrak{p}}$ at $g \in G$, then ψ maps to the vector field on G/H whose value at gH is $\pi_*\psi(g)$.

The bottom line is that the homogeneous manifold G/H carries a canonical (H,λ)-structure. So, as in Remark 16.4, the group H controls the geometry of G/H.

Example 16.26 (The sphere)**.** Consider the sphere as the homogeneous manifold $S^n \approx \mathrm{O}_{n+1} / \mathrm{O}_n$, as indicated in the first line of (16.13). In terms of block skew-symmetric $(n+1)\times(n+1)$ matrices the Lie subalgebra $\mathfrak{o}_n \subset \mathfrak{o}_{n+1}$ is

$$\mathfrak{o}_n = \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A \end{array}\right) : A \in \mathfrak{o}_n \right\} \tag{16.27}$$

and we choose the O_n-invariant complement

$$\mathfrak{p} = \left\{ \left(\begin{array}{c|c} 0 & v^* \\ \hline v & 0 \end{array}\right) : v \in \mathbb{R}^n \right\} \tag{16.28}$$

Note that $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{o}_n$. Write M_v for the matrix in (16.28). Then the curvature (16.20) is

$$\Omega(v,w) = -[M_v, M_w] = v \otimes w^* - w \otimes v^* \tag{16.29}$$

Here $w^* \in (\mathbb{R}^n)^*$ is the covector $w^*(u) = \langle w, u\rangle$, $u \in \mathbb{R}^n$, and $v \otimes w^* \in \mathbb{R}^n \otimes (\mathbb{R}^n)^* \cong \mathrm{End}(\mathbb{R}^n) = M_n\mathbb{R}$; the matrix in (16.29) is skew symmetric. In fact, under the identifications $\bigwedge^2\mathfrak{p} \cong \bigwedge^2\mathbb{R}^n$ and $\mathfrak{o}_n \cong \bigwedge^2\mathbb{R}^n$, the curvature $\Omega\colon \bigwedge^2\mathfrak{p} \to \mathfrak{o}_n$ is the identity map. This is equivalent to saying that S^n has *constant curvature* (a concept we will introduce later).

Connections on the frame bundle

(16.30) *Standard bases.* Let $e_1,\dots,e_n$ be the standard basis of $\mathbb{R}^n$; e_i is the column vector with 1 in the i^{th} position and 0 elsewhere. The dual basis of $(\mathbb{R}^n)^*$ consists of row vectors $e^1,\dots,e^n$. Set

$$E_i^j = e_i \otimes e^j = \begin{array}{c c} & j \\ i & \begin{pmatrix} & \vdots & \\ & 1 & \\ & \vdots & \end{pmatrix} \end{array} \in \mathbb{R}^n \otimes (\mathbb{R}^n)^* \cong M_n\mathbb{R} \cong \mathfrak{gl}_n\mathbb{R}. \tag{16.31}$$

Thus if $A \in \mathfrak{gl}_n\mathbb{R}$ we write $A = A^i_j\, E_i^j$.

(16.32) *Linear connections.* Let X be a smooth manifold and let $\varpi\colon \mathcal{B}(X)\to X$ be its principal $GL_n\mathbb{R}$-bundle of frames. Recall from (14.1) that associated to $A\in\mathfrak{gl}_n\mathbb{R}$ is a vertical vector field $\widehat{A}$. In particular, there is a basis of vertical vector fields $\widehat{E}^j_i$. The tautological $\mathbb{R}^n$-valued 1-form

$$\theta = \theta^i e_i \tag{16.33}$$

on $\mathcal{B}(X)$ generalizes (11.1). Namely, if $x\in X$ and $b\in\mathcal{B}(X)_x$ is a basis $b\colon \mathbb{R}^n\xrightarrow{\cong} T_xX$, then

$$\pi_*\xi = b\bigl(\theta(\xi)\bigr) = \theta^i(\xi)b(e_i),\qquad \xi\in T_b\mathcal{B}(X). \tag{16.34}$$

The form θ is sometimes called the *soldering form*: it solders the frame bundle to the base. The soldering form θ satisfies

$$\begin{aligned} \iota_{\widehat{A}}\theta &= 0, &\qquad & A\in\mathfrak{gl}_n\mathbb{R}\\ R_M^*\theta &= M^{-1}\theta & & M\in GL_n\mathbb{R}\end{aligned} \tag{16.35}$$

It follows that θ descends to an element of $\Omega^1_X(TX)$, and it is easy to see that the descent may be identified with $\mathrm{id}_{TX}\colon TX\to TX$.

Definition 16.36. A *linear connection* on X is a connection on $\varpi\colon\mathcal{B}(X)\to X$.

The connection form Θ is a $\mathfrak{gl}_n\mathbb{R}$-valued 1-form on X, which we write in our basis as

$$\Theta = \Theta^i_j E^j_i \tag{16.37}$$

Then as in Lemma 14.28 we have for $M\in GL_n\mathbb{R}$ that

$$\begin{aligned} \iota_{\widehat{E}^\ell_k}\Theta^i_j &= \delta^i_k\delta^\ell_j\\ R_M^*\Theta &= M^{-1}\Theta M\end{aligned} \tag{16.38}$$

The connection form Θ does not descend to X.

(16.39) *The global parallelism of* $\mathcal{B}(X)$. Suppose X has a linear connection. Then $\mathcal{B}(X)$ is canonically parallelized. We have already described a basis $\widehat{E}^j_i$ of vertical vector fields. There is a canonical horizontal vector field ∂_v associated to $v\in\mathbb{R}^n$ whose value at $b\colon\mathbb{R}^n\xrightarrow{\cong}T_xX$ in $\mathcal{B}(X)$ is the horizontal lift to $T_b\mathcal{B}(X)$ of $b(v)\in T_xX$; see Figure 55. For $v=e_k$ an element of the standard basis we write $\partial_{e_k}=\partial_k$, and we collate these into an $(\mathbb{R}^n)^*$-valued horizontal vector field

$$\partial = \partial_k e^k \tag{16.40}$$

The vector fields $\partial_k, \widehat{E}^j_i$ are a global parallelism of $\mathcal{B}(X)$. The dual basis of 1-forms is θ^k, Θ^i_j.

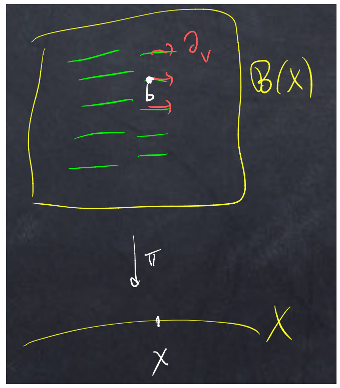

FIGURE 55. The canonical horizontal vector field

Remark 16.41.

(1) The global parallelism is a linear isomorphism

$$\underline{\mathfrak{aff}_n\mathbb{R}} \cong \underline{\mathbb{R}^n} \oplus \underline{\mathfrak{gl}_n\mathbb{R}} \longrightarrow T\big(\mathcal{B}(X)\big) \tag{16.42}$$

of vector bundles over $\mathcal{B}(X)$.

(2) The tautological 1-form θ is $\mathbb{R}^n$-valued and the connection 1-form Θ is $\mathfrak{gl}_n\mathbb{R}$-valued. The dual vector field ∂ is $(\mathbb{R}^n)^*$-valued and $A \mapsto A^i_j \widehat{E}^j_i$ is a vertical $(\mathfrak{gl}_n\mathbb{R})^*$-valued vector field.

(3) Recall that if X is an n-dimensional affine space with the linear connection that defines the global parallelism by translation, then $\mathcal{B}(X)$ is a right $\mathrm{Aff}_n\mathbb{R}$-torsor and θ, Θ are the Maurer–Cartan forms. They satisfy the Maurer–Cartan equations (8.18)

$$\begin{aligned} d\theta + \Theta \wedge \theta &= 0 \qquad\qquad & d\theta^i + \Theta^i_j \wedge \theta^j &= 0 \\ d\Theta + \Theta \wedge \Theta &= 0 & d\Theta^i_j + \Theta^i_k \wedge \Theta^k_j &= 0 \end{aligned} \tag{16.43}$$

which here are written first in matrix notation and then in terms of scalar differential forms. For a general manifold X with linear connection these structure equations are modified—the right hand side is in general nonzero, as we see below in (16.48), (16.50), and (16.51)

(16.44) *Linear connections and geometric structures.* Recall Definition 13.52 of a geometric structure on a smooth manifold X. Thus suppose (G_n, λ_n) is an n-dimensional symmetry type. An (G_n, λ_n)-structure on a smooth n-manifold X is a pair (π, φ) consisting of a principal G_n-bundle $\pi\colon P \to X$ and an isomorphism $\varphi\colon \mathcal{B}(X) \to \lambda_n(P)$ of principal $\mathrm{GL}_n\mathbb{R}$-bundles over X.

Definition 16.45. A linear connection Θ is compatible with the (G_n, λ_n)-structure if there exists a connection Θ' on $\pi\colon P \to X$ such that $\Theta = \varphi(\Theta')$.

The data of a compatible linear connection is simply a connection Θ' on $\pi\colon P \to X$.

Example 16.46. In Riemannian geometry, a compatible linear connection is called an *orthogonal connection*, and it is a connection on the orthonormal frame bundle $\mathcal{B}_O(X) \to X$ of a Riemannian manifold X.

Curvature and torsion

(16.47) *Definitions.* The curvature is defined for a connection on any principal bundle; see Definition 14.37 and Proposition 14.42. The torsion is only defined for linear connections.

The curvature $\Omega \in \Omega^2_{\mathcal{B}(X)}(\mathfrak{gl}_n\mathbb{R})$, written using matrix multiplication rather than Lie bracket, is

$$\Omega = d\Theta + \Theta \wedge \Theta = \Omega^i_j E^j_i, \qquad \Omega^i_j = d\Theta^i_j + \Theta^i_k \wedge \Theta^k_j. \tag{16.48}$$

The curvature descends to an element of $\Omega^2_X(\operatorname{End} TX)$, since the adjoint bundle to the bundle of frames is the bundle $\operatorname{End} TX \to X$ of endomorphisms of the tangent bundle.

Definition 16.49. The *torsion* of a linear connection Θ is

$$\tau = d\theta + \Theta \wedge \theta \quad \in \Omega^2_{\mathcal{B}(X)}(\mathbb{R}^n). \tag{16.50}$$

A linear connection is *torsionfree* if $\tau = 0$.

Write $\tau = \tau^i e_i$. Then

$$\tau^i = d\theta^i + \Theta^i_j \wedge \theta^j. \tag{16.51}$$

Lemma 16.52. *For $A \in \mathfrak{gl}_n\mathbb{R}$ and $M \in \mathrm{GL}_n\mathbb{R}$ we have*

$$\begin{aligned} \iota_{\hat{A}}\tau &= 0 \\ R^*_M\tau &= M^{-1}\tau \end{aligned} \tag{16.53}$$

It follows that τ descends to an element of $\Omega^2_X(TX)$.

Proof. Compute

$$\begin{aligned} \iota_{\hat{A}}\tau &= \iota_{\hat{A}}(d\theta + \Theta \wedge \theta) \\ &= -d\iota_{\hat{A}}\theta + \mathcal{L}_{\hat{A}}\theta + A\theta \\ &= \quad 0 \quad + \mathcal{L}_{\hat{A}}\theta + A\theta, \end{aligned} \tag{16.54}$$

and

$$\mathcal{L}_{\hat{A}}\theta = \frac{d}{dt}\bigg|_{t=0} R^*_{e^{tA}}\theta = \frac{d}{dt}\bigg|_{t=0} e^{-tA}\theta = -A\theta. \tag{16.55}$$

Now $\iota_{\hat{A}}\tau = 0$ follows. For the second equation in (16.53) we have

$$\begin{aligned} R^*_M\tau &= R^*_M(d\theta + \Theta \wedge \theta) \\ &= d(M^{-1}\theta) + M^{-1}\Theta M \wedge M^{-1}\theta \\ &= M^{-1}(d\theta + \Theta \wedge \theta) \\ &= M^{-1}\tau \end{aligned} \tag{16.56}$$

∎

(16.57) *The Bianchi identities.* From an identity in differential forms we learn new identities by differentiation.[31]

Proposition 16.58 (Bianchi identities)**.** *The differentials of torsion and curvature satisfy*

$$d\tau + \Theta \wedge \tau = \Omega \wedge \theta \tag{16.59}$$

$$d\Omega + [\Theta \wedge \Omega] = 0 \tag{16.60}$$

Later we will interpret the left hand sides as the *covariant* differentials $d_\Theta \tau$ and $d_\Theta \Omega$. Note that these equations descend to X.

Proof. For the first equation, apply d to (16.50):

$$\begin{aligned} d\tau &= d\Theta \wedge \theta - \Theta \wedge d\theta \\ &= \Omega \wedge \theta \;-\; \Theta \wedge \Theta \wedge \theta \;-\; \Theta \wedge \tau \;+\; \Theta \wedge \Theta \wedge \theta \\ &= \Omega \wedge \theta - \Theta \wedge \tau \end{aligned} \tag{16.61}$$

To prove (16.60), apply d to (16.48):

$$\begin{aligned} d\Omega &= d\Theta \wedge \Theta - \Theta \wedge d\Theta \\ &= \Omega \wedge \Theta \;-\; \Theta \wedge \Theta \wedge \Theta \;-\; \Theta \wedge \Omega \;+\; \Theta \wedge \Theta \wedge \Theta \\ &= \Omega \wedge \Theta - \Theta \wedge \Omega \end{aligned} \tag{16.62}$$

□

(16.63) *Indices galore!* Use the global basis θ^k, Θ^i_j of 1-forms on $\mathcal{B}(X)$ to define functions

$$T^i_{k\ell}, R^i_{jk\ell} \colon \mathcal{B}(X) \longrightarrow \mathbb{R} \tag{16.64}$$

that are coefficients of torsion and curvature:

$$\tau^i = \frac{1}{2} T^i_{k\ell}\, \theta^k \wedge \theta^\ell, \qquad T^i_{k\ell} = -T^i_{\ell k} \tag{16.65}$$

$$\Omega^i_j = \frac{1}{2} R^i_{jk\ell}\, \theta^k \wedge \theta^\ell \qquad R^i_{jk\ell} = -R^i_{j\ell k} \tag{16.66}$$

The coefficients of $\theta \wedge \Theta$ and $\Theta \wedge \Theta$ vanish since the contraction of these 2-forms with a vertical vector vanishes; see (14.44), (14.45), and (16.53).

The next computations follow directly from the structure equations (16.48) and (16.50).

[31] Equation (16.60) holds for connections on general principal bundles.

Proposition 16.67. *The bracketing relations among the basis vector fields on $\mathcal{B}(X)$ are:*

$$[\widehat{E}_i^j, \widehat{E}_k^\ell] = \delta_k^j \widehat{E}_i^\ell \;-\; \delta_i^\ell \widehat{E}_k^j \tag{16.68}$$

$$[\widehat{E}_i^j, \partial_k] = \delta_k^j \partial_i \tag{16.69}$$

$$[\partial_k, \partial_\ell] = -T_{k\ell}^i \partial_i \;-\; R_{jk\ell}^i \widehat{E}_i^j \tag{16.70}$$

Remark 16.71.

(1) The last relation (16.70) gives an important interpretation of curvature and torsion. Namely, up to sign, the torsion is the horizontal component of the Lie bracket of horizontal vector fields and the curvature is the vertical component. (This latter is the general interpretation of curvature in terms of the Frobenius tensor of the horizontal distribution.)

(2) The first two bracketing relations (16.68) and (16.69) have constant coefficients. In general, the torsion functions $T_{k\ell}^i$ and curvature functions $R_{jk\ell}^i$ are nonconstant. If they are constant, then (16.68)–(16.70) may be the structure equations of a Lie algebra. In particular, if $T = R = 0$ that Lie algebra is $\mathfrak{aff}_n\mathbb{R}$.

Lecture 17: Geodesics and torsion

In this lecture we explore the geometry of a smooth manifold equipped with a linear connection, focusing on two distinct concepts: geodesic motion and torsion. The linear connection induces parallelism along curves, and it is this parallelism that gives rise to these concepts.

A geodesic motion in affine space is one with constant velocity. The same definition applies on a smooth manifold with linear connection, since the parallelism along the motion leads to the concept of 'constant velocity'. (Soon we study infinitesimal parallel transport—*covariant differentiation*—and then can define *acceleration*; a geodesic motion has zero acceleration.) The equation for a geodesic is a system of second-order ODEs, and in **(17.5)** we use the standard device of rewriting it as a "doubled" system of first-order ODEs. Here that device appears in a geometric form using the frame bundle, where a geodesic becomes an integral curve of the standard horizontal vector fields. Existence and uniqueness of geodesics is thereby reduced to the standard theorems on integral curves of vector fields.

In the remainder of the lecture we give various geometric manifestations of the torsion of a linear connection. (As we have seen, the curvature is a more general concept that applies to connections on arbitrary principal bundles, and it has a cogent geometric interpretation in terms of the Frobenius tensor.) The first, in **(17.14)**, interprets torsion as the failure of a certain broken geodesic path to close to second order. The second interpretation brings us to the affine bundle of frames and the affine tangent bundle. A linear connection induces (Definition 17.32) an *affine connection*—a connection on the affine bundle of frames—and the torsion is the translational part of its curvature. Our third manifestation of torsion is an expression (17.50) for the de Rham operator in terms of differentiation by the horizontal vector fields on the total space of the frame bundle. This holds iff the linear connection is torsionfree, and it is one motivation for favoring torsionfree linear connections.

In our study of the affine connection induced from a linear connection, we introduce *development* **(17.42)** and *normal coordinates* **(17.44)**. The latter are canonical coordinate charts centered at each point $x \in X$ of a smooth manifold X equipped with a linear connection. The charts take values in the affine tangent space A_xX; the coordinate is the inverse of the *exponential map*. The exponential map does not generally intertwine the parallelisms on the affine space A_xX and on the curved manifold X, but it does if the linear connection is flat and torsionfree.

Geodesic motion

(17.1) *Geodesic motion in affine space.* Let A be an affine space over a vector space V. A *constant velocity motion* $\gamma\colon (a,b) \to A$ is one for which $\dot\gamma(t) \in V$ is independent of t, as the name suggests. The term *geodesic motion* is used synonymously for this class of motions. The parallelism of affine space is used to compare the velocities at different times t. Crucially, though, only the parallelism along the motion γ is used, not the global parallelism of affine space. This enables us to define geodesic motions in a smooth manifold equipped with a linear connection.

(17.2) *Geodesic motion in a smooth manifold.* Let X be a smooth manifold equipped with a linear connection, that is, a connection on the principal $\mathrm{GL}_n\mathbb{R}$-bundle of frames $\varpi\colon \mathcal{B}(X) \to X$. Recall that the tangent bundle $TX \to X$ is associated to ϖ via the standard representation of $\mathrm{GL}_n\mathbb{R}$ on $\mathbb{R}^n$. Suppose $\gamma\colon (a,b) \to X$ is a motion in X. Let $\tilde{\gamma}\colon (a,b) \to \mathcal{B}(X)$ be a horizontal lift; see **(14.61)**, **(14.67)**, and the surrounding text. A vector field along γ is a section of the pullback tangent bundle $\gamma^*TX \to (a,b)$. Present this pullback as the associated bundle

$$\gamma^*\mathcal{B}(X) \times_{\mathrm{GL}_n\mathbb{R}} \mathbb{R}^n \longrightarrow (a,b). \tag{17.3}$$

Then a section of $\gamma^*TX \to (a,b)$ is represented as $t \mapsto [\tilde{\gamma}(t), v(t)]$ for some curve $v\colon (a,b) \to \mathbb{R}^n$ of vectors; the square brackets denote the equivalence class in the balancing construction. According to **(14.79)**, the section is parallel iff $v(t)$ is a constant vector independent of t.

Definition 17.4. A motion $\gamma\colon (a,b) \to X$ is *geodesic* iff the velocity $\dot{\gamma}$ is parallel along γ.

Often the simpler term *geodesic* is used in place of 'geodesic motion'. Whereas the velocity of a motion on any smooth manifold is defined, we need the parallelism-along-curves of a linear connection to define acceleration. The geodesic equation, which equates the acceleration to zero, is a second-order ODE that we will see explicitly in local coordinates in the next lecture. It is second order since it is an equation for the time derivative of the velocity, and the velocity is the time derivative of position.

(17.5) *First-order formulation.* It is standard to rewrite a system of second-order differential equations as a system of first-order differential equations by introducing the derivatives as independent variables. Here we do so in a geometric form by passing to the frame bundle.

Recall that for each $v \in \mathbb{R}^n$ there is a horizontal vector field ∂_v on $\mathcal{B}(X)$; see **(16.39)**.

Proposition 17.6. *Fix $v \in \mathbb{R}^n$ and suppose $\tilde{\gamma}\colon (a,b) \to \mathcal{B}(X)$ is an integral curve of ∂_v. Then the projection $\gamma = \varpi \circ \tilde{\gamma}\colon (a,b) \to X$ is a geodesic motion in X.*

Proof. The motion $\tilde{\gamma}$ is horizontal, i.e., its velocity ∂_v is horizontal. Recall that for $x \in X$ and $b \in \mathcal{B}(X)_x$ a basis $b\colon \mathbb{R}^n \xrightarrow{\cong} T_xX$, the vector $\partial_v(b)$ is the horizontal lift of $b(v)$. In other words, as a section of (17.3) the velocity of γ is $t \mapsto [\tilde{\gamma}(t), v]$, which is parallel along γ. $\square$

Observe that

$$(R_M)_*\partial_v = \partial_{M^{-1}v}, \qquad M \in \mathrm{GL}_n\mathbb{R}, \tag{17.7}$$

which implies that $R_M \circ \tilde{\gamma}$ is an integral curve of $\partial_{M^{-1}v}$ that projects to the same integral motion on X as does $\tilde{\gamma}$.

(17.8) *Existence and uniqueness.* We use the existence and uniqueness of integral curves to prove the existence and uniqueness theorem for geodesics.

Theorem 17.9. *Let X be a smooth manifold equipped with a linear connection. Then for each $x \in X$ and $\xi \in T_xX$ there exists $\epsilon > 0$ and a unique geodesic motion $\gamma\colon (-\epsilon, \epsilon) \to X$ such that $\gamma(0) = x$ and $\dot\gamma(0) = \xi$.*

The uniqueness means that any two geodesic motions agree on their common domain.

Proof. For existence, fix $x \in X$ and $\xi \in T_xX$. Choose a frame b at x such that $b(e_1) = \xi$. Let $\tilde\gamma\colon (-\epsilon, \epsilon) \to \mathcal{B}(X)$ be an integral curve of ∂_1 with initial condition $\tilde\gamma(0) = b$. Then $\gamma = \varpi \circ \tilde\gamma$ is the desired geodesic motion on X.

For uniqueness, if $\beta\colon (-\delta, \delta) \to X$ is a geodesic motion with $\beta(0) = x$ and $\dot\beta(0) = \xi$, let $\tilde\beta\colon (-\delta, \delta) \to \mathcal{B}(X)$ be the horizontal lift with $\tilde\beta(0) = b$, where $b \in \mathcal{B}(X)$ is the same frame as in the previous paragraph. Then the velocity of $\tilde\beta$ at $t = 0$ is ∂_1, the horizontal vector that projects to ξ. Since β is a geodesic motion, the velocity of $\tilde\beta$ at time t is $\partial_1 = [\tilde\beta(t), e_1]$. That is, $\tilde\beta$ is an integral curve of ∂_1. Therefore, the motions $\tilde\beta$ and $\tilde\gamma$ agree on their common domain, as do their compositions with ϖ. $\square$

(17.10) *Geodesic completeness.* Geodesic motions may run off a manifold in finite time.

Definition 17.11. Let X be a smooth manifold equipped with a linear connection. Then X is *geodesically complete* if the maximal domain of every geodesic motion is $\mathbb{R}$.

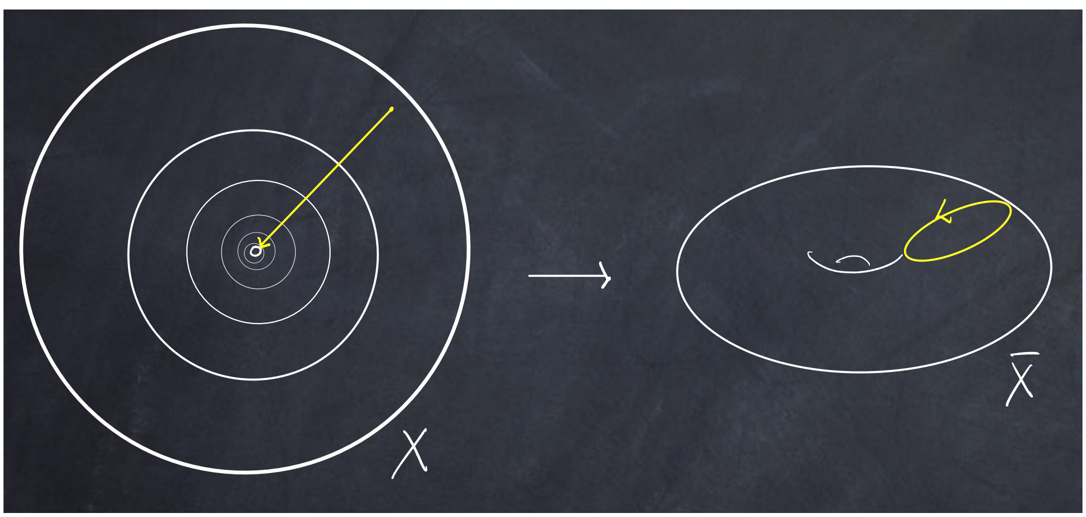

FIGURE 56. A noncomplete geodesic on an affine torus

Example 17.12. The manifold $X = \mathbb{A}^2 \setminus \{0, 0\}$ with its global parallelism is not geodesically complete: a constant velocity motion in $\mathbb{A}^2$ that passes through $(0, 0)$ defines two geodesic motions on X, neither of which has maximal domain $\mathbb{R}$. Now quotient X by the infinite cyclic group of homotheties generated by $(x, y) \mapsto (2x, 2y)$. The quotient manifold $\overline{X}$ is diffeomorphic to $S^1 \times S^1$, and since a homothety is an affine transformation—it maps parallel (constant) vector fields to parallel vector fields—it inherits a global parallelism. But $\overline{X}$ is not geodesically complete since its universal cover X is not. In particular, a ray that begins at a point $[c, c] \in \overline{X}$ with velocity $(-1, -1) \in \mathbb{R}^2$ spins around the torus and into oblivion in finite time; see Figure 56. (The homothety does not preserve any metric, so this parallelism is not compatible with a Riemannian structure. In particular, there is no notion of the speed of the geodesic.)

Remark 17.13. The Hopf–Rinow theorem tells that for the Levi–Civita connection on a Riemannian manifold, geodesic completeness is equivalent to completeness of the underlying metric space. In particular, a closed Riemannian manifold is geodesically complete.

Torsion and parallel transport

Next we give various geometric interpretations of the torsion of a linear connection (Definition 16.49). The reader may wish to review **(15.38)** for an analogous discussion of curvature.

Throughout X is a smooth manifold equipped with a linear connection, i.e., a connection on its frame bundle $\varpi\colon \mathcal{B}(X) \to X$.

(17.14) *Geodesics, parallel transport, and torsion.* Fix a point $x \in X$, a frame $b\colon \mathbb{R}^n \to T_xX$, and vectors $v, w \in \mathbb{R}^n$. Write $\xi = b(v)$ and $\eta = b(w)$. Let φ, ψ be local flows on $\mathcal{B}(X)$ generated by ∂_v, ∂_w, respectively. By Proposition 2.58 we have

$$[\partial_v, \partial_w]\big|_b = \frac{\partial^2}{\partial t \partial s}\bigg|_{t=s=0} \psi_{-s}\varphi_{-t}\psi_s\varphi_t(b). \tag{17.15}$$

Recall from (16.50) that the torsion of a linear connection Θ is

$$\tau = d\theta + \Theta \wedge \theta, \tag{17.16}$$

where θ is the tautological $\mathbb{R}^n$-valued 1-form on $\mathcal{B}(X)$. Evaluate on the horizontal vector fields ∂_v, ∂_w to deduce

$$\tau(\partial_v, \partial_w) = -\theta\big([\partial_v, \partial_w]\big). \tag{17.17}$$

Therefore, $-[\partial_v, \partial_w]\big|_b$ projects to $\tau_x(\xi, \eta)$ under ϖ_*.

Equation (17.15) computes the Lie bracket in terms of the flows, so it remains to interpret the projection of the flows to X. By Proposition 17.6 these projections are geodesics. Furthermore, the vector field ∂_w projects to the parallel vector field $[\varphi_t, w]$ along $\varpi \circ \varphi_t$. Hence the composition on the right hand side of (17.15) projects to the following sequence of motions, depicted in Figure 57:

(1) Follow the geodesic with initial position x and initial velocity ξ for time t to end up at x_1. Let $\xi_1 \in T_{x_1}X$ be the terminal velocity; it is the parallel transport of ξ along the geodesic.
(2) Parallel transport η along that geodesic to $\eta_1 \in T_{x_1}X$, and now follow the geodesic with initial position x_1 and initial velocity η_1 for time s to end up at x_2. Let $\eta_2 \in T_{x_2}X$ be the terminal velocity.
(3) Parallel transport ξ_1 along that geodesic to $\xi_2 \in T_{x_2}X$, and now follow the geodesic with initial position x_2 and initial velocity $-\xi_2$ for time t to end up at x_3.

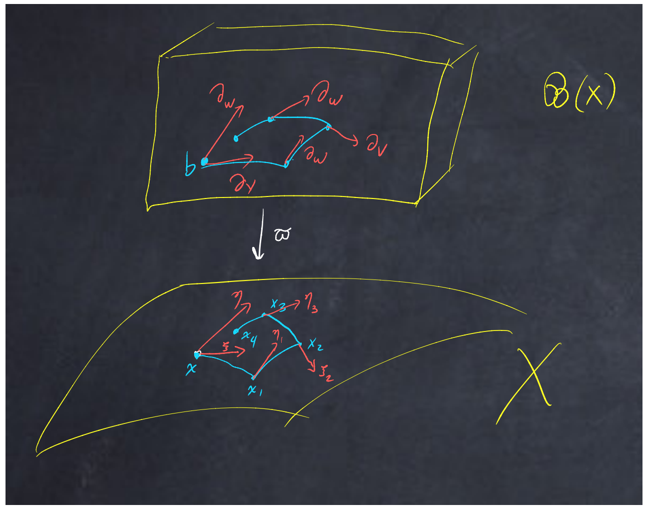

FIGURE 57. Torsion measures the failure of this "parallelogram" to close

(4) Parallel transport η_2 along that geodesic to $\eta_3 \in T_{x_3}X$, and now follow the geodesic with initial position x_3 and initial velocity $-\eta_3$ for time s to end up at x_4.

Then $x_4(t, s)$ is the function whose mixed second partial[32] computes $-\tau_x(\xi, \eta)$.

Remark 17.18. The horizontal component of $-[\partial_v, \partial_w]$ is the torsion; the vertical component is the curvature. The latter follows by evaluating

$$\Omega = d\Theta + \Theta \wedge \Theta \tag{17.19}$$

on ∂_v, ∂_w; see also (16.70).

(17.20) *The flows in affine space.* If $X = A$ is an affine space over a vector space V, then the geodesic motions sweep out line segments, and the "parallelogram" depicted in Figure 57 is an honest parallelogram. In particular, $x_4 = x$ for all t, s, and so the torsion of the linear connection vanishes. (The curvature vanishes as well.) Up on the total space $\mathcal{B}(A) \approx A \times \mathcal{B}(V)$ the flows of ∂_v, ∂_w commute.

Torsion and the affine tangent bundle

The torsion is an infinitesimal translation whereas curvature is an infinitesimal general linear transformation. This suggests that together they form an infinitesimal affine transformation. Hence given a smooth manifold X with linear connection, we are motivated to study parallel transport on the fiber bundle

$$AX \longrightarrow X \tag{17.21}$$

[32] Observe that $x_4(t, 0) = x_4(0, s) = x$ for all t, s. Hence the first partial with respect to s at $s = 0$ maps t to a vector in T_xX. The derivative at $t = 0$ is then a vector in T_xX, as desired.

whose fiber at $x \in X$ is the affine space A_xX formed by forgetting the origin of the tangent space T_xX. (Recall **(13.45)**.) We begin by examining this parallel transport on an affine space.

(17.22) *Affine parallel transport on affine space.* Let A be an affine space over V. Then the affine bundle (17.21) is the trivial bundle

$$\mathrm{pr}_1\colon A \times A \longrightarrow A \tag{17.23}$$

with fiber A. It has a global parallelism given by this product structure, which makes constant sections parallel, but that is not the parallelism we want to use. Rather, for $x \in A$ we view the "infinitesimal affine space A_x as a copy of A centered at x, so want $x \in A_x$ to parallel transport to $y \in A_y$ for all $y \in A$. In other words, the diagonal section of (17.23) is parallel.

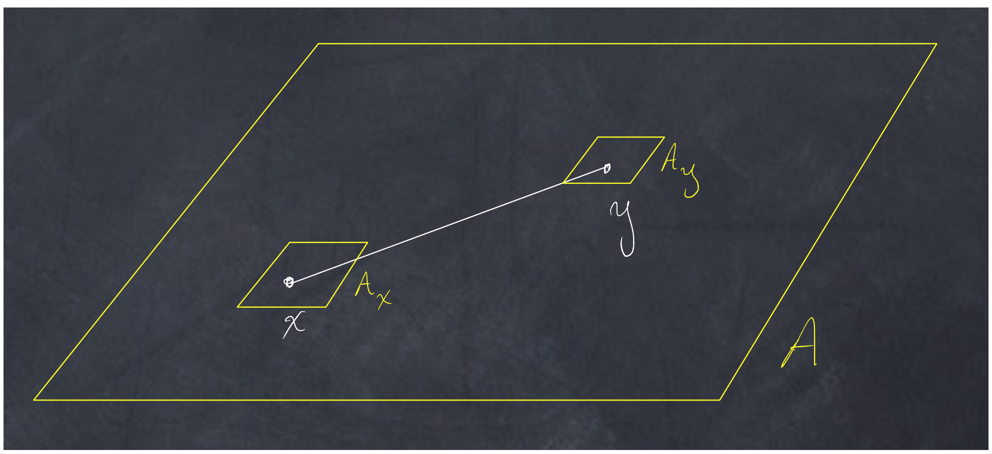

FIGURE 58. Affine parallel transport in affine space

(17.24) *Recollection of the affine group.* Recall from (1.25) that the affine group $\mathrm{Aff}_n\mathbb{R}$ is a split extension of the general linear group by translations:

$$1 \longrightarrow \mathbb{R}^n \xrightarrow{\;\iota\;} \mathrm{Aff}_n\mathbb{R} \underset{\sigma}{\overset{\delta}{\rightleftarrows}} \mathrm{GL}_n\mathbb{R} \longrightarrow 1 \tag{17.25}$$

The image of σ consists of affine transformations of $\mathbb{A}^n$ that fix the origin. The infinitesimal version of (17.25) is the split extension of Lie algebras[33]

$$1 \longrightarrow \mathbb{R}^n \xrightarrow{\;\dot\iota\;} \mathfrak{aff}_n\mathbb{R} \underset{\dot\sigma}{\overset{\dot\delta}{\rightleftarrows}} \mathfrak{gl}_n\mathbb{R} \longrightarrow 1 \tag{17.26}$$

which gives rise to a direct sum decomposition

$$\mathfrak{aff}_n\mathbb{R} = \mathfrak{gl}_n\mathbb{R} \oplus \mathbb{R}^n. \tag{17.27}$$

[33]The inclusion $\mathbb{R}^n \to \mathfrak{aff}_n\mathbb{R}$ is labeled "iota dot".

(17.28) *The affine bundle of frames.* We recall **(13.45)** as follows.

Definition 17.29. Let X be a smooth manifold. The *affine bundle of frames* of X is the principal $\mathrm{Aff}_n\mathbb{R}$-bundle $\varpi^{\mathrm{aff}}\colon \mathcal{A}(X) \to X$ associated to the principal $GL_n\mathbb{R}$-bundle of frames via the homomorphism $\sigma\colon GL_n\mathbb{R} \to \mathrm{Aff}_n\mathbb{R}$.

The linear and affine bundles of frames fit into the diagram

$$\begin{tikzcd} \mathcal{A}(X) \arrow[dd, "\varpi^{\mathrm{aff}}"'] \arrow[dr, shift left, "\delta"] & \\ & \mathcal{B}(X) \arrow[ul, shift left, "\sigma"] \arrow[dl, "\varpi"] \\ X & \end{tikzcd} \tag{17.30}$$

in which ϖ is a principal $GL_n\mathbb{R}$-bundle; ϖ^{aff} is a principal $\mathrm{Aff}_n\mathbb{R}$-bundle; and δ is an affine bundle of rank n (principal $\mathbb{R}^n$-bundle), the pullback of $AX \to X$ via ϖ. A point $b \in \mathcal{B}(X)_x$ in the fiber of ϖ over $x \in X$ is a linear isomorphism $b\colon \mathbb{R}^n \xrightarrow{\cong} T_xX$. A point $f \in \mathcal{A}(X)_x$ in the fiber of ϖ^{aff} over $x \in X$ is an affine isomorphism $f\colon \mathbb{A}^n \xrightarrow{\cong} A_xX$.

(17.31) *The affine connection associated to a linear connection.* Let $\Theta \in \Omega^1_{\mathcal{B}(X)}(\mathfrak{gl}_n\mathbb{R})$ be a linear connection on $\varpi\colon \mathcal{B}(X) \to X$. Recall the tautological $\mathbb{R}^n$-valued 1-form $\theta \in \Omega^1_{\mathcal{B}(X)}(\mathbb{R}^n)$

Definition 17.32. The *associated affine connection* on $\varpi^{\mathrm{aff}}\colon \mathcal{A}(X) \to X$ is

$$\widetilde{\Theta} = \delta^*(\dot{\sigma}\Theta + i\theta). \tag{17.33}$$

Use the decomposition (17.27) to locate $\widetilde{\Theta} \in \Omega^1_{\mathcal{A}(X)}(\mathfrak{aff}_n\mathbb{R})$.

Remark 17.34. For $X = A$ an affine space, the parallelism associated to (17.33) is the one discussed in **(17.22)**, as I leave you to verify in the homework.

Proposition 17.35. *The curvature $\widetilde{\Omega}$ of $\widetilde{\Theta}$ is*

$$\widetilde{\Omega} = \Omega \oplus \tau \tag{17.36}$$

as an element of $\Omega^2_{\mathcal{A}(X)}(\mathfrak{aff}_n\mathbb{R}) = \Omega^2_{\mathcal{A}(X)}(\mathfrak{gl}_n\mathbb{R} \oplus \mathbb{R}^n)$.

We streamline the notation by omitting 'δ^*' in the formulas. See the diagram in (17.30) to interpret. The curvature $\widetilde{\Omega}$ descends to X (hence also to $\mathcal{B}(X)$), where it takes values in the adjoint bundle of Lie algebras isomorphic to $\mathfrak{aff}_n\mathbb{R}$.

Proof. Compute:

$$\begin{aligned}\widetilde{\Omega} &= d\widetilde{\Theta} + \frac{1}{2}[\widetilde{\Theta} \wedge \widetilde{\Theta}] \\ &= d(\dot{\sigma}\Theta + i\theta) + \frac{1}{2}\big[(\dot{\sigma}\Theta + i\theta) \wedge (\dot{\sigma}\Theta + i\theta)\big] \\ &= \dot{\sigma}(d\Theta + \Theta \wedge \Theta) + i(d\theta + \Theta \wedge \theta) \\ &= \dot{\sigma}\Omega + i\tau\end{aligned} \tag{17.37}$$

To pass to the third line we use the fact that $[\mathbb{R}^n, \mathbb{R}^n] = 0$ in $\mathfrak{aff}_n\mathbb{R}$, from which $[i\theta, i\theta] = 0$. ☐

(17.38) *Parallel transport in the affine bundle.* The affine bundle $\pi\colon AX \to X$ is associated to $\varpi^{\mathrm{aff}}\colon \mathcal{A}(X) \to X$ via the affine action of $\mathrm{Aff}_n\mathbb{R}$ on $\mathbb{A}^n$. Thus π inherits a horizontal distribution and a parallel transport. The holonomy around a loop based at $x \in X$ is an affine transformation of A_xX. By Theorem 15.40 and Proposition 17.35 we see that the translational component of holonomy is related to torsion and the general linear component is related to curvature of the linear connection Θ. If Θ is torsionfree then there is no translational component of holonomy.

The torsionfree condition has a meaning for parallel transport in π, not just parallel transport around closed loops. Namely, observe that there is a section Δ of π,

$$AX \underset{\pi}{\overset{\Delta}{\rightleftarrows}} X \tag{17.39}$$

whose value at $x \in X$ is the origin in A_xX.

Proposition 17.40. *The linear connection Θ is torsionfree iff Δ is parallel.*

In the discussion **(17.22)** in affine space, the section Δ of (17.23) is parallel and in fact defines the parallelism used there. Of course, the parallelism of affine space is both torsionfree and flat.

Proof. First, observe that the section Δ is represented by the constant function $\mathcal{A}(X) \to \mathbb{A}^n$ with value $0 \in \mathbb{A}^n$, where we view $AX \to X$ as an associated affine bundle. The linear connection Θ is torsionfree iff the affine curvature $\widetilde{\Omega}$ has vanishing $\mathbb{R}^n$-component. Assume Θ is torsionfree. It follows from the Ambrose–Singer Theorem 15.48 that for any $f \in \mathcal{A}(X)$, the holonomy group $\mathrm{Hol}_f(\widetilde{\Theta}) \subset \mathrm{GL}_n\mathbb{R} \subset \mathrm{Aff}_n\mathbb{R}$. Hence $0 \in \mathbb{A}^n$ is a fixed point of the action of $\mathrm{Hol}_f(\widetilde{\Theta}) \subset \mathrm{Aff}_n\mathbb{R}$ on $\mathbb{A}^n$. Now apply Proposition 15.57 to conclude that Δ is parallel.

Conversely, if Δ is parallel, then by Proposition 15.54 at any $f \in \mathcal{A}(X)$ the holonomy group $\mathrm{Hol}_f(\widetilde{\Theta})$ fixes $0 \in \mathbb{A}^n$, hence $\mathrm{Hol}_f(\widetilde{\Theta}) \subset \mathrm{GL}_n\mathbb{R} \subset \mathrm{Aff}_n\mathbb{R}$. Now Theorem 15.40 implies that the translational component of $\widetilde{\Omega}$ vanishes, from which Θ is torsionfree by Proposition 17.35. ☐

Remark 17.41. The absence of translation in the affine parallel transport means there is no "slippage" or "sliding".

Development, the exponential map, and normal coordinates

(17.42) *Development.* Let $(a, b) \subset \mathbb{R}$ be an open interval that contains $0 \in \mathbb{R}$. Let $\gamma\colon (a, b) \to X$ be a motion whose initial position is denoted $\gamma(0) = x \in X$. For $t \in (a, b)$, let $\xi_t \in T_xX$ be the parallel transport of $\dot\gamma(t)$ to x along γ, using the parallel transport in the tangent bundle induced by the linear connection Θ.

Definition 17.43. The *development* $\delta\colon (a, b) \to A_xX$ of γ is the motion in A_xX whose initial position is $\delta(0) = 0$ and whose velocity is $\dot\delta(t) = \xi_t$.

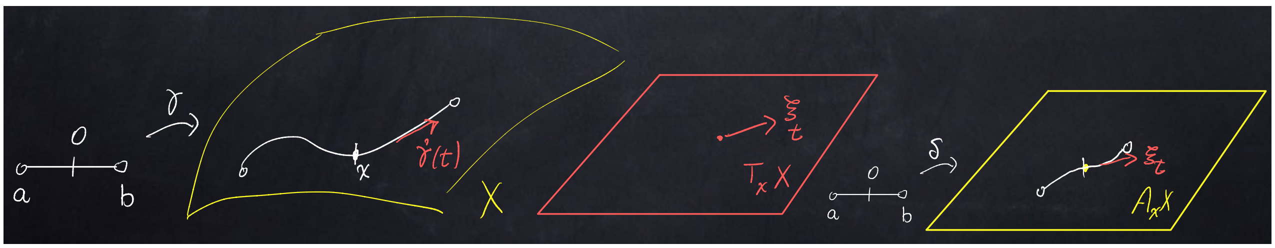

FIGURE 59. The development of a motion

The development is depicted in Figure 59. I leave it to a homework problem to prove that the development of γ is a constant velocity motion in A_xX iff γ is a geodesic motion.

(17.44) *Normal coordinates and the exponential map.* Fix $x \in X$. Define a map

$$\exp_x\colon U \longrightarrow X \tag{17.45}$$

whose domain is a suitable open neighborhood $U \subset A_xX$ of the origin as follows. For $p \in A_xX$ let $\xi \in T_xX$ be the displacement vector from the origin to p, and let γ_p denote the geodesic motion in X with initial position x and initial velocity ξ. If $1 \in \mathbb{R}$ lies in the domain of γ_p, then $p \in U$ and we set $\exp_x(p) = \gamma_p(1)$; see Figure 60. I will not prove here that U is open, but at least observe that compactness of the unit sphere in T_xX implies that U contains an open ball about the origin, and for our purposes it is sufficient to take U equal to that open ball.

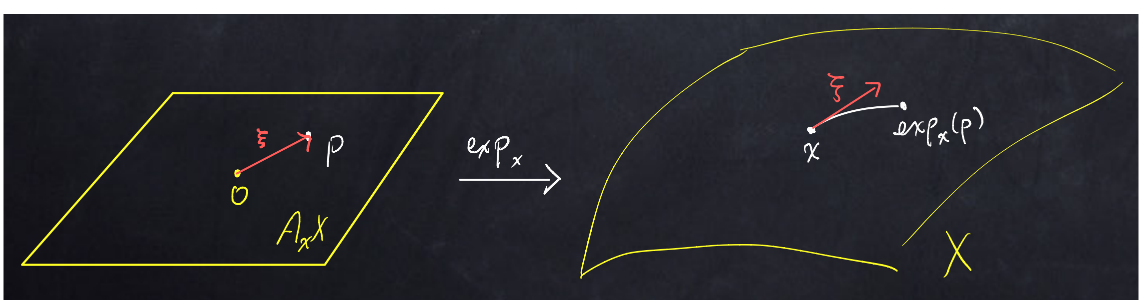

FIGURE 60. The exponential map

Remark 17.46. The exponential map is often defined on a neighborhood of the zero vector in T_xX. In that definition the geodesic with initial velocity ξ is denoted γ_ξ and $\exp_x(\xi) = \gamma_\xi(1)$.

Lemma 17.47. *$d(\exp_x)_0\colon T_xX \to T_xX$ equals id_{T_xX}.*

Proof. The value of $d(\exp_x)_0$ on $\xi \in T_xX$ is the velocity of the motion $t \mapsto \exp_x(0 + t\xi)$, which by definition is the initial velocity of the geodesic γ_ξ, which equals ξ. $\square$

The inverse function theorem implies that $\exp_x$ is a diffeomorphism of a neighborhood of the origin in A_xX with a neighborhood of x in X. Thus on a smooth manifold with a linear connection there are canonical coordinate systems about each point.[34]

Definition 17.48. A smooth manifold X with linear connection is *locally affine* if its torsion and curvature vanish.

In that case the exponential map at $x \in X$ is compatible with the parallelisms: the global affine parallelism on A_xX and the parallelism along curves given by the linear connection.

Torsion and differentiation

Now we take up our third interpretation of the torsion of a linear connection.

(17.49) *The Cartan d operator on the frame bundle.* Let X be a smooth manifold. A differential form $\omega \in \Omega^q_X$ lifts to a $GL_n\mathbb{R}$-equivariant function $\tilde\omega\colon \mathcal{B}(X) \to {\textstyle\bigwedge}^\bullet(\mathbb{R}^n)^*$. Now suppose X has a linear connection, and let ∂ be the $(\mathbb{R}^n)^*$-valued horizontal vector field on $\mathcal{B}(X)$. Consider the composition

$$\epsilon(\partial\tilde\omega)\colon \mathcal{B}(X) \xrightarrow{\partial\tilde\omega} (\mathbb{R}^n)^* \otimes {\textstyle\bigwedge}^\bullet(\mathbb{R}^n)^* \xrightarrow{\wedge} {\textstyle\bigwedge}^{\bullet+1}(\mathbb{R}^n)^* \tag{17.50}$$

in which ϵ is (left) exterior multiplication. The map (17.50) is $GL_n\mathbb{R}$-equivariant and descends to an element of $\Omega^{\bullet+1}_X$.

Proposition 17.51. *If Θ is torsionfree, then $\epsilon(\partial\tilde\omega)$ descends to $d\omega$ for all $\omega \in \Omega^\bullet_X$.*

Proof. As a preliminary, observe that for a torsionfree connection the Bianchi identity (16.59) reduces to $\Omega \wedge \theta = 0$. In terms of the functions $R^i_{jk\ell}\colon \mathcal{B}(X) \to \mathbb{R}$ defined in (16.66) we have

$$0 = \frac{1}{2} R^i_{jk\ell}\, \theta^k \wedge \theta^\ell \wedge \theta^j \tag{17.52}$$

for all i. It follows that the skew-symmetrization of $R^i_{jk\ell}$ in the variables j,k,ℓ vanishes.

[34] Well, the domain of the coordinate chart is not determined, so we have a canonical set of charts with connected domain such that any two agree on their common domain.

As a second preliminary, we compute the action of the vertical vector field $\widehat{E}_i^j$ on a $\mathrm{GL}_n\mathbb{R}$-equivariant function $\tilde{\omega}\colon \mathcal{B}(X) \to \bigwedge^\bullet(\mathbb{R}^n)^*$. The flow on $\mathcal{B}(X)$ generated by $\widehat{E}_i^j$ is the right action $t \mapsto R(e^{tE_i^j})$ by the one-parameter group generated by E_i^j. Hence by the transformation law (15.53)

$$\widehat{E}_i^j \cdot \tilde{\omega} = \frac{d}{dt}\Big|_{t=0} R^*_{e^{tE_i^j}}(\tilde{\omega}) = \frac{d}{dt}\Big|_{t=0} \rho\Big(e^{-tE_i^j}\Big)\,\tilde{\omega} = -\dot{\rho}(E_i^j)\tilde{\omega}, \tag{17.53}$$

where ρ is the action of $\mathrm{GL}_n\mathbb{R}$ on the exterior algebra $\bigwedge^\bullet(\mathbb{R}^n)^*$. It remains to compute the last expression: the (infinitesimal) action of $\mathfrak{gl}_n\mathbb{R}$ on the exterior algebra. Since $E_i^j\colon e_j \mapsto e_i$ by the action of $\mathfrak{gl}_n\mathbb{R}$ on $\mathbb{R}^n$, its action on the dual $(\mathbb{R}^n)^*$ maps $\theta^i \mapsto \theta^j$. Write $\epsilon^k = \epsilon(e^k)$ and $\iota_k = \iota(e_k)$ as endomorphisms of $\bigwedge^\bullet(\mathbb{R}^n)^*$. Then the action of E_i^j on $(\mathbb{R}^n)^*$ is $\epsilon^j\iota_i$, and this same expression expresses its action by derivations on the exterior algebra $\bigwedge^\bullet(\mathbb{R}^n)^*$. Hence (17.53) becomes

$$\widehat{E}_i^j \cdot \tilde{\omega} = -\epsilon^j\iota_i(\tilde{\omega}). \tag{17.54}$$

Note that this canonical vertical vector field acts *algebraically* on invariant functions; there is no differentiation in (17.54).

Now we proceed to the proof. The Cartan operator d is characterized as the unique order one linear differential operator on $\Omega_X^\bullet$ that obeys the Leibniz rule (with signs), agrees with the usual differential on Ω_X^0, and squares to zero. All by the last is evident for the operation defined by (17.50). The computation of the square of $\epsilon\partial$ is:

$$\begin{aligned}(\epsilon\partial)^2 &= \epsilon^k\partial_k\epsilon^\ell\partial_\ell \\ &= \epsilon^k\epsilon^\ell\partial_k\partial_\ell \\ &= \frac{1}{2}\epsilon^k\epsilon^\ell(\partial_k\partial_\ell + \partial_\ell\partial_k) + \frac{1}{2}\epsilon^k\epsilon^\ell[\partial_k, \partial_\ell] \\ &= -\frac{1}{2}R^i_{jk\ell}\epsilon^k\epsilon^\ell\widehat{E}_i^j \\ &= \frac{1}{2}R^i_{jk\ell}\epsilon^k\epsilon^\ell\epsilon^j\iota_i \\ &= 0\end{aligned} \tag{17.55}$$

To pass to the fourth line, use the fact that $\epsilon^k\epsilon^\ell$ is skew in k,ℓ whereas $(\partial_k\partial_\ell + \partial_\ell\partial_k)$ is symmetric in k,ℓ, hence the product vanishes. Also, use (16.70) and the vanishing of torsion. The penultimate step follows from (17.54). In the final step apply the conclusion following (17.52): the sum projects to the skew-symmetrization of $R^i_{jk\ell}$ in j,k,ℓ since $\epsilon^k\epsilon^\ell\epsilon^j$ is skew in j,k,ℓ. □

Remark 17.56. This formula for d can be taken as an additional motivation to prefer *torsionfree* linear connections.

(17.57) *Torsion and geometric structures.* Let (G_n, λ_n) be an n-dimensional symmetry type. Thus G_n is a Lie group and $\lambda_n\colon G_n \to \mathrm{GL}_n\mathbb{R}$ is a Lie group homomorphism. Definition 13.52

tells that a (G_n,λ_n)-structure on a smooth n-dimensional manifold X is a principal G_n-bundle $\pi\colon \mathcal{B}_G(X)\to X$ and an isomorphism ϕ of the frame bundle ϖ and the associated bundle $\lambda_n(\pi)$:

$$\begin{tikzcd} \mathcal{B}(X) \arrow[rr, "\phi"] \arrow[dr, "\varpi"'] & & \lambda_n\bigl(\mathcal{B}_G(X)\bigr) \arrow[dl, "\lambda_n(\pi)"] \\ & X & \end{tikzcd} \tag{17.58}$$

Definition 17.59. A (G_n,λ_n)-structure is *torsionfree* if π admits a connection Θ such that the induced linear connection $\phi^{-1}(\dot\lambda_n(\Theta))$ is torsionfree.

We will explore torsionfree geometric structures in future lectures.

Lecture 18: Local coordinates; the Levi–Civita connection

It is written: Whoever does not write expressions in local coordinates has not done his duty in a differential geometry course. This comes a bit late, perhaps, but not too late, so in the first part of this lecture we fulfill this duty. Namely, on a smooth manifold with linear connection we derive local expressions for the connection, curvature, torsion, and the equation for a geodesic.

At this point we will have completed our general study of parallelism on smooth manifolds, first for general principal bundles and then for the intrinsic bundle of frames. We turn to parallelism compatible with a geometric structure, an idea introduced in **(16.44)**. Our focus in this lecture, and in most of the rest of the course, is on Riemannian geometry. Historically, connections and the resulting parallelism came after the introduction of the curvature tensor. Thus we follow Riemann in his 1866 manuscript (which followed his 1854 Privatdocent lecture[35]) This is a concrete computation in local coordinates that seeks a local *isometry* of an arbitrary Riemannian manifold with a Euclidean space of the same dimension. The obstruction is the Riemann curvature tensor. Our discussion in 2 dimensions (Lecture 11) provides intuition for the geometry of the curvature tensor in general dimensions, which we develop in subsequent lectures.

To conclude, we construct the Levi–Civita connection on a Riemannian manifold. This provides an interpretation of the Cristoffel symbols that appear in the derivation of the Riemann curvature. We prove the stronger result that the map from connections compatible with a Riemannian metric to torsion tensors is an isomorphism. The proof generalizes that given in 2 dimensions (Theorem 11.4).

Computations with indices appear in this lecture. They may appear forbidding at first glance, but in fact they are not terribly difficult as these computations go. I highly recommend working through them carefully and independently, repeating them with pen and paper to develop facility.

Local coordinates

Let X be a smooth manifold equipped with a linear connection Θ on the bundle of frames $\varpi\colon \mathcal{B}(X) \to X$. Let $(U; x^1, \dots, x^n)$ be a chart on X. The chart induces a section

$$s = \left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) : U \longrightarrow \varpi^{-1}(U) \tag{18.1}$$

of the frame bundle over U.

(18.2) *Pullbacks to the base.* We write/claim:

$$s^*\theta^i = dx^i \tag{18.3}$$

$$s^*\Theta^i_j = \Gamma^i_{kj} dx^k \tag{18.4}$$

$$s^*\Omega^i_j = \frac{1}{2} R^i_{jk\ell}\, dx^k \wedge dx^\ell \tag{18.5}$$

$$s^*\tau^i = \frac{1}{2} T^i_{k\ell}\, dx^k \wedge dx^\ell \tag{18.6}$$

[35] I posted both on Canvas, in translation by Spivak. I highly recommend reading them, especially the 1854 lecture.

Recall that the tautological 1-form $\theta = \theta^i e_i$ on $\mathcal{B}(X)$ satisfies $\varpi_*\xi = \theta^i(\xi)b(e_i)$ at $b\colon \mathbb{R}^n \xrightarrow{\cong} T_xX$, from which (18.3) follows. Equation (18.4) defines the functions $\Gamma^i_{kj}\colon U \to \mathbb{R}$. Equations (18.5) and (18.6) follow from (16.65) and (16.66). The functions $R^i_{jk\ell}, T^i_{kj}\colon U \to \mathbb{R}$ are the pullbacks of the corresponding functions on $\mathcal{B}(X)$ via the section s.

(18.7) *Curvature.* To compute the curvature in local coordinates, apply s^* to the equation

$$\Omega^i_j = d\Theta^i_j + \Theta^i_m \wedge \Theta^m_j. \tag{18.8}$$

Using (18.5) and (18.4) we obtain

$$\frac{1}{2}R^i_{jk\ell}\,dx^k \wedge dx^\ell = \frac{\partial\Gamma^i_{kj}}{dx^\ell}dx^\ell \wedge dx^k + \Gamma^i_{km}\Gamma^m_{\ell j}\,dx^k \wedge dx^\ell. \tag{18.9}$$

Equate coefficients of the basis $dx^k \wedge dx^\ell$, $1 \le k < \ell \le n$, of 2-forms on U to obtain

$$\boxed{R^i_{jk\ell} = \frac{\partial\Gamma^i_{\ell j}}{dx^k} - \frac{\partial\Gamma^i_{kj}}{dx^\ell} + \Gamma^i_{km}\Gamma^m_{\ell j} - \Gamma^i_{\ell m}\Gamma^m_{kj}} \tag{18.10}$$

(18.11) *Torsion.* To compute the torsion in local coordinates, apply s^* to the equation

$$\tau^i = d\theta^i + \Theta^i_j \wedge \theta^j. \tag{18.12}$$

Noting $d\theta^i = 0$ we find

$$\frac{1}{2}T^i_{k\ell}\,dx^k \wedge dx^\ell = \Gamma^i_{\ell k}\,dx^k \wedge dx^\ell, \tag{18.13}$$

and upon equating coefficients over a basis we have

$$\boxed{T^i_{k\ell} = \Gamma^i_{k\ell} - \Gamma^i_{\ell k}} \tag{18.14}$$

Notice that in a local coordinate system, the torsion measures the deviation from symmetry in k, ℓ of the *Christoffel symbol*[36] $\Gamma^i_{k\ell}$.

[36]This may be a slightly expansive use of 'Christoffel symbol', which is often restricted to the Levi–Civita connection.

(18.15) *Preliminary.* The following is an important general formula in the theory of connections, one already used and that you proved in the homework. Nonetheless, here it is.

Lemma 18.16. *Let $\pi\colon P \to X$ be a principal G-bundle with a connection $\Theta \in \Omega^1_P(\mathfrak{g})$. Suppose $s\colon X \to P$ is a section of π and let $g\colon X \to G$ be a function. Then*

$$(sg)^*(\Theta) = \mathrm{Ad}_{g^{-1}} s^*\Theta + g^*\theta_G, \tag{18.17}$$

where $\theta_G \in \Omega^1_G(\mathfrak{g})$ is the Maurer–Cartan form.

Proof. The section $sg\colon X \to P$ of π is the composition

$$X \xrightarrow{s\times g} P \times G \xrightarrow{R} P, \tag{18.18}$$

where R is the right action of G on P. Now (18.17) follows from

$$R^*(\Theta) = \mathrm{Ad}_{g^{-1}} \Theta + \theta_G. \tag{18.19}$$

Equation (18.19) is an equality of $\mathfrak{g}$-valued 1-forms on $P \times G$, which can be verified by evaluating at $(p,g) \in P \times G$ on a tangent vector $\xi \in T_pP$ and then on a tangent vector $\zeta \in T_gG$. On the former we obtain (14.30) and on the latter we obtain (14.29), which are the defining equations of a connection. ∎

(18.20) *Geodesic equation.* Let $\gamma\colon (a,b) \to U$ be a motion in the domain U of our coordinate chart. It is a geodesic motion iff the velocity is parallel along γ. To compute the equation in local coordinates, form the pullback along γ of the frame bundle and its section s in (18.1):

$$\begin{tikzcd} \gamma^*\mathcal{B}(U) \arrow[r] \arrow[d, shift left, "\varpi'"] & \mathcal{B}(U) \arrow[d, shift right, "\varpi"'] \\ (a,b) \arrow[r, "\gamma"] \arrow[u, bend left, "s'"] & U \arrow[u, bend right, "s"'] \end{tikzcd} \tag{18.21}$$

Write

$$\gamma(t) = (x^1(t), \dots, x^n(t)), \tag{18.22}$$

and so

$$\dot\gamma = \dot x^1 \frac{\partial}{\partial x^1} + \cdots + \dot x^n \frac{\partial}{\partial x^n} = [s', \dot x]. \tag{18.23}$$

This last expression is a map $(a,b) \to \mathcal{B}(U) \times_{\mathrm{GL}_n\mathbb{R}} \mathbb{R}^n$, using the mixing construction presentation of the tangent bundle. By the discussion in **(14.61)** there exist functions

$$M\colon (a,b) \longrightarrow \mathrm{GL}_n\mathbb{R}, \tag{18.24}$$

unique up to a constant, such that $s'M$ is a parallel section of ϖ'. Fix such an M. Thus

$$0 = (s'M)^*(\Theta) = M^{-1}(s^*\Theta)M + M^{-1}\dot{M} = M^{-1}\Gamma_k \dot{x}^k M + M^{-1}\dot{M}, \tag{18.25}$$

where we write $\Gamma_k = (\,\Gamma^i_{kj}\,)_{i,j}$ and we have applied Lemma 18.16. Now $\dot{\gamma} = [s', \dot{x}] = [s'M, M^{-1}\dot{x}]$ is parallel along γ iff $M^{-1}\dot{x}$ is constant iff

$$\begin{aligned} 0 &= (M^{-1}\dot{x})^{\cdot} \\ &= -M^{-1}\dot{M}M^{-1}\dot{x} + M^{-1}\ddot{x} \\ &= M^{-1}\Gamma_k\dot{x}^k M M^{-1}\dot{x} + M^{-1}\ddot{x} \\ &= M^{-1}(\ddot{x} + \Gamma_k\dot{x}^k\dot{x}) \end{aligned} \tag{18.26}$$

where in the third line we substitute (18.25). This is the *geodesic equation*

$$\boxed{\frac{dx^i}{dt^2} + \Gamma^i_{k\ell}\dot{x}^k\dot{x}^\ell = 0}, \tag{18.27}$$

a second order ODE for the motion γ.

The Riemann curvature tensor

We follow Riemann and derive the curvature tensor as the local obstruction to putting a Riemannian metric into standard form.

Let X be a Riemannian manifold of dimension n, and suppose $x^1, \dots, x^n$ are local coordinates. Define local functions

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle \tag{18.28}$$

that encode the Riemannian metric. Introduce the inverse metric

$$g^{ij} = \langle dx^i, dx^j \rangle\,; \tag{18.29}$$

then $g_{ij}g^{jk} = \delta^k_i$. We seek a change of coordinates

$$y^a = y^a(x^1, \dots, x^n), \qquad a = 1, \dots, n, \tag{18.30}$$

such that the coordinate vector fields $\partial/\partial y^a$ form an *orthonormal* basis at each point in the local chart. Thus

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \left\langle \frac{\partial y^a}{\partial x^i}\frac{\partial}{\partial y^a}, \frac{\partial y^b}{\partial x^j}\frac{\partial}{\partial y^b} \right\rangle = \delta_{ab}\frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^j}. \tag{18.31}$$

This is a system of nonlinear PDE for the functions (18.30).

A basic theorem in calculus states the mixed partials commute for a function of several variables. This leads to obstructions, as we saw in our discussion of the local Frobenius theorem (see Example 5.22 and Proposition 5.39, for example). The Frobenius tensor measures the failure of this commutation, and is then a local obstruction to the existence of integral manifolds. This led us to curvature of a connection on a principal bundle. And now—once more in a concrete form—we will see that equality of mixed partials gives rise to the Riemannian curvature tensor as a local obstruction to solving (18.31).

Assuming (18.31) we derive consequences by repeated differentiation:

$$\begin{aligned} \frac{\partial g_{ij}}{\partial x^k} &= \sum_a \frac{\partial^2 y^a}{\partial x^k \partial x^i}\frac{\partial y^a}{\partial x^j} + \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^k \partial x^j} \\ \frac{\partial g_{ik}}{\partial x^j} &= \sum_a \frac{\partial^2 y^a}{\partial x^j \partial x^i}\frac{\partial y^a}{\partial x^k} + \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^j \partial x^k} \\ \frac{\partial g_{jk}}{\partial x^i} &= \sum_a \frac{\partial^2 y^a}{\partial x^i \partial x^j}\frac{\partial y^a}{\partial x^k} + \frac{\partial y^a}{\partial x^j}\frac{\partial^2 y^a}{\partial x^i \partial x^k} \end{aligned} \tag{18.32}$$

Combining these equations, and using equality of mixed partials, we deduce

$$\frac{1}{2}\left\{ \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right\} = \sum_a \frac{\partial y^a}{\partial x^i}\frac{\partial^2 y^a}{\partial x^j \partial x^k}. \tag{18.33}$$

Use the inverse metric to introduce

$$\Gamma^\ell_{jk} = \Gamma^\ell_{kj} = \frac{1}{2} g^{\ell i}\left\{ \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right\}. \tag{18.34}$$

An equivalent form of our PDE (18.31) is

$$\delta^{ab} = g^{ij}\frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^j}. \tag{18.35}$$

Multiply both sides of (18.33) by $\partial y^b/\partial x^\ell$ and conclude, after renaming dummy indices,

$$\Gamma^i_{jk}\frac{\partial y^b}{\partial x^i} = \sum_a \delta^{ab}\frac{\partial^2 y^a}{\partial x^i \partial x^j} = \frac{\partial^2 y^b}{\partial x^j \partial x^k}. \tag{18.36}$$

Equality of mixed partials in (18.36) still leads to a condition that involves the unknown functions. So... differentiate again:

$$\begin{aligned}\frac{\partial^3 y^b}{\partial x^\ell \partial x^j \partial x^k} &= \frac{\partial \Gamma^i_{jk}}{\partial x^\ell}\frac{\partial y^b}{\partial x^i} + \Gamma^i_{jk}\frac{\partial^2 y^b}{\partial x^\ell \partial x^i} \\ &= \frac{\partial \Gamma^i_{jk}}{\partial x^\ell}\frac{\partial y^b}{\partial x^i} + \Gamma^i_{jk}\Gamma^m_{i\ell}\frac{\partial y^b}{\partial x^m} \\ &= \left\{\frac{\partial \Gamma^i_{jk}}{\partial x^\ell} + \Gamma^m_{jk}\Gamma^i_{m\ell}\right\}\frac{\partial y^b}{\partial x^i}.\end{aligned} \tag{18.37}$$

We substitute (18.36) for the mixed second partial derivative to pass from the first to the second line. Now we are in good shape: the third partial derivative is symmetric in all three variables, so in particular it is symmetric in the indices k,ℓ. Introduce the *Riemann curvature tensor*

$$R^i_{jk\ell} = \frac{\partial \Gamma^i_{j\ell}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^\ell} + \Gamma^m_{j\ell}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{m\ell}. \tag{18.38}$$

The equality of mixed partials implies that solutions to (18.31) exist locally only if $R^i_{jk\ell} = 0$ for all i,j,k,ℓ.

In subsequent lectures we study the geometry of the Riemann curvature tensor. We conclude this lecture by interpreting the *Cristoffel symbols* Γ^i_{jk} as a special connection on the orthonormal frame bundle of a Riemannian manifold.

Torsion and the Levi–Civita connection

(18.39) *Torsion and geometric structures.* Let (G_n,λ_n) be an n-dimensional symmetry type. Thus G_n is a Lie group and $\lambda_n\colon G_n \to GL_n\mathbb{R}$ is a Lie group homomorphism. Recall Definition 13.52 of a (G_n,λ_n)-structure on a smooth n-dimensional manifold X: a principal G_n-bundle $\pi\colon \mathcal{B}_G(X)\to X$ and an isomorphism ϕ of the frame bundle ϖ and the associated bundle $\lambda_n(\pi)$:

$$\begin{tikzcd} \mathcal{B}(X) \arrow[rr, "\phi"] \arrow[dr, "\varpi"'] & & \lambda_n\big(\mathcal{B}_G(X)\big) \arrow[dl, "\lambda_n(\pi)"] \\ & X & \end{tikzcd} \tag{18.40}$$

Let A_π denote the set of connections on $\pi\colon \mathcal{B}_G(X)\to X$. A connection Θ induces a connection $\dot\lambda_n(\Theta)$ on ϖ, via the isomorphism ϕ, and it has an associated torsion, as in **(17.57)**. The torsion is a map

$$\tau\colon A_\pi \longrightarrow \Omega^2_X(TX) \tag{18.41}$$

from the *affine* space of connections to the *linear* space of 2-forms.

Proposition 18.42. *τ is an affine function with differential*

$$\begin{aligned} \Omega^1_X(\mathfrak{g}_{\mathcal{B}_G(X)}) &\longrightarrow \Omega^2_X(TX) \\ \alpha \quad &\longmapsto \dot\lambda_n(\alpha)\wedge\theta \end{aligned} \tag{18.43}$$

where $\theta\in\Omega^1_X(TX)$ is the identity map on TX viewed as a 1-form.

Recall that the pullback of this θ to $\mathcal{B}(X)$ is the tautological $\mathbb{R}^n$-valued 1-form.

Proof. The pullback of torsion to $\mathcal{B}(X)$ under the map $\varpi\colon \mathcal{B}(X)\to X$ is

$$\tau(\Theta) = d\theta + \dot\lambda_n(\Theta)\wedge\theta, \tag{18.44}$$

which is evidently an affine function of Θ with differential (18.43). □

Remark 18.45. The (G_n,λ_n)-structure is torsionfree (Definition 17.59) iff $0\in\Omega^2_X(TX)$ lies in the image of (18.41).

(18.46) *The Riemannian case.* Specialize to a Riemannian structure $(G_n,\lambda_n) = (\mathrm{O}_n, \text{inclusion})$.

Theorem 18.47. *Let X be a Riemannian manifold with orthonormal bundle of frames $\pi\colon\mathcal{B}_{\mathrm{O}}\to X$. Then the map (18.41) from orthogonal connections to torsion tensors is an isomorphism.*

This is an isomorphism of affine spaces, and since the codomain is a vector space we obtain a basepoint in the domain: the inverse image of the zero vector.

Corollary 18.48. *Let X be a Riemannian manifold. Then there exists a unique orthogonal torsionfree linear connection Θ_{LC}.*

Remark 18.49.

(1) The Levi–Civita connection, named after the Italian geometer Tulio Levi–Civita, is still mysterious in the sense that all known proofs are computational in one form or another. They all rely on the key maneuver in (18.59) below. There is a geometric construction of the Levi–Civita connection for Kähler manifolds, but not in general.
(2) Corollary 18.48 is sometimes called the *fundamental theorem of Riemannian geometry.*
(3) The tangent to an orthogonal connection is a 1-form with values in skew-symmetric endomorphisms of the tangent bundle, which in a local orthonormal frame $e_1,\dots,e_n$ and its dual coframe $e^1,\dots,e^n$ is

$$\alpha = \alpha^i_{kj}\,e^k\otimes\left(e_i\otimes e^j\right), \qquad \alpha^i_{kj} = -\alpha^j_{ki}. \tag{18.50}$$

The torsion tensor is a 2-form with values in the tangent bundle, so has the form

$$T = \frac{1}{2}T^i_{k\ell}\,(e^k\wedge e^\ell)\otimes e_i, \qquad T^i_{k\ell} = -T^i_{\ell k}. \tag{18.51}$$

In each case we have a set of functions with three indices satisfying a single skew-symmetry constraint, so the number of functions α^i_{kj} equals the number of functions $T^i_{k\ell}$. We are solving an affine equation for α given T with an equal number of equations and unknowns. The analog in finite dimensions is a matrix equation $Ax = b$ with b given and A a square matrix. In that case uniqueness and existence are equivalent. So too here.

(4) Let $\pi\colon P \to U$ be a principal G-bundle over a manifold U, and suppose $s\colon U \to P$ is a section of π. Then there is a canonical connection Θ_s on π that makes the global section s horizontal. This connection is flat and has trivial holonomy. It induces an isomorphism

$$\begin{aligned} A_\pi &\longrightarrow \Omega^1_U(\mathfrak{g}_P) \\ \Theta &\longmapsto s^*\Theta \end{aligned} \tag{18.52}$$

of the affine space of connections on π with the linear space of adjoint bundle valued 1-forms on U. This is the local form of a connection that we use in the proof.

(5) The proof for Riemannian 2-manifolds is given in **(11.17)**.

Proof of Theorem 18.47. We first prove uniqueness of an orthogonal connection with specified torsion, which can be done via a local argument. Then it suffices to prove local existence; global existence follows.

Let $U \subset X$ be an open set on which we have an orthonormal moving frame, i.e., a section $s\colon U \to \mathcal{B}_{\mathrm{O}}(U)$ of the orthonormal frame bundle $\pi\colon \mathcal{B}_{\mathrm{O}}(U) \to U$. For $\Theta \in A_\pi$ and $\tau \in \Omega^2_U(TU)$ write the local expressions

$$\begin{aligned} s^*\theta^i &= \overline{\theta}^i \\ s^*\Theta^i_j &= A^i_{kj}\,\overline{\theta}^k, &\qquad A^i_{kj} &= -A^j_{ki} \\ s^*\tau^i &= \frac12 T^i_{k\ell}\,\overline{\theta}^k\wedge\overline{\theta}^\ell, &\qquad T^i_{k\ell} &= -T^i_{\ell k} \end{aligned} \tag{18.53}$$

Then for some functions $S^i_{k\ell}\colon U \to \mathbb{R}$ we have

$$d\overline{\theta}^i = \frac12 S^i_{k\ell}\,\overline{\theta}^k\wedge\overline{\theta}^\ell, \qquad S^i_{k\ell} = -S^i_{\ell k}. \tag{18.54}$$

The equation that relates $T^i_{k\ell}$ to A^i_{kj} and $S^i_{k\ell}$ via the function (18.41) is obtained by applying s^* to $\tau^i = d\theta^i + \Theta^i_j \wedge \theta^j$:

$$\frac12 T^i_{k\ell}\,\overline{\theta}^k\wedge\overline{\theta}^\ell = \frac12 S^i_{k\ell}\,\overline{\theta}^k\wedge\overline{\theta}^\ell + A^i_{k\ell}\,\overline{\theta}^k\wedge\overline{\theta}^\ell \tag{18.55}$$

from which

$$A^i_{k\ell} - A^i_{\ell k} = T^i_{k\ell} - S^i_{k\ell}. \tag{18.56}$$

For local uniqueness, suppose for a given τ we have two solutions Θ_1, Θ_2, and let $\Delta = \Theta_2 - \Theta_1$. Then relative to the local moving frame on U we write

$$\Delta = \Delta^i_{kj}\,\overline{\theta}^k, \qquad \Delta^i_{kj} = -\Delta^j_{ki}, \tag{18.57}$$

for some functions $\Delta^i_{kj}\colon U \to \mathbb{R}$. The affine equation (18.56) for a single solution reduces to a linear equation for the difference of solutions:

$$\Delta^i_{k\ell} = \Delta^i_{\ell k}. \tag{18.58}$$

Combining (18.57) and (18.58) we see that $\Delta^i_{k\ell}$ is skew in i, ℓ and symmetric in k, ℓ. The symmetry and skew-symmetry are in complete tension which forces $\Delta = 0$:

$$\Delta^i_{k\ell} = \Delta^i_{\ell k} = -\Delta^k_{\ell i} = -\Delta^k_{i\ell} = \Delta^\ell_{ik} = \Delta^\ell_{ki} = -\Delta^i_{k\ell}. \tag{18.59}$$

For local existence, suppose given $T^i_{k\ell}, S^i_{k\ell}$ that are skew in k, ℓ. We solve for $A^i_{k\ell}$ using (18.56) and the skew-symmetry in (18.53):

$$\begin{aligned} A^i_{k\ell} &= T^i_{k\ell} - S^i_{k\ell} + A^i_{\ell k} \\ &= T^i_{k\ell} - S^i_{k\ell} - A^k_{\ell i} \\ &= T^i_{k\ell} - S^i_{k\ell} - T^k_{\ell i} + S^k_{\ell i} - A^k_{i\ell} \\ &= T^i_{k\ell} - S^i_{k\ell} - T^k_{\ell i} + S^k_{\ell i} + A^\ell_{ik} \\ &= T^i_{k\ell} - S^i_{k\ell} - T^k_{\ell i} + S^k_{\ell i} + T^\ell_{ik} - S^\ell_{ik} + A^\ell_{ki} \\ &= T^i_{k\ell} - S^i_{k\ell} - T^k_{\ell i} + S^k_{\ell i} + T^\ell_{ik} - S^\ell_{ik} - A^i_{k\ell} \end{aligned} \tag{18.60}$$

from which

$$A^i_{k\ell} = \frac{1}{2}\bigl(T^i_{k\ell} - S^i_{k\ell} - T^k_{\ell i} + S^k_{\ell i} + T^\ell_{ik} - S^\ell_{ik}\bigr). \tag{18.61}$$ $\square$

Lecture 19: Covariant derivatives

We first recall the directional derivative of a vector-valued function ψ on an affine space. The difference quotient for the derivative subtracts the value of ψ at distinct points of affine space, which is possible if ψ takes values in a fixed vector space. The directional derivatives combine to a vector-valued 1-form $d\psi$. A vector-valued function ψ on a smooth manifold X often take values on a variable vector space—a vector space that varies with the point on the manifold. In other words, ψ is a section of a vector bundle $\pi\colon E \to X$. A typical example is a vector field. To define the "directional derivative" we need to compare the values of ψ at distinct points of X, at least along curves, and those values lie in different vector spaces.

If π is associated to a principal bundle with connection, then there is an induced parallel transport along curves, and we use it to define the analog of the directional derivative, called the *covariant derivative*. These comprise a vector-valued 1-form $\nabla\psi$. As always, the calculus simplifies when pulled back to the total space of the principal bundle (Proposition 19.15).

The covariant derivative is a first-order differential operator: a linear map map that satisfies the Leibniz rule. This leads (Definition 19.24) to the concept of a covariant derivative on an *arbitrary* vector bundle. In the second part of the lecture we explore its properties and use it to develop a calculus of vector bundle-valued differential forms. A general covariant derivative has a curvature, and we give a formula (19.55) that expresses it as the failure of mixed partial derivatives to commute. Equivalently, curvature measures the failure of the twisted de Rham differential to square to zero.

We conclude by considering the intrinsic case of covariant derivatives on the tangent bundle, which are equivalent to linear connections. Since the tangent bundle is associated to the bundle of frames, there is a nice formula (19.82) for the covariant derivative, as there is for the torsion (19.87).

Review of the directional derivative

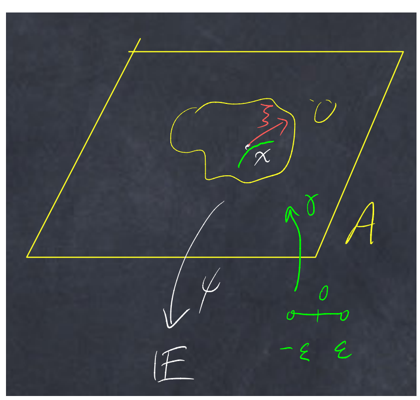

FIGURE 61. The directional derivative

Let A be an affine space over a real vector space V, let $U \subset A$ be an open subset, and let $\mathbb{E}$ be a (real or complex) vector space. Suppose $\psi\colon U \to \mathbb{E}$ is a function. Fix $x \in U$ and $\xi \in V$. Choose a smooth motion $\gamma\colon (-\epsilon, \epsilon) \to U$ such that $\gamma(t) \in U$ for all $t \in (-\epsilon, \epsilon)$, the initial position is $\gamma(0) = x$, and the initial velocity is $\dot\gamma(0) = \xi$; see Figure 61. There is a canonical choice of γ, namely the constant velocity motion $\gamma(t) = x + t\xi$. (There is no canonical choice of $\epsilon > 0$, but in any case we only need the *germ* of the motion, which is canonical.) The definition of the *directional derivative* uses this constant velocity motion. Namely, the directional derivative is

$$D_\xi\psi(x) = \frac{d}{dt}\bigg|_{t=0} \psi\bigl(\gamma(t)\bigr) = \lim_{t\to 0} \frac{\psi\bigl(\gamma(t)\bigr) - \psi(x)}{t} \tag{19.1}$$

If the limit exists for the constant velocity motion, then it exists for all smooth motions and the limit in (19.1) is independent of the choice of smooth motion. The numerator of the difference quotient is the difference of vectors in $\mathbb{E}$, and the entire difference quotient is an element of $\mathbb{E}$. The following is a basic theorem in calculus.

Theorem 19.2. *In this situation, if $D_\xi\psi(x)$ exists for all x,ξ and is a continuous function of x,ξ, then $\xi \mapsto D_\xi\psi(x)$ is a linear function of ξ.*

The covariant derivative on an associated vector bundle

Let X be a smooth manifold, let G be a Lie group, and suppose $P \to X$ is a principal G-bundle with connection $\Theta \in \Omega^1_P(\mathfrak{g})$. Let $\rho\colon G \to \mathrm{Aut}(\mathbb{E})$ be a linear representation of G, and denote the associated vector bundle as $\pi\colon E = \mathbb{E}_P \to X$. The connection on $P \to X$ induces an analog of the directional derivative on sections of $E \to X$: the *covariant derivative.*

(19.3) *Definition using the induced linear parallelism.* Recall that Θ induces a horizontal distribution on all associated fiber bundles, and there is a corresponding parallel transport along curves; see **(14.76)** and **(14.79)**. We apply this to the associated vector bundle $\pi\colon E \to X$.

Definition 19.4. Let $\psi\colon X \to E$ be a smooth section of π. Fix $x \in X$ and $\xi \in T_xX$. Let $\gamma\colon (-\epsilon,\epsilon) \to X$ be a smooth motion with initial position $\gamma(0) = x$ and initial velocity $\dot\gamma(0) = \xi$. Then the *covariant derivative* of ψ at x in the direction ξ is

$$\nabla_\xi\psi(x) = \frac{d}{dt}\bigg|_{t=0} \tau_t^{-1}\bigl(\psi\bigl(\gamma(t)\bigr)\bigr) = \lim_{t\to 0} \frac{\tau_t^{-1}\bigl(\psi\bigl(\gamma(t)\bigr)\bigr) - \psi(x)}{t} \tag{19.5}$$

where $\tau_t\colon E_x \to E_{\gamma(t)}$ is the parallel transport along γ induced by Θ.

The data is illustrated in Figure 62.

(19.6) *Computation on the total space of the principal bundle.* The pullback of the associated vector bundle $\pi\colon E \to X$ via the projection $P \to X$ is canonically the product fiber bundle over P with fiber $\mathbb{E}$: namely, $P \times \mathbb{E} \xrightarrow{\mathrm{pr}_1} P$. Thus the pullback of a section ψ of π is an $\mathbb{E}$-valued function that satisfies an equivariance condition:

$$\tilde\psi\colon P \longrightarrow \mathbb{E}, \qquad R_g^*\tilde\psi = \rho(g)^{-1}\tilde\psi, \quad g \in G. \tag{19.7}$$

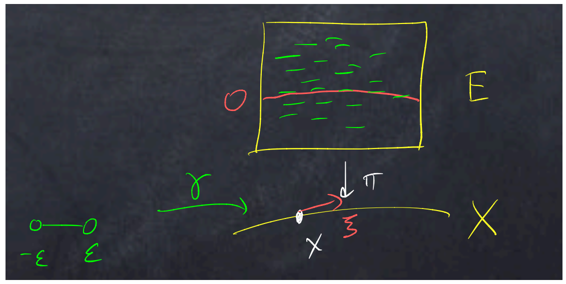

FIGURE 62. The horizontal distribution on a vector bundle and the induced covariant derivative

In particular, an element of the fiber E_x at a point $x \in X$ pulls back to an equivariant $\mathbb{E}$-valued function on P_x.

The pullback of the covariant derivative can be expressed in terms of the ordinary Cartan differential d on P. Resuming the setup of Definition 19.4, let $\widetilde{\xi}_p$ be the horizontal lift of ξ at $p \in P_x$.

Proposition 19.8. *The covariant derivative* $(\nabla_\xi \psi)(x) \in E_x$ *pulls back to the* G*-equivariant function*

$$\begin{aligned} P_x &\longrightarrow \mathbb{E} \\ p &\longmapsto d\tilde{\psi}_p(\widetilde{\xi}_p). \end{aligned} \tag{19.9}$$

Note that $d\tilde{\psi}_p(\widetilde{\xi}_p) = \widetilde{\xi}_p \cdot \tilde{\psi}_p$ is the directional derivative.

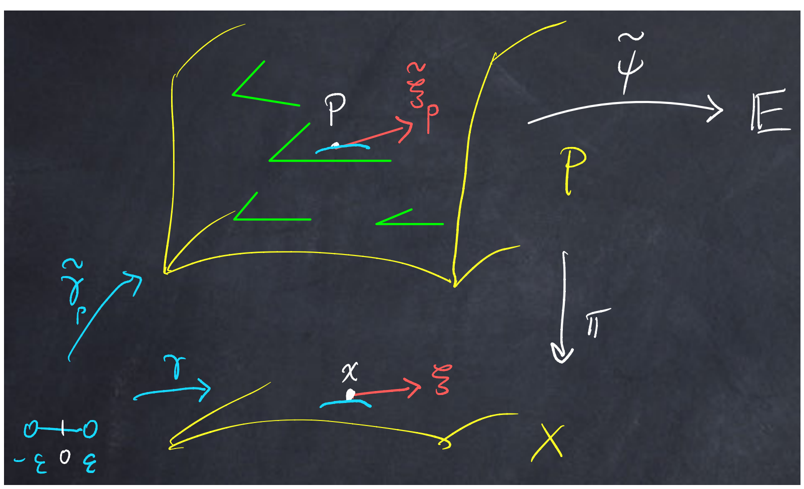

FIGURE 63. Computation of the covariant derivative

Proof. As depicted in Figure 63, let $\tilde{\gamma}_p$ be the horizontal lift of γ with initial position $p \in P_x$:

$$\begin{tikzcd} & P \arrow[d] \\ (-\epsilon,\epsilon) \arrow[ur, dashed, "\tilde{\gamma}_p"] \arrow[r, "\gamma"] & X \end{tikzcd} \tag{19.10}$$

Recall (Proposition 14.80) that the composition of a function $P \to \mathbb{E}$ with $\tilde{\gamma}_p$ is parallel iff that composition is constant. This means that the parallel transport in (19.5) drops out from $\tilde{\psi} \circ \tilde{\gamma}_p$, and thereby reduces the computation of covariant derivatives to the computation in affine space:

$$(\nabla_{\xi}\psi)(x) = \frac{d}{dt}\bigg|_{t=0} \tilde{\psi}\big(\tilde{\gamma}(t)\big) = d\tilde{\psi}_p(\tilde{\xi}_p). \tag{19.11}$$

(We conflate sections of $\pi\colon E \to X$ with their pullbacks to P.) ☐

Corollary 19.12. *The function $\xi \mapsto (\nabla_{\xi}\psi)(x)$ is a linear function of $\xi \in T_xX$.*

The pullback reduces the covariant derivative to the directional derivative, and so Corollary 19.12 reduces to Theorem 19.2.

(19.13) *The covariant derivative as a 1-form.* It follows from Corollary 19.12 that the equation

$$(\nabla\psi)_x(\xi) = (\nabla_{\xi}\psi)(x), \qquad x \in X, \quad \xi \in T_xX \tag{19.14}$$

determines a 1-form $\nabla\psi$ on X with values in E.

Proposition 19.15. *The pullback of $\nabla\psi$ to the total space P of $P \to X$ is the $\mathbb{E}$-valued 1-form*

$$\nabla\psi = d\tilde{\psi} + \dot{\rho}(\Theta)\tilde{\psi}. \tag{19.16}$$

On the left hand side of (19.16) we do not explicitly notate the pullback.

Proof. We first check that the right hand side descends to X. If $g \in G$ and $e \in \mathbb{E}$, then

$$\dot{\rho}(\Theta)\rho(g)^{-1}e = \rho(g)^{-1}\dot{\rho}(\Theta)\rho(g)\rho(g)^{-1}e = \rho(g)^{-1}\dot{\rho}(\Theta)e. \tag{19.17}$$

It follows that $R_g^*\big(d\tilde{\psi} + \dot{\rho}(\Theta)\tilde{\psi}\big) = \rho(g)^{-1}\big(d\tilde{\psi} + \dot{\rho}(\Theta)\tilde{\psi}\big)$. The contraction of the right hand side on a vertical vector vanishes by an argument similar to (16.54): if $\xi \in \mathfrak{g}$ and $\hat{\xi}$ is the induced vertical vector field on P, then

$$\iota_{\hat{\xi}}\big(d\tilde{\psi} + \dot{\rho}(\Theta)\tilde{\psi}\big) = \mathcal{L}_{\hat{\xi}}\tilde{\psi} + \dot{\rho}(\xi)\tilde{\psi}. \tag{19.18}$$

The Lie derivative is $\mathcal{L}_{\hat{\xi}}\tilde{\psi} = -\dot{\rho}(\xi)\tilde{\psi}$: differentiate the transformation law (19.7) along the motion $t \mapsto e^{t\xi}$ on P generated by $\hat{\xi}$. Hence (19.18) vanishes and therefore the right hand side descends.

Now it suffices to observe that on horizontal vectors the two sides agree, by Proposition 19.8. ☐

(19.19) *Properties of the covariant derivative.* Recall that to the vector bundle $\pi\colon E \to X$ we associate the $\mathbb{Z}$-graded vector space $\Omega^\bullet_X(E)$ of E-valued differential forms, i.e., the graded vector space of sections of the vector bundle $E \otimes \bigwedge^\bullet T^*X \to X$. These sections form a graded module over the graded algebra $\Omega^\bullet_X$ of differential forms on X. Locally, an element of $\Omega^\bullet_X(E)$ is a sum of products $\omega \otimes \psi$ in which $\omega \in \Omega^\bullet_X$ and $\psi \in \Omega^0_X(E)$. Corollary 19.12 implies that the covariant derivative is a map

$$\nabla\colon \Omega^0_X(E) \longrightarrow \Omega^1_X(E). \tag{19.20}$$

Proposition 19.21.
(1) ∇ *is a linear map.*
(2) *For* $f \in \Omega^0_X$ *and* $\psi \in \Omega^0_X(E)$ *we have*

$$\nabla(f\psi) = df \cdot \psi + f\nabla\psi. \tag{19.22}$$

Equation (19.22) is the Leibniz rule; it expresses that ∇ is a first order differential operator. Proposition 19.21 follows directly from (19.16), which reduces the Leibniz rule for the covariant derivative to the Leibniz rule for the Cartan d operator.

(19.23) *Remark on the horizontal distribution.* On an arbitrary vector bundle $\pi\colon E \to X$, not necessarily associated to a principal bundle with connection, one can reasonably ask which horizontal distributions give rise to covariant derivatives that satisfy the properties in (12.62). Since $\nabla 0 = 0$, we know that the 0-section must be flat, and this is a constraint that is illustrated in Figure 62: the horizontal distribution is tangent to the 0-section. But this observation hardly gets us toward the general properties. I do not know any geometric description of the "linear" horizontal distributions. Instead, we reverse the logic below and axiomatize a covariant derivative. From that we can deduce a horizontal distribution on the total space E, though we do not do so explicitly.

Covariant derivative on an arbitrary vector bundle

So far we have considered vector bundles associated to a principal bundle. The covariant derivative can also be defined on any vector bundle, as follows.

Definition 19.24. Let $\pi\colon E \to X$ be a vector bundle over a smooth manifold X. A *covariant derivative* on π is a linear operator

$$\nabla\colon \Omega^0_X(E) \longrightarrow \Omega^1_X(E) \tag{19.25}$$

that satisfies the Leibniz rule

$$\nabla(f\psi) = df \cdot \psi + f\nabla\psi, \qquad f \in \Omega^0_X, \quad \psi \in \Omega^0_X(E). \tag{19.26}$$

Denote the space of covariant derivatives on π as $\mathcal{D}_\pi$.

We prove below that $\mathcal{D}_\pi$ is nonempty.

(19.27) *Local expressions.* Suppose $\pi\colon E \to X$ has rank N. Let $U \subset X$ be an open subset on which we are given sections $e_1, \dots, e_N\colon U \to E$ of $\pi\colon E\big|_U \to U$ such that at each $x \in U$ the vectors $e_1(x), \dots, e_N(x)$ form a basis of E_x. Using the basis property define 1-forms A^i_j such that

$$\nabla e_j = A^i_j e_i, \qquad A^i_j \in \Omega^1_U. \tag{19.28}$$

Then for any section $\psi = e_j f^j$, $f^j \in \Omega^0_U$, we have

$$\begin{aligned} \nabla\psi = \nabla(e_j f^j) &= \nabla e_j \cdot f^j + e_j \cdot df^j \\ &= e_i A^i_j f^j + e_j df^j \\ &= e_i\big(df^i + A^i_j f^j\big) \end{aligned} \tag{19.29}$$

In matrix notation we write

$$e = \begin{pmatrix} e_1 & \dots & e_N \end{pmatrix} \qquad A = \begin{pmatrix} A^1_1 & \cdots & A^1_N \\ \vdots & \ddots & \vdots \\ A^N_1 & \cdots & A^N_N \end{pmatrix} \qquad f = \begin{pmatrix} f^1 \\ \vdots \\ f^N \end{pmatrix} \tag{19.30}$$

Then

$$\nabla(ef) = e(df + Af). \tag{19.31}$$

Observe that for any matrix of 1-forms $A \in \Omega^1_U(M_n)$, equation (19.31) defines a covariant derivative on the product bundle of rank N over U.

(19.32) *Existence of covariant derivatives.* Let $\pi\colon E \to X$ be a vector bundle and suppose $\nabla_0, \nabla_1 \in \mathcal{D}_\pi$. Then the difference $\Delta = \nabla_1 - \nabla_0$ is a linear operator $\Delta\colon \Omega^0_X(E) \to \Omega^1_X(E)$ that is also linear over functions:

$$\Delta(f\psi) = f\Delta\psi, \qquad f \in \Omega^0_X, \quad \psi \in \Omega^0_X(E) \tag{19.33}$$

as follows from (19.26). Hence Δ is given by a tensor $A \in \Omega^1_X(\operatorname{End} E)$:

$$\Delta(\psi) = A(\psi), \qquad \psi \in \Omega^0_X(E). \tag{19.34}$$

Therefore, $\mathcal{D}_\pi$ is empty or $\mathcal{D}_\pi$ is affine over $\Omega^1_X(\operatorname{End} E)$. In fact, the latter holds.

Proposition 19.35. *$\mathcal{D}_\pi$ is nonempty.*

Proof. Choose an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ together with trivializations of $\pi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha$. Let ∇^α denote a covariant derivative on π_α; see **(19.27)** for the existence proof. Choose a partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to $\mathcal{U}$. Then $\nabla = \sum\limits_{\alpha\in A} \rho_\alpha \nabla^\alpha$ is a covariant derivative on π. ☐

(19.36) *Duals and tensor products.* A covariant derivative ∇^E on a vector bundle $\pi\colon E \to X$ determines a covariant derivative ∇^{E^*} on the dual vector bundle $\pi\colon E^* \to X$ as follows. Let $\langle -,-\rangle\colon E^* \times E \to \underline{\mathbb{R}}$ be the duality pairing, and require

$$d\langle\theta,\psi\rangle = \langle\nabla^{E^*}\theta,\psi\rangle + \langle\theta,\nabla^E\psi\rangle, \qquad \theta\in\Omega^0_X(E^*), \quad \psi\in\Omega^0_X(E). \tag{19.37}$$

Note that (19.37) is an equation in Ω^1_X. The left hand side is the differential of the product of three factors: θ, ψ, and the pairing $\langle -,-\rangle$. The right hand side has only two terms, so the equation expresses that the pairing is covariant constant. Equation (19.37) determines ∇^{E^*} from ∇^E.

Similarly, let ∇',∇'' be covariant derivatives on vector bundles $\pi'\colon E'\to X$ and $\pi''\colon E''\to X$, respectively. Define a covariant derivative ∇ on the tensor product $\pi\colon E'\otimes E''\to X$ by

$$\nabla(\psi'\otimes\psi'') = \nabla'\psi'\otimes\psi'' + \psi'\otimes\nabla''\psi'', \qquad \psi'\in\Omega^0_X(E'), \quad \psi''\in\Omega^0_X(E''). \tag{19.38}$$

This equation expresses that the tensor product operation is covariant constant.

A covariant derivative on $E\to X$ induces one on $\operatorname{End}E\to X$ by combining these constructions, since $\operatorname{End}E\cong E\otimes E^*$.

(19.39) *Pullbacks.* Let $\pi\colon E\to X$ be a vector bundle and $\phi\colon X'\to X$ a smooth map of smooth manifolds. Form the pullback

$$\begin{tikzcd} E' \arrow[r, dashed] \arrow[d, dashed, "\pi'"'] & E \arrow[d, "\pi"] \\ X' \arrow[r, "\phi"] & X \end{tikzcd} \tag{19.40}$$

vector bundle π'. Suppose ∇ is a covariant derivative on π. Construct a pullback covariant derivative ∇' on π' as follows. Let $\psi'\colon X'\to E'$ be a section of π' and fix $x'\in X'$ and $\xi'\in T_{x'}X'$. Suppose $\gamma\colon(-\epsilon,\epsilon)\to X'$ is a motion with initial position $\gamma'(0)=x'$ and initial velocity $\dot\gamma'(0)=\xi'$. In the diagram

$$\begin{tikzcd} E'' \arrow[r, dashed] \arrow[d, dashed, "\pi''"] & E' \arrow[r, dashed] \arrow[d, dashed, "\pi'"] & E \arrow[d, "\pi"] \\ (-\epsilon,\epsilon) \arrow[r, "\gamma'"] \arrow[u, bend left, "\psi''"] & X' \arrow[r, "\phi"] \arrow[u, bend left, "\psi'"] & X \end{tikzcd} \tag{19.41}$$

the left square is a pullback from which it follows that the outer rectangle is a pullback. Let ψ'' be the pullback of ψ'. Use the outer pullback rectangle and the covariant derivative ∇ to define parallel transport $\tau_t\colon E''_0\to E''_t$. Set

$$(\nabla'_{\xi'}\psi')(x') = \frac{d}{dt}\bigg|_{t=0}\tau_{-t}\big(\psi''(t)\big). \tag{19.42}$$

This defines the pullback covariant derivative. We leave the reader to check that it satisfies the requisite properties.

The twisted de Rham complex and curvature

(19.43) *A differential graded module.* Recall from (19.19) that $\Omega_X^\bullet(E)$ is a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded algebra $\Omega_X^\bullet$. The Cartan differential d makes $\Omega_X^\bullet$ a DGA, or *differential* graded algebra. We now enhance $\Omega_X^\bullet(E)$ to a *differential* graded module over the DGA $\Omega_X^\bullet$.

Proposition 19.44. *There exists a unique sequence of operators d_∇ in*

$$\Omega_X^0(E) \xrightarrow{d_\nabla} \Omega_X^1(E) \xrightarrow{d_\nabla} \Omega_X^2(E) \xrightarrow{d_\nabla} \cdots \tag{19.45}$$

such that

(1) *d_∇ is linear;*
(2) *$d_\nabla(\omega\wedge\sigma) = d\omega\wedge\sigma + (-1)^q\omega\wedge d_\nabla\sigma$ for all $\omega\in\Omega_X^q$ and $\sigma\in\Omega_X^\bullet(E)$; and*
(3) *$d_\nabla = \nabla$ on $\Omega_X^0(E)$.*

Proof. The module $\Omega_X^\bullet(E)$ is generated by $\Omega_X^0(E)$: any submodule of $\Omega_X^\bullet(E)$ that contains $\Omega_X^0(E)$ equals $\Omega_X^\bullet(E)$. Therefore, d_∇ is determined by its value on products $\omega\cdot s$ with $\omega\in\Omega_X^\bullet$ and $s\in\Omega_X^0(E)$. But the value on such a product is determined by (2) and (3),

$$d_\nabla(\omega\cdot s) = d\omega\cdot s + (-1)^q\omega\wedge\nabla s, \qquad \omega\in\Omega_X^q, \quad s\in\Omega_X^0(E), \tag{19.46}$$

and the value on sums is determined by (1). This proves uniqueness.

For existence, use (19.46) to define d_∇, extend to be linear, and then check (2). We omit the details. $\square$

Remark 19.47. This is of course similar to the theorem that constructs d; see the Differential Topology notes. In that case we impose $d^2=0$ as part of the *characterization* of d. In Proposition 19.44 the Leibniz rule (2) is all that is needed to characterize d as a first order differential operator. The square d_∇ is computed, not prescribed.

(19.48) *Curvature.* The square d_∇^2 does not necessarily vanish, but it is tensorial.

Lemma 19.49. *$d_\nabla^2\colon \Omega_X^\bullet(E)\to\Omega_X^{\bullet+2}(E)$ is linear over functions.*

Proof. Fix $f\in\Omega_X^0$ and $\sigma\in\Omega_X^\bullet(E)$. Then

$$\begin{aligned} d_\nabla^2(f\sigma) &= d_\nabla(df\wedge\sigma + f d_\nabla\sigma)\\ &= d^2f\wedge\sigma - df\wedge d_\nabla\sigma + df\wedge d_\nabla\sigma + f d_\nabla^2\sigma\\ &= f d_\nabla^2\sigma. \end{aligned} \tag{19.50}$$

$\square$

Recall that linearity over functions implies tensoriality.

Corollary 19.51. *There exists $F_\nabla\in\Omega_X^2(\operatorname{End}E)$ such that*

$$d_\nabla^2(\sigma) = F_\nabla\wedge\sigma, \qquad \sigma\in\Omega_X^\bullet(E). \tag{19.52}$$

(19.53) *A formula for curvature.* The following is essentially the formula (4.48) for the differential of a 1-form, and it follows from it.

Proposition 19.54. *Let ∇ be a covariant derivative on a vector bundle $\pi\colon E\to X$ with curvature $F_\nabla\in\Omega^2_X(\operatorname{End}E)$. Then for vector fields $\xi_1,\xi_2\in\mathfrak{X}(X)$ we have*

$$F_\nabla(\xi_1,\xi_2)\psi=\nabla_{\xi_1}\nabla_{\xi_2}\psi-\nabla_{\xi_2}\nabla_{\xi_1}\psi-\nabla_{[\xi_1,\xi_2]}\psi,\qquad \psi\in\Omega^0_X(E). \tag{19.55}$$

We give the proof below in **(19.73)**.

Remark 19.56. Linearity over functions, as in Lemma 19.49, is a useful check on formulas, as is more generally the behavior of formulas when a tensor field, such as a vector field, is multiplied by a function. For example, the left hand side of (19.55) is linear over functions in both ξ_1,ξ_2, whereas on the right hand side we apparently see a first order differential operator in these variables. But substituting $f\xi_1$ for ξ_1 for some function f we find

$$\begin{aligned}\nabla_{f\xi_1}&\nabla_{\xi_2}\psi-\nabla_{\xi_2}\nabla_{f\xi_1}\psi-\nabla_{[f\xi_1,\xi_2]}\psi\\&=f\nabla_{\xi_1}\nabla_{\xi_2}\psi-(\xi_2f)\nabla_{\xi_1}\psi-f\nabla_{\xi_2}\nabla_{f\xi_1}\psi+(\xi_2f)\nabla_{\xi_1}\psi-f\nabla_{[\xi_1,\xi_2]}\psi\\&=f\bigl\{\nabla_{\xi_1}\nabla_{\xi_2}\psi-\nabla_{\xi_2}\nabla_{\xi_1}\psi-\nabla_{[\xi_1,\xi_2]}\psi\bigr\}.\end{aligned} \tag{19.57}$$

The terms that differentiate f cancel out. The verification that the right hand side is linear over functions in ξ_2 is similar.

(19.58) *Bianchi identity.* Recall the Bianchi identity (16.60) for a connection on a principal bundle.

Proposition 19.59. *Let ∇ be a covariant derivative on a vector bundle $\pi\colon E\to X$ with curvature $F_\nabla\in\Omega^2_X(\operatorname{End}E)$. Then*

$$[d_\nabla\wedge F_\nabla]=0 \tag{19.60}$$

as an operator $\Omega^\bullet_X(E)\to\Omega^{\bullet+3}_X(E)$.

Proof. For all $\sigma\in\Omega^\bullet_X(E)$ we have

$$\begin{aligned}d^3_\nabla\sigma&=F_\nabla\wedge d_\nabla\sigma\\&=d_\nabla(F_\nabla\wedge\sigma).\end{aligned} \tag{19.61}$$

But

$$d_\nabla(F_\nabla\wedge\sigma)=[d_\nabla\wedge F_\nabla]\wedge\sigma+F_\nabla\wedge d_\nabla\sigma, \tag{19.62}$$

from which

$$[d_\nabla\wedge F_\nabla]\wedge\sigma=0. \tag{19.63}$$

Let σ run over a local basis of sections to conclude. ∎

The bundle of frames

We began this lecture by passing from a connection on a principal G-bundle to a covariant derivative on every associated vector bundle. Now we execute the inverse construction, albeit for G the general linear group.

(19.64) *Connection on the bundle of frames.* To a vector bundle $\pi\colon E \to X$ we associate its principal $G = \mathrm{GL}_N\mathbb{R}$-bundle of frames $\varpi\colon \mathcal{B}(E) \to X$. Namely, a point in the fiber $\mathcal{B}(E)_x$ at $x \in X$ is an isomorphism $b\colon \mathbb{R}^N \to E_x$. The pullback of π over ϖ has N canonical sections:

(19.65)
$$\begin{tikzcd} E' \arrow[r, dashed] \arrow[d, dashed, "\pi'"'] & E \arrow[d, "\pi"] \\ \mathcal{B}(E) \arrow[r, "\varpi"] \arrow[u, bend right, "e_1,\dots,e_N"'] & X \end{tikzcd}$$

Namely, a point of E' is the data: a point $x \in X$, a basis $b = (e_1, \dots, e_N)$ of E_x, and a point $\epsilon \in E_x$. The projection π' forgets the point ϵ. The i^{th} section of π' sets $\epsilon = e_i$. In other words, the value of the section e_i at b is the i^{th} element of the basis b.

Suppose that ∇ is a covariant derivative on π, and let ∇' be the pullback covariant derivative on π'. Define 1-forms Θ^i_j on $\mathcal{B}(E)$ by

(19.66)
$$\nabla' e_j = e_i \Theta^i_j, \qquad \Theta^i_j \in \Omega^1_{\mathcal{B}(E)}.$$

It is convenient to rewrite (19.66) in matrix notation, as in (19.30):

(19.67)
$$\nabla' e = e\,\Theta, \qquad \Theta = \bigl(\Theta^i_j\bigr) \in \Omega^1_{\mathcal{B}(E)}(\mathfrak{gl}_N\mathbb{R}).$$

Lemma 19.68. *Θ is a connection on $\varpi\colon \mathcal{B}(E) \to X$.*

Proof. We check the two conditions in Lemma 14.28. First, for each $x \in X$ and basis b of E_x, an element $A \in \mathfrak{gl}_N\mathbb{R}$ generates a motion $t \mapsto b \cdot e^{tA}$ in $\mathcal{B}(E)_x$. Since each e_i takes values in the fixed vector space E_x in the fiber $\mathcal{B}(E)_x$, the covariant derivative of e along this motion is the ordinary directional derivative. Since essentially $e = b$, this is

(19.69)
$$\left.\frac{d}{dt}\right|_{t=0} b \cdot e^{tA} = b \cdot A,$$

from which the value of Θ on the vertical vector generated by A is A. This proves (14.29). To check (14.30), for $M \in \mathrm{GL}_N\mathbb{R}$ compute

(19.70)
$$\nabla'(eM) = \nabla' e \cdot M = e\,\Theta M = (eM)(M^{-1}\Theta M).$$ ∎

Remark 19.71. The constructions in **(19.3)** and **(19.64)** define equivalences of categories

(19.72)
$$\left\{\begin{matrix}\text{principal } \mathrm{GL}_N\mathbb{R}\text{-bundles over } X \\ \text{with connection}\end{matrix}\right\} \rightleftarrows \left\{\begin{matrix}\text{rank } N \text{ vector bundles over } X \\ \text{with covariant derivative}\end{matrix}\right\}.$$

(19.73) *An application.* We have found it convenient to compute in a fiber bundle $E \to X$ with fiber F that is associated with a principal bundle $P \to X$ by pulling back to the total space P: the fiber bundle untwists and sections of $E \to X$ become functions $P \to F$. Here, then, is an application of the correspondence (19.72).

Proof of Proposition 19.54. Let $\varpi\colon \mathcal{B}(E) \to X$ be the bundle of frames of $\pi\colon E \to X$. The pullback under ϖ of an element $\sigma \in \Omega^\bullet_X(E)$ is a differential form $\tilde\sigma \in \Omega^\bullet_{\mathcal{B}(E)}(\mathbb{R}^N)$. A small variation of Proposition 19.15 computes the pullback of $d_\nabla\sigma$ under ϖ:

$$d_\nabla\sigma = d\tilde\sigma + \Theta\wedge\tilde\sigma, \tag{19.74}$$

where $\Theta \in \Omega^1_{\mathcal{B}(E)}(\mathfrak{gl}_N\mathbb{R})$ is the connection on ϖ constructed in Lemma 19.68. Let $\Omega \in \Omega^2_{\mathcal{B}(E)}(\mathfrak{gl}_N\mathbb{R})$ be the curvature of Θ. Then Ω descends to F_∇.

Let $\widetilde{\xi_1}, \widetilde{\xi_2}$ be the horizontal lifts of ξ_1,ξ_2 to $\mathcal{B}(E)$. Then the lift of $F_\nabla(\xi_1,\xi_2)$ to $\mathcal{B}(E)$ is

$$\begin{aligned} F_\nabla(\xi_1,\xi_2) &= \Omega(\widetilde{\xi_1},\widetilde{\xi_2}) \\ &= d\Theta(\widetilde{\xi_1},\widetilde{\xi_2}) + (\Theta\wedge\Theta)(\widetilde{\xi_1},\widetilde{\xi_2}) \\ &= -\Theta\bigl([\widetilde{\xi_1},\widetilde{\xi_2}]\bigr) \end{aligned} \tag{19.75}$$

which is a $\mathfrak{gl}_N\mathbb{R}$-valued function. Let $\tilde\psi\colon \mathcal{B}(E) \to \mathbb{R}^N$ be the lift of the section ψ. Recall that it satisfies the equivariance condition

$$\tilde\psi(bM) = M^{-1}\tilde\psi(b), \qquad b\in\mathcal{B}(E), \quad M\in GL_N\mathbb{R}. \tag{19.76}$$

Differentiate in M to see that $-\Theta([\widetilde{\xi_1},\widetilde{\xi_2}])\tilde\psi$ is the directional derivative of $\tilde\psi$ along the vertical component of $[\widetilde{\xi_1},\widetilde{\xi_2}]$.

From (19.74) we see that

$$\begin{aligned} \nabla_{\xi_2}\psi &= \widetilde{\xi_2}\tilde\psi \\ \nabla_{\xi_1}\nabla_{\xi_2}\psi &= \widetilde{\xi_1}\widetilde{\xi_2}\tilde\psi \end{aligned} \tag{19.77}$$

meaning, as usual, that the lift of the left hand sides to $\mathcal{B}(E)$ is the right hand side. Thus the right hand side of (19.55) lifts to

$$\widetilde{\xi_1}\widetilde{\xi_2}\tilde\psi - \widetilde{\xi_2}\widetilde{\xi_1}\tilde\psi - \widetilde{[\xi_1,\xi_2]}\tilde\psi = \bigl\{[\widetilde{\xi_1},\widetilde{\xi_2}] - \widetilde{[\xi_1,\xi_2]}\bigr\}\tilde\psi \tag{19.78}$$

which is the directional derivative of $\tilde\psi$ along the vertical component of $[\widetilde{\xi_1},\widetilde{\xi_2}]$. $\square$

(19.79) *Linear connections and covariant derivatives.* As a special case of the preceding—the intrinsic case—a covariant derivative on the tangent bundle $TX \to X$ of a smooth manifold X is equivalent to a linear connection: a connection on the principal bundle of frames $\varpi\colon \mathcal{B}(X) \to X$.

Remark 19.80. The term 'connection' is often used interchangeably with 'covariant derivative', especially in the intrinsic context.

Let $\mathbb{E}$ be a vector space equipped with a representation $\rho\colon \mathrm{GL}_n\mathbb{R} \to \mathrm{Aut}(\mathbb{E})$. Sections of the associated vector bundle $\pi\colon E = \mathbb{E}_{\mathcal{B}(X)} \to X$ are *tensor fields* on X. One expression for the induced covariant derivative of tensor fields is given in Proposition 19.15. Here is another. Recall the $(\mathbb{R}^n)^*$-valued horizontal vector field $\partial = \partial_k e^k$ on $\mathcal{B}(X)$. The following very useful formula is a direct consequence of Proposition 19.8.

Proposition 19.81. *Let ψ be a section of $\pi\colon E \to X$ and let $\tilde{\psi}\colon \mathcal{B}(X) \to \mathbb{E}$ be its pullback under ϖ. Then the covariant derivative pulls back to*

$$\nabla\psi = \partial\tilde{\psi}, \tag{19.82}$$

a $\mathrm{GL}_n\mathbb{R}$-equivariant function $\mathcal{B}(X) \to (\mathbb{R}^n)^* \otimes \mathbb{E}$.

More general differential operators on X can often be written in a similar form on the frame bundle; see **(17.49)** for an example.

Remark 19.83. The following observation may be useful to make explicit. A covector $\omega \in T_x^*X$ has a lift in two different senses to the frame bundle $\varpi\colon \mathcal{B}(X) \to X$. (The same applies to differential forms of arbitrary degree.) On the one hand, $\varpi^*(\omega) \in \Omega^1_{\mathcal{B}(X)}$ is the lift as a 1-form. On the other hand, there is a lift $\tilde{\omega}\colon \mathcal{B}(X) \to (\mathbb{R}^n)^*$ to an $(\mathbb{R}^n)^*$-valued function. They are related by

$$\varpi^*(\omega)(\partial_v) = \langle\tilde{\omega}, v\rangle, \qquad v \in \mathbb{R}^n. \tag{19.84}$$

(19.85) *Torsion in terms of covariant derivatives.* The following formula expresses the torsion directly in terms of covariant derivatives.

Proposition 19.86. *Let X be a smooth manifold with a linear connection and associated covariant derivative ∇ on $TX \to X$. Let $\tau \in \Omega^2_X(TX)$ be the torsion. Then*

$$\tau(\xi_1, \xi_2) = \nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1 - [\xi_1, \xi_2], \qquad \xi_1, \xi_2 \in \mathcal{X}(X). \tag{19.87}$$

Proof. Let $\widetilde{\xi_1}, \widetilde{\xi_2}$ denote the horizontal lifts of ξ_1, ξ_2, respectively. Plug into $\tau = d\theta + \Theta \wedge \theta$ and use Proposition 19.81:

$$\begin{aligned} \tau(\xi_1, \xi_2) &= \widetilde{\xi_1}\theta(\widetilde{\xi_2}) - \widetilde{\xi_2}\theta(\widetilde{\xi_1}) - \theta\big([\widetilde{\xi_1}, \widetilde{\xi_2}]\big) \\ &= \nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1 - [\xi_1, \xi_2]. \end{aligned} \tag{19.88}$$

$\square$

Lecture 20: The Levi–Civita covariant derivative and the Riemann curvature tensor

We constructed the Levi–Civita connection of a Riemannian manifold in **(18.46)** as a connection on the orthonormal frame bundle $\varpi\colon \mathcal{B}_{\mathrm{O}}(X) \to X$. Our first task in this lecture is to construct the corresponding Levi–Civita covariant derivative on the tangent bundle. We end up with an *6-term explicit* formula (20.15) in terms of the metric and Lie bracket, a formula that is very useful in many situations. It is not necessary to memorize the precise formula, but rather one should know its properties so it can be quickly recovered (Remark 20.16(2)).

In the second half of the lecture we explore the symmetries of the Riemann curvature tensor, which are enumerated in Theorem 20.36.

The Levi–Civita covariant derivative

(20.1) *Covariant derivatives and geometric structures.* Let X be a smooth manifold. In (19.72) we constructed a 1:1 correspondence (which is an equivalence of categories)

$$\{\text{covariant derivatives on } TX \longrightarrow X\} \longleftrightarrow \{\text{connections on } \mathcal{B}(X) \longrightarrow X\}\,. \tag{20.2}$$

Now suppose (G_n, λ_n) is a symmetry type and X has a (G_n, λ_n)-structure encoded in a principal G_n-bundle $\mathcal{B}_G(X) \to X$ (and an isomorphism of the associated principal $\mathrm{GL}_n\mathbb{R}$-bundle with the frame bundle). We ask for additional data/conditions on the covariant derivative on $TX \to X$ such that the corresponding connection has a natural lift to $\mathcal{B}_G(X) \to X$. We do not address that problem in general, but only treat the case of a Riemannian structure.

Remark 20.3. The setting generalizes to arbitrary vector bundles, and in particular Proposition 20.5 applies to vector bundles equipped with an inner product.

(20.4) *Orthogonal covariant derivatives on the tangent bundle.* Let X be a Riemannian manifold.

Proposition 20.5. *Suppose ∇ is a covariant derivative on the tangent bundle $TX \to X$. Then the induced connection on the frame bundle $\mathcal{B}(X) \to X$ restricts to a connection on the orthonormal frame bundle $\mathcal{B}_{\mathrm{O}}(X) \to X$ iff*

$$d\langle \xi_1, \xi_2\rangle = \langle \nabla\xi_1, \xi_2\rangle + \langle \xi_1, \nabla\xi_2\rangle \tag{20.6}$$

for all vector fields $\xi_1, \xi_2 \in \mathfrak{X}(X)$.

Proof. Let Θ be a connection 1-form on $\mathcal{B}_{\mathrm{O}}(X) \to X$. Suppose given vector fields $\xi_1, \xi_2 \in \mathfrak{X}(X)$, and let $\widetilde{\xi}_1, \widetilde{\xi}_2\colon \mathcal{B}_{\mathrm{O}}(X) \to \mathbb{R}^n$ be the lifts to O_n-equivariant functions. Then since Θ takes values in skew-symmetric matrices, we have

$$\begin{aligned} d\langle \widetilde{\xi}_1, \widetilde{\xi}_2\rangle &= \langle d\widetilde{\xi}_1, \widetilde{\xi}_2\rangle + \langle \widetilde{\xi}_1, d\widetilde{\xi}_2\rangle \\ &= \langle d\widetilde{\xi}_1 + \Theta\widetilde{\xi}_1, \widetilde{\xi}_2\rangle + \langle \widetilde{\xi}_1, d\widetilde{\xi}_2 + \Theta\widetilde{\xi}_2\rangle \\ &= \langle \nabla\xi_1, \xi_2\rangle + \langle \xi_1, \nabla\xi_2\rangle, \end{aligned} \tag{20.7}$$

where at the last stage we write the descent to X using Proposition 19.15.

Conversely, suppose ∇ is a covariant derivative on $TX \to X$ that satisfies (20.6). Let Θ be the corresponding linear connection on $\mathcal{B}(X) \to X$, constructed in **(19.64)**. Similar to (19.65), construct the pullback of the tangent bundle over the orthonormal frame bundle

(20.8)
$$\begin{tikzcd} E' \arrow[r, dashed] \arrow[d, dashed, "\pi'"'] & TX \arrow[d, "\pi"] \\ \mathcal{B}_{\mathrm{O}}(X) \arrow[u, bend right, "e_1,\dots,e_n"'] \arrow[r, "\varpi"] & X \end{tikzcd}$$

let ∇' be the pullback covariant derivative, and let Θ be the 1-form defined as in (19.66):

(20.9)
$$\nabla' e_j = e_i\Theta^i_j, \qquad \Theta^i_j \in \Omega^1_{\mathcal{B}_{\mathrm{O}}(X)}.$$

The proof of Lemma 19.68 shows that the extension of Θ to the principal $\mathrm{GL}_n\mathbb{R}$-bundle of frames is a connection. To prove that Θ is a connection on $\mathcal{B}_{\mathrm{O}}(X) \to X$ we show that it takes values in skew-symmetric matrices. Namely, since $e_1, \dots, e_n$ are orthonormal,

(20.10)
$$\begin{aligned} 0 &= d\langle e_j, e_k\rangle \\ &= \langle \nabla e_j, e_k\rangle + \langle e_j, \nabla e_k\rangle \\ &= \langle e_i\Theta^i_j, e_k\rangle + \langle e_j, \Theta^i_k e_i\rangle \\ &= \Theta^k_j + \Theta^j_k \end{aligned}$$

∎

(20.11) *The 6-term formula.* Let X be a Riemannian manifold. Then according to (20.6) and (19.87) the Levi–Civita covariant derivative, which is orthogonal and torsionfree, satisfies

(20.12)
$$\xi\langle \eta, \zeta\rangle = \langle \nabla_\xi \eta, \zeta\rangle + \langle \eta, \nabla_\xi \zeta\rangle$$

(20.13)
$$[\xi, \eta] = \nabla_\xi \eta - \nabla_\eta \xi$$

for all vector fields $\xi, \eta, \zeta \in \mathfrak{X}(X)$.

Theorem 20.14. *The Levi–Civita covariant derivative on a Riemannian manifold X satisfies*

(20.15)
$$\langle \nabla_\xi \eta, \zeta\rangle = \frac{1}{2}\Big\{\xi\langle \eta, \zeta\rangle + \eta\,\langle \zeta, \xi\rangle - \zeta\langle \xi, \eta\rangle + \langle [\xi, \eta], \zeta\rangle - \langle [\eta, \zeta], \xi\rangle + \langle [\zeta, \xi], \eta\rangle\Big\}$$

for all vector fields $\xi, \eta, \zeta \in \mathfrak{X}(X)$.

Remark 20.16.

(1) For $x \in X$ let $\zeta(x)$ run over an orthonormal basis of T_xX to determine $(\nabla_\xi \eta)(x) \in T_xX$.

(2) The form of the 6-term formula (20.15) is easy to remember, but the signs are not. One can quickly derive them by using the fact that the left hand side is linear over functions in ξ and ζ but is a first order differential operator on η:

$$\langle \nabla_{f\xi}(g\eta), h\zeta\rangle = fgh\,\langle \nabla_{\xi}\eta, \zeta\rangle + f(\xi g)h\langle \eta, \zeta\rangle, \qquad f, g, h \in \Omega^0_X. \tag{20.17}$$

The other input is that the terms $\xi\langle \eta, \zeta\rangle$ and $\langle [\xi, \eta], \zeta\rangle$ have $+$ signs, which follows immediately from (20.12) and (20.13).

Proof. Alternately apply (20.12) and (20.13), each three times:

$$\begin{aligned} \langle \nabla_{\xi}\eta, \zeta\rangle &= \xi\langle \eta, \zeta\rangle - \langle \eta, \nabla_{\xi}\zeta\rangle \\ &= \xi\langle \eta, \zeta\rangle - \langle \eta, [\xi, \zeta]\rangle - \langle \eta, \nabla_{\zeta}\xi\rangle \\ &= \xi\langle \eta, \zeta\rangle - \langle \eta, [\xi, \zeta]\rangle - \zeta\langle \eta, \xi\rangle + \langle \nabla_{\zeta}\eta, \xi\rangle \\ &= \xi\langle \eta, \zeta\rangle - \langle \eta, [\xi, \zeta]\rangle - \zeta\langle \eta, \xi\rangle + \langle [\zeta, \eta], \xi\rangle + \langle \nabla_{\eta}\zeta, \xi\rangle \\ &= \xi\langle \eta, \zeta\rangle - \langle \eta, [\xi, \zeta]\rangle - \zeta\langle \eta, \xi\rangle + \langle [\zeta, \eta], \xi\rangle + \eta\langle \zeta, \xi\rangle - \langle \zeta, \nabla_{\eta}\xi\rangle \\ &= \xi\langle \eta, \zeta\rangle - \langle \eta, [\xi, \zeta]\rangle - \zeta\langle \eta, \xi\rangle + \langle [\zeta, \eta], \xi\rangle + \eta\langle \zeta, \xi\rangle - \langle \zeta, [\eta, \xi]\rangle - \langle \zeta, \nabla_{\xi}\eta\rangle \end{aligned} \tag{20.18}$$

(Compare to (18.60).) Rearrange the terms to derive (20.15). ☐

(20.19) *The Levi–Civita covariant derivative in local coordinates.* Let $(U; x^1, \dots, x^n)$ be a chart on X. Then on U we have the framing $\dfrac{\partial}{\partial x^1}, \dots, \dfrac{\partial}{\partial x^n}$ and the dual coframing $dx^1, \dots, dx^n$. The metric on the tangent space with respect to this framing and the dual metric on the cotangent space with respect to this coframing define functions $g_{ij}, g^{ij}\colon U \to \mathbb{R}$ by

$$g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle, \qquad g^{ij} = \langle dx^i, dx^j\rangle, \tag{20.20}$$

which make up inverse matrices in the sense that $g_{ij}g^{jk} = \delta^k_i$ for $1 \le i, k \le n$. Define functions $\Gamma^i_{kj}\colon U \to \mathbb{R}$ that express the Levi–Civita covariant derivative relative to this framing:

$$\nabla_{\partial/\partial x^k} \frac{\partial}{\partial x^j} = \Gamma^i_{kj} \frac{\partial}{\partial x^i}. \tag{20.21}$$

Use the fact that $\left[\dfrac{\partial}{\partial x^k}, \dfrac{\partial}{\partial x^\ell}\right] = 0$ to deduce from (20.15) that

$$\langle \nabla_{\partial/\partial x^k} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^\ell}\rangle = \Gamma^m_{kj} g_{m\ell} = \frac{1}{2}\left(\frac{\partial g_{j\ell}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} - \frac{\partial g_{kj}}{\partial x^\ell}\right) \tag{20.22}$$

from which we obtain the classical formula

$$\Gamma^i_{kj} = \frac{1}{2} g^{i\ell} \left(\frac{\partial g_{j\ell}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} - \frac{\partial g_{kj}}{\partial x^\ell} \right) \tag{20.23}$$

for the *Christoffel symbols*. We previously introduced this formula as an notational convenience (18.34) in our derivation of the Riemann curvature tensor. Here we see its geometric meaning in terms of the Levi–Civita connection.

(20.24) *The Riemann curvature in local coordinates.* In Lecture 18 we derived the classical formula for the Riemann curvature tensor in local coordinates twice. In (18.10) it was derived for a general linear connection. Then in (18.38) it was derived, following Riemann, as the obstruction to finding local coordinates in which the coordinate vector fields are orthonormal.

The Riemann curvature tensor and its symmetries

We continue with the Riemannian manifold X.

(20.25) *The global parallelism of $\mathcal{B}_{\mathrm{O}}(X)$.* We modify **(16.30)** and **(16.39)** slightly for the Lie subalgebra $\mathfrak{o}_n \subset \mathfrak{gl}_n\mathbb{R}$. Namely, for $1 \le i, j \le n$ set

$$S^j_i = E^j_i - E^i_j. \tag{20.26}$$

The skew-symmetric matrices S^j_i form a basis of $\mathfrak{o}_n$ if we restrict $1 \le i < j \le n$. But it is convenient to use the spanning set in which i, j are unrestricted and impose skew-symmetry on the coefficients, all compensated by a factor of $1/2$. Let $\widehat{S}^j_i$ denote the corresponding vertical vector field on the total space $\mathcal{B}_{\mathrm{O}}(X)$ of the orthonormal frame bundle $\varpi\colon \mathcal{B}_{\mathrm{O}}(X) \to X$. The horizontal vector fields ∂_k are defined as before.

(20.27) *The Riemann tensor.* Apply (16.66) to define functions $R^i_{jk\ell}\colon \mathcal{B}_{\mathrm{O}}(X) \to \mathbb{R}$:

$$\Omega^i_j = \frac{1}{2} R^i_{jk\ell}\, \theta^k \wedge \theta^\ell. \tag{20.28}$$

Since $\Omega = \left(\Omega^i_j\right)$ is a 2-form with values in skew-symmetric matrices, we have

$$R^i_{jk\ell} = -R^j_{ik\ell}. \tag{20.29}$$

The dual to (20.28) is (16.70) specialized to the Riemannian case:

$$[\partial_k, \partial_\ell] = -R^i_{jk\ell}\, \widehat{E}^j_i = -\frac{1}{2} R^i_{jk\ell}\, \widehat{S}^j_i. \tag{20.30}$$

The collection of functions $R^i_{jk\ell}$ satisfy equivariance conditions for the O_n-action on $\mathcal{B}_{\mathrm{O}}(X)$; they are the components of the pullback to $\mathcal{B}_{\mathrm{O}}(X)$ of a tensor

$$R \in \Omega^2_X(\mathrm{SkewEnd}\,TX), \tag{20.31}$$

a 2-form with values in the vector bundle of skew-symmetric endomorphisms of the tangent bundle. Fix $x \in X$ and let $b \in \mathcal{B}_{\mathrm{O}}(X)$ be an orthonormal basis $e_1, \dots, e_n$ of T_xX. Then

$$R(e_k, e_\ell)e_j = R^i_{jk\ell}e_i. \tag{20.32}$$

R is the *Riemann curvature tensor*.

(20.33) *Symmetries.* Define $R_{ijk\ell} \colon \mathcal{B}_{\mathrm{O}}(X) \to \mathbb{R}$ to be the function with value

$$R_{ijk\ell}(b) := \langle e_i, R(e_k, e_\ell)e_j\rangle = R^i_{jk\ell}(b). \tag{20.34}$$

at the basis $b = (e_1, \dots, e_n) \in \mathcal{B}_{\mathrm{O}}(X)_x$. These descend to a quadrilinear form

$$\widetilde{R} \colon TX \times TX \times TX \times TX \longrightarrow \underline{\mathbb{R}}. \tag{20.35}$$

It obeys symmetry conditions expressed by the first three of the following constraints on $R_{ijk\ell}$.

Theorem 20.36. *For all $1 \le i, j, k, \ell \le n$ we have*

$$R_{ijk\ell} = -R_{jik\ell} \tag{20.37}$$

$$R_{ijk\ell} = -R_{ij\ell k} \tag{20.38}$$

$$R_{ijk\ell} = \quad R_{k\ell ij} \tag{20.39}$$

$$R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0 \tag{20.40}$$

Equations (20.37)–(20.39) imply that $\widetilde{R}$ factors down to a *symmetric* bilinear form

$$\widetilde{R} \colon {\textstyle\bigwedge}^2 TX \times {\textstyle\bigwedge}^2 TX \longrightarrow \underline{\mathbb{R}}. \tag{20.41}$$

Put differently,

$$\widetilde{R} \in \mathrm{Sym}^2\bigl({\textstyle\bigwedge}^2 T^*X\bigr). \tag{20.42}$$

The formula for $\widetilde{R}$ in terms of R is: for tangent vectors $\xi, \eta, \zeta, \lambda$ we have

$$\widetilde{R}(\xi \wedge \eta, \zeta \wedge \lambda) = \langle \xi, R(\zeta, \lambda)\eta\rangle. \tag{20.43}$$

Proof. The skew-symmetry (20.37) is (20.29), and the skew-symmetry (20.38) is (16.66). To prove (20.40) we use the Bianchi identity (16.59):

$$\begin{aligned} 0 = d_\Theta \tau = d\tau + \Theta \wedge \tau = \Omega \wedge \theta \\ = \frac{1}{2} R^i_{jk\ell}\, \theta^k \wedge \theta^\ell \wedge \theta^j\, e_i \end{aligned} \tag{20.44}$$

since the Levi–Civita connection Θ is torsionfree. Fix i, j, k, ℓ. Then the coefficient of $\theta^k \wedge \theta^\ell \wedge \theta^j\, e_i$ in (20.44) is

$$\frac{1}{2}\left(R^i_{jk\ell} + R^i_{k\ell j} + R^i_{\ell jk}\right) - \frac{1}{2}\left(R^i_{j\ell k} + R^i_{\ell kj} + R^i_{kj\ell}\right) = R^i_{jk\ell} + R^i_{k\ell j} + R^i_{\ell jk} \tag{20.45}$$

and it vanishes, which is (20.40).

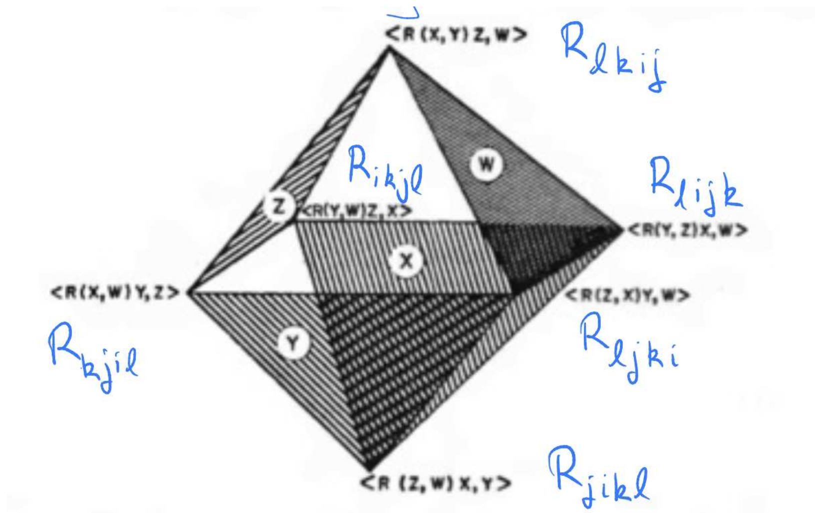

FIGURE 64. Proof of (20.39)

The symmetry (20.39) follows from (20.37), (20.38), and (20.40) by a beautiful argument of Milnor depicted in Figure 64 (which is lifted from his *Morse Theory* book). The six vertices of the octahedron are each labeled by a component of the Riemann tensor, and (20.40) implies that the sum of the components at the vertices of the faces labeled X, Y, Z, and W vanish. Apply the proved symmetries (20.37) and (20.38) to deduce

$$\begin{aligned} 0 &= R_{ikj\ell} + R_{jik\ell} + R_{\ell ijk} \\ &= R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} \\ 0 &= R_{kji\ell} + R_{\ell jki} + R_{jik\ell} \\ &= R_{jk\ell i} + R_{j\ell ik} + R_{jik\ell} \\ 0 &= R_{kji\ell} + R_{ikj\ell} + R_{\ell kij} \\ &= R_{kji\ell} + R_{ki\ell j} + R_{k\ell ji} \\ 0 &= R_{\ell kij} + R_{\ell ijk} + R_{\ell jki} \end{aligned} \tag{20.46}$$

Add these four equations to find

$$0 = 2(R_{\ell kij} + R_{ji\ell k}) \tag{20.47}$$

from which (20.39) follows. □

Remark 20.48. An equivalent expression for the Bianchi identity (20.40) is

$$R(\xi_1, \xi_2)\xi_3 + R(\xi_2, \xi_3)\xi_1 + R(\xi_3, \xi_1)\xi_2 = 0, \tag{20.49}$$

where $\xi_1, \xi_2, \xi_3 \in T_xX$ are tangent vectors at a point $x \in X$. This follows from (20.32).

Remark 20.50. Note that (20.40) does *not* follow from (20.37)–(20.39). In other words, (20.42) is not the definitive location of the Riemann tensor; one must also impose the Bianchi identity.

Lecture 21: The Gauss–Bonnet theorem

In the next few lectures we prove the *Gauss–Bonnet–Chern* theorem, which gives a formula for the Euler number of a closed Riemannian manifold in terms of the Riemann curvature.[37] Along the way we introduce *Chern–Weil* and *Chern–Simons* invariants of connections, each of which is an important topic in its own right. A fundamental ingredient in the proof is (a special case of) the Poincaré–Hopf theorem, which is Theorem 28.33 in the Differential Topology notes.

In this lecture we begin with the classical Gauss–Bonnet theorem for 2-manifolds. This classical Gauss–Bonnet theorem[38] is one of the first theorems to relate Riemannian geometry to the topology of the underlying smooth manifold (or, in this case, the topology of the underlying topological space). There are many, many theorems that do so, including quite recent ones; as an illustration of the genre, we state a theorem of Kazdan–Warner. The proof of the classical Gauss–Bonnet theorem is completed at the end of the lecture.

The main ingredients of the proof are presented in arbitrary dimensions. The Euler number is computed by counting with signs the zeros of a vector field ξ with simple zeros. Away from the zeros the (normalized) vector field defines a section of the sphere bundle of the tangent bundle. We extend this over a compact manifold with boundary, obtained by "blowing up" the points at which ξ vanishes. In dimension two we conclude by identifying the sphere bundle with the oriented orthonormal frame bundle, assuming the underlying Riemannian surface is oriented. Then we apply Stokes' theorem to the connection form. In higher dimensions we need to find an analogous *transgressing differential form* on the total space of the sphere bundle. The construction of that form, due to Chern, takes up the next few lectures.

I posted some papers, including Chern's original paper, on Canvas. They are well worth reading.

The classical Gauss–Bonnet theorem

(21.1) *Statement and remarks.* Let (Σ, g) be a closed Riemannian 2-manifold. The Riemannian metric determines a smooth density $d\mu_g$; at each point it gives the area of parallelograms in the tangent space. Let $K_g\colon \Sigma \to \mathbb{R}$ be the Gauss curvature (Corollary 11.9). The classical Gauss–Bonnet theorem is the following.

Theorem 21.2. *The Euler number of Σ is*

$$\operatorname{Euler}(\Sigma) = \frac{1}{2\pi}\int_\Sigma K_g\, d\mu_g. \tag{21.3}$$

We make several remarks.

[37] A simple version of the Gauss–Bonnet theorem for surfaces is treated in **(15.36)**.

[38] The "Gauss–Bonnet" theorem for closed manifolds, which is the version we discuss here, is due to Walther von Dyck. Daniel Gottlieb, in an article about the history and sociology of Gauss–Bonnet (posted on Canvas), writes, "Part of this story shows that the name of a theorem is not really for an attribution. It is very convenient to have a name for important theorems, and the main point is that people should know approximately what theorem is meant by the name rather than who gets the credit. Still, one can reflect that Bonnet's name is famous and Dyck's is virtually unknown these days."

Remark 21.4.

(1) The theorem does not require that Σ be oriented. For example, the theorem applies to the real projective plane and to the Klein bottle. In the proofs below we assume an orientation to simplify the presentation. Straightforward arguments with the orientation double cover extend the proofs to unoriented manifolds.

(2) Let $\Sigma = S^2(R)$ be the 2-sphere with the round metric induced by the standard embedding into Euclidean space $\mathbb{E}^3$ as a sphere of radius $R > 0$. Then the Gauss curvature is the constant $1/R^2$; see (11.33) and the text that follows. The area is $4\pi R^2$ and so (21.3) is

$$\mathrm{Euler}(S^2) = \frac{1}{2\pi}\,\frac{1}{R^2}\,(4\pi R^2) = 2. \tag{21.5}$$

Notice that the radius R cancels out as it must: the Euler number is a topological invariant independent of the Riemannian metric.

(3) *Geometry computes topology.* Namely, in (21.3) the left hand side is a global topological invariant. It is computed by the right hand side, which is the integral of a local geometric invariant. The local geometric invariant depends on extra local structure—the Riemannian metric—but its integral is independent of it. The local structure is chosen from a continuous space (of Riemannian metrics), whereas the global invariant is discrete. This means the local geometric invariant (Gauss curvature times area density) carries hidden integrality.

(4) *Topology constrains geometry.* The Gauss–Bonnet theorem implies constraints on the possible Gauss curvatures of a Riemannian surface. For example, S^2 does not admit a metric with nonpositive Gauss curvature. Similarly the torus and Klein bottle, which have vanishing Euler numbers, do not admit metrics with either positive or negative Gauss curvature. And connected oriented closed surfaces of genus ≥ 2, which have negative Euler numbers, do not admit metrics with positive Gauss curvature.

(21.6) *An existence theorem.* In the other direction, and as an illustration of the interplay between topology and local geometry, we state without proof the following theorem of Kazdan–Warner. It is an existence theorem for Riemannian metrics, the only constraints being those identified in Remark 21.4(4).

Theorem 21.7 (Kazdan–Warner)**.** *Let Σ be a closed connected 2-manifold. Suppose $K\colon \Sigma \to \mathbb{R}$ is a function that satisfies*

(i) *if* $\mathrm{Euler}(\Sigma) > 0$, *then there exists $x \in \Sigma$ such that $K(x) > 0$;*
(ii) *if* $\mathrm{Euler}(\Sigma) = 0$, *then K is either identically zero or takes on both positive and negative values; and*
(iii) *if* $\mathrm{Euler}(\Sigma) < 0$, *then there exists $x \in \Sigma$ such that $K(x) < 0$.*

Then there exists a Riemannian metric g on Σ whose Gauss curvature satisfies $K_g = K$.

In fact, Kazdan–Warner prove the more precise existence of metrics with prescribed Gauss curvature *conformally equivalent* to a given Riemannian metric. This involves proving existence of solutions to a nonlinear partial differential equation on Σ. It is typical that natural questions in differential geometry quickly lead to nonlinear PDEs; their study is the field of *global analysis*.

(21.8) *The oriented orthonormal frame bundle.* Assume that the Riemannian 2-manifold Σ is also oriented. Then a unit tangent vector $e_1 \in T_x\Sigma$ at a point $x \in \Sigma$ determines an oriented orthonormal frame e_1, e_2 at x: the second basis vector e_2 is obtained from e_1 by a $\pi/2$-rotation in the direction determined by the orientation. So the principal SO_2-bundle of oriented orthonormal frames

$$\varpi\colon \mathcal{B}_{\mathrm{SO}}(\Sigma) \longrightarrow \Sigma \tag{21.9}$$

is identified with the unit circle bundle of the tangent bundle

$$\pi\colon S(T\Sigma) \longrightarrow \Sigma, \tag{21.10}$$

which in this case is a principal bundle with structure group the abelian Lie group SO_2.

Remark 21.11. If X is a Riemannian manifold of dimension $n \geq 3$, then the unit sphere bundle $S(TX) \to X$ is *not* a principal bundle, whereas the oriented orthonormal frame bundle is a principal bundle with structure group SO_n.

Recall from Theorem 11.4 that the 3-manifold $\mathcal{B}_{\mathrm{SO}}(\Sigma)$ has a global coframing $\theta^1, \theta^2, \Theta^2_1$ that satisfies the following properties. First, the Levi–Civita connection is the kernel of the connection 1-form Θ^2_1. In particular,

$$\iota_{\widehat{A}}\Theta^2_1 = 1, \tag{21.12}$$

where $\widehat{A}$ is the vertical vector field on $\mathcal{B}_{\mathrm{SO}}(\Sigma)$ induced from $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{so}_2$. Next, the 2-form $\theta^1 \wedge \theta^2$ descends to a 2-form $\Omega_g \in \Omega^2_\Sigma$ that is the area 2-form defined by the Riemannian metric and orientation. (The associated density is $d\mu_g = |\Omega_g|$.) Finally, the curvature[39] descends to the base:

$$d\Theta^2_1 = \pi^*\omega. \tag{21.14}$$

The 2-form ω has an expression in terms of the Gauss curvature; see (11.27):

$$\omega = -K_g\,\Omega_g. \tag{21.15}$$

Observe that ω is the integrand in the Gauss–Bonnet formula (21.3).

[39] Since SO_2 is abelian, in matrix notation we have

$$d\Theta + \Theta \wedge \Theta = d\Theta + \frac{1}{2}[\Theta \wedge \Theta] = d\Theta. \tag{21.13}$$

(21.16) *Transgressing differential forms.* Equation (21.14) and its generalizations are the key to the proof of Gauss–Bonnet and much more, so we isolate the relationship between Θ_1^2 and ω.

Definition 21.17. Let $\pi\colon E \to X$ be a fiber bundle. Then $\alpha \in \Omega_E^q$ is a *transgressing differential form* if

$$d\alpha = \pi^*\omega \tag{21.18}$$

for some $\omega \in \Omega_X^{q+1}$.

Differentiating (21.18) we deduce that $d\omega = 0$, since π^* is injective. Definition 21.17 simply says that α is a transgressing form if $d\alpha$ descends to the base. The restriction of α to any fiber of π is a closed differential q-form. So both this restriction to the fiber and the form ω on the base are closed, hence represent de Rham cohomology classes. These cohomology classes are said to be related by transgression, a notion introduced by Borel. The geometric version here was introduced by Chern, and it tracks the differential forms, not just the underlying cohomology classes.

(21.19) *Special case: parallelizable surfaces.* Suppose $\xi \in \mathfrak{X}(\Sigma)$ is a nowhere vanishing vector field on the Riemannian surface Σ. Then $s := \frac{\xi}{\|\xi\|}\colon \Sigma \to S(T\Sigma) \approx \mathcal{B}_{\mathrm{SO}}(\Sigma)$ is a global section of $\varpi\colon \mathcal{B}_{\mathrm{SO}}(\Sigma) \to \Sigma$. Apply Stokes' theorem to $s^*\Theta_1^2$ and use (21.14) and (21.15) to deduce

$$0 = \int_\Sigma ds^*\Theta_1^2 = \int_\Sigma s^*d\Theta_1^2 = \int_\Sigma \omega = -\int_\Sigma K_g\,\Omega_g. \tag{21.20}$$

In other words, the right hand side of (21.3) vanishes. In this case the Euler number of Σ also vanishes, as we recall below. If Σ is connected, then Σ is either a torus or a Klein bottle. (We are assuming for convenience that Σ is oriented, but as mentioned before with a bit of elaboration our arguments apply to unoriented surfaces as well.)

On a general surface, we will use a vector field with isolated transverse zeros to compute the Euler number. Then we modify (21.20) to integrate on a blow-up of Σ at the zeros, which is a manifold with boundary. The boundary term in Stokes' theorem then gives the Euler number. For this last step we need the relationship between Euler numbers and vector fields, which we now recall.

Euler number and the Poincaré–Hopf theorem

This section is a recollection of Lecture 28 in the Differential Topology notes, to which I refer for details and proofs.

(21.21) *Definition of the Euler number and elementary properties.* Let X be a closed oriented manifold. Write Δ for the diagonal submanifold of $X \times X$; it is diffeomorphic to X.

Definition 21.22. The *Euler number* of X is the self-intersection number of the diagonal:

$$\mathrm{Euler}(X) = \#^{X\times X}(\Delta, \Delta). \tag{21.23}$$

This integer is computed via oriented intersection theory. In fact, it is independent of the orientation on X and can be defined on unoriented manifolds.

The normal bundle to $\Delta \subset X \times X$ is isomorphic to the tangent bundle $TX \to X$, and by the tubular neighborhood theorem the self-intersection number (21.23) can be computed inside the normal bundle. This leads to the following theorem.

Theorem 21.24. *Let X be a closed oriented manifold, and write $Z \subset TX$ for the image of the zero section. Then*

$$\mathrm{Euler}(X) = \#^{TX}(Z, Z). \tag{21.25}$$

The zero section $0\colon X \to TX$ assigns the zero vector to each point of X.

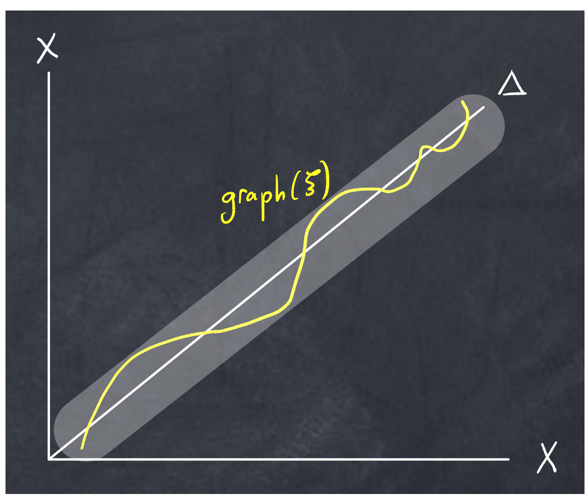

FIGURE 65. Computing the self-intersection number of $\Delta \subset X \times X$

The self-intersection number (21.25) is computed by writing the first copy of Z as the image of a function and then perturbing that function to be transverse to the second copy of Z. Lemma 28.12 in the Differential Topology notes shows that it suffices to perturb through sections of $TX \to X$. In other words, there exists a vector field $\xi \in \mathfrak{X}(X)$ with a finite set $\{x_1, \dots, x_N\} \subset X$ of transverse zeros, as depicted in Figure 65. Then the intersection number is the count of zeros with sign.

(21.26) *The local sign.* The differential of $\xi\colon X\to TX$ at a zero $x\in X$ is a linear map

$$T_xX \to T_0(TX). \tag{21.27}$$

Two transverse submanifolds pass through $0\in T_xX$: the zero section and the fiber. This gives a direct sum splitting of $T_0(TX)$, and the differential $d\xi_x$ is identified as the self-map of T_xX obtained by composing (21.27) with projection onto the tangent space to the fiber. At a transverse zero this is an isomorphism $d\xi_x\colon T_xX\to T_xX$. The sign tracks whether it is orientation-preserving or orientation-reversing.

Remark 21.28. No orientation of T_xX is necessary to know if an automorphism preserves or reverses orientation.

An expression for the sign is then

$$\epsilon = \pm 1 = \operatorname{sign}\det d\xi_x. \tag{21.29}$$

We need an equivalent expression, computed in the following lemma.

Lemma 21.30. *Let V be a finite dimensional real vector space, and write $S(V)=V/\mathbb{R}^{>0}$ for the quotient by scalar multiplication by positive real numbers. Let $T\colon V\to V$ be an invertible linear map; it induces a diffeomorphism $S(T)\colon S(V)\to S(V)$. Then*

$$\operatorname{sign}\det T = \deg S(T). \tag{21.31}$$

The degree ±1 of the diffeomorphism tells whether it preserves or reverses orientation.

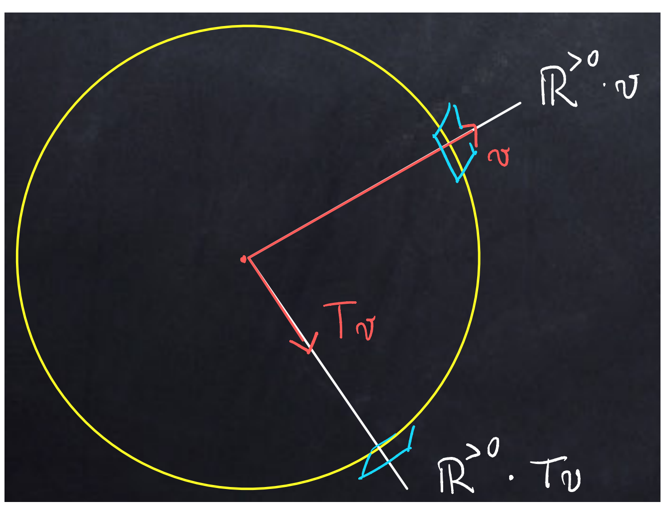

FIGURE 66. Computation of the sign

Proof. Fix a nonzero vector $v \in V$. In the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R}\cdot v \arrow[r] \arrow[d, "T|_{\mathbb{R}\cdot v}"] & V \arrow[r] \arrow[d, "T"] & T_{[v]}S(V) \arrow[r] \arrow[d, "dS(T)_{[v]}"] & 0 \\ 0 \arrow[r] & \mathbb{R}\cdot Tv \arrow[r] & V \arrow[r] & T_{[Tv]}S(V) \arrow[r] & 0 \end{tikzcd} \tag{21.32}$$

the vertical maps are isomorphisms. Orient the rays $\mathbb{R}^{>0}\cdot v$ and $\mathbb{R}^{>0}\cdot Tv$ to be "outgoing" from the origin, so that the first vertical map is orientation preserving. Fix an orientation of V and so induce an orientation of the quotient spaces in (21.32). The latter are consistent with the induced orientation of $S(V)$. It now follows that T is orientation-preserving iff $dS(T)$ is orientation-preserving. ☐

(21.33) *A special case of the Poincaré–Hopf theorem.*

Theorem 21.34. *Let X be a closed manifold, and suppose ξ is a vector field that is transverse to the zero section of $TX\to X$. Let $x_1,\dots,x_N$ be the zeros of ξ, and define $\epsilon_i = \operatorname{sign}\det d\xi_{x_i}$. Then*

$$\mathrm{Euler}(X) = \sum_i \epsilon_i. \tag{21.35}$$

The general Poincaré–Hopf is for vector fields with isolated, but not necessarily transverse, zeros. It is treated in Lecture 29 of the Differential Topology notes.

The blow-up

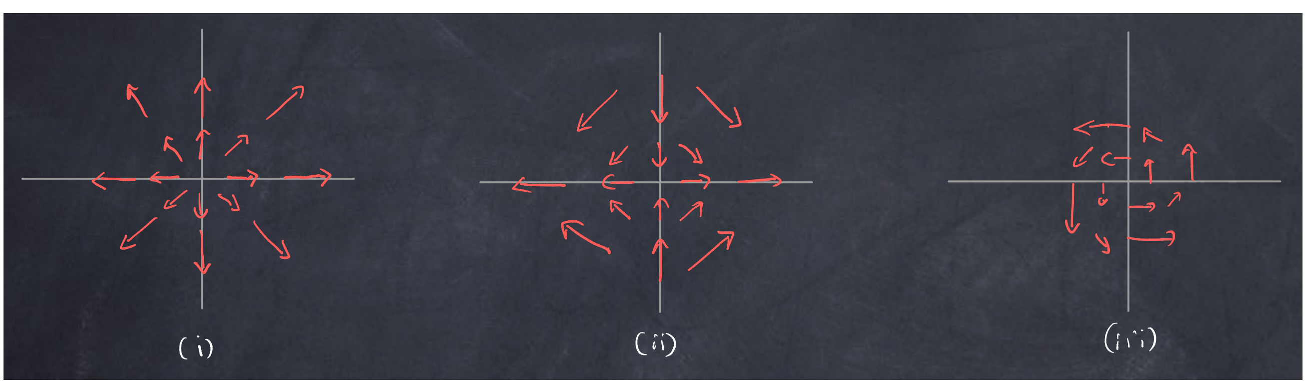

Figure 67. Vector fields (i) $x\dfrac{\partial}{\partial x}+y\dfrac{\partial}{\partial y}$ (ii) $x\dfrac{\partial}{\partial x}-y\dfrac{\partial}{\partial y}$ (iii) $x\dfrac{\partial}{\partial y}-y\dfrac{\partial}{\partial x}$

(21.36) *Intuition.* Return to dimension 2 and let's consider local models for an isolated transverse zero of a vector field, so a vector field ξ in $\mathbb{A}^2_{x,y}$. Three such vector fields are depicted in Figure 67. After normalization, each defines a section of the unit circle bundle of the tangent bundle away from the origin. Cut out a disk of radius δ about the origin and consider the behavior of the section as $\delta \to 0$. You should conclude that in the limit the section "winds" around the circle fiber over the origin. In fact, each of these vector fields is linear, and in the limit one obtains the restriction of the vector field to the unit circle, which is a map from the circle to itself. Furthermore, you should conclude that the degree of that map is the sign of the zero of each vector field; see Lemma 21.30. For a nonlinear vector field the $\delta \to 0$ limit is the normalized differential of ξ at the origin, restricted to the unit circle.

We construct a compactification of the plane minus the origin which is a blow-up: it replaces the origin by the circle of directions at the origin. Then we extend the section of the circle bundle to a map of the blow-up into the circle bundle. Finally, we apply Stokes' theorem as in (21.20), but now with boundary terms that produce the signs in (21.35). We carry this out in arbitrary dimensions.

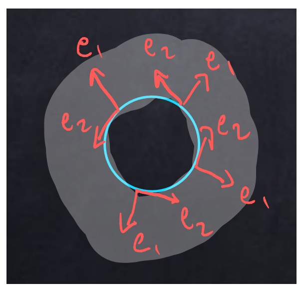

FIGURE 68. The boundary orientation and Θ^2_1

Remark 21.37 (orientations). Recall that the 1-form Θ^2_1 restricts on a fiber of the oriented orthonormal frame bundle to the Maurer–Cartan form on SO_2 (or rather on the right SO_2-torsor that is the fiber). It measures the instantaneous rate of turning of e_1 toward e_2 in a moving frame. In Figure 68 the circle is oriented clockwise as the boundary of its exterior, and the value of Θ^2_1 on a unit vector in the clockwise direction is -1. That sign reappears at the end of the lecture in a more general context; see (21.50).

(21.38) *The sphere bundle of a Riemannian manifold.* As a preliminary, let X be an n-dimensional oriented Riemannian manifold, let $\varpi\colon \mathcal{B}_{SO}(X) \to X$ be its oriented orthonormal frame bundle, and let $\pi\colon S(TX) \to X$ be the unit sphere bundle of its tangent bundle. Then ϖ is a principal SO_n-bundle and the fiber bundle π is associated via the action of SO_n on S^{n-1}. It inherits a horizontal distribution from the Levi–Civita connection on ϖ. Identify S^{n-1} as the homogeneous manifold SO_n / SO_{n-1}, where $SO_{n-1} \subset SO_n$ is the subgroup of $n \times n$ orthogonal matrices of the form $\begin{pmatrix} 1 & \\ & M \end{pmatrix}$, $M \in SO_{n-1}$. We can also identify $S(TX)$ as the quotient $\mathcal{B}_{SO}(X)/ SO_{n-1}$. Recall the canonical horizontal vector fields $\partial_1, \dots, \partial_n$. The vector field ∂_1 drops to a horizontal vector field ∂ on $S(TX)$; these vector fields are q-related (Definition 2.65), where $q\colon \mathcal{B}_{SO}(X) \to S(TX)$ is the quotient map. Recall that the integral curves of ∂_1 in $\mathcal{B}_{SO}(X)$ project to geodesic motions on X. Since the vector fields are q-related, the same is true for the integral curves of ∂ in $S(TX)$. The construction is a blow-up of the Riemannian exponential map; see **(17.44)**.

(21.39) *The blow-up.* Let X be a closed n-dimensional Riemannian manifold, and suppose ξ is a vector field with isolated zeros at $x_1, \dots, x_N$. Let $\pi\colon S(TX) \to X$ be the unit sphere bundle of the tangent bundle. We construct a compact manifold with boundary $\widehat{X}$ and a smooth map $p\colon \widehat{X} \to X$ such that

(i) the boundary is $\partial\widehat{X} = \bigsqcup_i p^{-1}(x_i)$;

(ii) the restriction of p to the interior of $\widehat{X}$ is a diffeomorphism $\operatorname{int}(\widehat{X}) \to X \setminus \{x_1, \dots, x_N\}$; and

(iii) there is a canonical diffeomorphism $p^{-1}(x_i) \approx S(T_{x_i}X)$.

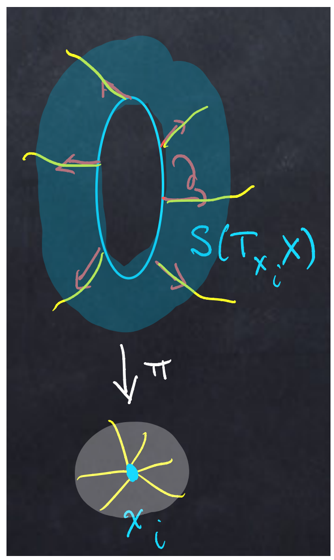

FIGURE 69. Integral curves of ∂

For the construction, at each $i \in \{1, \dots, N\}$ consider the integral curves of ∂ that begin on the fiber $S(T_{x_i}X)$ of $\pi\colon S(TX) \to X$ over x_i. These project to geodesics in X that emanate from x_i, as in Figure 69. Since $S(T_{x_i}X)$ is compact, there exists $\epsilon_i > 0$ such that the flow exists for time ϵ_i uniformly in the initial position. This gives a diagram

$$\begin{tikzcd} {[0,\epsilon_i) \times S(T_{x_i}X)} \arrow[r] \arrow[dr, "p"'] & S(TX) \arrow[d, "\pi"] \\ & X \end{tikzcd} \tag{21.40}$$

Furthermore, it follows from Lemma 17.47 and the inverse function theorem that the restriction of p to $(0, \epsilon_i) \times S(T_{x_i}X)$ is a diffeomorphism onto its image, possible after reducing ϵ_i. Now construct $\widehat{X}$ from the disjoint union

$$\bigsqcup_i \Bigl([0,\epsilon_i) \times S(T_{x_i}X)\Bigr) \;\sqcup\; \Bigl(X \setminus \{x_1, \dots, x_n\}\Bigr) \tag{21.41}$$

using the diffeomorphism on each $(0,\epsilon_i)\times S(T_{x_i}X)$ to glue.

(21.42) *Extension of the vector field ξ.* Assume now that ξ has *transverse* zeros at each x_i, or equivalently that $d\xi_{x_i}\colon T_{x_i}X\to T_{x_i}X$ is an isomorphism for each i.

Proposition 21.43. *The section $s=\frac{\xi}{\|\xi\|}\colon X\setminus\{x_1,\dots,x_N\}\to S(TX)$ of $\pi\colon S(TX)\to X$ extends to the map f_ξ in the commutative diagram*

(21.44)
$$\begin{tikzcd} \widehat{X} \arrow[r, "f_\xi"] \arrow[drr, "p"] & & S(TX) \arrow[d, "\pi"] \\ X\setminus\{x_1,\dots,x_N\} \arrow[u, hook] \arrow[urr, "s"] \arrow[rr, hook] & & X \end{tikzcd}$$

Proof. It suffices to define f_ξ on $\partial\widehat{X}=\bigsqcup_i S(T_{x_i}X)$ and prove that the extension is smooth. The definition is

$$f_\xi\big|_{S(T_{x_i}X)}=S(d\xi_{x_i}), \tag{21.45}$$

where for $T\colon V\to V$ an invertible linear map on a real inner product space we set

$$S(T)(v)=\frac{T(v)}{\|T(v)\|},\qquad v\in V. \tag{21.46}$$

Near x_i use parallel transport along radial geodesics to identify each tangent space in a neighborhood with $V=T_{x_i}X$. Then the smoothness of the extension is a consequence of the following lemma.

Lemma 21.47. *Let V be a finite dimensional inner product space, suppose $F\colon B_\epsilon(0)\to V$ is a smooth map with $F(0)=0$ and dF_0 invertible. Then for all $v\in V$,*

$$\lim_{t\to 0}\frac{F(tv)}{\|F(tv)\|}=\frac{dF_0(v)}{\|dF_0(v)\|}. \tag{21.48}$$

Proof. Use $dF_0(v)=\lim_{t\to0}\dfrac{F(tv)}{t}$. □

□

Gauss–Bonnet and transgressing forms

Continue with the closed oriented Riemannian n-dimensional manifold X, the vector field ξ, the blow-up $\widehat{X}$, and the map $f_\xi\colon\widehat{X}\to S(TX)$.

Proposition 21.49. *Suppose $\alpha\in\Omega^{n-1}_{S(TX)}$ satisfies*

(i) $\int_{S(T_xX)}\alpha=1$ *for all $x\in X$; and*

(ii) $d\alpha = \pi^*\omega$ *for some closed* $\omega \in \Omega^n_X$.

Then

$$\mathrm{Euler}(X) = -\int_X \omega. \tag{21.50}$$

In (i) the sphere $S(T_xX) \subset X$ is oriented as the boundary of the ball it bounds.

Proof. Apply Stokes' theorem to $f_\xi^*\alpha$. Then

$$\int_{\partial\widehat{X}} f_\xi^*\alpha = \int_{\widehat{X}} f_\xi^*\pi^*\omega = \int_{\mathrm{int}(\widehat{X})} \omega = \int_{\widehat{X}} \omega. \tag{21.51}$$

Here we use that $\partial\widehat{X} \subset \widehat{X}$ has measure zero. On the other hand, the integral of the pullback of α computes the degree of f_ξ on each boundary component, which by Lemma 21.30 is the sign of the determinant of the differential of ξ. The orientation of $S_{T_{x_i}X}$ as the boundary of $\widehat{X}$ is opposite to that as the boundary of the ball in $T_{x_i}X$, and so

$$\int_{\partial\widehat{X}} f_\xi^*\alpha = \sum_i -\deg\left(\frac{\xi}{\|\xi\|}\bigg|_{S(T_{x_i}X)}\right) = -\sum_i \epsilon_i. \tag{21.52}$$

Apply Theorem 21.34 to conclude. □

(21.53) *Proof of Theorem 21.2.* It only remains to observe that $\alpha = \Theta_1^2/2\pi$ satisfies the conditions in Proposition 21.49.

Lecture 22: Chern–Weil and Chern–Simons forms

This lecture is an exposition of parts of the classical paper Shiing-Shen Chern, James Simons, *Characteristic forms and geometric invariants*, Annals of Math., **99**(1974), pp. 48–69.

We begin with Chern–Weil forms, which predate this paper, though I don't know that they appeared in print in a joint paper. Given a Lie group G we seek a universal procedure that associates to each principal G-bundle $\pi\colon P\to X$ with connection Θ a differential form $\omega(\Theta)\in\Omega^\bullet_X$. The association should be *natural* under pullbacks of connections. It turns out that every such natural differential form is determined by an Ad-invariant polynomial on the Lie algebra $\mathfrak{g}$. Furthermore, every such form $\omega(\Theta)$ is closed. We construct these Chern–Weil forms (22.12), but we do not prove that they are the only natural differential forms.

There are two types of Chern–Simons forms. The first (22.29) is associated to two connections on $\pi\colon P\to X$, and it lives on the base X. It is a *secondary* form in the sense that its differential (22.31) is the difference of two *primary* Chern–Weil forms. A corollary is that the de Rham cohomology class of a Chern–Weil form is independent of the connection. The second type of Chern–Simons form (22.43) is associated to a single connection on $\pi\colon P\to X$, and it lives on the total space P. It is a transgressing form in the sense of Definition 21.17: it transgresses the Chern–Weil form to a bi-invariant form on G.

In the next lecture we extend this discussion to transgression in certain associated fiber bundles, and then we complete the proof of the Chern–Gauss–Bonnet theorem.

Chern–Weil forms

(22.1) *Universal data and universal forms.* Let G be a Lie group. (For topological purposes one usually assumes that G has finitely many components, but for our purposes that is not strictly necessary.) We seek *natural* or *universal* constructions of differential forms from G-connections. That is, to every principal G-bundle $\pi\colon P\to X$ with connection Θ we associate a differential form $\omega(\Theta)\in\Omega^\bullet_X$ without making any choices. More precisely, *naturality* means that if

(22.2)
$$\begin{array}{ccc} P' & \xrightarrow{\widetilde{\varphi}} & P \\ {\scriptstyle\pi'}\downarrow & & \downarrow{\scriptstyle\pi} \\ X' & \xrightarrow{\varphi} & X \end{array}$$

is a pullback diagram and $\Theta\in\Omega^1_P(\mathfrak{g})$ is a connection form on π, then

(22.3)
$$\omega(\widetilde{\varphi}^*\Theta)=\varphi^*\omega(\Theta).$$

The construction we give associates a natural form to a symmetric multilinear G-invariant map

(22.4)
$$h\colon \mathfrak{g}\times\cdots\times\mathfrak{g}\longrightarrow\mathbb{R}$$

If there are $k \in \mathbb{Z}^{>0}$ arguments, then the G-invariance is

$$h(\mathrm{Ad}_g\,\xi_1, \dots, \mathrm{Ad}_g\,\xi_k) = h(\xi_1, \dots, \xi_k), \qquad g \in G, \quad \xi_1, \dots, \xi_k \in \mathfrak{g}. \tag{22.5}$$

The vector space of such symmetric multilinear k-forms is $(\mathrm{Sym}^\bullet\,\mathfrak{g}^*)^G$. There is a theorem that the construction below produces all natural assignments of a differential form to a G-connection.[40]

Example 22.6. Let $G = \mathrm{GL}_N\mathbb{R}$ for some $N \in \mathbb{Z}^{>0}$, or more generally suppose that G is a subgroup of $\mathrm{GL}_N\mathbb{R}$, i.e., G is a matrix group. (We can replace $\mathbb{R}$ by $\mathbb{C}$—or, for the invariant theory discussion, by any field.) Then for $k = 2$ the bilinear form

$$h(A_1, A_2) = \mathrm{trace}(A_1A_2) \tag{22.7}$$

is symmetric and G-invariant. For $k \geq 3$ one can symmetrize the k-linear form

$$A_1, \dots, A_k \longmapsto \mathrm{trace}(A_1 \cdots A_k) \tag{22.8}$$

to obtain a symmetric G-invariant form. For $G = \mathrm{GL}_N\mathbb{R}$ and $k = N$ there is an invariant form h whose value on the diagonal is

$$h(\underbrace{A, \dots, A}_{N}) = \det(A), \tag{22.9}$$

which is a polynomial of degree N in N^2 variables. *Classical invariant theory* is, in part, the study of such invariant forms.

Remark 22.10. For $G = \mathrm{SO}_{2m}$ there is a slightly exotic G-invariant symmetric m-linear form, the *Pfaffian*, which is what we use for the Gauss–Bonnet theorem in the next lecture.

(22.11) *The Chern–Weil form.* For the rest of this lecture fix G and $h \in (\mathrm{Sym}^\bullet\,\mathfrak{g}^*)^G$. Then for a connection Θ on a principal G-bundle $\pi\colon P \to X$ with curvature $\Omega \in \Omega^2_P(\mathfrak{g})$, define

$$\omega = \omega(\Theta) = \omega_h(\Theta) = h(\underbrace{\Omega \wedge \cdots \wedge \Omega}_{k}) = h(\Omega^{\wedge k}) \ \in \Omega^{2k}_P. \tag{22.12}$$

This is the *Chern–Weil form* associated to h and the connection Θ.

Lemma 22.13.

(1) *ω descends to X.*

[40]In fact, there is a "generalized smooth manifold" $B_\nabla G$ such that a map $X \to B_\nabla G$ is a G-connection, and a strong form of the theorem alluded to proves that the Chern–Weil construction is an isomorphism from $(\mathrm{Sym}^\bullet\,\mathfrak{g}^*)^G$ to the de Rham complex of $B_\nabla G$; the differential is zero. See D. S. Freed, M. J. Hopkins, *Chern–Weil forms and abstract homotopy theory*, Bull. Amer. Math. Soc., **50** (2013), no. 3, pp. 431–468, (`arXiv:1301.5959`).

(2) $d\omega = 0$.

As usual, we denote the descended form as $\omega \in \Omega^{2k}_X$.

Proof. For (1) we check that ω is G-invariant and that the interior product with a vertical vector vanishes. Thus if $g \in G$, then

$$R_g^*\omega = R_g^*\big(h(\Omega^{\wedge k})\big) = h\big((R_g^*\Omega)^{\wedge k}\big) = h\big((\mathrm{Ad}_{g^{-1}}\,\Omega)^{\wedge k}\big) = h(\Omega^{\wedge k}) = \omega. \tag{22.14}$$

At the penultimate stage we use the G-invariance (22.5). Now if $\zeta \in \mathfrak{g}$ and $\widehat{\zeta}$ is the induced vertical vector field on P, then

$$\iota_{\widehat{\zeta}}\,\omega = \sum h(\Omega \wedge \cdots \wedge \iota_{\widehat{\zeta}}\Omega \wedge \cdots \wedge \Omega) = 0. \tag{22.15}$$

Here we use that $\iota_{\widehat{\zeta}}\Omega = 0$. This completes the proof of (1).

The proof of (2) uses the Bianchi identity (16.60), which asserts $d\Omega = -[\Theta \wedge \Omega]$. Hence

$$\begin{aligned} d\omega &= \sum h(\Omega \wedge \cdots \wedge d\Omega \wedge \cdots \wedge \Omega) \\ &= -\sum h(\Omega \wedge \cdots \wedge [\Theta \wedge \Omega] \wedge \cdots \wedge \Omega) \\ &= 0, \end{aligned} \tag{22.16}$$

since it suffices to evaluate on horizontal vectors and Θ vanishes on horizontal vectors. (Alternatively, the infinitesimal version of G-invariance (22.5) can be used to deduce the vanishing.) $\square$

The universal connection

To study the Chern–Weil form $\omega(\Theta)$ on families of connections we introduce a universal connection on a fixed principal G-bundle $\pi\colon P \to X$.

(22.17) *Universal objects and moduli spaces.* The construction here is ubiquitous in geometry and beyond. As an example, let V be a vector space and fix a nonnegative integer ℓ. The collection of all ℓ-dimensional subspaces $W \subset V$ forms a manifold $\mathrm{Gr}_\ell(V)$, the *Grassmannian*. Each point of $\mathrm{Gr}_\ell(V)$ represents a subspace $W \subset V$, and these subspaces collate to a vector bundle $S \to \mathrm{Gr}_\ell(V)$ whose fiber at $W \in \mathrm{Gr}_\ell(V)$ is the vector space W. (In fact, it comes to us as a subbundle of the constant vector bundle $\underline{V} \to \mathrm{Gr}_\ell(V)$ with fiber V.) In this example, $\mathrm{Gr}_\ell(V)$ is the moduli space and $S \to \mathrm{Gr}_\ell(V)$ is the universal object that the moduli space parametrizes.

Remark 22.18. Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of fixed genus $g \in \mathbb{Z}^{\ge 0}$. Riemann surfaces can have automorphisms, and this complicates the construction of a universal Riemann surface $\mathcal{C}_g \to \mathcal{M}_g$. In this situation one should construct a moduli *stack* rather than a moduli space, and in that world the universal object exists.

Our example—connections on a fixed bundle—is more analogous to the case of the Grassmannian, albeit infinite dimensional.

Remark 22.19. In the following we freely use infinite dimensional vector spaces and infinite dimensional affine spaces, at least in a formal way. The construction of Chern–Simons forms below only uses finite dimensional families of connections—in fact a 1-dimensional family—so we do not attempt to justify the infinite dimensional calculus we use now.

(22.20) *The universal 1-form.* Recall that the space $\mathcal{A}_\pi$ of connections on $\pi\colon P \to X$ is an affine subspace of the vector space of 1-forms $\Omega^1_P(\mathfrak{g})$. We begin, then, with the universal $\mathfrak{g}$-valued 1-form on P, which is a 1-form Θ^u on the Cartesian product manifold $\Omega^1_P(\mathfrak{g}) \times P$. Fix $\Theta \in \Omega^1_P(\mathfrak{g})$, $p \in P$, $\xi, \xi_1, \xi_2 \in T_pP$, and $\Lambda, \Lambda_1, \Lambda_2 \in \Omega^1_P(\mathfrak{g})$. Then the universal 1-form is defined by

$$\begin{aligned} \Theta^u_{(\Theta,p)}(\Lambda) &= 0 \\ \Theta^u_{(\Theta,p)}(\xi) &= \Theta_p(\xi) \end{aligned} \tag{22.21}$$

A short computation shows that the differential is

$$\begin{aligned} d\Theta^u_{(\Theta,p)}(\Lambda_1,\Lambda_2) &= 0 \\ d\Theta^u_{(\Theta,p)}(\Lambda,\xi) &= \Lambda\cdot\Theta^u(\xi) - \xi\cdot\Theta^u(\Lambda) - \Theta^u\bigl([\Lambda,\xi]\bigr) = \Lambda(\xi) \\ d\Theta^u_{(\Theta,p)}(\xi_1,\xi_2) &= d\Theta_p(\xi_1,\xi_2) \end{aligned} \tag{22.22}$$

In the intermediate step in the second formula, Λ is extended to a constant vector field on $\Omega^1_P(\mathfrak{g})$ and ξ to a vector field on P, then each is extended to the Cartesian product $\Omega^1_P(\mathfrak{g}) \times P$ to be constant along the other factor.

(22.23) *The universal connection.* The 1-form restricts to a universal connection form Θ^u on the principal G-bundle

$$A_\pi \times P \longrightarrow A_\pi \times X \tag{22.24}$$

Its curvature Ω^u can be computed from (22.22) to be

$$\begin{aligned} \Omega^u_{(\Theta,p)}(\Lambda_1,\Lambda_2) &= 0 \\ \Omega^u_{(\Theta,p)}(\Lambda,\xi) &= \Lambda(\xi) \\ \Omega^u_{(\Theta,p)}(\xi_1,\xi_2) &= \Omega_\Theta(\xi_1,\xi_2) \end{aligned} \tag{22.25}$$

where now $\Lambda, \Lambda_1, \Lambda_2 \in \Omega^1_X(\mathfrak{g}_P)$.

(22.26) *The universal Chern–Weil form.* Finally, apply (22.12) to the universal connection to obtain the universal Chern–Weil form

$$\omega(\Theta^u) = h(\underbrace{\Omega^u \wedge \cdots \wedge \Omega^u}_{k}) = h\bigl((\Omega^u)^{\wedge k}\bigr) \ \in \Omega^{2k}_{A_\pi \times X}. \tag{22.27}$$

The universal property is: the restriction of $\omega(\Theta^u)$ to $\{\Theta\} \times X$ is $\omega(\Theta)$ for each $\Theta \in A_\pi$.

Chern–Simons forms

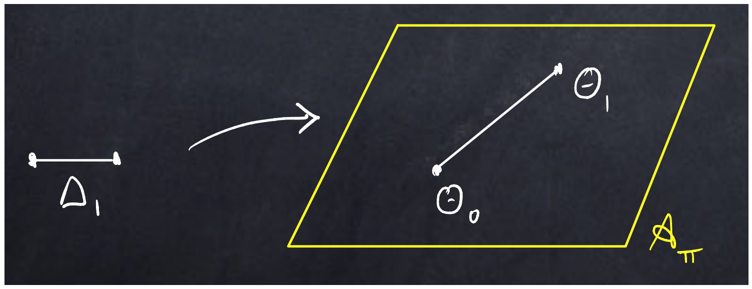

FIGURE 70. The affine path connecting two connections

(22.28) *The Chern–Simons form of two connections.* Continuing with the principal G-bundle $\pi\colon P\to X$, suppose that $\Theta_0,\Theta_1\in A_\pi$ are two connections. Since A_π is an affine space, there is a unique affine map $\Delta^1\to A_\pi$ from the 1-simplex $\Delta^1=[0,1]$ which begins at Θ_0 and terminates at Θ_1; see Figure 70. The *Chern–Simons form* is the integral of the universal Chern–Weil form (22.27) along this 1-simplex:

$$\alpha(\Theta_0,\Theta_1)=\int_{\Delta^1}\omega(\Theta^u)\ \in\Omega_X^{2k-1}. \tag{22.29}$$

Proposition 22.30. *The differential of the Chern–Simons form is*

$$d\alpha(\Theta_0,\Theta_1)=\omega(\Theta_1)-\omega(\Theta_0). \tag{22.31}$$

Proof. Stokes' theorem. ☐

Corollary 22.32. *The de Rham cohomology class of the Chern–Weil form of a connection on $\pi\colon P\to X$ is independent of the connection.*

Therefore, the bundle π and the invariant polynomial h determine a cohomology class

$$h(\pi)\ \in H^{2k}_{dR}(X)\cong H^{2k}(X;\mathbb{R}). \tag{22.33}$$

This is an example of a *characteristic class*. It satisfies a naturality property: given a pullback diagram (22.2),

$$h(\pi')=\varphi^*h(\pi). \tag{22.34}$$

The map[41]

$$\big(\mathrm{Sym}^\bullet\,\mathfrak{g}^*\big)\longrightarrow H^{2\bullet}(B_\bullet G;\mathbb{R}) \tag{22.35}$$

that assigns to an invariant polynomial a characteristic class is an algebra homomorphism. Under favorable circumstances it is an isomorphism.

[41]As in footnote [40] there is a classifying object $B_\bullet G$ for principal G-bundles: a map $X\to B_\bullet G$ is a principal G-bundle over X.

Remark 22.36. The Chern–Weil construction of a characteristic class from a connection is an example of the *Geometry computes Topology* principle in Remark 21.4(3), and it is a vast generalization of the example discussed there.

(22.37) *Explicit formula.* We now compute an explicit formula for the Chern–Simons form $\alpha(\Theta_0,\Theta_1)$. Set $\Lambda=\Theta_1-\Theta_0$, which descends to the base X as an element of $\Omega^1_X(\mathfrak{g}_P)$. The universal curvature restricts to

$$\Omega^u\big|_{\Delta^1\times P}= dt\wedge\Lambda+\Omega_t, \tag{22.38}$$

where t is the coordinate along $\Delta^1=[0,1]$ and

$$\Omega_t=\Omega_0+td\Lambda+t[\Theta_0\wedge\Lambda]+\frac{t^2}{2}[\Lambda\wedge\Lambda]. \tag{22.39}$$

Then a straightforward computation gives

$$\alpha(\Theta_0,\Theta_1)=k\int_0^1 dt\,h(\Lambda\wedge\Omega_t^{\wedge(k-1)})\ \in\Omega_X^{2k-1}. \tag{22.40}$$

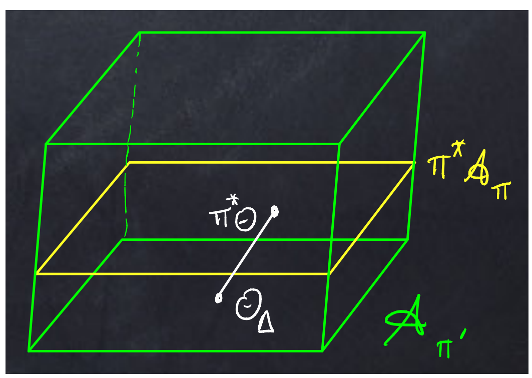

FIGURE 71. The product connection Θ_Δ and the pullback connection $\pi^*\Theta$

(22.41) *The Chern–Simons form of a single connection.* The pullback principal G-bundle

$$\begin{tikzcd} \pi^*P \arrow[r] \arrow[d, shift right, "\pi'"'] & P \arrow[d, "\pi"] \\ P \arrow[u, shift right, bend right, "\Delta"'] \arrow[r, "\pi"] & X \end{tikzcd} \tag{22.42}$$

has total space $\pi^*P\approx P\times_X P$, and so there is a canonical diagonal section Δ. The G-translates of Δ provide a trivialization of π' (as in Proposition 13.4), and there is a corresponding integrable connection Θ_Δ for which these translates are integral manifolds. In particular, Θ_Δ is flat.

Now suppose $\Theta \in A_\pi$ is a connection on $\pi\colon P \to G$. It pulls back to a connection on $\pi'\colon \pi^*P \to P$; see Figure 71. The *Chern–Simons form* of the connection Θ is

$$\alpha(\Theta) = \alpha(\Theta_\Delta, \pi^*\Theta) \ \in \Omega_P^{2k-1}. \tag{22.43}$$

Notice that the Chern–Simons form (22.43) of a single connection is a scalar differential form on the total space P of $\pi\colon P \to X$, whereas the Chern–Simons form (22.29) of two connections is a scalar differential form on the base space X.

(22.44) *Explicit formula.* Use Δ to identify $\pi^*P \approx P \times G$, and then (22.40) becomes

$$\begin{tikzcd} P \times G \arrow[r, "R"] \arrow[d, shift right, "\mathrm{pr}_1"'] & P \arrow[d, "\pi"] \\ P \arrow[r, "\pi"] \arrow[u, shift right, "\mathrm{id} \times e"'] & X \end{tikzcd} \tag{22.45}$$

Under this identification, and using $\Lambda = \pi^*\Theta - \Theta_\Delta$, we have

$$\begin{aligned} \Theta_\Delta &= \theta_G \\ \pi^*\Theta = R^*\Theta &= \mathrm{Ad}_{g^{-1}}\Theta + \theta_G \\ \Lambda &= \mathrm{Ad}_{g^{-1}}\Theta \end{aligned} \tag{22.46}$$

where $\theta_G \in \Omega_G^1(\mathfrak{g})$ is the Maurer–Cartan form on G. Specialize (22.39) to this case, pulling back by the section $\mathrm{id} \times e$ to compute on P:

$$\begin{aligned} \Omega_t &= t d\Theta + \frac{t^2}{2}[\Theta \wedge \Theta] \\ &= t\Omega + \frac{t^2 - t}{2}[\Theta \wedge \Theta]. \end{aligned} \tag{22.47}$$

where Ω is the curvature of Θ. So we obtain the explicit formula

$$\alpha(\Theta) = k\int_0^1 dt\, h(\Theta \wedge (\Omega_t)^{k-1}) \ \in \Omega_P^{2k-1}. \tag{22.48}$$

Example 22.49. Let $k = 2$ and write

$$h(\xi_1, \xi_2) = \langle \xi_1, \xi_2 \rangle, \qquad \xi_1, \xi_2 \in \mathfrak{g}. \tag{22.50}$$

Then the Chern–Simons form is

$$\begin{aligned} \alpha(\Theta) &= 2\int_0^1 dt\, \left[t\langle \Theta \wedge \Omega \rangle + \left(\frac{t^2 - t}{2} \right) \langle \Theta \wedge [\Theta \wedge \Theta] \rangle \right] \\ &= \langle \Theta \wedge \Omega \rangle - \frac{1}{6}\langle \Theta \wedge [\Theta \wedge \Theta] \rangle. \end{aligned} \tag{22.51}$$

This scalar 3-form is the starting point of the *Chern-Simons theory* that constructs both classical and quantum invariants in low-dimensional geometry and topology.

(22.52) *The Chern–Simons form as a transgressing form.* Recall Definition 21.17 of a transgressing form in a fiber bundle. The following theorem tells that the Chern–Simons form is a transgressing form for the Chern–Weil form. Let Θ be a connection on $\pi\colon P \to X$ with curvature Ω, let θ be the Maurer–Cartan form on G, fix $x \in X$, let $i_x\colon P_x \hookrightarrow P$ denote the inclusion, fix $g \in G$, let $\alpha = \alpha(\Theta)$ be the Chern–Simons form (22.48), and let $\omega = \omega(\Theta)$ be the Chern–Weil form (22.12).

Proposition 22.53.

(1) $i_x^*\alpha = c_k\, h(\theta \wedge [\theta \wedge \theta]^{\wedge(k-1)})$, *where*[42]

$$c_k = k\int_0^1 dt\,\left(\frac{t^2-t}{2}\right)^{k-1} = \frac{(-1)^{k-1}}{2^{k-1}\binom{2k-1}{k}} = \frac{(-1)^{k-1}k!(k-1)!}{2^{k-1}(2k-1)!}. \tag{22.54}$$

(2) $R_g^*\alpha = \alpha$.

(3) $d\alpha = \pi^*\omega$.

Proof. Use the explicit formulas (22.48) and (22.47). For (1), observe that $i_x^*\Omega = 0$ and $i_x^*\Theta = \theta_{P_x} = \theta$, where we identify the Maurer–Cartan form on P_x with the Maurer–Cartan form on G. Hence

$$\begin{aligned} i_x^*\alpha &= k\int_0^1 dt\, h\left(\theta \wedge \left(\frac{t^2-t}{2}[\theta\wedge\theta]\right)^{\wedge(k-1)}\right) \\ &= \left[k\int_0^1 dt\,\left(\frac{t^2-t}{2}\right)^{k-1}\right] h(\theta\wedge[\theta\wedge\theta]^{\wedge(k-1)}) \\ &= c_k\, h(\theta\wedge[\theta\wedge\theta]^{\wedge(k-1)}) \end{aligned} \tag{22.55}$$

For (2) use that $R_g^*\Theta = \mathrm{Ad}_{g^{-1}}\Theta$, $R_g^*\Omega = \mathrm{Ad}_{g^{-1}}\Omega$, and h is G-invariant. The transgression equation (3) is a corollary of (22.31). $\square$

(22.56) *Bi-invariant differential forms on G and transgression.* The differential form

$$\eta = c_k\, h(\theta\wedge[\theta\wedge\theta]^{\wedge(k-1)}) \;\in \Omega_G^{2k-1} \tag{22.57}$$

in Proposition 22.53(1) is a left-invariant form on G, since the Maurer–Cartan form θ is left-invariant, and it is also right invariant since $R_g^*\theta = \mathrm{Ad}_{g^{-1}}\theta$, $g \in G$, and h is G-invariant. So η is bi-invariant.

The left-invariant forms comprise the vector space $\bigwedge^\bullet\mathfrak{g}^*$ and the bi-invariant forms comprise the subspace $\left(\bigwedge^\bullet\mathfrak{g}^*\right)^G$ of Ad-invariant forms on $\mathfrak{g}$. Suppose η is bi-invariant, and let $\iota\colon G \to G$ be inversion $g \mapsto g^{-1}$. The differential at $e \in G$ is $d\iota = -\,\mathrm{id}_{\mathfrak{g}}$, from which $d\iota$ acts on a homogeneous

[42] The first few values are $c_1 = 1$, $c_2 = -1/6$, $c_3 = 1/40$, and $c_4 = -1/280$.

differential form η at e by $(-1)^{\deg \eta}$. Hence for a homogeneous bi-invariant form η on G we deduce at $g \in G$ that

$$\begin{aligned} (\iota^*\eta)_g &= (\iota^* L^*_{g^{-1}}\eta_e) \\ &= R^*_g \iota^* \eta_e \\ &= (-1)^{\deg \eta} R^*_g \eta_e \\ &= (-1)^{\deg \eta} \eta_g \end{aligned} \tag{22.58}$$

from which $\iota^*\eta = (-1)^{\deg \eta}\eta$. Then

$$(-1)^{\deg \eta + 1} d\eta = \iota^* d\eta = d\iota^*\eta = (-1)^{\deg \eta} d\eta \tag{22.59}$$

which implies $d\eta = 0$. In other words, every bi-invariant form is closed. Put differently, the de Rham complex on bi-invariant forms is the vector space $\left(\bigwedge^{\bullet}\mathfrak{g}^*\right)^G$ with zero differential. The Chern–Simons form effects a transgression homomorphism

$$(\mathrm{Sym}^{\bullet}\,\mathfrak{g}^*)^G \longrightarrow \left(\bigwedge^{\bullet}\mathfrak{g}^*\right)^G. \tag{22.60}$$

Lecture 23: Transgression in associated bundles; proof of Chern–Gauss–Bonnet

In this lecture we conclude the proof of the Chern–Gauss–Bonnet formula. Sadly, we did not get the constants to work out at the end of the lecture. Dear Reader, please find the error and email me at `dafr@math.harvard.edu`.

We begin by generalizing the discussion of transgression in principal bundles, which begins in **(22.41)**, to transgressions in associated bundles of reductive homogeneous manifolds. Once more, we use the basic Chern–Simons construction **(22.28)** to produce the transgressing form.

The invariant polynomial relevant to the Euler number is the *pfaffian*, which is a square root of the determinant on skew-symmetric matrices. Determinants are best located in exterior algebra, and so too are pfaffians.

In our application the reductive homogeneous manifold is the sphere $S^{n-1} \approx \mathrm{SO}_n / \mathrm{SO}_{n-1}$; see Lecture 21. In particular, in Proposition 21.49 we must compute the integral of the transgressing form over the sphere. We do so, but presumably that is where some constants go astray... Nonetheless, we indicate how to derive the formula for the Euler number as an integral of the pfaffian of the Riemann curvature tensor.

Transgression in associated bundles

(23.1) *Notation.* Fix G a Lie group, $h \in (\mathrm{Sym}^k \mathfrak{g}^*)^G$ an invariant polynomial of degree $k \in \mathbb{Z}^{>0}$ on the Lie algebra of G, $\pi\colon P \to X$ a principal G-bundle, Θ a connection 1-form on π, and let Ω be the curvature of Θ.

(23.2) *An associated bundle of reductive homogeneous spaces.* Suppose $H \subset G$ is a closed Lie subgroup, and assume given a reductive structure

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h} \tag{23.3}$$

on the Lie algebra, i.e., an H-invariant decomposition of the adjoint representation. Let $\rho\colon \mathfrak{g} \to \mathfrak{h}$ denote the projection map with kernel $\mathfrak{p}$. Then there is a pullback diagram

$$\begin{tikzcd} f^*P \arrow[r, shift left, "\tilde{f}"] \arrow[d, "\pi'"'] & P \arrow[l, shift left, "\Delta"] \arrow[dl, "\varpi"] \arrow[d, "\pi"] \\ P/H \arrow[r, "f"] & X \end{tikzcd} \tag{23.4}$$

in which π, π' are principal G-bundles, ϖ is a principal H-bundle, and $f, \tilde{f}$ are fiber bundles with fiber G/H. The map Δ is the diagonal $\Delta(p) = (pH, p)$, $p \in P$, where we embed $f^*P \subset P/H \times P$.

Remark 23.5. The special case $H = \{e\}$ is (22.42). In fact, the entire discussion specializes to **(22.41)** ff.

Observe that π' is the principal G-bundle associated to ϖ via the inclusion $H \hookrightarrow G$: there is an isomorphism

$$f^*P \longrightarrow P \times_H G \tag{23.6}$$

of right G-manifolds. To construct it, fix $x \in X$ and $p, p' \in P_x$. Write $p = p'g$ for a unique $g \in G$. The map (23.6) sends $(p'H, p) \in (f^*P)_x$ to the equivalence class of $(p', g) \in P_x \times G$ in $(P_x) \times_H G$.

(23.7) *The induced connection on ϖ.* Decompose the connection Θ on π as

$$\Theta = \Theta_{\mathfrak{p}} + \Theta_{\mathfrak{h}}, \qquad \Theta_{\mathfrak{h}} = \rho(\Theta). \tag{23.8}$$

Lemma 23.9. *$\Theta_{\mathfrak{h}} = \rho(\Theta) \in \Omega^1_P(\mathfrak{h})$ is a connection form on ϖ.*

Proof. For $h \in H$ we compute

$$R_h^*\big(\rho(\Theta)\big) = \rho(R_h^*\Theta) = \rho(\operatorname{Ad}_{h^{-1}}\Theta) = \operatorname{Ad}_{h^{-1}}\rho(\Theta), \tag{23.10}$$

since ρ is H-invariant. Also, if $\zeta \in \mathfrak{h}$ and $\widehat{\zeta}$ is the induced vertical vector field on P, then

$$\iota_{\widehat{\zeta}}\Theta_{\mathfrak{h}} = \iota_{\widehat{\zeta}}\rho(\Theta) = \rho(\iota_{\widehat{\zeta}}\Theta) = \rho(\zeta) = \zeta. \tag{23.11}$$

Together, (23.10) and (23.11) imply that $\Theta_{\mathfrak{h}}$ is a connection form. $\square$

(23.12) *Two connections on π'.* The principal G-bundle $\pi' \colon f^*P \approx P \times_H G \to P/H$ in (23.4) has two connection forms:

$$\begin{aligned} \Xi_0 &= \text{connection induced from } \Theta_{\mathfrak{h}} \\ \Xi_1 &= \tilde{f}^*\Theta \end{aligned} \tag{23.13}$$

The first, Ξ_0, is induced from the connection $\Theta_{\mathfrak{h}}$ on ϖ, using the isomorphism (23.6).

Remark 23.14. For $H = \{e\}$ the connection Ξ_0 is the connection Θ_Δ in **(22.41)**.

Introduce the Chern-Simons form

$$\alpha = \alpha(\Xi_0, \Xi_1) \ \in \Omega^{2k-1}_{P/H} \tag{23.15}$$

as an application of the general formula (22.29). The form α is the protagonist of this lecture.

(23.16) *Explicit formula for α.* To compute α, expand (23.4) into the pullback diagram

$$\begin{tikzcd} P \times G \arrow[r, "q"] \arrow[d, "\mathrm{pr}_1"] & P \times_H G \approx f^*P \arrow[r, "\tilde{f}"] \arrow[r, bend left, "R"] \arrow[d, "\mathrm{pr}_1 = \pi'"] & P \arrow[d, "\pi"] \arrow[dl, "\varpi"'] \\ P \arrow[r, "\varpi"] \arrow[u, bend left, "s"] & P/H \arrow[r, "f"] & X \end{tikzcd} \tag{23.17}$$

Here $R\colon P \times_H G \to P$ is the right action of G on P and the section s is $s(p) = (p, e)$, $p \in P$. Then

$$\begin{aligned} q^*\Xi_0 &= \mathrm{Ad}_{g^{-1}}\,\Theta_{\mathfrak{h}} + \theta_G \\ q^*\Xi_1 &= q^*R^*\Theta = \mathrm{Ad}_{g^{-1}}\,\Theta + \theta_G \end{aligned} \tag{23.18}$$

where θ_G is the Maurer-Cartan form on G. Introduce the difference

$$\widetilde{\Lambda} = q^*\Xi_1 - q^*\Xi_0 = \mathrm{Ad}_{g^{-1}}(\Theta = \Theta_{\mathfrak{h}}) = \mathrm{Ad}_{g^{-1}}(\Theta_{\mathfrak{p}}). \tag{23.19}$$

Pull back the computation to P via the section s. Then

$$\begin{aligned} \Lambda := s^*\widetilde{\Lambda} &= \Theta_{\mathfrak{p}} \\ s^*\Xi_0 &= \Theta_{\mathfrak{h}} \end{aligned} \tag{23.20}$$

The curvature of $\Theta = \Theta_{\mathfrak{h}} + \Theta_{\mathfrak{p}}$ is

$$\Omega = \Omega_{\mathfrak{h}} + d\Theta_{\mathfrak{p}} + [\Theta_{\mathfrak{h}} \wedge \Theta_{\mathfrak{p}}] + \frac{1}{2}[\Theta_{\mathfrak{p}} \wedge \Theta_{\mathfrak{p}}]. \tag{23.21}$$

Following the procedure in **(22.37)**, the curvature of $\Theta_{\mathfrak{h}} + t\Lambda$ is

$$\begin{aligned} \Omega_t &= \Omega_{\mathfrak{h}} + t\,d\Theta_{\mathfrak{p}} + t[\Theta_{\mathfrak{h}} \wedge \Theta_{\mathfrak{p}}] + \frac{t^2}{2}\,[\Theta_{\mathfrak{p}} \wedge \Theta_{\mathfrak{p}}] \\ &= t\Omega + (1-t)\Omega_{\mathfrak{h}} + \left(\frac{t^2 - t}{2}\right)[\Theta_{\mathfrak{p}} \wedge \Theta_{\mathfrak{p}}] \end{aligned} \tag{23.22}$$

Then from (22.40) we have

$$\alpha = k\int_0^1 dt\, h(\Lambda \wedge \Omega_t^{\wedge(k-1)}). \tag{23.23}$$

As written, this is a $(2k-1)$-form on P, and it descends to P/H.

(23.24) *α as a transgression form.* We introduce a crucial hypothesis in the statement of the next proposition; it guarantees that α is a transgression form in the fiber bundle $f\colon P/H \to X$. Let $j_x\colon (P/H)_x \hookrightarrow P/H$ denote the inclusion. Write the Maurer–Cartan form on G as $\theta_G = \theta_{\mathfrak{p}} + \theta_{\mathfrak{h}}$, using the decomposition $\mathfrak{g} = \mathfrak{p}+\mathfrak{h}$. Let $\omega = \omega(\Theta) \in \Omega^{2k}_X$ be the Chern–Weil form of the connection Θ. Recall the constant c_k defined in (22.54).

Proposition 23.25. *Assume h maps to zero under the restriction map $(\mathrm{Sym}^k\,\mathfrak{g}^*)^G \to (\mathrm{Sym}^k\,\mathfrak{h}^*)^H$.*

(1) $j_x^*\alpha = c_k\, h(\theta_{\mathfrak{p}} \wedge [\theta_{\mathfrak{p}} \wedge \theta_{\mathfrak{p}}]^{\wedge(k-1)})$.
(2) $d\alpha = f^*\omega$.

Proof. For (1) use the explicit formula formulas in **(23.16)** and compute on P. Let $i_x\colon P_x \hookrightarrow P$ denote the inclusion. Then

$$\begin{aligned} i_x^*\Lambda &= \theta_{\mathfrak{p}} \\ i_x^*\Omega_t &= \left(\frac{t^2-t}{2}\right)[\theta_{\mathfrak{p}} \wedge \theta_{\mathfrak{p}}]. \end{aligned} \tag{23.26}$$

Now (1) follows from (23.23). For (2) we apply Proposition 22.30 to the definition (23.15) of α to deduce

$$d\alpha = \omega(\Xi_1) = \omega(\Xi_0). \tag{23.27}$$

Now $\omega(\Xi_1) = \omega$, so it suffices to prove $\omega(\Xi_0) = 0$. But $s^*(\Xi_0) = \Theta_{\mathfrak{h}}$, so $s^*(\Omega_0) = \Omega_{\mathfrak{h}}$, and the conclusion follows since by hypothesis $h(\Omega_{\mathfrak{h}}^{\wedge k}) = 0$. $\square$

(23.28) *The transgressed form on G/H.* The form $j_x^*\alpha$ in Proposition 23.25 is the transport to $(P/H)_x$ of the G-invariant form

$$\eta = c_k\, h(\theta_{\mathfrak{p}} \wedge [\theta_{\mathfrak{p}} \wedge \theta_{\mathfrak{p}}]^{\wedge(k-1)}) \ \in \Omega^{2k-1}_{G/H} \tag{23.29}$$

on the homogeneous manifold G/H. The left-invariance follows since $L_g^*\theta_G = \theta_G$ from which $L_g^*\theta_{\mathfrak{p}} = \theta_{\mathfrak{p}}$. The form η is determined by its value at the basepoint $[H] \in G/H$. Since $T_{[H]}G/H \cong \mathfrak{p}$, this value is an element of $\bigwedge^{2k-1}\mathfrak{p}^*$ and it is H-invariant. Hence transgression is a linear map

$$\ker\bigl((\mathrm{Sym}^k\,\mathfrak{g}^*)^G \to (\mathrm{Sym}^k\,\mathfrak{h}^*)^H\bigr) \longrightarrow (\textstyle\bigwedge^{2k-1}\mathfrak{p}^*)^H. \tag{23.30}$$

The pfaffian

For the Chern–Gauss–Bonnet theorem on an n-dimensional manifold, we have $G = \mathrm{SO}_n$, $H = \mathrm{SO}_{n-1}$, and h is a multiple of the *pfaffian*, which is an invariant polynomial on skew-symmetric matrices. We give a brief introduction here. We assume $n = 2m$ is even.

(23.31) *Algebraic definition.* Let V be an oriented real inner product space of dimension $2m$, which we eventually take to be $V=\mathbb{R}^{2m}$ with its standard orientation and dot product. Then V carries a volume form $\nu \in \operatorname{Det} V^* = \bigwedge^{2m} V^*$. Associated to a skew-symmetric endomorphism $T\colon V \to V$ is the 2-form $\omega_T \in \bigwedge^2 V^*$ defined by

$$\omega_T(\xi_1,\xi_2)=\langle \xi_1, T\xi_2\rangle, \qquad \xi_1,\xi_2\in V. \tag{23.32}$$

Definition 23.33. $\operatorname{pfaff} T \in \mathbb{R}$ is defined by

$$\frac{(\omega_T)^{\wedge m}}{m!}=(\operatorname{pfaff} T)\nu. \tag{23.34}$$

This is the diagonal of a multilinear symmetric form $h\in \operatorname{Sym}^m \mathfrak{o}(V)^*$ defined by

$$h(T_1,\dots,T_m)\nu=\frac{\omega_{T_1}\wedge\cdots\wedge\omega_{T_m}}{m!}. \tag{23.35}$$

Example 23.36. Take $V=\mathbb{R}^2$ with standard basis e_1,e_2, dual basis e^1,e^2, and volume form $\nu=e^1\wedge e^2$. The skew-symmetric matrix

$$T=\begin{pmatrix}0&-a\\a&0\end{pmatrix} \tag{23.37}$$

corresponds to the 2-form

$$\omega_T=-a\,e^1\wedge e^2. \tag{23.38}$$

Then $\operatorname{pfaff} T=-a$. Observe that $\det T=(\operatorname{pfaff} T)^2=a^2$.

Remark 23.39.

(1) The pfaffian changes sign if we reverse the orientation of V.
(2) The form ω_T is nondegenerate iff T is invertible iff $\operatorname{pfaff} T\neq 0$.
(3) There is a sign choice in (23.32): one can put instead $\langle T\xi_1,\xi_2\rangle$ on the right hand side.

(23.40) *Pfaffians and determinants.*

Lemma 23.41. $(\operatorname{pfaff} T)^2=\det T$.

We do not give a proof here, but do mention a more general statement. Namely, if V is a vector space of dimension $2m$ (no inner product or orientation, any base field of characteristic zero), then there is a notion of a skew transformation $T\colon V\to V^*$, namely $T^*=-T$. The determinant of T is the induced map $\det T\colon \operatorname{Det} V \to \operatorname{Det} V^*$ on the $(2m)^{\text{th}}$ exterior power. In other words, $\det T \in (\operatorname{Det} V^*)^{\otimes 2}$. On the other hand, T determines $\omega_T \in {\bigwedge}^2 V^*$ and

$$\frac{(\omega_T)^{\wedge m}}{m!} \in \operatorname{Det} V^*. \tag{23.42}$$

The more general claim is

$$\left(\frac{(\omega_T)^{\wedge m}}{m!}\right)^{\otimes 2} = \det T \tag{23.43}$$

in $(\operatorname{Det} V^*)^{\otimes 2}$.

(23.44) *The pfaffian on* $\mathrm{SO}_n / \mathrm{SO}_{n-1}$. Henceforth set $V = \mathbb{R}^{2m}$ and write $n = 2m$. The decomposition

$$\begin{aligned} \mathfrak{o}_n &= \mathbb{R}^{n-1} \oplus \mathfrak{o}_{n-1} \\ &= \quad \mathfrak{p} \quad \oplus \quad \mathfrak{h} \end{aligned} \tag{23.45}$$

is illustrated by the matrix

$$\left(\begin{array}{c|c} 0 & -{}^t v \\ \hline v & A \end{array}\right), \qquad v \in \mathbb{R}^{n-1}, \qquad A \in \mathfrak{so}_{n-1}. \tag{23.46}$$

Observe that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$, i.e., $\mathrm{SO}_n / \mathrm{SO}_{n-1}$ is a symmetric space.

Lemma 23.47. *The restriction of* h *to* $\mathfrak{so}_{n-1}$ *vanishes.*

Here h is the symmetric multilinear form (23.35) built from the pfaffian.

Proof. The pfaffian vanishes on odd dimensional vector spaces. Or, more concretely, if $A \in \mathfrak{so}_{n-1}$, then ω_A does not involve e^1, so the wedge product in (23.35) vanishes. $\square$

Integral of the transgressing form over the fiber

(23.48) *The identification of* $\mathrm{SO}_n / \mathrm{SO}_{n-1}$ *with the sphere.* Let $S^{n-1} \subset \mathbb{R}^n$ be the sphere of unit norm vectors, and let the north pole $(1, 0, \dots, 0) \in \mathbb{R}^n$ be the basepoint. The motion

$$\begin{pmatrix} \cos t & -\sin t & 0 & \cdots \\ \sin t & \cos t & 0 & \cdots \\ 0 & 0 & & \\ \vdots & \vdots & & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \tag{23.49}$$

in S^n has initial velocity the vector at the north pole induced by the matrix

$$e_2 = \left(\begin{array}{cc|c|c} 0 & -1 & & \\ 1 & 0 & & \\ \hline & & & \\ \hline & & & \ddots \end{array}\right) \in \mathfrak{so}_n \tag{23.50}$$

Here we use the standard basis $e_2, e_3, \dots, e_n$ of $\mathfrak{p} = \mathbb{R}^{n-1}$ as embedded in skew-symmetric $n \times n$ matrices; see (23.46). Conclusion: the standard dot product on $\mathfrak{p} = \mathbb{R}^{n-1}$ corresponds to the unit radius round metric on $S^{n-1} \approx \mathrm{SO}_n / \mathrm{SO}_{n-1}$.

(23.51) *The integral over S^{n-1}.* Next, we compute the transgressed differential form η in (23.29), which is a form of degree $n-1$ on $S^{n-1} \approx \mathrm{SO}_n / \mathrm{SO}_{n-1}$. By the general theory, η is an SO_n-invariant form on S^{n-1}, and there is a 1-dimensional real line of such forms; they form the vector space $(\operatorname{Det}\mathfrak{p}^*)^{\mathrm{SO}_{n-1}}$, and the action of SO_{n-1} on the determinant line of $\mathfrak{p} = \mathbb{R}^{n-1}$ is trivial. So η is the volume form of the round metric on a sphere of some radius.

We now compute

$$h\big(e_2, [e_3, e_4], [e_5, e_6], \dots\big), \tag{23.52}$$

where h is given in (23.35). The matrix e_2 is (23.50), and

$$[e_3, e_4] = \left(\begin{array}{c|cc|c} & & & \\ \hline & 0 & -1 & \\ & 1 & 0 & \\ \hline & & & \ddots \end{array}\right) \tag{23.53}$$

and so on. According to Example 23.36 the corresponding 2-forms on $\mathbb{R}^n$ are

$$\begin{aligned} e_2 &\longleftrightarrow -e^1 \wedge e^2 \\ [e_3, e_4] &\longleftrightarrow -e^3 \wedge e^4 \end{aligned} \tag{23.54}$$

Hence from (23.35) we deduce

$$h(e_2, [e_3, e_4], \dots) = \frac{(-1)^m}{m!}. \tag{23.55}$$

Next, we compute the value of the form η in (23.29), viewed as an element of $\operatorname{Det}\mathfrak{p}^*$, on the basis $e_2, \dots, e_n$. We must sum (with sign) over the $(n-1)! = (2m-1)!$ permutations of the basis elements, and a little thought shows that each term in that sum contributes $(-1)^m/m!$, as in (23.55). From (22.54) we deduce

$$\eta(e_2,\dots,e_{2m}) = \frac{c_m\,(-1)^m(2m-1)!}{m!} = -\,\frac{(m-1)!}{2^{m-1}}. \tag{23.56}$$

For the unit radius sphere the basis $e_2,\dots,e_n$ is orthonormal, so the right hand side of (23.56) tells which multiple η is of the standard volume form. The volume of the unit radius sphere in Euclidean space of dimension $2m$ is $2\pi^m/(m-1)!$, so with the standard orientation we obtain

$$\int_{S^{n-1}} \eta = -\,\frac{(m-1)!}{2^{m-1}}\,\frac{2\pi^m}{(m-1)!} = -\,\frac{\pi^m}{2^{m-2}}. \tag{23.57}$$

Application to Chern–Gauss–Bonnet

We resume the argument at the end of Lecture 21. Recall that X is a closed oriented n-manifold ($n=2m$), and $S(TX)\to X$ is the unit sphere bundle of its tangent bundle. We seek a differential form of degree $n-1$ on the total space $S(TX)$ that satisfies the two conditions in Proposition 21.49.

Proposition 23.58. *The Chern–Simons form $-\alpha/(2\pi)^m$ in (23.23) satisfies these conditions.*

Here we take $\pi\colon P\to X$ to be the oriented orthonormal SO_n frame bundle of X, so $S(TX) = P/\mathrm{SO}_{n-1}$. Sadly, in this version of the notes we are off by a power of 2.

Proof. According to (23.57), the integral of $-\alpha/(2\pi)^m$ over the sphere fiber is

$$\frac{\pi^m}{2^{m-2}}\,\frac{1}{(2\pi)^m} = \frac{1}{2^{2m-2}}. \tag{23.59}$$

(If the constants were working out, it should be 1.) The second condition in Proposition 21.49 follows from Proposition 23.25(2). □

If we get the constants to work out, then we have completed the proof of the following.

Theorem 23.60 (Chern–Gauss–Bonnet)**.** *Let X be a closed oriented Riemannian manifold of dimension $n=2m$. Let R denote the Riemann curvature tensor. Then*

$$\mathrm{Euler}(X) = \int_X \frac{1}{(2\pi)^m}\,\mathrm{pfaff}(R). \tag{23.61}$$

Lecture 24: Sectional curvature of a Riemannian manifold

(Notes from Lectures 22 and 23 will appear in the next week or two as part of a collated version of all notes from the semester.)

An alternative way to present the information in the Riemann curvature tensor is as a function on 2-dimensional subspaces of the tangent bundle: the *sectional curvature*. This is a generalization to arbitrary dimensions of the Gauss curvature. We begin this lecture with a definition of the sectional curvature. The reader will be well-advised to first review the material in Lecture 20, especially the 6-term formula and the symmetries of the Riemann tensor. We prove (Proposition 24.22) that the sectional curvature equals the Gauss curvature of an embedded surface obtained by the Riemannian exponential map.

The simplest Riemannian manifolds have *constant* sectional curvature κ. On such manifolds the Lie brackets of the canonical global vector fields on the total space of the orthonormal frame bundle have constant coefficients. Those bracketing relations define a finite dimensional Lie algebra, whose isomorphism type depends on the sign of κ; see Proposition 24.33. If the constant curvature manifold is connected and *complete*, then the total space of the orthonormal frame bundle is a torsor for a Lie group. In **(24.40)** we give standard models for constant curvature manifolds: spheres, Euclidean space, and hyperbolic space. Constant curvature manifolds are unique in both a local (Theorem 24.54) and global (Theorem 24.55) sense. Our proof of the local theorem uses the Maurer–Cartan equations to construct a map of orthonormal frame bundles, which is an application of the general Theorem 7.8 (or rather the variation Theorem 9.2).

We conclude by stating some important theorems in Riemannian geometry—old and fairly new—that illustrate relationships between sectional curvature and topology. These only scratch the surface of a time-honored line of research.

Throughout this lecture X denotes a Riemannian manifold.

Sectional curvature

(24.1) *Curvature on 2-planes.* The symmetries of the Riemann curvature (Theorem 20.36) are important here. Recall from (20.41) that the Riemann curvature tensor may be regarded as a real-valued symmetric bilinear form on "bivectors":

$$\widetilde{R}\colon \textstyle\bigwedge^2 TX \times \bigwedge^2 TX \longrightarrow \underline{\mathbb{R}} \tag{24.2}$$

More concretely, for $x \in X$ and $\xi, \eta, \zeta, \lambda \in T_xX$,

$$\widetilde{R}(\xi\wedge\eta, \zeta\wedge\lambda) = \langle \xi, R(\zeta,\lambda)\eta\rangle, \tag{24.3}$$

where $R \in \Omega^2_X(\operatorname{SkewEnd} X)$ is the Riemann tensor. Let

$$\begin{aligned} q_{\widetilde{R}}\colon \textstyle\bigwedge^2 TX &\longrightarrow \mathbb{R} \\ \Xi &\longmapsto \widetilde{R}(\Xi,\Xi) \end{aligned} \tag{24.4}$$

be the associated quadratic form.

If V is a vector space, then the *Plücker embedding* is the map

$$\varphi\colon \operatorname{Gr}_2(V) \longrightarrow \mathbb{P}\left(\textstyle\bigwedge^2 V\right) \tag{24.5}$$

that sends a 2-plane $\pi \subset V$ to the line in $\bigwedge^2 V$ spanned by $\xi \wedge \eta$ for any basis ξ, η of π. Said differently, the inclusion $\pi \hookrightarrow V$ induces an inclusion $\operatorname{Det}\pi = \bigwedge^2 \pi \hookrightarrow \bigwedge^2 V$ whose image is $\varphi(\pi)$. Let $\operatorname{Gr}_2(TX) \to X$ denote the fiber bundle whose fiber at $x \in X$ is the Grassmannian of 2-dimensional subspaces $\pi \subset T_xX$. Then (24.5) induces a map of fiber bundles

$$\begin{tikzcd} \operatorname{Gr}_2(TX) \arrow[rr, "\varphi"] \arrow[dr] & & \mathbb{P}\left(\bigwedge^2 TX\right) \arrow[dl] \\ & X & \end{tikzcd} \tag{24.6}$$

Use the induced inner product on $\bigwedge^2 TX \to X$ to define its sphere bundle, which is the total space of a double cover:

$$\begin{tikzcd} S\left(\bigwedge^2 TX\right) \arrow[r, hook] \arrow[d, "2:1"] & \bigwedge^2 TX \\ \mathbb{P}\left(\bigwedge^2 TX\right) & \end{tikzcd} \tag{24.7}$$

The restriction of the quadratic function (24.4) to the total space of the sphere bundle drops to the total space of the projective bundle.

Definition 24.8. Let X be a Riemannian manifold. The composition

$$K = K_X\colon \operatorname{Gr}_2(TX) \xrightarrow{\varphi} \mathbb{P}\left(\textstyle\bigwedge^2 TX\right) \xrightarrow{q_{\widetilde{R}}} \mathbb{R} \tag{24.9}$$

is the *sectional curvature* of X.

More concretely, from (24.3) we see that if ξ, η is an *orthonormal* basis of $\pi \in \operatorname{Gr}_2(TX)$, then

$$K(\pi) = \langle \xi, R(\xi,\eta)\eta \rangle. \tag{24.10}$$

In particular, for an orthonormal frame $e_1, \dots, e_n$ the sectional curvature of a "basis 2-plane" is

$$K(e_i \wedge e_j) = R_{ijij}. \tag{24.11}$$

Remark 24.12. Scalar curvature has units $1/\text{Length}^2$.

Proposition 24.13. *Let X be a Riemannian manifold. Then the sectional curvature determines the Riemann curvature tensor.*

Proof. We are given all functions R_{ijij} and must solve for the general $R_{ijk\ell}$. First, all curvatures $\langle \xi, R(\xi,\zeta)\eta\rangle$ with a single vector repeated are determined. To prove this, choose i,j,ℓ distinct and expand

$$2K\bigl(e_i\wedge(e_j+e_\ell)\bigr) = \langle e_i, R(e_i, e_j+e_\ell)(e_j+e_\ell)\rangle = R_{ijij} + 2R_{iji\ell} + R_{i\ell i\ell} \tag{24.14}$$

to solve for $R_{iji\ell}$ in terms of sectional curvatures. Then expand

$$\langle e_i+e_k, R(e_i+e_k, e_\ell)e_j\rangle \tag{24.15}$$

to solve for $R_{ijk\ell} - R_{i\ell kj}$ in terms of known curvatures. Similarly, $R_{ik\ell j} - R_{ij\ell k}$ can be expressed in terms of known curvatures. Now add the last two expressions and apply Bianchi (20.40) to solve for $R_{ijk\ell}$. ∎

Remark 24.16. If $\dim X = 2$, then $K\colon \mathrm{Gr}_2(TX) = X \to \mathbb{R}$ is the Gauss curvature.

(24.17) *Preliminary: Levi–Civita covariant derivative of a submanifold.* We discussed the Riemannian geometry of a surface in Euclidean 3-space in Lectures 8–11. Here we discuss the induced covariant derivative of a submanifold of a Riemannian manifold in arbitrary dimensions.

Let X be a Riemannian manifold and let $\Sigma\subset X$ be a submanifold. Then Σ inherits a Riemannian metric: at any $x\in\Sigma$ the tangent space $T_x\Sigma$ is a subspace of T_xX and so inherits an inner product. Let ∇ be the Levi–Civita covariant derivative on X and let ∇' be the Levi–Civita covariant derivative on Σ.

Proposition 24.18. *For vector fields $\xi,\eta\in\mathfrak{X}(\Sigma)$ we have*

$$\nabla'_\xi\eta = \Pi\bigl(\nabla_\xi\eta\bigr), \tag{24.19}$$

where at $x\in X$ the map $\Pi_x\colon T_xX\to T_x\Sigma$ is orthogonal projection.

Proof. First, if $\widehat{\xi},\widehat{\eta}$ are vector fields on X that are tangent to Σ, so restrict to vector fields ξ,η on Σ, then the Lie bracket $[\widehat{\xi},\widehat{\eta}]$ on X is also tangent to Σ and it restricts to the Lie bracket $[\xi,\eta]$ computed on Σ. This follows from the definition of the Lie derivative (Definition 2.18) since the flow on X generated by $\widehat{\xi}$ preserves Σ and restricts on Σ to the flow generated by ξ.

Let ξ,η,ζ be vector fields on Σ, and choose extensions to vector fields $\widehat{\xi},\widehat{\eta},\widehat{\zeta}$ on X. The 6-term formula (20.15) implies

$$\begin{aligned}\langle\nabla'_\xi\eta,\zeta\rangle &= \frac12\Bigl\{\xi\langle\eta,\zeta\rangle + \eta\,\langle\zeta,\xi\rangle - \zeta\langle\xi,\eta\rangle + \langle[\xi,\eta],\zeta\rangle - \langle[\eta,\zeta],\xi\rangle + \langle[\zeta,\xi],\eta\rangle\Bigr\}\\ \langle\nabla_{\widehat{\xi}}\widehat{\eta},\widehat{\zeta}\rangle &= \frac12\Bigl\{\widehat{\xi}\langle\widehat{\eta},\widehat{\zeta}\rangle + \widehat{\eta}\,\langle\widehat{\zeta},\widehat{\xi}\rangle - \widehat{\zeta}\langle\widehat{\xi},\widehat{\eta}\rangle + \langle[\widehat{\xi},\widehat{\eta}],\widehat{\zeta}\rangle - \langle[\widehat{\eta},\widehat{\zeta}],\widehat{\xi}\rangle + \langle[\widehat{\zeta},\widehat{\xi}],\widehat{\eta}\rangle\Bigr\}\end{aligned} \tag{24.20}$$

where the first equation is computed on Σ and the second is computed on X. The restriction of the right hand side of the second equation to Σ equals the right hand side of the first equation. The left hand side of the second equation equals $\langle\nabla_\xi\eta,\zeta\rangle$. Hence (24.19) follows. ∎

(24.21) *Geometric interpretation of sectional curvature.* Let X be a Riemannian manifold, $x \in X$ a point, and $\xi, \eta \in T_xX$ orthonormal tangent vectors that span a 2-plane $\pi \subset T_xX$. We give a formula for the sectional curvature $K_X(\pi)$ in terms of Gauss curvature. Recall the exponential map defined in **(17.44)**. Let $U \subset A_xX$ be an open neighborhood of the origin of the affine tangent space on which $\exp_x$ is a diffeomorphism onto its image, let $U' = U \cap \pi$, and let $\Sigma = \exp_x(U')$; see Figure 72. Then $\Sigma \subset X$ is a 2-dimensional submanifold, so Σ inherits a Riemannian metric.

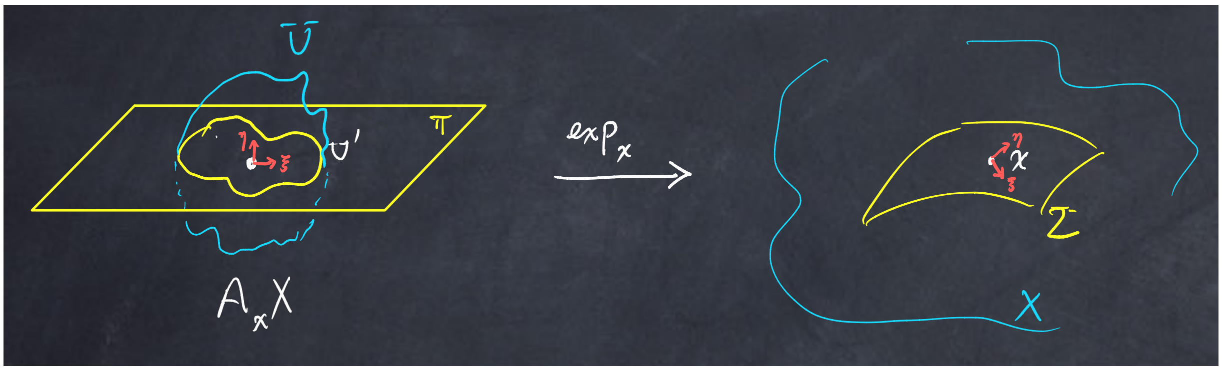

FIGURE 72. The plane π exponentiates to the local surface Σ

Proposition 24.22. $K_X(\pi) = K_\Sigma(x)$.

In other words, the sectional curvature of X at x on the 2-plane π is the Gauss curvature of Σ at x.

Proof. Extend the vectors ξ, η to vector fields $\hat\xi, \hat\eta$ in a neighborhood of x by parallel transport along geodesics in X emanating from x. These geodesics are the exponential images of rays in A_xX emanating from 0, so in particular those with initial velocity in π lie in Σ. It follows that $\hat\xi, \hat\eta$ restrict to vector fields ξ, η on Σ. Let ∇ be the Levi–Civita covariant derivative on X. The geodesic equation implies that $\nabla_{\hat\xi}\hat\xi = \nabla_{\hat\eta}\hat\eta = 0$ in the entire neighborhood. Furthermore, $\nabla\hat\xi(x) = \nabla\hat\eta(x) = 0$ by the parallel transport equation, and therefore $[\hat\xi, \hat\eta](x) = 0$ by the torsionfree condition (20.13). Hence by (24.10) and (19.55) we have, evaluating at the point x,

$$\begin{aligned} K_X(\pi) &= \langle\hat\xi, R(\hat\xi,\hat\eta)\hat\eta\rangle \\ &= \langle\hat\xi, \nabla_{\hat\xi}\nabla_{\hat\eta}\hat\eta\rangle - \langle\hat\xi, \nabla_{\hat\eta}\nabla_{\hat\xi}\hat\eta\rangle \\ &= -\langle\hat\xi, \nabla_{\hat\eta}\nabla_{\hat\xi}\hat\eta\rangle \\ &= -\hat\eta\,\langle\hat\xi, \nabla_{\hat\xi}\hat\eta\rangle + \langle\nabla_{\hat\eta}\hat\xi, \nabla_{\hat\xi}\hat\eta\rangle \\ &= -\hat\eta\,\langle\hat\xi, \nabla_{\hat\xi}\hat\eta\rangle \end{aligned} \tag{24.23}$$

We can make the same manipulation restricted to Σ to derive that

$$K_\Sigma(x) = -\eta\,\langle\xi, \nabla'_\xi\eta\rangle, \tag{24.24}$$

where ∇' is the Levi–Civita covariant derivative on Σ. Use Proposition 24.18 to conclude. ∎

Constant curvature

(24.25) *The Riemann tensor of a constant curvature manifold.* The term 'constant curvature' for a Riemannian manifold refers to *sectional* curvature.

Definition 24.26. Let X be a Riemannian manifold of dimension $n \ge 2$. Then X has *constant curvature* if $K_X\colon \mathrm{Gr}_2(TX) \to \mathbb{R}$ is a constant function.

We use the procedure in the proof of Proposition 24.13 to determine the Riemann tensor.

Proposition 24.27. *Let X be a Riemannian manifold of constant curvature κ. Then, relative to an orthonormal frame, $R^i_{jk\ell}$ vanishes unless (1) $i = k$ and $j = \ell$ are distinct or (2) $i = \ell$ and $j = k$ are distinct, in which case $R^i_{jij} = \kappa$. Equivalently, for any tangent vectors $\xi, \eta, \zeta, \lambda$,*

$$\langle \xi, R(\zeta,\lambda)\eta\rangle = \kappa\bigl[\langle \xi,\zeta\rangle\langle \eta,\lambda\rangle - \langle \xi,\lambda\rangle\langle \eta,\zeta\rangle\bigr]. \tag{24.28}$$

Proof. From (24.14) we deduce that $R^i_{jk\ell}$ vanishes when the indices take three values. Then by expanding (24.15) we deduce that $R^i_{jk\ell}$ vanishes for four distinct indices. For i, j distinct, $R^i_{jij} = R_{ijij} = \kappa$ is the constant sectional curvature. $\square$

(24.29) *The controlling Lie algebra.* Let X be a Riemannian manifold with orthonormal frame bundle $\varpi\colon \mathcal{B}_{\mathrm{O}}(X) \to X$. Recall from **(20.25)** the global framing of $\mathcal{B}_{\mathrm{O}}(X)$ given by the vector fields $\partial_k, \widehat{S}^j_i$, where $1 \le k \le n$ and $1 \le i < j \le n$. The bracketing relations (16.68)–(16.70) specialize to

$$[\widehat{S}^j_i, \widehat{S}^\ell_k] = \delta^j_k \widehat{S}^\ell_i \;-\; \delta^\ell_i \widehat{S}^j_k \;-\; \delta^j_\ell \widehat{S}^k_i \;+\; \delta^k_i \widehat{S}^j_\ell \tag{24.30}$$

$$[\widehat{S}^j_i, \partial_k] = \delta^j_k \partial_i \;-\; \delta^i_k \partial_j \tag{24.31}$$

$$[\partial_k, \partial_\ell] = -\frac{1}{2} R^i_{jk\ell} \widehat{S}^j_i \tag{24.32}$$

where $R^i_{jk\ell}\colon \mathcal{B}_{\mathrm{O}}(X) \to X$ are the coefficients of the Riemann curvature tensor. (The sum on the right hand side of (24.32) runs over all $1 \le i, j \le n$.) If X has constant curvature κ, then $R^i_{jk\ell} = \kappa$ or $R^i_{jk\ell} = 0$, by Proposition 24.27, so all $R^i_{jk\ell}$ are constant functions. In that case (24.30)–(24.32) are the structure equations for a Lie algebra $\mathfrak{g}_n(\kappa)$ of dimension $\dim \mathcal{B}_{\mathrm{O}}(X) = n(n+1)/2$.

Proposition 24.33. *The isomorphism type of $\mathfrak{g}_n(\kappa)$ is*

$$\mathfrak{g}_n(\kappa) \cong \begin{cases} \mathfrak{so}_{n+1}, & \kappa > 0; \\ \mathfrak{euc}_n, & \kappa = 0; \\ \mathfrak{o}_{n,1}, & \kappa < 0. \end{cases} \tag{24.34}$$

Proof. On $\mathbb{R}^{n+1}$ define the diagonal symmetric bilinear form

$$\langle e_i, e_j\rangle = \begin{cases} \kappa, & i=j=0; \\ 1, & 1\le i=j\le n; \\ 0, & i\neq j. \end{cases} \tag{24.35}$$

The isomorphism with the Lie algebra spanned by $\partial_k, \widehat{S}_i^j$ is expressed by the matrix

$$\left(\begin{array}{c|c} 0 & -\kappa\partial_i \\ \hline \partial_i & \widehat{S}_i^j \end{array}\right) \tag{24.36}$$

in which the upper left block is $n\times n$. □

(24.37) *A better model.* We give a useful description of $\mathfrak{g}_n(\kappa)$. Let $V=\mathbb{R}^n$ and then $\mathfrak{o}(V)$ is the Lie algebra of $n\times n$ skew-symmetric real matrices. Then

$$\mathfrak{g}_n(\kappa)\cong \mathfrak{o}(V)\oplus V. \tag{24.38}$$

The bracketing is as follows. $\mathfrak{o}(V)$ is a sub Lie algebra. The Lie bracket $\mathfrak{o}(V)\times V\to V$ is matrix multiplication, the action of $\mathfrak{o}(V)$ on V. The Lie bracket $V\times V\to\mathfrak{o}(V)$ is given by the formula

$$[\xi,\eta](\zeta) = \bigl[[\xi,\eta],\zeta\bigr] = \kappa\bigl[\langle\xi,\zeta\rangle\eta - \langle\eta,\zeta\rangle\xi\bigr], \qquad \xi,\eta,\zeta\in V. \tag{24.39}$$

(24.40) *Standard models.* For $\kappa\in\mathbb{R}$ and $n\in\mathbb{Z}^{\ge 2}$ we introduce a complete[43] simply connected Riemannian n-manifold $X_n(\kappa)$ of constant curvature κ.

The following standard models are homogeneous Riemannian manifolds, and their curvature is computed in **(26.59)**.

$\underline{\kappa>0}$: Let $\mathbb{E}^{n+1}$ denote standard Euclidean space with coordinates $x^0,x^1,\dots,x^n$ and standard Euclidean metric $ds^2=(dx^0)^2+(dx^1)^2+\cdots+(dx^n)^2$. (The underlying manifold is the standard affine space $\mathbb{A}^{n+1}$.) Define $X_n(\kappa)$ to be the sphere of radius $1/\sqrt{\kappa}$ with the induced Riemannian metric:

$$X_n(\kappa>0) = \bigl\{(x^0,x^1,\dots,x^n)\in\mathbb{E}^{n+1} : (x^0)^2+(x^1)^2+\cdots+(x^n)^2 = 1/\kappa\bigr\}. \tag{24.41}$$

[43]We have not discussed completeness in this course. One interpretation is geodesic completeness (Definition 17.11). Another is completeness of the metric space in which the distance between two points is the infimum of the length of a smooth path connecting them. The Hopf–Rinow theorem asserts that the two notions of completeness are equivalent.

$\underline{\kappa = 0}$: Simply let

$$X_n(\kappa = 0) = \mathbb{E}^n \tag{24.42}$$

be Euclidean n-space.

$\underline{\kappa < 0}$: Let $\mathbb{M}^{n+1}$ denote standard Minkowski spacetime with coordinates $x^0, x^1, \dots, x^n$ and standard Lorentz metric $ds^2 = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^n)^2$. (The underlying manifold is the standard affine space $\mathbb{A}^{n+1}$.) This is *not* a positive definite metric, but it is nondegenerate, a so-called *semi-Riemannian* metric. Define $X_n(\kappa)$ to be the hyperboloid of "radius" $1/\sqrt{-\kappa}$ with the induced metric:

$$X_n(\kappa < 0) = \bigl\{(x^0, x^1, \dots, x^n) \in \mathbb{M}^{n+1} : -(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 = -1/\kappa, \quad x^0 > 0\bigr\}. \tag{24.43}$$

Nonzero tangent vectors to $X_n(\kappa)$ are *spacelike* in that they have positive norm square, so the induced metric on $X_n(\kappa)$ is indeed positive definite, i.e., is a Riemannian metric.

(24.44) *Isometries.* The following christens maps that are compatible with Riemannian metrics.

Definition 24.45. Let X, X' be Riemannian manifolds. A smooth map $f\colon X' \to X$ is an *isometric immersion* if it preserves the metrics:

$$\langle f_*\xi', f_*\eta' \rangle_X = \langle \xi', \eta' \rangle_{X'}, \qquad x' \in X', \quad \xi', \eta' \in T_{x'}X'. \tag{24.46}$$

If, in addition, f is an embedding, then we say f is an *isometric embedding.* If f is a diffeomorphism, then we say f is an *isometry.*

Observe that a linear map f_* that satisfies (24.46) is necessarily injective.

Example 24.47. An isometric immersion $\gamma\colon (a,b) \to X$ from an open interval in $\mathbb{R}$ (with the standard metric) to a Riemannian manifold X is a unit speed motion.

Let $f\colon X' \to X$ be an isometry. Then f is covered by a diffeomorphism of orthonormal frame bundles:

$$\begin{tikzcd} \mathcal{B}_{\mathrm{O}}(X') \arrow[r, "\tilde f"] \arrow[d, "\varpi'"'] & \mathcal{B}_{\mathrm{O}}(X) \arrow[d, "\varpi"] \\ X' \arrow[r, "f"] & X \end{tikzcd} \tag{24.48}$$

The lift $\tilde f$ of an orthonormal basis $b' = (e_1, \dots, e_n)$ of $T_{x'}X'$ at some $x' \in X'$ is the orthonormal basis $(f_*e_1, \dots, f_*e_n)$ of $T_{f(x')}X$.

(24.49) *Lie groups of isometries.* The model constant curvature manifolds $X_n(\kappa)$ admit transitive Lie groups of isometries. In fact, isometries act simply transitively on the total space $\mathcal{B}_{\mathrm{O}}(X_n(\kappa))$ of the orthonormal frame bundle: so $\mathcal{B}_{\mathrm{O}}(X_n(\kappa))$ is a torsor over a Lie group. We find it convenient to work with *connected* Lie groups, and so we orient $X_n(\kappa)$ and work with the connected manifold of *oriented* orthonormal frames $\mathcal{B}_{\mathrm{SO}}(X_n(\kappa)) \subset \mathcal{B}_{\mathrm{O}}(X_n(\kappa))$. It is the total space of a principal SO_n-bundle $\mathcal{B}_{\mathrm{SO}}(X_n(\kappa)) \to X_n(\kappa)$. The following Lie groups act simply transitively on $\mathcal{B}_{\mathrm{SO}}(X_n(\kappa))$:

$$G_n(\kappa) \cong \begin{cases} \mathrm{SO}_{n+1}, & \kappa > 0; \\ \mathrm{SEuc}_n, & \kappa = 0; \\ \mathrm{SO}^{\uparrow}_{n,1}, & \kappa < 0. \end{cases} \tag{24.50}$$

Here SEuc_n is the component of the identity of the Euclidean group; it consists of orientation-preserving Euclidean transformations. Similarly, $\mathrm{SO}^{\uparrow}_{n,1}$ is the component of the identity of the group of isometries of a Lorentz signature metric.[44]

Remark 24.51. See Remark 1.29 for the connection between Lie groups and frame bundles in the case of affine space.

(24.52) *Uniqueness theorem.* We prove a theorem spelling out in what sense the model constant curvature Riemannian manifolds $X_n(\kappa)$ are models. Orient $X_n(\kappa)$ and let

$$\pi\colon \mathcal{B}_n(\kappa) \longrightarrow X_n(\kappa) \tag{24.53}$$

be the principal SO_n-bundle of oriented orthonormal frames.

Theorem 24.54. *Let X be a simply connected Riemannian n-manifold with constant curvature $\kappa \in \mathbb{R}$. Then there exists an isometric immersion $f\colon X \to X_n(\kappa)$.*

There is also a global version of the theorem, which we will not prove here.

Theorem 24.55. *Let X be a complete simply connected Riemannian n-manifold with constant curvature $\kappa \in \mathbb{R}$. Then there exists an isometry $f\colon X \to X_n(\kappa)$.*

Proof of Theorem 24.54. The manifold X is orientable, since it is simply connected. Orient X, let $\varpi\colon \mathcal{B}_{\mathrm{SO}}(X) \to X$ be the principal SO_n-bundle of oriented orthonormal frames, and let $\widetilde{\mathcal{B}}_{\mathrm{SO}}(X) \to \mathcal{B}_{\mathrm{SO}}(X)$ be the universal cover of the total space. The tautological 1-forms θ^i and the Levi–Civita 1-forms Θ^i_j on $\mathcal{B}_{\mathrm{SO}}(X)$ satisfy the Maurer–Cartan structure equations of $G_n(\kappa)$, the dual equations to (24.30)–(24.32). The pullbacks of θ^i, Θ^i_j to $\widetilde{\mathcal{B}}_{\mathrm{SO}}(X)$ satisfy the same equations. Let $\overline{\theta}^i, \overline{\Theta}^i_j$ be the Maurer–Cartan forms on the $G_n(\kappa)$-torsor $\mathcal{B}_n(\kappa)$. Apply Theorem 7.8(2) to construct a map

$$\varphi\colon \widetilde{\mathcal{B}}_{\mathrm{SO}}(X) \longrightarrow \mathcal{B}_n(\kappa) \tag{24.56}$$

[44] For $n \geq 2$, which we assume, the group $\mathrm{O}_{n,1}$ has four components, and the component group $\pi_0\mathrm{O}_{n,1}$ is isomorphic to the Klein group $\mu_2 \times \mu_2$. One homomorphism to μ_2 tracks whether $g \in \mathrm{O}_{n,1}$ is orientation-preserving. Another tracks whether g preserves the orientation of time. (There are two components of timelike vectors, and g either preserves the components or exchanges them.)

that satisfies

$$\begin{aligned} \varphi^*\overline{\theta}^i &= \theta^i \\ \varphi^*\overline{\Theta}^i_j &= \Theta^i_j \end{aligned} \tag{24.57}$$

We claim that φ preserves fibers, i.e., that there is a map $f\colon X \to X_n(\kappa)$ that fits into

$$\begin{tikzcd} \widetilde{\mathcal{B}}_{\mathrm{SO}}(X) \arrow[r, "\varphi"] \arrow[d, "\widetilde{\varpi}"'] & \mathcal{B}_n(\kappa) \arrow[d, "\pi"] \\ X \arrow[r, "f"] & X_n(\kappa) \end{tikzcd} \tag{24.58}$$

To verify this, fix $x \in X$ and let $\widetilde{F} = \widetilde{\mathcal{B}}_{\mathrm{SO}}(X)_x$. We use some elementary topology to prove that $\widetilde{F}$ is connected, or equivalently that $\widetilde{F}$ is path connected. Namely, the universal cover $\widetilde{\mathcal{B}}_{\mathrm{SO}}(X) \to \mathcal{B}_{\mathrm{SO}}(X)$ restricts to a covering map $\widetilde{F} \to F := \mathcal{B}_{\mathrm{SO}}(X)_x$, and $F \approx \mathrm{SO}_n$ is path connected. Our claim follows if we prove that $\widetilde{F} \to F$ is a universal covering map. By the construction of the universal cover, it suffices to show that any path in $\mathcal{B}_{\mathrm{SO}}(X)$ that connects the images $b_0, b_1 \in F$ of any two points $\tilde{b}_0, \tilde{b}_1 \in \widetilde{F}$ is homotopic to a path in F. That follows from the fact that X is simply connected. Next, the (connected) submanifold $\widetilde{F}$ is a leaf of the foliation cut out by the equations $\theta^i = 0$, $i = 1, \dots, n$. Equations (24.57) imply that φ is a local diffeomorphism, so in particular it restricts to an immersion on $\widetilde{F}$. Those equations also imply that the image of $d\varphi$ restricted to $T\widetilde{F}$ lies in $\ker d\pi$. It follows that $\varphi(\widetilde{F})$ is contained in a fiber of π, which proves the existence of f. Note that f is an immersion, since φ is a local diffeomorphism.

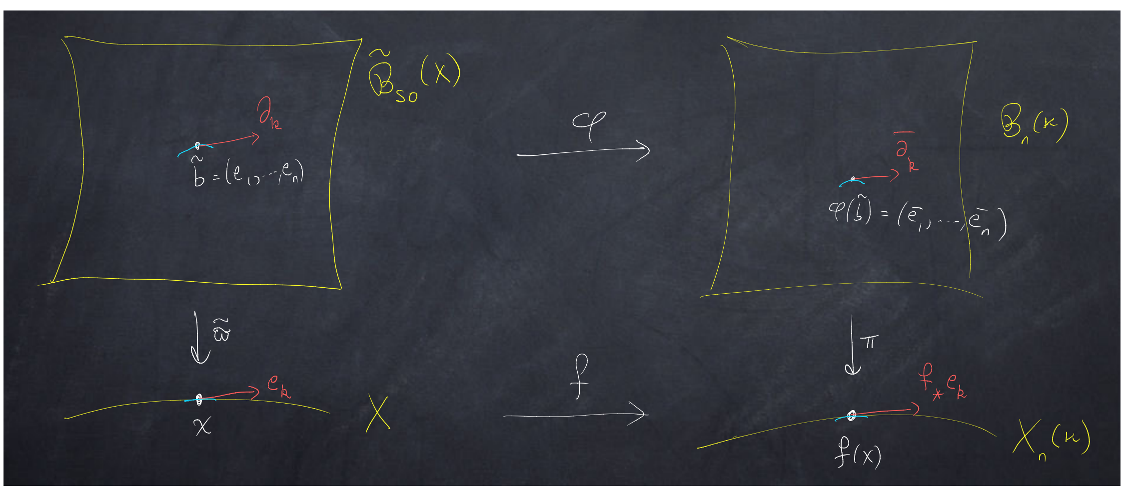

FIGURE 73. Computation of df

Now we claim that φ is a lift to $\widetilde{\mathcal{B}}_{\mathrm{SO}}(X)$ of the map $\tilde{f}\colon \mathcal{B}_{\mathrm{SO}}(X) \to \mathcal{B}_n(\kappa)$ on oriented orthonormal frames induced by f; see (24.48) for the definition of $\tilde{f}$. We illustrate the proof in Figure 73. Fix $x \in X$, let $b = (e_1, \dots, e_n) \in \mathcal{B}_{\mathrm{SO}}(X)_x$ be an oriented orthonormal frame at x, let $\tilde{b} \in \widetilde{\mathcal{B}}_{\mathrm{SO}}(X)$ be a lift of b. For $1 \le i \le n$, lift the horizontal vector field ∂_k on $\mathcal{B}_{\mathrm{SO}}(X)$ to a vector field $\widetilde{\partial}_k$ on the cover $\widetilde{\mathcal{B}}_{\mathrm{SO}}(X)$. An integral curve of $\widetilde{\partial}_k$ projects to a geodesic motion on X with initial position x and initial velocity e_k. Write $\varphi(\tilde{b}) = \bar{b} = (\bar{e}_1, \dots, \bar{e}_n)$, which is an oriented orthonormal frame at $f(x)$. Now from (24.57) the differential of φ maps $\widetilde{\partial}_k$ to the corresponding horizontal vector field $\overline{\partial}_k$ on $\mathcal{B}_n(\kappa)$, hence φ maps integral curves of $\widetilde{\partial}_k$ to integral curves of $\overline{\partial}_k$, hence f maps the $\widetilde{\varpi}$-image of integral curves of $\widetilde{\partial}_k$ to the π-image of integral curves of $\overline{\partial}_k$. So finally the differential of f maps the initial velocity of the resulting geodesic on X to the initial velocity of the resulting geodesic on $X_n(\kappa)$. In other words, $f_*(e_k) = \bar{e}_k$. It follows that $\varphi = \tilde{f}$, as claimed.

Finally, f is an isometric immersion since its differential maps orthonormal frames to orthonormal frames. □

Theorems about sectional curvature

We say the curvature of a Riemannian manifold *has a sign* if either $K_X > 0$, $K_X = 0$, or $K_X < 0$: the sectional curvature is either everywhere positive, everywhere zero, or everywhere negative. The theorems quoted here without proof are for curvatures with a definite sign.

(24.59) *Compact manifolds.* This first theorem[45] is for compact Riemannian manifolds.

Theorem 24.60. *Let X be a compact manifold. Suppose X has Riemannian metrics g, g' whose sectional curvatures K, K' have a sign. Then either $KK' > 0$ or $K = K' = 0$.*

In other words, K and K' are both positive, both zero, or both negative.

If $\dim X = 2$, then Theorem 24.60 is a consequence of the Gauss–Bonnet theorem. In higher dimensions it is a distillation of several different theorems in Riemannian geometry.

(24.61) *Positive curvature.* The first result is due to Bonnet–Myers.[46]

Theorem 24.62. *Let X be a complete Riemannian manifold of positive sectional curvature. Then X is compact with finite fundamental group.*

As a consequence, the universal cover of X is also compact.

The next result is a much stronger restriction on the fundamental group if $\dim X$ is even.

Theorem 24.63. *Suppose X is a compact connected even-dimensional Riemannian manifold with positive sectional curvature. Then if X is orientable, it is simply connected. If X is not orientable, then $\pi_1 X \cong \mathbb{Z}/2\mathbb{Z}$.*

Note that $\mathbb{RP}^n$, n even, is an example of a nonorientable even-dimensional Riemannian manifold of positive sectional curvature.

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[45] I learned the statement from a mathoverflow posting of Misha Kapovich:
https://mathoverflow.net/questions/133007/why-dont-mathbbtn-mathbbsn-mathbbhn-admit-other-metrics-of-cons

[46] There is a stronger version about positive Ricci curvature. The version stated with positive sectional curvature is essentially due to Bonnet.

Remark 24.64. In odd dimensions the theorem fails: lens spaces have metrics of positive sectional curvature and can have a fundamental group that is cyclic of any order. Even more exotically, any finite subgroup $\Gamma \subset \mathrm{SO}_3 \approx \mathbb{RP}^3$ acts freely by left translation, and the action is by isometries for the standard metric on $\mathbb{RP}^3$. Hence the quotient is a compact Riemannian 3-manifold X with positive (in fact, constant) sectional curvature. The fundamental group is isomorphic to the double cover of Γ in SU_2. For example, Γ can be the order 60 group of symmetries of the icosahedron, in which case the fundamental group is nonabelian of order 120.

The next result is a *pinching theorem*. The homeomorphism statement was proved by Berger, and the diffeomorphism statement fairly recently by Brendle–Schoen. It is called the *differentiable sphere theorem*.

Theorem 24.65. *Let X be a complete simply connected Riemannian manifold whose sectional curvature is* 1/4*-pinched:* $1/4 < K_X \le 1$*. Then X is diffeomorphic to S^n.*

Remark 24.66.

(1) The same conclusion—that X is diffeomorphic to a sphere—holds if the pinching is pointwise: at each point $x \in X$ the max and min of K_X restricted to $\mathrm{Gr}_2(T_xX)$ have ratio < 4.
(2) The proof by Brendle and Schoen uses the Ricci flow introduced by Hamilton and used by Perelman to prove the Poincaré conjecture in dimension three.
(3) If we allow $1/4 \le K_X \le 1$, then X could also be $\mathbb{CP}^n$, $\mathbb{HP}^n$, or $\mathbb{OP}^2$.

(24.67) *Negative curvature.* The most basic theorem is for nonpositive curvature and is due to Cartan–Hadamard.

Theorem 24.68. *Let X be a complete n-dimensional Riemannian manifold of nonpositive sectional curvature. Then the universal cover of X is diffeomorphic to $\mathbb{A}^n$.*

In particular, the universal cover is contractible. Such a topological space is called *aspherical*: its homotopy groups π_q vanish for $q > 1$.

Lecture 25: More Riemannian curvature

We continue our discussion of the Riemann curvature tensor by distilling from it simpler quantities: Ricci curvature and scalar curvature. The Ricci form is of the same type as the metric, so it makes sense to ask that the metric and Ricci curvature be proportional. That proportionality condition is a Riemannian variation of Einstein's equation in relativity, so Riemannian manifolds that satisfy it are called *Einstein manifolds*. We prove that if the proportionality is only imposed pointwise, then as long as the dimension is at least 3 the proportionality is locally constant. The fact that the Ricci curvature and metric are tensors of the same type is also the basis of the Ricci flow equation, which plays an important role in modern developments that we do not cover here.

We conclude the lecture by reciting the general structure of the curvature tensor from the representation theory of the orthogonal group. There is a noteworthy extra decomposition in dimension 4 on oriented Riemannian manifolds. Self-duality of curvature is an important condition in 4-dimensional differential geometry and its applications to topology and physics.

Ricci and scalar curvature

(25.1) *Review.* Let X be a Riemannian manifold. The Riemann tensor is defined by global functions on the total space $\mathcal{B}_{\mathrm{O}}(X)$ of the orthonormal frame bundle as in (16.66):

$$\Omega^i_j = \frac{1}{2} R^i_{jk\ell}\theta^i \wedge \theta^j, \qquad R^i_{jk\ell} = R_{ijk\ell}. \tag{25.2}$$

The lowered index is made with no change since we work with *orthonormal* frames. Theorem 20.36 lists the symmetries of $R_{ijk\ell}$.

(25.3) *Definitions.* We give the definitions in terms of bases.

Definition 25.4. Let X be a Riemannian manifold.

(1) The *Ricci form* $\mathrm{Ric}\colon TX \times TX \to \mathbb{R}$ is the symmetric bilinear form defined by the functions

$$R_{j\ell} = R^i_{ji\ell} = \sum_i R_{iji\ell} \tag{25.5}$$

on $\mathcal{B}_{\mathrm{O}}(X)$.

(2) The *scalar curvature* $r\colon X \to \mathbb{R}$ is the real-valued function

$$r = \sum_j R_{jj} = \sum_{i,j} R_{ijij}. \tag{25.6}$$

Use the equivariance properties of $R^i_{jk\ell}$ under O_n to deduce that (25.5) descends to a symmetric bilinear form on TX and that (25.6) descends to a real-valued function on X. On the other hand, having descended $R^i_{jk\ell}$ to the curvature 2-form (20.31), we can write

$$\mathrm{Ric}_x(\xi_1,\xi_2)=\mathrm{Tr}\big(\eta\longmapsto R(\eta,\xi_2)\xi_1\big),\qquad x\in X,\quad \xi_1,\xi_2\in T_xX, \tag{25.7}$$

and

$$r(x)=\sum_i \mathrm{Ric}_x(e_i,e_i),\qquad x\in X, \tag{25.8}$$

where e_i runs over an *orthonormal* basis of T_xX.

(25.9) *The Ricci form and the metric.* The Riemannian metric g and the Ricci form Ric are both symmetric bilinear forms, an important concurrence. First, we use it to give a more intrinsic form of (25.8). Namely, since the metric g is nondegenerate—in fact, positive definite—we define the *Ricci operator* $S\colon TX\to TX$ as the ratio of Ric and g:

$$\mathrm{Ric}_x(\xi,\eta)=\langle \xi,S_x(\eta)\rangle_x,\qquad x\in X,\quad \xi,\eta\in T_xX. \tag{25.10}$$

Then (25.8) becomes

$$r(x)=\mathrm{Tr}(S_x). \tag{25.11}$$

More significantly, we can impose a proportionality condition.

Definition 25.12. A Riemannian manifold X with metric g is *Einstein* if its Ricci form satisfies

$$\mathrm{Ric}=\lambda g \tag{25.13}$$

for some constant $\lambda\in\mathbb{R}$.

Manifolds of constant sectional curvature **(24.40)** are Einstein, but there are many more examples, such as the projective spaces $\mathbb{CP}^n$ and $\mathbb{HP}^n$ in their standard metrics.

Remark 25.14 (Ricci flow)*.* Another manifestation of the concurrence of types is the Ricci flow equation. Namely, if X is a Riemannian manifold with metric g_0, then we can write an equation for a path g_t, $t>0$, of Riemannian metrics:

$$\frac{dg_t}{dt}=-2\,\mathrm{Ric}(g_t). \tag{25.15}$$

The absolute value of the constant is a bit of a convenience; its sign is crucial. Also, the Ricci curvature behaves as the negative Laplacian of the metric; the constant -2 in (25.15) is chosen to maximize the similarity to the heat equation. On the infinite dimensional manifold of Riemannian metrics, the right hand side defines a vector field, and (25.15) is the equation for an integral curve. But there is no such cute trickery to turn a PDE (here related to the heat equation) into an ODE: even existence for small times t is highly nontrivial.

The Ricci flow equation was introduced by Richard Hamilton in the early 1980s, and he and Yau speculated that it might be useful in attacking Thurston's geometrization conjecture for 3-manifolds, a special case of which is the Poincaré conjecture. That turned out to be the case, and this program was completed by Perelman in the early 2000s. The Ricci flow has been applied to many other problems in Riemannian and Kähler geometry, for example Theorem 24.65.[47]

(25.16) *Variable proportionality.* One could contemplate generalizing Definition 25.12 to allow the proportionality λ to vary from point to point. However, we have the following.

Proposition 25.17. *Suppose X is a Riemannian manifold whose metric g and Ricci form* Ric *satisfy*

$$\mathrm{Ric}_x = \lambda(x) g_x, \qquad x \in X, \tag{25.18}$$

for some function $\lambda\colon X \to \mathbb{R}$. Then if $\dim X \geq 3$ the function λ is locally constant.

Proof. Write $n = \dim X$. Work on the total space $\mathcal{B}_{\mathrm{O}}(X)$ of the orthonormal frame bundle with its global framing $\partial_i, \widehat{E}_i^j$ and dual coframing θ^i, Θ^i_j. We eventually evaluate the expressions below on horizontal vector fields ∂_i, so work modulo the ideal generated by Θ^i_j. Use '$+$ ——' to denote terms that involve Θ^i_j. Then the Bianchi identity (16.60) asserts:

$$0 = d_\Theta \Omega = d\Omega + [\Theta \wedge \Omega], \tag{25.19}$$

and so for all $1 \leq i, j \leq n$ we have

$$\begin{aligned} 0 &= d\Omega^i_j + \Theta^i_k \wedge \Omega^k_j - \Omega^i_k \wedge \Theta^k_j \\ &= d\Omega^i_j + \text{——} \\ &= d\big(\frac12 R^i_{jk\ell}\theta^k \wedge \theta^\ell\big) + \text{——} \end{aligned} \tag{25.20}$$

Evaluate on ∂_i and sum over i, using the hypothesis (25.18) in the form $R_{j\ell} = \lambda \delta_{j\ell}$:

$$\begin{aligned} 0 &= \frac12 (\partial_i R^i_{jk\ell})\theta^k \wedge \theta^\ell - \frac12 d(R^i_{ji\ell}) \wedge \theta^\ell + \frac12 d(R^i_{jk\ell}) \wedge \theta^k + \text{——} \\ &= \frac12 (\partial_i R^i_{jk\ell})\theta^k \wedge \theta^\ell - dR_{j\ell} \wedge \theta^\ell + \text{——} \\ &= \frac12 (\partial_i R^i_{jk\ell})\theta^k \wedge \theta^\ell - d\lambda \wedge \theta^j + \text{——} \end{aligned} \tag{25.21}$$

[47] Hamilton recently gave an overview lecture on Ricci flow: `https://www.youtube.com/watch?v=lcCBJVLFZ_c`

This equation in 2-forms holds for each j. Evaluate on ∂_m, ∂_j and sum over j:

$$
\begin{aligned}
0 &= \partial_i R^i_{jmj} - (\partial_m\lambda)\delta^j_j + (\partial_j\lambda)\delta^j_m \\
&= \sum_i \partial_i R^j_{ijm} - (n-1)\partial_m\lambda \\
&= \sum_i \partial_i R_{im} - (n-1)\partial_m\lambda \\
&= \partial_m\lambda - (n-1)\partial_m\lambda \\
&= (2-n)\partial_m\lambda
\end{aligned}
\tag{25.22}
$$

Hence if $n \geq 2$ we conclude $\partial_m\lambda = 0$ for all m, i.e., $d\lambda = 0$. □

(25.23) *Examples of Einstein manifolds.* This is too vast an area to do any justice in these notes. In the next lecture we will mention some homogeneous examples. There are constant sectional curvature examples that are not simply connected—the simply connected constant sectional curvature examples are homogeneous—already vast in the case of compact manifolds: e.g. the theory of spherical space forms (positive curvature), flat manifolds (Bieberbach etc.), and hyperbolic manifolds (negative curvature). Or we could consider the Kähler case: *Kähler–Einstein metrics.* (We introduce Kähler geometry in the last lecture of the course.) The theory depends very much on the sign of the proportionality constant λ. For example, a *Calabi–Yau manifold* is the case $\lambda = 0$. All of these examples, or their universal covers, fit one or both of two classes of special Riemannian manifolds: homogeneous and Kähler. The intersection of those classes is also a very interesting set of Riemannian manifolds. But there is much outside of these classes as well. You might have a look at the book *Einstein Manifolds*, a collation by the fictional Arthur Besse from the mid 1980s.

(25.24) *Scalar curvature and topology.* This is another subject we did not have time to discuss in the course. One famous theorem of Lichnerowicz gives an obstruction to positive scalar curvature in terms of a special Pontrjagin number: the $\hat{A}$ genus. In another direction, every manifold of dimension ≥ 3 admits a metric of constant negative scalar curvature. In dimension 2 this is false, for example none of S^2, $\mathbb{RP}^2$, $S^1 \times S^1$, and the Klein bottle admit metrics of negative (scalar=Gauss up to a positive factor) curvature: apply the Gauss–Bonnet theorem. There is much work on compact manifolds of positive scalar curvature.

Decomposition of the Riemann curvature tensor

Let V be an n-dimensional real inner product space. The curvature tensor at a point x in a Riemannian manifold X lies in a subspace of $\operatorname{SkewEnd} V \otimes \bigwedge^2 V^*$, where $V = T_xX$; see (20.31). That subspace is cut out by the symmetries in Theorem 20.36, and it is invariant under the orthogonal group $\mathrm{O}(V)$. This leads to the problem, in any dimension, of describing the representation of $\mathrm{O}(V)$ in which the Riemann curvature tensor transforms, where "describing" means expressing it as a sum of irreducible representations. Here we give an indication of the answer without proof.

(25.25) *A few representations.* For the general linear group $\mathrm{Aut}(V)$, relevant representations are the trivial representation on scalars $\mathbb{R}$ and the representation on the vector space $\mathrm{Sym}^2 V^*$ of symmetric bilinear forms $V \times V \to \mathbb{R}$, which is also irreducible. However, the restriction of the latter to $\mathrm{O}(V)$ is reducible: the inner product spans an invariant line in $\mathrm{Sym}^2 V^*$. Its orthogonal complement is an irreducible representation that we denote $\mathrm{Sym}^2_0 V^*$. On a Riemannian manifold this induces a decomposition of the Ricci form Ric into two pieces:

$$\mathrm{Ric} = r \oplus \mathrm{Ric}_0 \,. \tag{25.26}$$

(There is a factor of $1/n$ that one might put in, but here I summarize the "information" in the Ricci form.) Ric_0 is called the *traceless Ricci form*; the trace in (25.11) vanishes if we use Ric_0 in (25.10).

(25.27) $n = 2$. In 2 dimensions the Riemann tensor function $R_{ijk\ell}$ is possibly nonzero only if the indices $i, j, k, \ell \in \{1, 2\}$ are equal in pairs $i = k$, $j = \ell$ or $i = \ell$, $j = k$, and both possible values are realized. One can easily work out that there is only one independent number, so the curvature reduces to a single scalar. That can be taken to be the Gauss curvature K or the scalar curvature $r = 2K$. We schematically write

$$R = r. \tag{25.28}$$

(25.29) $n = 3$. Now there is more:

$$R = r \ \oplus \ \mathrm{Ric}_0 \,. \tag{25.30}$$

In other words, the Riemann curvature reduces to the Ricci form.

It is apposite at this point to note the following.

Proposition 25.31. *A connected 3-dimensional Einstein manifold has constant sectional curvature.*

Proof. Let X be such a manifold, so that on $\mathcal{B}_{\mathrm{O}}(X)$ we have $R_{j\ell} = \lambda\delta_{j\ell}$ for some $\lambda \in \mathbb{R}$. Let

$$K_{ij} = K_{ji} = R_{ijij} \tag{25.32}$$

be the sectional curvature in the i, j plane (relative to an orthonormal basis which is a point of $\mathcal{B}_{\mathrm{O}}(X)$). Then from (25.5)

$$\begin{aligned} R_{11} &= K_{12} + K_{13} \\ R_{22} &= K_{21} + K_{23} \\ R_{33} &= K_{31} + K_{33} \end{aligned} \tag{25.33}$$

Hence

$$2K_{12} = R_{11} + R_{22} - R_{33} = \lambda, \tag{25.34}$$

and similarly we conclude $K_{23} = K_{13} = \lambda/2$. $\square$

(25.35) $n \geq 4$. Once we arrive to dimension 4 and beyond the Ricci form does not capture all of the information in the curvature tensor. Rather, there is a single $\mathrm{O}(V)$-irreducible piece left after subtracting off Ric; it is called the *Weyl curvature* W:

$$R = r \;\oplus\; \mathrm{Ric}_0 \;\oplus W. \tag{25.36}$$

The Weyl curvature is encoded in functions $W^i_{jk\ell}\colon \mathcal{B}_{\mathrm{O}}(X) \to \mathbb{R}$ that satisfy the same symmetries as do $R^i_{jk\ell}$. One significant fact about the Weyl tensor is that it is *conformally* invariant: it is unchanged under $g \mapsto e^{2\phi} g$ for some function $\phi\colon X \to \mathbb{R}$.

(25.37) $n = 4$. Dimension 4 is quite special in Riemannian geometry and in differential topology—those specialnesses are related. This is due to *self-duality*, which enters in the presence of an orientation. Thus suppose V is an oriented 4-dimensional inner product space. Then the representation $\bigwedge^2 V^*$ of 2-forms decomposes under the *special* orthogonal group $\mathrm{SO}(V)$:

$$\textstyle\bigwedge^2 V^* = \bigwedge^2_+ V^* \;\oplus\; \bigwedge^2_- V^* \tag{25.38}$$

2-forms in the first summand are *self-dual*; 2-forms in the second summand are *anti-self-dual.* The decomposition is effected using the *Hodge star operator*, which is an automorphism of $\bigwedge^2 V^*$ that squares to $\mathrm{id}_{\bigwedge^2 V^*}$; (25.38) is the decomposition into $+1$ and -1 eigenspaces.

On an *oriented* Riemannian 4-manifold, the Weyl curvature decomposes into self-dual and anti-self-dual parts:

$$R = r \;\oplus\; \mathrm{Ric}_0 \;\oplus W_+ \;\oplus\; W_-. \tag{25.39}$$

Lecture 26: Homogeneous Riemannian manifolds

In these last two lectures we can only scratch the surface of three topics: homogeneous Riemannian manifolds, complex manifolds, and Kähler manifolds. We introduced homogeneous manifolds in **(16.1)**, which is a good preliminary to the discussion here of the Riemannian case.

We begin by proving that an isometry of Riemannian manifolds with connected domain is determined by its value and first differential at one point. This implies that the group of isometries of a connected Riemannian manifold is finite dimensional. It is also a Lie group by a theorem of Myers–Steenrod that we do not prove here. As another preliminary, we give the easy proof of the existence of Haar measure on a compact Lie group, and we use it to prove elementary theorems about invariant complements and invariant symmetric bilinear forms on linear representations.

The main topic is the geometry of a Riemannian homogeneous manifold G/H. We prove that if G acts effectively, then the homogeneous manifold is reductive and H is compact. There is a canonical linear connection on G/H, from the reductive structure, but it may not be the Levi–Civita connection. We compute a formula for the Levi–Civita connection (Proposition 26.51), and then report formulas for various of its curvatures. In particular, we verify that the model manifolds in **(24.40)** do have constant sectional curvature. We conclude with some remarks about geodesics and rigid bodies.

Some preliminaries

(26.1) *Isometries are determined by the 1-jet at a point.* Recall Definition 24.45 of an isometry.

Proposition 26.2. *Let $f_0, f_1\colon X' \to X$ are isometries of Riemannian manifolds, and suppose that X' is connected. Let $\tilde f_0, \tilde f_1\colon \mathcal{B}_O(X') \to \mathcal{B}_O(X)$ be the lifts (24.48) to orthonormal frame bundles. Suppose for some $x' \in X'$ we have $\tilde f_0\big|_{\mathcal{B}_O(X')_{x'}} = \tilde f_1\big|_{\mathcal{B}_O(X')_{x'}}$. Then $f_0 = f_1$.*

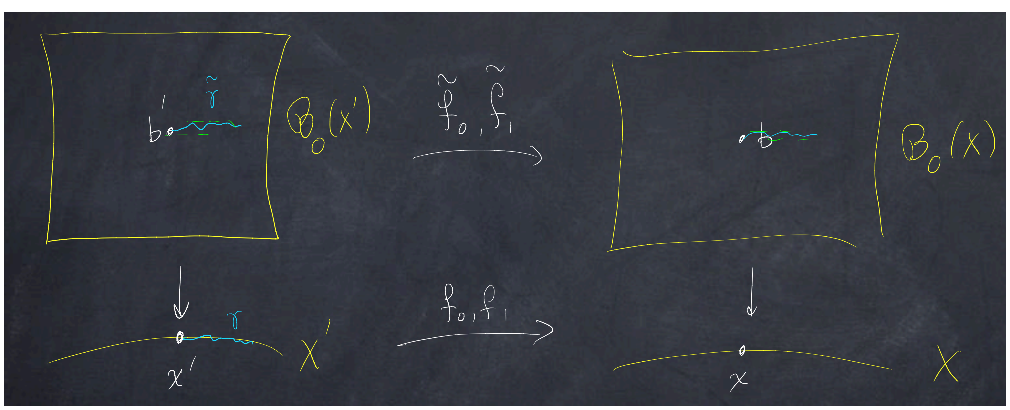

FIGURE 74. Equality of isometries that agree to 1st order at a point

Proof. See Figure 74 for an illustration of the proof. Set $n = \dim X' = \dim X$. The lifts $\tilde f_0, \tilde f_1$ preserve Levi–Civita connections, since the Levi–Civita connection is canonically determined from the metric, and so they map the tautological horizontal vector fields $\partial_1',\dots,\partial_n'$ on $\mathcal{B}_O(X')$ to the tautological horizontal vector fields $\partial_1,\dots,\partial_n$ on $\mathcal{B}_O(X)$. Fix a basis $b'=(e_1',\dots,e_n')\in\mathcal{B}_O(X')$ of $T_{x'}X'$. Let $\gamma\colon[0,1]\to X'$ be a smooth motion with initial position x', and let $\tilde\gamma\colon[0,1]\to\mathcal{B}_O(X')$ be its horizontal lift with initial position b'. Then $\tilde\gamma(t)=\bigl(e_1'(t),\dots,e_n'(t)\bigr)$ is the parallel transport of the initial frame b' along γ. Let $v\colon[0,1]\to\mathbb{R}^n$ be smooth curve such that $\dot\gamma(t)=\tilde\gamma(t)\cdot v(t)=v^i(t)e_i'(t)$. Then $\tilde\gamma$ is the integral curve of the time-varying vector field $v^i(t)\partial_i'$ with initial position b'. By assumption we have $\tilde f_0(b')=\tilde f_1(b')=b$ for some $b\in\mathcal{B}_O(X)_x$, where $f_0(x')=f_1(x')=x\in X$. It follows that the images of $\tilde\gamma$ under $\tilde f_0$ and $\tilde f_1$ are integral curves of $v^i(t)\partial_i$ with initial position b. Hence they are equal. In particular, $f_0\bigl(\gamma(1)\bigr)=f_1\bigl(\gamma(1)\bigr)$. Since X' is connected, it follows that $f_0=f_1$. $\square$

(26.3) *Bi-invariant measures on a compact Lie group.* Let V be a finite dimensional real vector space and let $\mathcal{B}(V)=\{b\colon\mathbb{R}^n\xrightarrow{\cong}V\}$ be its right $\mathrm{GL}_n\mathbb{R}$-torsor of bases. The real line

$$|\operatorname{Det}V^*|=\bigl\{\mu\colon\mathcal{B}(V)\to\mathbb{R}:\mu(bg)=|det(g)|\mu(b),\qquad b\in\mathcal{B}(V),\quad g\in\mathrm{GL}_n\mathbb{R}\bigr\}\tag{26.4}$$

is the line of *densities* on V. Decompose

$$|\operatorname{Det}V^*|=|\operatorname{Det}V^*|_+\ \amalg\ |\operatorname{Det}V^*|_0\ \amalg\ |\operatorname{Det}V^*|_-\tag{26.5}$$

according as the function μ is positive, zero, or negative. Then $|\operatorname{Det}V^*|_+$ is the ray of positive volume functions on V. (If A is an affine space over V, then $|\operatorname{Det}V^*|_+$ is the ray of translation-invariant positive measures on A.)

Let X be a smooth manifold. Associated to $\varpi\colon\mathcal{B}(X)\to X$ via the representation $g\mapsto|\det g|$ is the real line bundle $|\det T^*X|\to X$, and it decomposes as in (26.5). A section of

$$|\det T^*X|_+\longrightarrow X\tag{26.6}$$

is a *smooth measure* on X. Since the fibers of (26.6) are contractible, smooth measures exist. One can integrate continuous compactly supported functions against a smooth measure.

Theorem 26.7. *Let G be a compact Lie group. Then there exists a smooth bi-invariant measure μ on G.*

Since G is compact, the total measure $\mu(G)\in\mathbb{R}^{>0}$ is finite. There is a unique choice of μ such that $\mu(G)=1$.

Proof. The ray $|\operatorname{Det}\mathfrak{g}^*|_+$ consists of left-invariant smooth measures on G. Pullback by right translation gives an action of G on this ray, and the action is by a homomorphism $\rho\colon G\to\mathbb{R}^{>0}$, where $\mathbb{R}^{>0}$ is the group of positive real numbers under multiplication. Since G is compact, $\rho(G)$ is a compact subgroup of $\mathbb{R}^{>0}$. The only such is the identity $\{1\}\subset\mathbb{R}^{>0}$. Hence any element of $|\operatorname{Det}\mathfrak{g}^*|_+$ is a bi-invariant smooth measure. $\square$

(26.8) *Invariant complements.* In a linear representation of a Lie group, not every invariant subspace has an invariant complement. For example, let $n \in \mathbb{Z}$ act on $\mathbb{R}^2$ via the matrix $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. The 1-dimensional subspace of vectors $\left(\begin{smallmatrix} * \\ 0 \end{smallmatrix}\right)$ is invariant, but there is no invariant complement. This does not happen for representations of *compact* Lie groups.

Proposition 26.9. *Let G be a compact Lie group, suppose G acts linearly on a vector space V, and let $V' \subset V$ be an invariant subspace. Then there exists an invariant complement $V'' \subset V$, i.e., $V = V' \oplus V''$ as a representation of G.*

Proof. The space $\Pi \subset \operatorname{End}(V)$ of linear projections $V \to V'$ is an affine space over $\operatorname{Hom}(V/V', V')$. The linear action of G on V induces an affine action of G on Π by conjugation. Let μ be the smooth bi-invariant measure on G with $\mu(G) = 1$. Choose any $\pi \in \Pi$. Then

$$\pi' = \int_G (g \circ \pi \circ g^{-1})\, \mu(g) \tag{26.10}$$

is a G-invariant element of Π. The kernel $V'' = \ker(\pi')$ is an invariant complement. □

(26.11) *Invariant metrics.* The same argument proves that every real representation of a compact Lie group is orthogonalizable, and every complex representation is unitarizable.

Proposition 26.12. *Let G be a compact Lie group and suppose G acts linearly on a real vector space V. Then there exists a G-invariant inner product on V.*

Proof. The space $\mathcal{M}$ of (positive definite) inner products on V is an open convex cone in the vector space $\operatorname{Sym}^2 V^*$, and the linear action of G on $\operatorname{Sym}^2 V^*$ preserves $\mathcal{M}$. Thus if $m \in \mathcal{M}$ is any element, for $g \in G$ the transform is $m^g(\xi, \eta) = m(g\xi, g\eta)$, $\xi, \eta \in G$. Then

$$m' = \int_G m^g\, \mu(g) \tag{26.13}$$

is a G-invariant inner product. □

(26.14) *Invariant symmetric bilinear forms.* A representation $\rho\colon G \to \operatorname{Aut}(V)$ on a vector space V is *irreducible* if V does not contain any G-invariant subspace.

Proposition 26.15. *Let G be a compact Lie group equipped with an irreducible representation on a real vector space V. Then the vector space of G-invariant symmetric bilinear forms on V is 1-dimensional.*

In other words, any two G-invariant symmetric bilinear forms on V are proportional.

Proof. By Proposition 26.12 we can and do choose a G-invariant inner product $\langle -, - \rangle$ on V. Now suppose B is a G-invariant symmetric bilinear form on V. Define $T \in \operatorname{End}(V)$ by

$$B(\xi, \eta) = \langle \xi, T(\eta) \rangle, \qquad \xi, \eta \in V. \tag{26.16}$$

Then T is self-adjoint, so it is diagonalizable, and it is also G-invariant. If T is not a multiple of the identity, then any eigenspace of T is a G-invariant subspace of V, which contradicts the irreducibility of the representation. □

Homogeneous Riemannian manifolds

(26.17) *Definition and examples.*

Definition 26.18. Let X be a Riemannian manifold. Then X is *homogeneous Riemannian* if for every pair of points $x, x' \in X$, there exists an isometry $f\colon X \to X$ with $f(x) = f(x')$.

It is a theorem of Myers–Steenrod that the group of all isometries of a Riemannian manifold carries a natural Lie group structure. For our purposes we assume given a Lie group G that acts transitively on X (on the left) by isometries. Furthermore, we assume that G acts *effectively*, i.e., for $g^{\neq e} \in G$ there exists $x \in X$ such that $gx \neq x$.

Example 26.19 (spheres)**.** The chart (16.13) tells various *compact* Lie groups G that act transitively on various spheres X. Similar to (26.10) we can average Riemannian metrics over a compact Lie group to construct an invariant Riemannian metric. These spheres are homogeneous Riemannian manifolds. Note that some spheres, such as S^7, have several different Lie groups listed that act transitively and isometrically. The G-invariant geometry differs depending on G. (The larger the G, the fewer invariants.)

Example 26.20 (projective spaces)**.** The real, complex, and quaternionic projective spaces $\mathbb{RP}^n$, $\mathbb{CP}^n$, and $\mathbb{HP}^n$ are each homogeneous Riemannian under a compact Lie group. So too is the octonionic (Cayley) plane $\mathbb{OP}^2 = \mathrm{F}_4 \,/\, \mathrm{Spin}_9$.

There are more exotic compact examples, such as $\mathrm{E}_8 \,/\, \mathrm{Spin}_{16}$, which has dimension 128. There are also many noncompact examples, such as hyperbolic space $\mathbb{H}^n$ (see (24.43)). A Lie group—compact or not—with a left-invariant metric is a homogeneous Riemannian manifold.

(26.21) *Reductivity.* For the rest of this lecture, let G be a Lie group that acts transitively and effectively by isometries on a *connected* Riemannian manifold X. Furthermore, fix $x \in X$ and let

$$H = \{h \in G : hx = x\} \tag{26.22}$$

be the isotropy group.

Lemma 26.23. *H is a compact Lie subgroup of G.*

Proof. Since (26.22) is a closed condition, $H \subset G$ is a *closed* subgroup of G, hence by Theorem 4.4 is a Lie subgroup. The main point is to prove compactness. Each $h \in H$ fixes x, and the differential of its action at x is a linear isometry of T_xX. In other words, the isotropy representation factors through the orthogonal group:

$$\rho\colon H \longrightarrow \mathrm{O}(T_xX). \tag{26.24}$$

Any $h \in \ker \rho$ fixes x and fixes each orthonormal frame at x, hence by Proposition 26.2 it acts as the identity on X. Since G acts effectively, we conclude $h = e$. In other words, ρ is injective, H is a subgroup of the compact Lie group $\mathrm{O}(T_xX)$, and therefore H is compact. $\square$

Corollary 26.25. *There exists an H-invariant decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}, \tag{26.26}$$

where $H \subset G$ acts on the Lie algebra by the restriction of the adjoint action of G on $\mathfrak{g}$.

This follows from Proposition 26.9. From now on we fix a choice of $\mathfrak{p}$. (It may not be unique.)

(26.27) *Symmetry.* The H-invariance of the decomposition (26.26) implies that

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &\subset \mathfrak{h} \\ [\mathfrak{h}, \mathfrak{p}] &\subset \mathfrak{p} \end{aligned} \tag{26.28}$$

Definition 26.29. The homogeneous Riemannian manifold $X \approx G/H$ is *locally symmetric* if

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}. \tag{26.30}$$

Briefly, the term 'local symmetry' means that about each $y \in X$ there exists a local isometry (an isometry on a neighborhood of y) that fixes y and has differential $-\operatorname{id}_{T_yX}$. At the basepoint $y = x$, this isometry is induced from the automorphism $\operatorname{id}_{\mathfrak{h}} \oplus -\operatorname{id}_{\mathfrak{p}}$ of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. The manifold is *symmetric* if each of these local symmetries extends to a global isometry. For example, the simply connected constant curvature Riemannian manifolds are symmetric. So too are projective spaces. A closed Riemann surface of constant curvature -1 is locally symmetric but not symmetric.

The Levi–Civita connection on a homogeneous Riemannian manifold

(26.31) *The canonical connection.* We recall **(16.17)**. The map

$$\begin{aligned} \pi\colon G &\longrightarrow X \\ g &\longmapsto gx \end{aligned} \tag{26.32}$$

is a principal H-bundle: $H \subset G$ acts simply transitively on the fibers of π by right multiplication. Left translation gives a trivialization

$$TG \cong \underline{\mathfrak{g}} \cong \underline{\mathfrak{h}} \oplus \underline{\mathfrak{p}}. \tag{26.33}$$

We have $\underline{\mathfrak{h}} = \ker \pi_*$ is the relative tangent bundle to π, the vertical tangent bundle of the principal bundle π. Since $\mathfrak{p}$ is H-invariant under the adjoint action, the distribution $\underline{\mathfrak{h}} \subset TG$ is H-invariant under right multiplication, hence defines a connection on π. This is the *canonical connection*. In particular, the differential of π induces a linear isomorphism

$$\mathfrak{p} \cong \underline{\mathfrak{p}}_e \xrightarrow{\pi_*} T_xX. \tag{26.34}$$

Use it to transport the inner product on T_xX to $\mathfrak{p}$, which we denote as $\langle -,-\rangle$.

More generally, elements of $\mathfrak{g}$ induce vector fields on X:

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathfrak{X}(X) \\ \xi &\longmapsto \widehat{\xi} \end{aligned} \tag{26.35}$$

The value of $\widehat{\xi}$ at $y \in X$ is $\frac{d}{dt}\big|_{t=0} (e^{t\xi}y)$. Setting $y = gx$ we see that the *right* invariant vector field $\widetilde{\xi}$ on G is π-related (recall Definition 2.65) to $\widehat{\xi}$. Note that at $g \in G$ we have

$$\widetilde{\xi}_g = \frac{d}{dt}\bigg|_{t=0} e^{t\xi}g = \frac{d}{dt}\bigg|_{t=0} g(g^{-1}e^{t\xi}g) = (\mathrm{Ad}_{g^{-1}}\,\xi)_g. \tag{26.36}$$

It follows that the map (26.35) is an antihomomorphism: $\widehat{[\xi,\eta]} = -[\widehat{\xi},\widehat{\eta}]$ for $\xi,\eta \in \mathfrak{g}$.

Recall also **(16.21)**, but in a variation using the Riemannian metric. Namely, use the isotropy homomorphism (26.24), together with the isomorphism (26.34), to construct a principal $\mathrm{O}(\mathfrak{p})$-bundle associated to π:

$$\varpi\colon G \times_H \mathrm{O}(\mathfrak{p}) \longrightarrow X. \tag{26.37}$$

If we fix an orthonormal frame at x, then ϖ is identified with the orthonormal frame bundle of X. Note that G acts on the total space of ϖ by left translation, and this action covers the action of G on X. Under the isomorphism with the orthonormal frame bundle, this is the usual lift (24.48) of isometries. Also, the bundle π is a subbundle of ϖ, since ρ in (26.24) is injective: there is an induced inclusion $G \hookrightarrow G \times_H \mathrm{O}(\mathfrak{p})$ that sends $g \in G$ to the equivalence class of $(g,e) \in G \times \mathrm{O}(\mathfrak{p})$. See Figure 75 for an illustration.

The canonical connection $\underline{\mathfrak{p}} \subset TG$ on π induces a connection on ϖ. In general, it is not the Levi–Civita connection since it is not in general torsionfree.

(26.38) *The Levi–Civita connection.* A connection on ϖ is *homogeneous* if it is G-invariant. The canonical connection is homogeneous, and any other homogeneous connection on ϖ has horizontal space at $(e,e) \in G \times_H \mathrm{O}(\mathfrak{p})$ given as the graph $\Gamma \subset \mathfrak{p} \oplus \mathfrak{o}(\mathfrak{p})$ of an H-invariant linear map

$$\Lambda\colon \mathfrak{p} \longrightarrow \mathfrak{o}(\mathfrak{p}). \tag{26.39}$$

Translate Γ by the left action of G and the right action of $\mathrm{O}(\mathfrak{p})$ to produce the corresponding homogeneous connection on ϖ. Our task is to compute Λ so that this is the Levi–Civita connection, and for that we compute the torsion.[48]

[48] In lecture I took a different route and computed a formula for the Levi–Civita covariant derivative: I used the 6-term formula (20.15) to derive an expression for $\nabla_{\widehat{\xi}}\widehat{\eta}(x)$, $\xi,\eta \in \mathfrak{p}$.

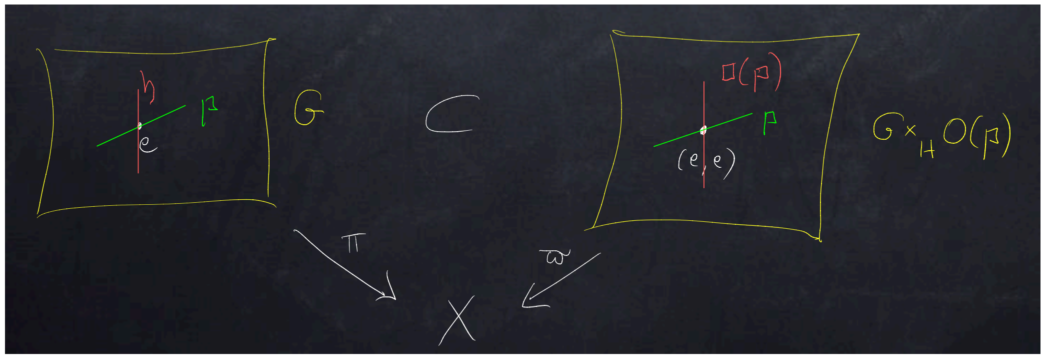

FIGURE 75. The canonical connection extends to an orthogonal linear connection

First, we compute the torsion of the canonical connection on $\pi\colon G\to X$. Use the decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{p}$ to decompose the Maurer–Cartan form $\theta\in\Omega^1_G(\mathfrak{g})$ as $\theta=\theta_{\mathfrak{h}}+\theta_{\mathfrak{p}}$, where $\theta_{\mathfrak{h}}\in\Omega^1_G(\mathfrak{h})$ and $\theta_{\mathfrak{p}}\in\Omega^1_G(\mathfrak{p})$. The form $\theta_{\mathfrak{h}}$ is the connection form of the canonical connection. Also, $\dot\rho(\theta_{\mathfrak{p}})$ is the restriction of the tautological 1-form[49] on $G\times_H \mathrm{O}(\mathfrak{p})$ to G. Now the Maurer–Cartan equation for G implies

$$0=d\theta+\frac12[\theta\wedge\theta]=\Bigl\{d\theta_{\mathfrak{h}}+\frac12[\theta_{\mathfrak{h}}\wedge\theta_{\mathfrak{h}}]+\frac12[\theta_{\mathfrak{p}}\wedge\theta_{\mathfrak{p}}]\Bigr\}\;+\;\Bigl\{d\theta_{\mathfrak{p}}+[\theta_{\mathfrak{h}}\wedge\theta_{\mathfrak{p}}]\Bigr\}. \tag{26.40}$$

The last expression is a decomposition into $\mathfrak{h}$- and $\mathfrak{p}$-components if X is locally symmetric. In general, we deduce from (26.40) that

$$d\theta_{\mathfrak{p}}+[\theta_{\mathfrak{h}}\wedge\theta_{\mathfrak{p}}]+\frac12[\theta_{\mathfrak{p}}\wedge\theta_{\mathfrak{p}}]_{\mathfrak{p}}=0, \tag{26.41}$$

since

$$d\theta_{\mathfrak{h}}+\frac12[\theta_{\mathfrak{h}},\theta_{\mathfrak{h}}]=0 \tag{26.42}$$

by the Maurer–Cartan equation of the subgroup H. (The restriction of $\theta_{\mathfrak{h}}$ to $H\subset G$ is the Maurer–Cartan form of H, and by left invariance (26.42) holds on all of G.) Now it follows from (26.41) that the torsion of the canonical connection is

$$\tau=-\frac12[\theta_{\mathfrak{p}}\wedge\theta_{\mathfrak{p}}]_{\mathfrak{p}}. \tag{26.43}$$

The right hand side is an H-equivariant element of $\Omega^2_G(\mathfrak{p})$, and by Remark 16.24 it descends to an element of $\Omega^2_X(TX)$. Note that for $\xi,\eta\in\mathfrak{p}$,

$$\tau(\xi,\eta)=-[\xi,\eta]_{\mathfrak{p}}. \tag{26.44}$$

[49] Here the tautological 1-form is $\mathfrak{p}$-valued rather than $\mathbb{R}^n$-valued, unless we fix a basis of $\mathfrak{p}$.

Now we take up the connection on ϖ given by the graph of Λ in (26.39). Write the connection form as $\dot\rho\theta_{\mathfrak{h}}+\alpha$ for $\alpha\colon \mathfrak{p}\to\mathfrak{o}(\mathfrak{p})$, and use $(\dot\rho\theta_{\mathfrak{h}}+\alpha)(\xi+\Lambda(\xi))=0,\ \xi\in\mathfrak{p}$, to deduce $\alpha=-\Lambda$. From Proposition 18.42 we see that the torsion of $\dot\rho\theta_{\mathfrak{h}}-\Lambda$ is

$$\tau(\Lambda)=-\frac{1}{2}[\theta_{\mathfrak{p}}\wedge\theta_{\mathfrak{p}}]_{\mathfrak{p}}-\Lambda\wedge\theta_{\mathfrak{p}}, \tag{26.45}$$

and so the condition $\tau(\Lambda)=0$ is the equation

$$-[\xi,\eta]_{\mathfrak{p}}-\Lambda(\xi)\eta-\Lambda(\eta)\xi=0,\qquad \xi,\eta\in\mathfrak{p}. \tag{26.46}$$

Since Λ maps to skew-symmetric endomorphisms of $\mathfrak{p}$,

$$\langle\Lambda(\xi)\eta,\zeta\rangle+\langle\eta,\Lambda(\xi)\zeta\rangle=0,\qquad \xi,\eta,\zeta\in\mathfrak{p}. \tag{26.47}$$

Use (26.46) and (26.47) to play the golden oldie (see (18.60) and (20.18)):

$$\begin{aligned}\langle\Lambda(\xi)\eta,\zeta\rangle&=-\langle\eta,\Lambda(\xi)\zeta\rangle\\&=-\langle\eta,[\xi,\zeta]_{\mathfrak{p}}\rangle-\langle\eta,\Lambda(\zeta)\xi\rangle\\&=-\langle\eta,[\xi,\zeta]_{\mathfrak{p}}\rangle+\langle\Lambda(\zeta)\eta,\xi\rangle\\&=-\langle\eta,[\xi,\zeta]_{\mathfrak{p}}\rangle+\langle[\zeta,\eta]_{\mathfrak{p}},\xi\rangle+\langle\Lambda(\eta)\zeta,\xi\rangle\\&=-\langle\eta,[\xi,\zeta]_{\mathfrak{p}}\rangle+\langle[\zeta,\eta]_{\mathfrak{p}},\xi\rangle-\langle\zeta,\Lambda(\eta)\xi\rangle\\&=-\langle\eta,[\xi,\zeta]_{\mathfrak{p}}\rangle+\langle[\zeta,\eta]_{\mathfrak{p}},\xi\rangle-\langle\zeta,[\eta,\xi]_{\mathfrak{p}}\rangle-\langle\zeta,\Lambda(\xi)\eta\rangle\end{aligned} \tag{26.48}$$

from which

$$2\langle\Lambda(\xi)\eta,\zeta\rangle=-\langle\eta,[\xi,\zeta]_{\mathfrak{p}}\rangle+\langle\xi,[\zeta,\eta]_{\mathfrak{p}}\rangle-\langle\zeta,[\eta,\xi]_{\mathfrak{p}}\rangle. \tag{26.49}$$

Introduce

$$\begin{aligned}\alpha\colon \mathfrak{p}&\longrightarrow \operatorname{End}(\mathfrak{p})\\ \xi&\longmapsto(\eta\mapsto[\xi,\eta]_{\mathfrak{p}})\end{aligned} \tag{26.50}$$

and let $\alpha^*_\xi\in\operatorname{End}(\mathfrak{p})$ be its adjoint with respect to $\langle-,-\rangle$. Then (26.49) is the following.

Proposition 26.51. *The Levi–Civita connection on the Riemannian homogeneous space X is the shift of the canonical connection by the linear map $\Lambda\colon\mathfrak{p}\to\mathfrak{o}(\mathfrak{p})$ defined as*

$$\Lambda(\xi)\eta=\frac{1}{2}\Big\{[\xi,\eta]_{\mathfrak{p}}-\alpha^*_\xi(\eta)-\alpha^*_\eta(\xi)\Big\},\qquad \xi,\eta\in\mathfrak{p}. \tag{26.52}$$

Remark 26.53. In the locally symmetric case $\Lambda=0$; the canonical connection is the Levi–Civita connection.

Curvature of Riemannian homogeneous manifolds

At this point we leave detailed computations to the reader and merely report some results.

(26.54) *Sectional curvature.* Set

$$U(\xi,\eta)=\frac{1}{2}\bigl(\alpha^*_\xi(\eta)+\alpha^*_\eta(\xi)\bigr),\qquad \xi,\eta\in\mathfrak{p}. \tag{26.55}$$

Proposition 26.56. *Let X be a homogeneous Riemannian manifold as above. Then for $\xi,\eta\in\mathfrak{p}\cong T_xX$ at the basepoint x we have*

$$\begin{aligned}\langle\xi,R(\xi,\eta)\eta\rangle=-\frac{3}{4}\|[\xi,\eta]_\mathfrak{p}\|^2-\frac{1}{2}\langle\bigl[\xi,[\xi,\eta]\bigr]_\mathfrak{p},\eta\rangle-\frac{1}{2}\langle\bigl[\eta,[\eta,\xi]\bigr]_\mathfrak{p},\xi\rangle\\ +\|U(\xi,\eta)\|^2-\langle U(\xi,\xi),U(\eta,\eta)\rangle.\end{aligned} \tag{26.57}$$

If X is locally symmetric, then

$$\langle\xi,R(\xi,\eta)\eta\rangle=-\langle\bigl[\xi,[\xi,\eta]\bigr],\eta\rangle. \tag{26.58}$$

These formulas at the basepoint give formulas on all of X via the transitive G-action.

(26.59) *Constant curvature manifolds.* We use Proposition 26.56 to verify that the supposed constant curvature manifold $X_n(\kappa)$ in **(24.40)** indeed has constant curvature $k\in\mathbb{R}$. To do so we apply the model in **(24.37)** for its global symmetry group $G_n(\kappa)$. The Lie algebra $\mathfrak{g}_n(\kappa)$ is decomposed in (24.38) as $\mathfrak{h}\oplus\mathfrak{p}$, and $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{h}$. From (24.39) we find for $\xi,\eta,\zeta\in\mathfrak{p}=V$ that

$$\bigl[[\xi,\eta],\zeta\bigr]=k\bigl[\langle\xi,\zeta\rangle\eta-\langle\eta,\zeta\rangle\xi\bigr], \tag{26.60}$$

and so (24.39) reduces to

$$\langle\xi,R(\xi,\eta)\eta\rangle=k\bigl[\langle\xi,\xi\rangle\langle\eta,\eta\rangle-\langle\xi,\eta\rangle^2\bigr]. \tag{26.61}$$

By (24.28) this is the constant curvature equation.

(26.62) *Homogeneous Einstein manifolds.* The formula for Ricci and scalar curvatures can be deduced from Proposition 26.56. We do not report these, nor do we investigate general homogeneous Einstein metrics. Instead, we observe the following.

Proposition 26.63. *Let X be a homogeneous Riemannian manifold as above, and suppose the isotropy representation $\rho\colon H\to \mathrm{O}(\mathfrak{p})$ is irreducible. Then X is Einstein.*

A Riemannian manifold whose isotropy representation is irreducible is termed *isotropy irreducible.*

Proof. The Ricci form at the basepoint is an H-invariant symmetric bilinear form on $\mathfrak{p}$, hence by Proposition 26.15 it is a multiple of the metric. ☐

Notice that the Riemannian metric on an isotropy irreducible homogeneous space is unique up to a positive scalar.

Example 26.64. The constant curvature manifolds $S^n, \mathbb{E}^n, \mathbb{H}^n$ are Einstein. So to are the projective spaces $\mathbb{CP}^n, \mathbb{HP}^n, \mathbb{OP}^2 = \mathrm{F}_4 / \mathrm{Spin}_9$. There are many more, such as $\mathrm{E}_8 / \mathrm{Spin}_{16}$.

Further topics

I am certain that *my* dual variables of time and energy are both low right now, so I will only indicate a few directions for *you* to explore.

(26.65) *Geodesic equation.* In our general setup we ask for a motion $g\colon (a,b) \to G$ that projects under π to a geodesic motion on X. You can work out that if

$$\eta(t) = [\dot{g}g^{-1}]_{\mathfrak{p}}(t), \tag{26.66}$$

then the geodesic equation is

$$\frac{d\eta}{dt} - \alpha^*_\eta(\eta) = 0. \tag{26.67}$$

(26.68) *Left-invariant metrics on Lie groups.* As remarked earlier, if G is a Lie group, then G is a homogeneous manifold under the action of G by left multiplication, and any inner product on the Lie algebra $\mathfrak{g}$ transports to a homogeneous metric on G. In this case $\alpha_\xi(\eta) = \mathrm{ad}_\xi(\eta)$, and so $\alpha^*_\xi = \mathrm{ad}^*_\xi$. The formulas for the Levi–Civita connection and curvature simplify. You can find that $g(t)$ is a geodesic motion iff $\xi := g^{-1}\dot{g}$ satisfies

$$\frac{d\xi}{dt} - \mathrm{ad}^*_\xi(\xi) = 0. \tag{26.69}$$

If the inner product on $\mathfrak{g}$ is invariant under the adjoint action of G, in which case the Riemannian metric on G is bi-invariant, then the formulæ simplify even further. In particular, 1-parameter groups are geodesics.

The analysis extends to invariant metrics on left G-torsors.

(26.70) *Rigid bodies.* In the 1960s, Vladimir Arnold made beautiful connections of this geometry to classical physics. Suppose A is an affine space over a vector space V, and we consider a rigid body moving in A. The moments of inertia of the rigid body form an inner product on V. The position and orientation of the rigid body in space is a point in $\mathcal{B}_{\mathrm{O}}(A)$, the manifold of orthonormal bases in the Euclidean space A. As in Remark 1.29(1), $\mathcal{B}_{\mathrm{O}}(A)$ is a left $\mathrm{Euc}(A)$-torsor, and it has an invariant metric. Geodesic motion in that metric is the motion of a rigid body.

More strikingly, Arnold observed that the Euler equation for an ideal fluid (incompressible, non-viscous) moving on a Riemannian manifold M—these are nonlinear PDE—can (formally) be expressed as geodesic motion for a certain metric on the infinite dimensional Lie group of volume-preserving diffeomorphisms of M.

Lecture 27: Complex and Kähler manifolds

In this final lecture of the course, we provide an introduction to (almost) complex manifolds, (almost) Hermitian manifolds, and Kähler manifolds.[50] Almost complex structures and almost Hermitian structures are geometric structures based on *symmetry types*, as introduced in **(1.34)**, **(1.36)**, **(1.53)**, and **(13.51)**. Our treatment emphasizes the underlying symmetry groups, which fit into a diagram (27.107). Each of these geometric structures has associated *integrability conditions*, uniformly expressed as the existence of a torsionfree connection; see **(17.57)**. Complex manifolds, Hermitian manifolds, and Kähler manifolds satisfy different variants of integrability.

We begin with background linear algebra. The category of complex vector spaces W and complex linear maps is equivalent to the category of real vector spaces $W_{\mathbb{R}}$ equipped with an automorphism I that squares to minus the identity and real linear maps that commute with I. The complexification of $W_{\mathbb{R}}$ is canonically the direct sum of two subspaces, and this leads to a decomposition of complex differential forms as well.

An almost complex manifold is a smooth manifold whose tangent spaces are equipped with such an automorphism I. The linear algebra applies pointwise and leads to a decomposition of differential forms and of the de Rham differential d. We are led to an integrability condition that takes several equivalent forms. An manifold with integrable almost complex structure has the structure of a *complex manifold*. We introduce complex manifolds and state this fundamental *Newlander–Nirenberg* Theorem 27.67. Its proof involves analysis of PDEs that we do not undertake here.

In almost complex geometry the analog of a Riemannian metric is a hermitian structure. We first recall the linear algebra of Hermitian inner products. An almost complex manifold whose tangent spaces have Hermitian inner products is an *almost Hermitian manifold*. There are two integrability conditions: one for the underlying almost complex structure and one for the Hermitian structure. In the latter case an integrable almost Hermitian manifold is called a *Kähler* manifold. The Kähler condition is naturally expressed as the torsionfree condition for a geometric structure. Calculus on Kähler manifolds is also very pleasant in terms of the frame bundle, as we briefly indicate.

We conclude with a beautiful geometric construction of the Chern connection due to Singer. In the case of a Kähler manifold, this leads to a "synthetic" construction of the Levi–Civita connection on a Kähler manifold: we direction construct the horizontal distribution. Recall that in general the Levi–Civita connection of a Riemannian manifold is produced one way or another by a formula: see (18.61) and (20.15). In the Kähler case, at least, we have a direct construction.

There is quite a bit of material in these notes that was not covered in lecture, and I would have like to tell much more...

Some linear algebra

There is *not* a canonical $i \in \mathbb{C}$ such that $i^2 = -1$: the Galois automorphism of $\mathbb{C}/\mathbb{R}$ exchanges the two possibilities. As is typical, we break that symmetry by fixing a choice of $i = \sqrt{-1}$.

[50] Ideally, we would have first provided an introduction to (almost) symplectic manifolds. Instead, we are reduced to a few footnotes on the subject.

(27.1) *Complexified complex vector spaces.* The real vector space $W_{\mathbb{R}}$ that underlies a complex vector space W is equipped with an endomorphism $I\colon W_{\mathbb{R}}\to W_{\mathbb{R}}$ that satisfies $I^2=-\mathrm{id}_{W_{\mathbb{R}}}$; it is scalar multiplication on W by i. There is an equivalence of categories between complex vector spaces W and pairs (V,I) of a real vector space and an endomorphism $I\colon V\to V$ such that $I^2=-\mathrm{id}_V$. Then $W\longleftrightarrow(W_{\mathbb{R}},I)$ under that equivalence, and the *complex conjugate* vector space $\overline{W}$ corresponds to the other choice of $\sqrt{-1}$: under the equivalence $\overline{W}\longleftrightarrow(W_{\mathbb{R}},-I)$.

The endomorphism I of $W_{\mathbb{R}}$ does not have any (real) eigenvectors, but it does diagonalize if we complexify:

$$W_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}\cong W'\oplus W'', \tag{27.2}$$

where W' is the $(+i)$-eigenspace of I with scalars extended to $\mathbb{C}$, and W'' is its $(-i)$-eigenspace. There are complex linear isomorphisms

$$\begin{aligned} W &\longrightarrow W' \\ \xi &\longmapsto \frac{\xi_{\mathbb{R}}-iI\xi_{\mathbb{R}}}{2} \end{aligned} \tag{27.3}$$

and

$$\begin{aligned} \overline{W} &\longrightarrow W'' \\ \bar{\xi} &\longmapsto \frac{\bar{\xi}_{\mathbb{R}}+iI\bar{\xi}_{\mathbb{R}}}{2} \end{aligned} \tag{27.4}$$

In these formulas $\xi_{\mathbb{R}}\in W_{\mathbb{R}}$ equals $\xi\in W$; the sets that underlie W and $W_{\mathbb{R}}$ are equal.

(27.5) *Determinants and orientations.* Under the equivalence $W\longleftrightarrow(W_{\mathbb{R}},I)$, a complex endomorphism $A\in\mathrm{End}_{\mathbb{C}}W$ has an underlying real endomorphism $A_{\mathbb{R}}\in\mathrm{End}_{\mathbb{R}}(W_{\mathbb{R}})$ that commutes with the complex structure: $A_{\mathbb{R}}\circ I=I\circ A_{\mathbb{R}}$.

Lemma 27.6. *Suppose W is a finite dimensional complex vector space and let $A\in\mathrm{End}\,W$. Then* $\det A_{\mathbb{R}}=|\det A|^2$.

Proof. The complexification of $A_{\mathbb{R}}$ preserves the decomposition (27.2), and under the isomorphisms (27.3) and (27.4) its restrictions to W',W'' are $A,\overline{A}$, respectively. It follows that

$$\det A_{\mathbb{R}}=(\det A)(\det\overline{A})=(\det A)\overline{(\det A)}=|\det A|^2. \tag{27.7}$$

□

Corollary 27.8. *For W a finite dimensional complex vector space there is a canonical isomorphism*

$$\mathrm{Det}\,W_{\mathbb{R}}\xrightarrow{\cong}\mathrm{Det}\,W\otimes\overline{\mathrm{Det}\,W}. \tag{27.9}$$

In particular, $W_{\mathbb{R}}$ has a canonical orientation.

For any nonzero $\omega \in \operatorname{Det} W$, the image of $\omega \otimes \bar{\omega}$ under the inverse of (27.9) is positively oriented. Observe that $\operatorname{Det} W \otimes \overline{\operatorname{Det} W}$ has a canonical real structure.

Remark 27.10. If $f_1, \dots, f_m$ is a basis of W, then $f_1 \wedge \cdots \wedge f_m \wedge \bar{f}_1 \wedge \cdots \wedge \bar{f}_m \in \operatorname{Det} W_{\mathbb{R}}$ is positively oriented in this convention. If we change to a new complex basis of W, then the corresponding element of $\operatorname{Det} W_{\mathbb{R}}$ multiplies by a positive number (Lemma 27.6).

(27.11) *Complex differential forms.* Recall that the exterior algebra of a direct sum of vector spaces is canonically the ($\mathbb{Z}$-graded) tensor product of their exterior algebras. (This is developed in detail in Lecture 21 of the Multivariable Analysis notes.) Then

$$\textstyle\bigwedge^k(W_{\mathbb{R}}^* \otimes_{\mathbb{R}} \mathbb{C}) \cong \bigwedge^k(W^* \oplus \overline{W}^*) \cong \bigoplus_{p+q=k} \bigwedge^p W^* \otimes \bigwedge^q \overline{W}^*. \tag{27.12}$$

Elements of $\bigwedge^p W^* \otimes \bigwedge^q \overline{W}^*$ are called *(p,q)-forms.*

Almost complex manifolds

We assume the dimension $n = 2m$ is an even integer.

(27.13) *The complex symmetry type.* The group $\mathrm{GL}_m\mathbb{C}$ of invertible $m \times m$ complex matrices is a subgroup of the group $\mathrm{GL}_n\mathbb{R}$ of invertible $m \times m$ real matrices: a complex automorphism of $W = \mathbb{C}^m$ induces a real automorphism of $W_{\mathbb{R}} = \mathbb{R}^n$. Define the complex symmetry type as the pair $(\mathrm{GL}_m\mathbb{C}, \mathrm{incl})$; see **(1.36)**. Recall Definition 13.52 of a general geometric structure.

Definition 27.14. An *almost complex manifold* X is a smooth manifold equipped with a $(\mathrm{GL}_m\mathbb{C}, \mathrm{incl})$-structure.

The almost complex structure is a reduction of the principal $\mathrm{GL}_n\mathbb{R}$-bundle of frames to a principal $\mathrm{GL}_m\mathbb{C}$-bundle:

$$\begin{tikzcd} \mathcal{B}'(X) \arrow[rr, hook] \arrow[dr, "\pi'", "\mathrm{GL}_m\mathbb{C}"'] & & \mathcal{B}(X) \arrow[dl, "\varpi"', "\mathrm{GL}_n\mathbb{R}"] \\ & X & \end{tikzcd} \tag{27.15}$$

Such reductions are in 1:1 correspondence with sections of the associated fiber bundle

$$\mathcal{B}(X)/\operatorname{GL}_m\mathbb{C} \longrightarrow X \tag{27.16}$$

with fiber the homogeneous space $\mathrm{GL}_n\mathbb{R}/\operatorname{GL}_m\mathbb{C}$. One can apply algebro-topological methods to address the existence and uniqueness of sections.

(27.17) *The complexified tangent bundle; an alternative definition.* The tangent bundle of an almost complex manifold is associated to the complex frame bundle $\pi'\colon \mathcal{B}'(X) \to X$ via the representation $\mathrm{Aut}(\mathbb{C}^m) = \mathrm{GL}_m\mathbb{C} \hookrightarrow \mathrm{GL}_n\mathbb{R} = \mathrm{Aut}(\mathbb{R}^n)$. Use the correspondence $\mathbb{C}^m \longleftrightarrow (\mathbb{R}^n, I_0)$ for $I_0 = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$ described in **(27.1)** to deduce the following.

Proposition 27.18. *An almost complex structure on a smooth manifold X is equivalent to an endomorphism $I \in \mathrm{End}(TX)$ such that $I^2 = -\mathrm{id}_{TX}$.*

The decomposition (27.2) induces a decomposition

$$TX \otimes_{\mathbb{R}} \mathbb{C} \cong T'X \oplus T''X \tag{27.19}$$

of the complexified tangent bundle to an almost complex manifold. Vectors in $T'X$ are of type $(1,0)$; vectors in $T''X$ are of type $(0,1)$. The isomorphisms (27.3) and (27.4) induce isomorphisms

$$TX \xrightarrow{\cong} T'X \tag{27.20}$$

$$\overline{TX} \xrightarrow{\cong} T''X \tag{27.21}$$

of *complex* vector bundles.

Remark 27.22. Sections of $T'X \to X$ and sections of $T''X \to X$ are *complex* vector fields. They are not tangent to integral curves, for example. However, we can identify sections with *real* vector fields using the isomorphisms (27.20) and (27.21).

(27.23) *Orientation.* Corollary 27.8 implies that an almost complex manifold carries a canonical orientation. Another way to see that is as follows. For any smooth n-dimensional manifold X, the orientation double cover is the fiber bundle associated to the action of $\mathrm{GL}_n\mathbb{R}$ on $\{\pm 1\}$ via multiplication by the sign of the determinant. The restriction of that action to $\mathrm{GL}_m\mathbb{C}$ is trivial by Lemma 27.6. The fixed point $+1 \in \{\pm 1\}$ induces a global section of the orientation double cover.

(27.24) *Examples and non-examples.* Obviously, an odd dimensional manifold does not admit an almost complex structure, nor does a nonorientable manifold. Here are a few less trivial remarks.

Example 27.25. An orientable 2-manifold admits an almost complex structure. Indeed, the structure group of an oriented 2-manifold is $\mathrm{GL}_2^+\mathbb{R}$, and it deformation retracts onto $\mathrm{GL}_1\mathbb{C} \subset \mathrm{GL}_2^+\mathbb{R}$. Hence in that case the fibers of (27.16) are contractible, so it admits a section.

Example 27.26. The 4-sphere $X = S^4$ does not admit an almost complex structure. We can argue for this using characteristic classes: if there is an almost complex structure, then $c_2(T'X) = e(X)$ is the nonzero Euler class of the 4-sphere, but also $c_2(T'X) = -p_1(X)$ vanishes.

Example 27.27. The 6-sphere does admit an almost complex structure. In fact, S^6 is a homogeneous space for the exceptional Lie group G_2 with stabilizer group SU_3. The group $\mathrm{SU}_3 \subset \mathrm{GL}_6\mathbb{R}$ preserves the standard complex structure on $\mathbb{R}^6$, and this induces a homogeneous almost complex structure on S^6.

(27.28) *Complex differential forms.* Let X be an almost complex manifold. The decomposition (27.12) induces a decomposition

$$\Omega_X^k(\mathbb{C}) \cong \bigoplus_{p+q=k} \Omega_X^{p,q} \tag{27.29}$$

of complex differential forms on X. An element of $\Omega_X^{p,q}$ is a complex differential form of *type* (p,q).

From the direct sum decomposition we obtain projections $\pi^{p,q}\colon \Omega_X^k(\mathbb{C}) \to \Omega_X^{p,q}$, and so first order differential operators

$$\begin{aligned} \partial\colon \Omega_X^{p,q} &\hookrightarrow \Omega_X^k(\mathbb{C}) \xrightarrow{\;d\;} \Omega_X^{k+1}(\mathbb{C}) \xrightarrow{\;\pi^{p+1,q}\;} \Omega_X^{p+1,q} \\ \overline{\partial}\colon \Omega_X^{p,q} &\hookrightarrow \Omega_X^k(\mathbb{C}) \xrightarrow{\;d\;} \Omega_X^{k+1}(\mathbb{C}) \xrightarrow{\;\pi^{p,q+1}\;} \Omega_X^{p,q+1} \end{aligned} \tag{27.30}$$

The map $\overline{\partial}$ is the *d-bar operator*. Evidently $d = \partial + \overline{\partial}$ on $\Omega_X^1(\mathbb{C})$. However, that need not be the case on $\Omega_X^k(\mathbb{C}),\ k \geq 2$. Namely the red arrows $\pi^{0,2}\circ d$ and $\pi^{2,0}\circ d$ in the diagram

$$\begin{tikzcd}
 & & \Omega_X^{2,0} \arrow[r, "\partial"] \arrow[dr, "\overline{\partial}"'] & \cdots \\
 & \Omega_X^{1,0} \arrow[ur, "\partial"] \arrow[dr, "\overline{\partial}"'] \arrow[ddr, red] & \oplus & \\
\Omega_X^{0,0} \arrow[ur, "\partial"] \arrow[dr, "\overline{\partial}"'] & \oplus & \Omega_X^{1,1} \arrow[ur, "\partial"] \arrow[dr, "\overline{\partial}"'] & \cdots \\
 & \Omega_X^{0,1} \arrow[ur, "\partial"] \arrow[dr, "\overline{\partial}"'] \arrow[uur, red] & \oplus & \\
 & & \Omega_X^{0,2} \arrow[ur, "\partial"] \arrow[r, "\overline{\partial}"'] & \cdots
\end{tikzcd} \tag{27.31}$$

could be nonzero. They are complex conjugate, so either both are nonzero or both vanish. Vanishing is equivalent to a torsionfree condition on the almost complex structure, as we see below.

(27.32) *The Nijenhaus tensor.* The subbundle $T'X \subset TX \otimes_{\mathbb{R}} \mathbb{C}$ of tangent vectors of type $(1,0)$ is a *complex* distribution on an almost complex manifold X. Just as complex tangent vectors do not have integral curves, complex distributions do not have integral manifolds. However, the notion of integrability still makes sense. Namely, the Frobenius tensor (Definition 5.18) is defined. In this case, the Frobenius tensor is a skew-symmetric map

$$\phi_{T'X}\colon T'X \times T'X \longrightarrow (TX \otimes_{\mathbb{R}} \mathbb{C})/T'X \cong T''X, \tag{27.33}$$

which under the isomorphisms (27.20) and (27.21) is identified with a real tensor.

Definition 27.34. Let X be an almost complex manifold. The *Nijenhaus tensor* $N \in \Omega_X^2(TX)$ is the real form of the Frobenius tensor (27.33).

This prescription leads to the formula

$$N(\xi,\eta)=[\xi,\eta]-[I\xi,I\eta]+I[I\xi,\eta]+I[\xi,I\eta],\qquad \xi,\eta\in TX. \tag{27.35}$$

Lemma 27.36. *The map*

$$\Omega^{0,1}_X\hookrightarrow\Omega^1_X(\mathbb{C})\xrightarrow{\;d\;}\Omega^2_X(\mathbb{C})\xrightarrow{\;\pi^{2,0}\;}\Omega^{2,0}_X \tag{27.37}$$

vanishes iff $N=0$.

Proof. For $\bar\omega\in\Omega^{0,1}_X$ and ξ,η vector fields of type $(1,0)$ we have

$$d\bar\omega(\xi,\eta)=\xi\bar\omega(\eta)-\eta\bar\omega(\xi)-\bar\omega([\xi,\eta])=-\bar\omega([\xi,\eta]) \tag{27.38}$$

and the $(0,1)$-component of $[\xi,\eta]$ vanishes for all ξ,η iff $N=0$. ☐

Example 27.39. The Nijenhaus tensor of the almost complex structure (Example 27.27) on $S^6=\mathrm{GL}_6\mathbb{R}/\mathrm{SU}_3$ is nonzero. Whether S^6 admits a *torsionfree* almost complex structure is an open problem.

(27.40) *The torsionfree condition.* Recall from Definition 17.59 that the almost complex structure on X is torsionfree iff the complex frame bundle $\mathcal{B}'(X)\to X$ admits a torsionfree connection.

Proposition 27.41. *An almost complex structure on X is torsionfree iff its Nijenhaus tensor N vanishes.*

Proof. If Θ' is a connection on $\mathcal{B}'(X)\to X$, and ∇ is the associated covariant derivative on $TX\to X$, then $\nabla I=0$. (Conversely, a covariant derivative on the tangent bundle is induced from a connection on the complex frame bundle iff the almost complex structure I is parallel.) Now, if Θ' is torsionfree, then so too is ∇ and hence

$$[\xi,\eta]=\nabla_\xi\eta-\nabla_\eta\xi,\qquad \xi,\eta\in\mathfrak{X}(X). \tag{27.42}$$

Since ∇ preserves the decomposition (27.19) of $TX\otimes_{\mathbb{R}}\mathbb{C}$, if we take ξ,η complex of type $(1,0)$ in (27.42), then the Lie bracket $[\xi,\eta]$ is also of type $(1,0)$. Therefore, $N=0$.

Conversely, suppose $N=0$. About any point of X there exists an open neighborhood U and a coframing $\vartheta^1,\dots,\vartheta^m\in\Omega^{1,0}_U$ by *complex* 1-forms[51] of type $(1,0)$. By (the complex conjugate of) Lemma 27.36, their differentials do not have any component of type $(0,2)$. Thus

$$d\vartheta^i=\frac12A^i_{jk}\vartheta^j\wedge\vartheta^k\;+\;A^i_{j\bar k}\vartheta^j\wedge\vartheta^{\bar k},\qquad A^i_{jk}=-A^i_{kj} \tag{27.43}$$

[51] A real coframing is then given by $(\vartheta^1+\overline{\vartheta}^1)/2,\dots,(\vartheta^m+\overline{\vartheta}^m)/2,-i(\vartheta^1-\overline{\vartheta}^1)/2,\dots,-i(\vartheta^m-\overline{\vartheta}^m)/2$

for complex functions $A^i_{jk}, A^i_{j\bar k} \in \Omega^0_U(\mathbb{C})$. Define the complex 1-form

$$(\Theta')^i_j = \frac{1}{2} A^i_{jk}\vartheta^k \;+\; A^i_{j\bar k}\overline{\vartheta}^k. \tag{27.44}$$

Then the connection form

$$\Theta' = \begin{pmatrix} (\Theta')^i_j & \\ & (\overline{\Theta}')^i_j \end{pmatrix} \in \Omega^1_U\bigl(\mathrm{End}(T'X) \oplus \mathrm{End}(T''X)\bigr) \tag{27.45}$$

is real, preserves the almost complex structure I, and is torsionfree:

$$\begin{aligned} d\vartheta^i + (\Theta')^i_j \wedge \vartheta^j &= 0 \\ d\overline{\vartheta}^i + (\overline{\Theta}')^i_j \wedge \overline{\vartheta}^j &= 0 \end{aligned} \tag{27.46}$$

Each of these conditions is affine in Θ', hence averaging using a partition of unity constructs a global torsionfree connection on $\pi' \colon \mathcal{B}'(X) \to X$. $\square$

(27.47) *Calculus on torsionfree almost complex manifolds.* Let X be an almost complex manifold with vanishing Nijenhaus tensor, and choose a torsionfree connection Θ' on $\pi' \colon \mathcal{B}'(X) \to X$. Then $\mathcal{B}'(X)$ has horizontal complex vector fields

$$\begin{aligned} &\partial'_1, \dots, \partial'_m \\ &\partial''_1, \dots, \partial''_m \end{aligned} \tag{27.48}$$

that are complex versions of the real vector fields in (16.40). Note that $\partial''_k = \overline{\partial'_k}$. The corresponding real vector fields, restricted to $\mathcal{B}'(X) \subset \mathcal{B}(X)$, are

$$(\partial'_1 + \partial''_1)/2, \dots, (\partial'_m + \partial''_m)/2, -i(\partial'_1 - \partial''_1)/2, \dots, -i(\partial'_m - \partial''_m)/2. \tag{27.49}$$

Let

$$\begin{aligned} &f_1, \dots, f_m \\ &f^1, \dots, f^m \end{aligned} \tag{27.50}$$

denote the standard dual bases of $\mathbb{C}^m$ and $(\mathbb{C}^m)^*$. The differential operators $\partial, \overline{\partial}$ in (27.30) on X lift to operators

$$\begin{aligned} \partial &= \epsilon(f^k)\partial'_k \\ \overline{\partial} &= \bar\epsilon(\bar f^k)\partial''_k \end{aligned} \tag{27.51}$$

that act on lifts of complex differential forms to $\mathrm{GL}_m\mathbb{C}$-equivariant functions

$$\mathcal{B}'(X) \longrightarrow {\textstyle\bigwedge}^p(\mathbb{C}^m)^* \otimes {\textstyle\bigwedge}^q\overline{(\mathbb{C}^m)^*}. \tag{27.52}$$

Recall that $\epsilon(f)$ is exterior multiplication by the complex 1-form f.

Proposition 27.53. *Let X be a torsionfree almost complex manifold. Then*

$$d = \partial + \overline{\partial}. \tag{27.54}$$

We have already proved this on 1-forms, but not in general.

Proof. This follows from Proposition 17.51. □

Complex manifolds

(27.55) *Holomorphic functions.* Let A be an affine space over a *complex* vector space V, let B be an affine space over a complex vector space W, and suppose $U \subset A$ is an open set.

Definition 27.56. A smooth function $f\colon U \to B$ is *holomorphic* if its differential $df_x\colon V \to W$ is complex linear for all $x \in U$.

This condition on the first differential is a first order PDE, the system of *Cauchy–Riemann equations*. For a complex-valued function f on an open subset of standard finite dimensional affine space $\mathbb{A}^m_{\mathbb{C}}$ with coordinates $z^1, \dots, z^m$, the Cauchy–Riemann equations are

$$\frac{\partial f}{\partial \bar{z}^1} = \cdots = \frac{\partial f}{\partial \bar{z}^m} = 0. \tag{27.57}$$

(27.58) *Complex manifolds.* A smooth manifold is pasted together from open subsets of real affine space by smooth functions. Similarly, a complex manifold is pasted together from open subsets of complex affine space by holomorphic functions.

Definition 27.59. Let X be a topological manifold. Suppose $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X, and for each $\alpha \in A$ suppose we are given a continuous map $x_\alpha\colon U_\alpha \to A_\alpha$ into a finite dimensional complex affine space A_α such that x_α is a homeomorphism onto its image. Then if for each $\alpha, \beta \in A$ the map

$$x_\alpha(U_\alpha \cap U_\beta) \longrightarrow x_\beta(U_\alpha \cap U_\beta) \hookrightarrow A_\beta \tag{27.60}$$

is holomorphic, we say that this data gives X the structure of a *complex manifold*.

The definition is completely parallel to that of a smooth structure on a topological manifold.[52]

Remark 27.61. Since holomorphic maps are smooth, a complex manifold X has a canonical real atlas and so a canonical smooth manifold structure. Denote[53] the underlying smooth manifold as $X_{\mathbb{R}}$.

[52] A smooth structure is usually taken to be a *maximal* atlas. Definition 27.59 is the definition of a complex atlas, and we should take a maximal atlas to define a complex manifold.

[53] This is appropriate for an introduction, but with experience one does not use a separate notation.

The basic theory develops in parallel to that of smooth manifolds. For example, at each point $x \in X$ of a complex manifold there is a complex tangent space T_xX, or equivalently a complex structure on the real tangent space $T_xX_{\mathbb{R}}$. In other words, $X_{\mathbb{R}}$ has a canonical almost complex structure.

The following could be a theorem or a definition.

Definition 27.62. Let X, X' be complex manifolds. A smooth map $f\colon X'_{\mathbb{R}} \to X_{\mathbb{R}}$ is *holomorphic* if its differential is complex linear at each point of X'.

A *standard chart* on a complex manifold takes values in the standard complex affine space $\mathbb{A}^m_{\mathbb{C}}$. The coordinates $z^1, \dots, z^m$ are holomorphic functions on an open set $U \subset X$, and

$$\begin{aligned} &dz^1, \dots, dz^m && \in \Omega^{1,0}_U \\ &d\bar{z}^1, \dots, d\bar{z}^m && \in \Omega^{0,1}_U \\ &\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^m} && \in \Gamma(T'U) \\ &\frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^m} && \in \Gamma(T''U) \end{aligned} \tag{27.63}$$

are pointwise bases. The dz^i are holomorphic 1-forms, and the $\partial/\partial z^i$ are holomorphic vector fields.

Proposition 27.64. *The Nijenhaus tensor of a complex manifold vanishes.*

Hence by Proposition 27.41 the almost complex structure on a complex manifold is torsionfree.

Proof. We must show that the Lie bracket of vector fields of type $(1,0)$ is a vector field of type $(1,0)$, and it suffices to do so locally. In a standard local chart on an open set $U \subset X$ we verify

$$\left[\xi^i \frac{\partial}{\partial z^i}, \eta^j \frac{\partial}{\partial z^j}\right] = \left\{\xi^j \frac{\partial \eta^i}{\partial z^j} - \eta^j \frac{\partial \xi^i}{\partial z^j}\right\} \frac{\partial}{\partial z^i}, \qquad \xi^i, \eta^i \in \Omega^0_U(\mathbb{C}). \tag{27.65}$$

□

(27.66) *The Newlander–Nirenberg theorem.* The following basic theorem is a converse to Proposition 27.64.

Theorem 27.67 (Newlander–Nirenberg)**.** *A torsionfree almost complex manifold admits a unique compatible complex structure.*

Remark 27.68. Thus if X is a torsionfree almost complex manifold, the theorem asserts the existence of local holomorphic complex-valued functions, and what's more about any point a collection $z^1, \dots, z^m$ of local holomorphic functions whose differentials are linearly independent.

Summarizing Proposition 27.64 and Theorem 27.67, there is an equivalence between the category of torsionfree almost complex manifolds and the category of complex manifolds.

(27.69) *Examples.*

Example 27.70. Let W be a finite dimensional complex vector space. Then the Grassmannian $\mathrm{Gr}_k(W)$ is a complex manifold. In particular, the projective space $\mathbb{P}(W)$ is a complex manifold. For $W = \mathbb{C}^{m+1}$ we obtain the standard projective space $\mathbb{CP}^m$.

Example 27.71. A finite set of homogeneous polynomials in $m+1$ variables cuts out a complex submanifold $X \in \mathbb{CP}^m$ if they are transverse. Chow's Theorem asserts that a complex submanifold of $\mathbb{CP}^m$ is algebraic, i.e., it is cut out this way. This basic result explains why algebraic techniques are so powerful in complex geometry.

Example 27.72. A *complex Lie group* is a complex manifold equipped with a group structure such that the group operators are holomorphic. (This is the holomorphic version of Definition 3.6.) A prime example is the complex general linear group $\mathrm{GL}_m\mathbb{C}$ of $m \times m$ invertible complex matrices. The unitary group U_m is a closed Lie subgroup of $\mathrm{GL}_m\mathbb{C}$. There are complex versions of the orthogonal and symplectic groups as well.

The next two examples use more of the theory of Lie groups than we have discussed in this course, but they are worth mentioning nonetheless.

Example 27.73. Let G be a connected compact Lie group and $T \subset G$ a maximal torus. Then the homogeneous manifold G/T admits G-invariant complex structures.

Example 27.74. Let G be an even dimensional compact Lie group. Then G admits a left invariant complex structure. The proof of this assertion uses a bit of structure theory. Namely, under the adjoint action restricted to a maximal torus $T \subset G$, the Lie algebra of G decomposes as

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\bar{\alpha}} \mathfrak{g}_{\bar{\alpha}}, \tag{27.75}$$

where $\mathfrak{t}$ is the Lie algebra of T (on which T acts trivially) and each $\mathfrak{g}_{\bar{\alpha}}$ is a real 2-dimensional irreducible representation of T. (The sum is finite and runs over the set of roots up to sign.) Each representation $\mathfrak{g}_{\bar{\alpha}}$ admits two opposite T-invariant complex structures (which correspond to a choice of positive root in $\bar{\alpha} = \{\pm\alpha\}$), and the even-dimensional real vector space $\dim \mathfrak{t}$ also admits a T-invariant complex structure since T acts trivially. The vanishing of the Nijenhaus tensor is a pleasant exercise in the structure theory.

Example 27.76. As a special case of Example 27.74, the 4-dimensional Lie group $G = \mathrm{U}_2$ has a double cover whose total space is diffeomorphic to $S^1 \times S^3$, and a complex structure on U_2 lifts to a complex structure on the double cover. In fact, the product of any two odd-dimensional spheres admits a complex structure. (It is even true that the product of a circle and some exotic odd-dimensional spheres admits a complex structure.)

We remark that the complex manifolds in Example 27.74 and Example 27.76 do not admit Kähler metrics. (We discuss Kähler metrics later in this lecture.)

(27.77) *The complex frame bundle of a complex manifold.*

Definition 27.78. Let X be a complex manifold and let G be a complex Lie group. Then a *holomorphic principal bundle* $\pi\colon P \to X$ is a complex manifold P equipped with a free right holomorphic action of G such that π is a quotient map and π is holomorphic.

Let X be a complex manifold of complex dimension m. As in (27.15) it has a principal $\mathrm{GL}_m\mathbb{C}$-bundle of frames

$$\pi'\colon \mathcal{B}'(X) \longrightarrow X \tag{27.79}$$

A point in the fiber $\mathcal{B}'(X)_x$ over $x \in X$ is a complex basis $b' = (f_1, \dots, f_m)$ of the complex tangent space $T'_x X$.

Proposition 27.80. *There is a complex manifold structure on $\mathcal{B}'(X)$ such that π' is a holomorphic principal $\mathrm{GL}_m\mathbb{C}$-bundle.*

Proof. Let $U \subset X$ be a coordinate patch with local holomorphic coordinates $z^1, \dots, z^m$. Define functions $P^i_j\colon (\pi')^{-1}(U) \to \mathbb{C}$ by

$$f_j\bigg|_x = P^i_j(b')\frac{\partial}{\partial z^i}\bigg|_x, \qquad b' = (f_1, \dots, f_m) \in \mathcal{B}'(X)_x, \quad x \in U. \tag{27.81}$$

The smooth functions $\{(\pi')^* z^i, P^i_j\}$ are smooth coordinates on $\pi^{-1}(U)$, and we declare them to be holomorphic. We leave the reader to check that overlaps under change of coordinates on X are holomorphic functions. It is straightforward to verify in these local coordinates that π' is a holomorphic map and the $\mathrm{GL}_m\mathbb{C}$-action is holomorphic. $\square$

(27.82) *The tautological holomorphic 1-forms.* Analogous to the tautological 1-forms (16.34) on the real frame bundle of a smooth manifold, there are complex 1-forms

$$\vartheta^1, \dots, \vartheta^m \in \Omega^{1,0}_{\mathcal{B}'(X)}. \tag{27.83}$$

With respect to the local holomorphic coordinates just introduced, we have dual to (27.81) that

$$\vartheta^i = (P^{-1})^i_j dz^j, \tag{27.84}$$

and since the coefficients $(P^{-1})^i_j$ are rational functions of the holomorphic functions P^i_j, it follows that

$$\bar\partial\vartheta^i = 0. \tag{27.85}$$

In other words, ϑ^i is a holomorphic 1-form.

(27.86) *Connections on a holomorphic principal bundle.* Let $\pi\colon P \to X$ be a holomorphic principal bundle with structure group a complex Lie group G, as in Definition 27.78. Over P we have a short exact sequence

$$0 \longrightarrow T(P/X) \longrightarrow TP \xrightarrow{\pi_*} \pi^*TX \longrightarrow 0 \tag{27.87}$$

of *holomorphic* vector bundles equipped with a holomorphic G-action. Recall that a connection is a G-equivariant splitting of (27.87), so a G-equivariant direct sum decomposition $TP = H \oplus T(P/X)$.

Definition 27.88. Let H be a connection on $\pi\colon P \to X$.

(1) We say H *is compatible with the complex structure* if $I_P(H) = H$.
(2) We say H *is a holomorphic connection* if $H \to P$ is a holomorphic vector bundle.

Holomorphic connections may not exist; a famous early paper of Atiyah introduces the *Atiyah class*, which obstructs their existence. On the other hand, connections compatible with the complex structure do exist.

Proposition 27.89. *If H is compatible with the complex structure, then its curvature form Ω has type* $(1,1)+(2,0)$.

In other words, $\Omega^{0,2} = 0$.

Proof. Let $\Theta \in \Omega^1_P(\mathfrak{g})$ be the connection form of a connection H on $\pi\colon P \to X$. Since $\mathfrak{g}$ is complex, there is a decomposition

$$\Omega^1_P(\mathfrak{g}) = \Omega^{1,0}_P(\mathfrak{g}) \oplus \Omega^{0,1}_P(\mathfrak{g}). \tag{27.90}$$

A connection Θ is compatible with the complex structure iff $\Theta \in \Omega^{1,0}_P(\mathfrak{g})$. Then by inspection the curvature

$$\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta] \tag{27.91}$$

has type $(1,1)+(2,0)$; the first term has this mixed type and the second has type $(2,0)$. $\square$

More linear algebra

As preparation for our introduction to Hermitian and Kähler geometry, we discuss inner products on a complex vector space.

(27.92) *Hermitian inner products.*

Definition 27.93. Let W be a complex vector space. A *Hermitian inner product* on W is a bilinear form

$$h = \langle -, - \rangle \colon \overline{W} \times W \longrightarrow \mathbb{C} \tag{27.94}$$

which for all $\xi, \eta \in W$ satisfies

$$\langle \bar{\xi}, \eta \rangle = \overline{\langle \bar{\eta}, \xi \rangle} \tag{27.95}$$

$$\langle \bar{\xi}, \xi \rangle > 0 \qquad \text{if } \xi \neq 0 \tag{27.96}$$

Condition (27.95) is the Hermitian symmetry; condition (27.96) is positive definiteness.

(27.97) *The induced real inner product and symplectic form.* Let h be a Hermitian inner product on a complex vector space W, and decompose h into its real and imaginary parts:

$$h = g + i\omega. \tag{27.98}$$

Then g is a real inner product on the underlying real vector space $W_{\mathbb{R}}$, and ω is a *symplectic form* on $W_{\mathbb{R}}$, that is, a nondegenerate skew-symmetric bilinear form. Recall also that $W_{\mathbb{R}}$ carries a complex structure I. These three pieces of structure are related by the equation

$$\omega(\xi, \eta) = g(I\xi, \eta), \qquad \xi, \eta \in W_{\mathbb{R}}. \tag{27.99}$$

This equation implies the following.

Proposition 27.100. *Any two of I, ω, g determine the third.*

Remark 27.101. Proposition 27.100 is too terse: there are compatibility conditions among I, ω, g. For example, $I \colon W_{\mathbb{R}} \to W_{\mathbb{R}}$ preserves both ω and g. On the other hand, given ω and g we can define I using (27.99), and then I is skew-symmetric, but it may not square to $-\mathrm{id}_{W_{\mathbb{R}}}$, so we need to add that assumption as a compatibility condition between ω and g.

(27.102) *Standard model.* On the model finite dimensional space $W = \mathbb{C}^m$ with standard basis $f_1, \dots, f_m$ and dual basis $f^1, \dots, f^m$, define the standard Hermitian inner product

$$h = \sum_{i=1}^{m} \bar{f}^i \otimes f^i. \tag{27.103}$$

The basis $f_1, \dots, f_m$ is *unitary* in the sense that

$$h(\bar{f}_i, f_j) = \delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases} \tag{27.104}$$

The form h induces the standard inner product and symplectic form on $W_{\mathbb{R}} = \mathbb{R}^n$, where $n = 2m$.

(27.105) *Symmetry groups and symmetry types.* We arrange the data in the diagram

$$\begin{tikzcd} & I \\ (h,I) \arrow[ur] \arrow[r] \arrow[dr] & \omega \\ & g \end{tikzcd} \tag{27.106}$$

The symmetry groups that preserve the data fit into a diagram of inclusions:

$$\begin{tikzcd} & \mathrm{GL}_m\mathbb{C} \arrow[dr, hook] & \\ \mathrm{U}_m \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & \mathrm{Sp}_n\mathbb{R} \arrow[r, hook] & \mathrm{GL}_n\mathbb{R} \\ & \mathrm{O}_n \arrow[ur, hook] & \end{tikzcd} \tag{27.107}$$

Each of the groups in (27.107) induces a symmetry type via its inclusion into $\mathrm{GL}_n\mathbb{R}$. In the remainder of the lecture we focus on the symmetry type $(\mathrm{U}_m, \mathrm{incl})$.

(Almost) Hermitian and Kähler manifolds

(27.108) *Definitions.* There are several complex analogs of a real Riemannian manifold.

Definition 27.109. Let X be a smooth manifold.

(1) An *almost Hermitian* structure is a $(\mathrm{U}_m, \mathrm{incl})$-structure.
(2) It is a *Hermitian* structure if the underlying almost complex manifold is torsionfree.
(3) It is a *Kähler* structure if the $(\mathrm{U}_m, \mathrm{incl})$-structure is torsionfree.

Equivalently, an almost Hermitian structure is a pair (I, h) of an almost complex structure I and Hermitian metric h on the resulting complex tangent bundle. By Theorem 27.67, a Hermitian manifold is a complex manifold equipped with a Hermitian metric. It is Kähler iff there exists a torsionfree covariant derivative ∇ on $TX \to X$ such that

$$\nabla I = \nabla h = 0. \tag{27.110}$$

Notice from the diagram (27.106) that an almost Hermitian structure on a smooth manifold X induces simultaneously an almost complex structure, an almost symplectic structure, and a Riemannian structure. Any Riemannian structure is torsionfree. The almost Hermitian structure is Hermitian iff the induced almost complex structure is torsionfree.

(27.111) *The frame bundles.* Let X be an almost Hermitian manifold. Then corresponding to the diagram (27.107) of symmetry groups is a diagram of principal bundles over X, whose total spaces fit into the diagram of inclusions

$$\begin{tikzcd} & \mathcal{B}'(X) \arrow[dr, hook] & \\ \mathcal{B}_{\mathrm{U}}(X) \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & \mathcal{B}_{\mathrm{Sp}}(X) \arrow[r, hook] & \mathcal{B}(X) \\ & \mathcal{B}_{\mathrm{O}}(X) \arrow[ur, hook] & \end{tikzcd} \tag{27.112}$$

Recall that if $G \to G'$ is a homomorphism of Lie groups, then a connection on a principal G-bundle induces a connection on the associated G'-bundle.

Proposition 27.113. *If X is Kähler, then $\mathcal{B}_{\mathrm{U}}(X) \to X$ admits a unique torsionfree connection Θ, and it induces the Levi–Civita connection Θ_{LC} on $\mathcal{B}_{\mathrm{O}}(X) \to X$.*

We call Θ the *Kähler connection.*

Proof. Any torsionfree Θ induces Θ_{LC}, since Θ_{LC} is the unique torsionfree connection on $\mathcal{B}_{\mathrm{O}}(X) \to X$. But then Θ is the restriction of Θ_{LC} to $\mathcal{B}_{\mathrm{U}}(X) \subset \mathcal{B}_{\mathrm{O}}(X)$. This characterizes it uniquely. $\square$

Remark 27.114. On a general almost Hermitian manifold, the restriction of Θ_{LC} to $\mathcal{B}_{\mathrm{U}}(X)$ is not valued in the unitary algebra $\mathfrak{u}_m \subset \mathfrak{o}_n$, so is not a connection on $\mathcal{B}_{\mathrm{U}}(X) \to X$.

Proposition 27.115. *If X is Kähler, then the almost complex structure $\mathcal{B}'(X) \to X$ is torsionfree.*

Proof. A torsionfree connection on $\mathcal{B}_{\mathrm{U}}(X) \to X$ induces a torsionfree connection on the (almost) complex frame bundle $\mathcal{B}'(X) \to X$. $\square$

(27.116) *The Chern connection.* The following theorem—due to Chern and Nakano—holds much more generally, for holomorphic principal G-bundles equipped with a reduction to a principal K-bundle, where $K \subset G$ is the maximal compact subgroup. We write here for frame bundles and $\mathrm{U}_m \subset \mathrm{GL}_m\mathbb{C}$, but the proof—due to Singer—is the same in the general case.

Recall that a Hermitian manifold X is complex, and that on a complex manifold X the complex frame bundle $\mathcal{B}'(X) \to X$ is holomorphic.

Proposition 27.117. *Suppose that X is a Hermitian manifold. Then there exists a unique connection Θ on $\mathcal{B}_{\mathrm{U}}(X) \to X$ such that the induced connection Θ' on $\mathcal{B}'(X) \to X$ is compatible with the complex structure.*

In other words, the horizontal distribution on $\mathcal{B}'(X)$ is invariant under the complex structure I. The unique connection in Proposition 27.117 is called the *Chern connection.*

Proof. We explicitly construct the horizontal distribution H on $\mathcal{B}_{\mathrm{U}}(X)$. Fix $b \in \mathcal{B}_{\mathrm{U}}(X)$, let $W_b = T_b\mathcal{B}_{\mathrm{U}}(X)$, and let $V_b = T_b\mathcal{B}'(X)$. Set

$$H_b = W_b \cap I_b(W_b), \tag{27.118}$$

where $I_b \colon V \to V$ is the complex structure on $\mathcal{B}'(X)$. Then $H_b \subset V_b$ is a complex subspace—it is evidently stable under I_b—and it is U_m-invariant since I is $\mathrm{GL}_m\mathbb{C}$-invariant on $\mathcal{B}'(X)$. Finally, we claim that $H_b \subset W_b$ is horizontal. First, restrict (27.118) to the vertical tangent space to $\mathcal{B}'(X)$ at b: the right hand side is then $\mathfrak{u}_m \cap \sqrt{-1}\mathfrak{u}_m \subset \mathfrak{gl}_m\mathbb{C}$. But the only complex matrix that is simultaneously skew-Hermitian and Hermitian is the zero matrix. Hence H_b does not contain any nonzero vertical vectors. Since $W_b + I_b(W_b) = V_b$ we compute

$$\dim H_b = 2\dim W_b - \dim V_b = 2(2m+m^2) - (2m+2m^2) = 2m, \tag{27.119}$$

which equals the dimension of the base X. It follows that H_b is horizontal. $\square$

Proposition 27.120. *The curvature of the Chern connection has type* $(1,1)$*.*

Proof. By Proposition 27.89 the curvature Ω on $\mathcal{B}'(X)$ has type $(2,0)+(1,1)$. The restriction of Ω to $\mathcal{B}_{\mathrm{U}}(X)$ takes values in skew-Hermitian matrices, so $\Omega + {}^t\overline{\Omega} = 0$. It follows that the $(2,0)$-component of Ω vanishes, since ${}^t\overline{\Omega}$ has type $(0,2)+(1,1)$. $\square$

(27.121) *The Chern connection and the Kähler condition.*

Proposition 27.122. *Let X be a Hermitian manifold. Then X is Kähler iff the Chern connection Θ' on $\mathcal{B}'(X) \to X$ is torsionfree. In that case Θ' restricts to the Kähler connection on $\mathcal{B}_{\mathrm{U}}(X) \to X$.*

Proof. The reverse implication is the definition of a Kähler manifold. Suppose then that X is Kähler and Θ is the Kähler connection on $\mathcal{B}_{\mathrm{U}}(X) \to X$. Recall the tautological *holomorphic* $\mathbb{C}^m$-valued 1-form $\vartheta = \vartheta^i f_i$ on $\mathcal{B}'(X)$; see **(27.82)**. Since Θ is torsionfree, we have

$$d\vartheta + \Theta \wedge \vartheta = 0. \tag{27.123}$$

Since $d\vartheta$ has type $(2,0)$ by (27.85), it follows that Θ has type $(1,0)$. Hence Θ induces the Chern connection on $\mathcal{B}'(X) \to X$, and therefore the Chern connection is torsionfree. $\square$

(27.124) *A synthetic construction of the Kähler connection.* In summary, if X is a Kähler manifold then its Levi–Civita connection agrees with the Chern connection: they both induce the same connection on $\mathcal{B}(X) \to X$. (It helps to refer to the chart (27.112).) The "synthetic construction" (27.118) of the Chern connection—a simple geometric prescription—is then a synthetic construction of the Levi–Civita connection.

(27.125) *Calculus on a Kähler manifold.* The Kähler connection gives horizontal vector fields (27.48) on $\mathcal{B}_{\mathrm{U}}(X)$ as well as expressions (27.51) for the differential operators $\partial, \overline{\partial}$. Now in the Kähler case, on the manifold $\mathcal{B}_{\mathrm{U}}(X)$, there are also adjoints $\partial^*, \overline{\partial}^*$, and these too have nice expressions in terms of contraction with a vector field:

$$\begin{aligned} \partial^* &= -\sum_k \iota(f_k)\partial'_k \\ \overline{\partial}^* &= -\sum_k \iota(\bar{f}_k)\partial''_k \end{aligned} \tag{27.126}$$

This is the starting point for proving Hodge identities and carrying out other computations in Kähler geometry.

Problem Set # 1

Math230a: Differential Geometry

Due: September 11

Please form working groups of 3 ± 1.5 (round up) people to go over lectures and tackle problem sets. The problem sets are a crucial part of the class, and I strongly suggest that you engage with them. You can also use Discord to discuss problems, but please do not spoil others' joy by giving solutions; hints are better. The problems are as much about the journey as the destination. Those getting a grade in the course (this includes all undergraduates) should regularly hand in solutions. Do take advantage of office hours and problem sessions.

I tend to give *many* problems. They are at different levels and some are open-ended. Do as many as you can do well. I'd rather have you do fewer problems better than have you do more problems worse. Don't expect to turn in solutions to every problem every week. Again, better to turn in fewer well-written solutions than many poorly written ones.

The notes on Multivariable Analysis, Lectures 1 and 2, have material about affine space that you may want to read through. If you need reviews of exterior algebra and differential forms, they appear later in those notes and also in the notes on Differential Topology.

It is also a good idea to look over the material on ODE in Lectures 17 and 18 of the Multivariable Analysis notes. There is a global version in the middle of Lecture 14 of the Differential Topology notes. That material is good preparation for Lie derivatives, which we cover in the second lecture of the course.

For Problem 4 you may want to review Proposition 19.16 in the Differential Topology notes.

This problem set is a bit heavy on foundational problems, which is natural at the beginning to give you some review of techniques we will use right away in the course. As an antidote, a bit of classical geometry is here too. Enjoy!

1. Prove the following theorem of Menelaus. Let A be an affine space and p_0, p_1, p_2 independent points, i.e. points whose affine span is 2-dimensional. Let q_0 be a point on the line spanned by p_1, p_2; q_1 be a point on the line spanned by p_2, p_0; and q_2 be a point on the line spanned by p_0, p_1. Then q_0, q_1, q_2 are collinear if and only if
$$\frac{q_2p_0}{q_2p_1}\,\frac{q_0p_1}{q_0p_2}\,\frac{q_1p_2}{q_1p_0} = 1.$$
(Each ratio is a real number: what is the definition? No Euclidean structure is necessary, but if one is present can you define it in terms of distances? Can the ratio be 0 or ∞? Is the statement true if so?)

2. Let A, B be affine spaces over vector spaces V, W, respectively, and let $f\colon A \to B$ be a function. Do not assume finite dimensionality.

   (a) Proof or counterexample: f is affine iff for all $p_0, p_1 \in A$ and $\lambda^0, \lambda^1 \in \mathbb{R}$ with $\lambda^0 + \lambda^1 = 1$,
$$f(\lambda^i p_i) = \lambda^i f(p_i).$$

Here and always we use the summation convention: sum over indices which appear once as a subscript and once as a superscript (in an expression, i.e., on one side of an equation).

(b) Suppose that for every finite subset $S \subset A$ the *center of mass* (define!) of $f(S)$ is the image of the center of mass of S. Does it follow that f is affine?

(c) Define what it means for f to be (affine) quadratic.

(d) Define precisely an equivalence relation on quadratic functions $\mathbb{A}^2 \to \mathbb{R}$ by, roughly, considering two to be equivalent if they are related by a "change of coordinates". How many equivalence classes are there? For each equivalence class of quadratics $q\colon \mathbb{A}^2 \to \mathbb{R}$ describe the shape of $q^{-1}(0)$.

3. This problem gives practice with the index notation and summation convention that I use and strongly recommend to you. Note carefully the placement (superscript vs. subscript) of the indices in what follows. The actual name of the index (i or j or α) is arbitrary, though as always a judicious choice of notation helps you and your readers.

Let V be an n dimensional (real) vector space. Suppose $\{e_j\}$ and $\{f_i\}$ are two bases for V which are related by the equation

$$e_j = P^i_j f_i,$$

where P is an invertible matrix. In the matrix $P = \left(P^i_j\right)$, i is the row index and j the column index.

(a) Suppose $\xi \in V$ is a vector. Then we can find real numbers ξ^j and $\tilde{\xi}^i$ such that $\xi = \xi^j e_j = \tilde{\xi}^i f_i$. Express $\tilde{\xi}^i$ in terms of the ξ^j.

(b) Suppose $T\colon V \to V$ is a linear transformation. Relative to the basis $\{e_j\}$ it is expressed as the matrix A defined by $Te_j = A^i_j e_i$, and relative to the basis $\{f_i\}$ it is expressed as the matrix B defined by $Tf_i = B^j_i f_j$. What is the relationship between A and B?

(c) The *dual space* V^* is the vector space of all linear functionals $V \to \mathbb{R}$; it is also n dimensional. Every basis of V gives rise to a *dual basis* of V^*. For the basis $\{e_j\}$ of V the dual basis $\{e^i\}$ of V^* is defined by the equation

$$e^i(e_j) = \delta^i_j = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

(This equation defines the symbol δ^i_j.) The dual basis $\{f^j\}$ is defined similarly. Express f^j in terms of the e^i.

(d) Suppose $\omega \in V^*$. Then we define its components relative to the basis $\{e^i\}$ by the equation $\omega = \omega_i e^i$ and its components relative to the basis $\{f^j\}$ by the equation $\omega = \tilde{\omega}_j f^j$. Express the ω_i in terms of the $\tilde{\omega}_j$.

(e) The tensor and exterior powers of V have natural induced bases. For example, what is the basis for $\bigotimes^2 V^* = V^* \otimes V^*$, the space of bilinear forms on V? What about $\bigwedge^2 V^*$, the space of alternating bilinear forms? What about $\mathrm{Sym}^2 V^*$, the space of symmetric bilinear forms?

4. Let X be a smooth manifold. Recall that a *vector field* is a section of the tangent bundle $TX \to X$.
   (a) Suppose $\xi\colon C^\infty(X) \to C^\infty(X)$ is a *derivation* on the algebra of smooth functions on X, i.e., ξ is a linear map and
   $$\xi(fg) = (\xi f)g + f(\xi g), \qquad f, g \in C^\infty(X).$$
   Prove that ξ defines a unique vector field whose directional derivative on functions is the given derivation. (Define these terms carefully. Henceforth we identify vector fields and the associated derivation.)
   (b) Define the *Lie bracket* $[\xi_1, \xi_2]$ of vector fields ξ_1, ξ_2 on X as the commutator of derivations:
   $$[\xi_1, \xi_2]f = \xi_1\xi_2 f - \xi_2\xi_1 f.$$
   Prove that $[\xi_1, \xi_2]$ is a derivation, hence is a vector field.
   (c) The vector space of vector fields on X is often denoted $\mathfrak{X}(X)$. For $\xi_1, \xi_2, \xi_3 \in \mathfrak{X}(X)$ prove
   $$\begin{aligned} [\xi_1, \xi_2] + [\xi_2, \xi_1] &= 0 \\ [\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]] &= 0 \end{aligned}$$
   These equations express that $\big(\mathfrak{X}(X), [-,-]\big)$ is a *Lie algebra*.

5. Let X be a smooth manifold and $\Omega^\bullet(X)$ the $\mathbb{Z}$-graded commutative algebra of differential forms. Let ξ, ξ_1, ξ_2 be vector fields on X. In differential calculus we have three basic (graded) derivations on $\Omega^\bullet(X)$: $d, \mathcal{L}_\xi, \iota_\xi$, where d is the *exterior derivative* of degree $+1$, $\mathcal{L}_\xi$ is the *Lie derivative* of degree 0, and ι_ξ is the *contraction* of degree -1. We define the Lie derivative $\mathcal{L}_\xi$ in Lecture 2. In case I don't get to it, I indicate the definition of contraction in part (a). Note that d and $\mathcal{L}_\xi$ are first-order differential operators, whereas ι_ξ is algebraic. In this problem you will compute the commutators of these operations. These are the basic equations of differential calculus.
   (a) Let V be a vector space. For $\xi \in V$ let $\iota_\xi\colon V^* \to \mathbb{R}$ be the linear map which evaluates a functional $\alpha \in V^*$ on the vector ξ. Prove that there is a unique extension $\iota_\xi\colon \bigwedge^\bullet V^* \to \bigwedge^\bullet V^*$ of degree -1 which satisfies the derivation (Leibniz) formula
   $$\iota_\xi(\omega_1 \wedge \omega_2) = \iota_\xi\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \iota_\xi\omega_2, \qquad \omega_1 \in {\textstyle\bigwedge^k} V^*, \quad \omega_2 \in {\textstyle\bigwedge^\bullet} V^*.$$
   (What are the corresponding derivation formulæ for d and $\mathcal{L}_\xi$?)
   (b) Verify $[d, \iota_\xi] = \mathcal{L}_\xi$. This is the Cartan formula for the Lie derivative on differential forms.
   (c) Show $[d, \mathcal{L}_\xi] = 0$. What does this formula express about the operator d?
   (d) Note $[d, d] = 0$ is one of the defining properties of d. What is $[\iota_{\xi_1}, \iota_{\xi_2}]$?
   (e) Check $[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{[\xi_1, \xi_2]}$.
   (f) Compute the remaining commutator $[\mathcal{L}_{\xi_1}, \iota_{\xi_2}] = \iota_{[\xi_1, \xi_2]}$.
   (g) Let α be a 1-form. Expand $d\alpha(\xi_1, \xi_2) = \iota_{\xi_1}\iota_{\xi_2} d\alpha$ using the above formulas. The only differentiation in the final answer should be directional derivatives of functions and Lie brackets of vector fields. Try the next case: $d\beta$ for a 2-form β.

6. (a) Prove the statement that follows Definition 1.52 in the notes: For the symmetry type $O_n \hookrightarrow GL_n\mathbb{R}$ and a vector space V, the data $(\mathcal{B}_G(V), \theta)$ in the definition is equivalent to a choice of positive definite inner product on V.

   (b) What is the symmetry type of a complex structure on an $n = 2m$ dimensional vector space? Show that a vector space with this structure, in the sense of Definition 1.52, is a complex vector space. State and prove a converse as well.

   (c) What is the symmetry type of an n-dimensional vector space equipped with a codimension one subspace? What about the same for vector spaces equipped with an inner product?

   (d) Consider the following two 2-dimensional symmetry types. The first is $SO_2 \times \mu\!\!\!\mu_2 \to GL_2\mathbb{R}$, where $\mu\!\!\!\mu_2 = \{\pm 1\}$ is the cyclic group of order two and the map is projection onto the first factor followed by inclusion. The second is $\mathrm{Spin}_2 \to GL_2\mathbb{R}$, which is the composition of the double cover map $\mathrm{Spin}_2 \to SO_2$ followed by the inclusion $SO_2 \hookrightarrow GL_2\mathbb{R}$. (Both Spin_2 and SO_2 are isomorphic to the circle group $\mathbb{T}$ of unit norm complex numbers; the double cover is the map $\lambda \mapsto \lambda^2$, $\lambda \in \mathbb{T}$.) The internal symmetry group in each case is $\mu\!\!\!\mu_2$. Describe as best you can the geometry of a 2-dimensional affine space equipped with each geometric structure.

Problem Set # 2

Math230a: Differential Geometry

Due: September 18

Throughout we use '$\mathfrak{g}$' to denote the Lie algebra of a Lie group G.

1. Let V be a finite dimensional real vector space and $B\colon V \times V \to \mathbb{R}$ a nondegenerate bilinear form. Define

$$\mathrm{Aut}_B(V) = \{P \in \mathrm{Aut}(V) : B(P\xi_1, P\xi_2) = B(\xi_1, \xi_2) \text{ for all } \xi_1, \xi_2 \in V\}$$

and

$$\mathrm{End}_B(V) = \{A \in \mathrm{End}(V) : B(A\xi_1, \xi_2) + B(\xi_1, A\xi_2) = 0 \text{ for all } \xi_1, \xi_2 \in V\}.$$

(a) Prove that $\mathrm{Aut}_B(V)$ is a Lie group with Lie algebra $\mathrm{End}_B(V)$.

(b) Let $V = \mathbb{R}^n$ for some $n \in \mathbb{Z}^{>0}$. Suppose B is the standard (symmetric) inner product. Identify $\mathrm{Aut}_B(\mathbb{R}^n)$ with the group O_n of orthogonal matrices.

(c) Let $V = \mathbb{R}^{2m}$ for some $m \in \mathbb{Z}^{>0}$. Suppose B is a nondegenerate skew-symmetric form: for the standard basis $e_1, \dots, e_{2m}$ of $\mathbb{R}^{2m}$, set

$$B(e_i, e_{i+m}) = -B(e_{i+m}, e_i) = 1, \qquad i \in \{1, \dots, m\},$$

and set $B(e_i, e_j) = 0$ otherwise. Identify the group $\mathrm{Aut}_B(\mathbb{R}^{2m})$ explicitly in terms of block 2×2 matrices in which the blocks have size $m \times m$. This is the *symplectic group* $\mathrm{Sp}_{2m}\mathbb{R}$.

2. Let $\alpha\colon G' \to G$ be a homomorphism of Lie groups.

(a) Prove that the differential α_* induces a linear map from left invariant vector fields on G' to left invariant vector fields on G. Furthermore, prove that this linear map is a Lie algebra homomorphism, which we denote $\dot\alpha\colon \mathfrak{g}' \to \mathfrak{g}$.

(b) Prove that the diagram

$$\begin{tikzcd} \mathfrak{g}' \arrow[r, "\dot\alpha"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G' \arrow[r, "\alpha"] & G \end{tikzcd}$$

commutes.

3. Let G be a Lie group, let X be a smooth manifold, and suppose G acts on X. There is an induced linear map $\mathfrak{g} \to \mathfrak{X}(X)$ which maps $\xi \in \mathfrak{g}$ to the vector field $\hat{\xi}$ on X.

   (a) If G acts on the *right*, show that for all $\xi_1, \xi_2 \in \mathfrak{g}$,

   $$[\hat{\xi}_1, \hat{\xi}_2] = \widehat{[\xi_1, \xi_2]}.$$

   (b) If G acts on the *left*, show that for all $\xi_1, \xi_2 \in \mathfrak{g}$,

   $$[\hat{\xi}_1, \hat{\xi}_2] = -\widehat{[\xi_1, \xi_2]}.$$

4. Answer for each of the following Lie groups G:

   A finite dimensional real vector space V

   $$\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$$

   $$\mathrm{SU}_2 = \left\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C},\ AA^* = I,\ \det A = 1\right\}$$

   $$\mathrm{SL}_2\mathbb{R}$$

   $$\mathrm{GL}_n\mathbb{R}$$

   $$\mathrm{O}_n$$

   (a) Is G abelian?

   (b) Is G compact?

   (c) Is G connected?

   (d) Is G simply connected?

   (e) What is the Lie algebra of G? (Identify in concrete terms, for example as a Lie algebra of matrices.)

5. Let G be a Lie group.

   (a) As a preliminary, let V be a finite dimensional real vector space. Define a real line $|\operatorname{Det} V|$ such that an ordered n-tuple $\xi_1, \dots \xi_n \in V$ define an element $|\xi_1 \wedge \cdots \wedge \xi_n| \in |\operatorname{Det} V|$, and elements for two ordered bases are related by the absolute value of the determinant of the change of basis matrix. Identify $|\operatorname{Det} V^*|$ as a certain space of functions $V^{\times n} \to \mathbb{R}$. Show that positive functions determine an orientation of $|\operatorname{Det} V^*|$. Interpret a positive function as a notion of volume for n-dimensional parallelepipeds in V. Does this induce a notion of volume for lower dimensional parallelepipeds? Identify positive elements as translationally invariant positive measures on V. Construct such a positive element from an inner product on V.

   (b) Apply to the tangent bundle of a smooth manifold. Define the notion of a smooth positive measure on a smooth manifold. Do they always exist?

(c) The real line $|\operatorname{Det}\mathfrak{g}^*|$ consists of left-invariant measures on G. Define an action of G on this line. Compute the action in case G is compact. Compute it for $G = \mathrm{GL}_n\mathbb{R}$ and $G = \mathrm{SL}_n\mathbb{R}$.

(d) A *Haar measure* on G is a bi-invariant positive smooth measure on G. Prove that a Haar measure exists if G is compact. Normalize it so the total volume of G is 1.

(e) Write a formula for Haar measure on the circle group $\mathbb{T} \subset \mathbb{C}$; the formula should be in terms of $\lambda \in \mathbb{T}$. On the multiplicative group $\mathbb{R}^\times$ of nonzero real numbers. On the additive group $\mathbb{R}$ of real numbers. On the orthogonal group O_2.

6. Suppose G is a connected compact Lie group.

(a) Let $\Omega^\bullet_{\mathrm{linv}}(G) \subset \Omega^\bullet(G)$ denote the vector subspace of left-invariant differential forms. Show that $\Omega^\bullet_{\mathrm{linv}}(G)$ is in fact a sub-differential graded algebra, i.e., it is closed under multiplication and the differential d.

(b) Construct an isomorphism

$$\textstyle\bigwedge^\bullet \mathfrak{g}^* \to \Omega^\bullet_{\mathrm{linv}}(G).$$

Transfer the differential on $\Omega^\bullet_{\mathrm{linv}}(G)$ to $\bigwedge^\bullet \mathfrak{g}^*$ and write a formula for it. In this way you obtain a differential graded complex defined directly from the Lie algebra $\mathfrak{g}$. Observe that your definition works for *any* Lie algebra (it needn't be the Lie algebra of a compact Lie group).

(c) Prove that the inclusion in part (a) induces an isomorphism on cohomology. A map of cochain complexes with this property is called a *quasi-isomorphism*. So you can compute the de Rham cohomology of G from this Lie algebra complex. (Hint: Average over G using Haar measure to construct a left-invariant form from an arbitrary form.)

(d) Use the inverse map $g \mapsto g^{-1}$ to show that the differential of a *bi-invariant* differential form vanishes. Show that the de Rham cohomology of G is isomorphic to the algebra of bi-invariant forms.

(e) Use these ideas to compute $H^\bullet_{\mathrm{dR}}(SU_2)$.

(f) (for those who know some Riemannian geometry) Endow G with a bi-invariant metric. Is there a relationship between harmonic forms and bi-invariant forms?

Problem Set # 3

Math230a: Differential Geometry

Due: September 25

1. (a) Let M be a 3-manifold and α a nonzero 1-form. Prove that the 2-dimensional distribution determined by α is integrable if and only if $\alpha \wedge d\alpha = 0$.

   (b) The Hopf fibration $\pi\colon S^3 \to S^2$ may be constructed by identifying S^3 as the unit sphere in $\mathbb{C}^2$ and S^2 as $\mathbb{CP}^1$; then the map is $\pi(z^1, z^2) = [z^1, z^2]$, where $(z^1)^2 + (z^2)^2 = 1$ and $[z^1, z^2]$ is the equivalence class in the projective line. The kernel of the differential π_* is an (integrable) one-dimensional distribution on S^3. Let $E \subset TS^3$ be the 2-dimensional distribution whose fiber at $p \in S^3$ is the orthogonal complement of $\ker \pi_*$ relative to the standard round metric on S^3. Is E integrable? Find a nonzero 1-form α which generates the ideal $\mathcal{I}(E)$ associated to E. Compute $d\alpha$ and $\alpha \wedge d\alpha$.

2. Suppose M is a smooth manifold and $E \subset TM$ a distribution. Define

$$\mathcal{I}(E) = \{\omega \in \Omega^\bullet_M : \omega\big|_\Delta = 0\}.$$

   (a) Prove that $\mathcal{I}(E) \subset \Omega^\bullet_M$ is an ideal.

   (b) Prove that if E has corank r—that is, if $\dim E_m + r = \dim_m M$ for all $m \in M$—then E is locally generated by r independent 1-forms.

   (c) Prove that $\mathcal{I}(E)$ is closed under d if and only if E is integrable.

   (d) Consider the distribution E on $\mathbb{A}^3_{x,y,z}$ spanned by the vector fields $\partial/\partial x$ and $x\,\partial/\partial y + \partial/\partial z$. Show that E is not integrable. Show that any point $(x, y, z) \in \mathbb{A}^3$ may be joined to $(0, 0, 0)$ by a piecewise smooth curve whose tangent line belongs to E.

3. Example or proof of nonexistence: A codimension 1 foliation on the sphere S^4.

4. (a) Let $P, Q\colon \mathbb{A}^2 \to \mathbb{R}$ be smooth functions. Define the 2-dimensional distribution E on $\mathbb{A}^2_{x,y} \times \mathbb{R}_z$ with

$$E_{(x,y,z)} = \operatorname{span}\left\{\frac{\partial}{\partial x} + P\frac{\partial}{\partial z},\ \frac{\partial}{\partial y} + Q\frac{\partial}{\partial z}\right\}.$$

   Compute the Frobenius tensor of E.

(b) Suppose X is a manifold and G a Lie group. Let θ^i, $i = 1, \dots, n$ be a basis of left-invariant 1-forms on G and suppose

$$d\theta^i + \frac{1}{2}c^i_{jk}\theta^j \wedge \theta^k = 0$$

for constants c^i_{jk}. Let θ^i_X, $i = 1, \dots, n$ be 1-forms on X. Consider the ideal of differential forms on $X \times G$ generated by $\pi_2^*\theta^i - \pi_1^*\theta^i_X$, where $\pi_1 \colon X \times G \to X$ and $\pi_2 \colon X \times G \to G$ are projections. Prove that this ideal is closed under d if and only if

$$d\theta^i_X + \frac{1}{2}c^i_{jk}\theta^j_X \wedge \theta^k_X = 0$$

(c) Compute the Frobenius tensor of the distribution in (b) defined as the simultaneous kernel of the 1-forms $\pi_2^*\theta^i - \pi_1^*\theta^i_X$.

5. This is a collection of exercises on the Maurer-Cartan form.

(a) Let G be a Lie group with Maurer-Cartan form θ. Compute $R_g^*\theta$ for $g \in G$. Do this first for a matrix group, where you can write $\theta = g^{-1}dg$ for $g \colon G \to M_n\mathbb{R}$ the natural matrix-valued function on a matrix group. ($M_n\mathbb{R}$ is a vector space, so the differential of the function g is defined as a $M_n\mathbb{R}$-valued 1-form.)

(b) Let G be a Lie group and suppose T is a *right* G-torsor. Show that the Maurer-Cartan form on G transports to a canonical element of $\Omega^1_T(\mathfrak{g})$. Can you give a prose definition of this Maurer-Cartan 1-form on the torsor: "its value at a point $t \in T$ on the vector $\zeta \in T_tT$ is..."? What is the Maurer-Cartan equation? What is the pullback of the Maurer-Cartan 1-form by an element of G acting on T?

(c) Let V be an n-dimensional real vector space and $\mathcal{B}(V)$ the right $\mathrm{GL}_n(\mathbb{R})$-torsor of bases. (Recall that a basis is an isomorphism $b \colon \mathbb{R}^n \to V$.) Let Θ^i_j be the Maurer-Cartan forms in the standard basis of the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$. Suppose $b(t)$ is a smooth curve in $\mathcal{B}(V)$. Write the basis $b(t)$ as $\{e_1(t), \dots, e_n(t)\}$ and the dual basis as $\{e^1(t), \dots, e^n(t)\}$. Prove that

$$\Theta^i_j(\dot{b}) = \langle e^i(0), \dot{e}_j(0)\rangle.$$

Heuristically, then, Θ^i_j measures the instantaneous motion of e_j in the direction of e_i, where 'direction' is determined by the entire basis $e_1, \dots, e_n$. This interpretation is important!

(d) Let A be an n-dimensional real affine space and $\mathcal{B}(A)$ the right $\mathrm{Aff}_n(\mathbb{R})$-torsor of bases of the underlying vector space at all points of A. So a point of $\mathcal{B}(A)$ is an affine isomorphism $\mathbb{A}^n \to A$. Let θ^i, Θ^i_j be the Maurer-Cartan forms in the standard basis of the Lie algebra of $\mathrm{Aff}_n(\mathbb{R})$. (Define this basis: the single index is for infinitesimal translations, and the double index for infinitesimal linear transformations, as in (c).) Suppose $b(t)$ is a smooth curve in $\mathcal{B}(A)$ which projects to the curve $x(t)$ in A, and write the underlying basis of V as in (c). Prove that

$$\theta^i(\dot{b}) = \langle e^i(0), \dot{x}(0)\rangle.$$

Interpret this in prose terms.

Problem Set # 4

Math230a: Differential Geometry

Due: October 2

1. Let X be a smooth manifold and $f\colon Y \hookrightarrow X$ an injective immersion. Suppose Z is a smooth manifold, $g\colon Z \to X$ is a smooth map, and assume that g factors through f: there exists a map (of sets) $h\colon Z \to Y$ such that $g = f \circ h$. Prove that h is smooth iff h is continuous.

2. Let $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ denote the division algebra of quaternions. (Recall $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$ and real numbers commute with i, j, k.) Let $G \subset \mathbb{H}$ denote the subset of quaternions $g = a + bi + cj + dk$ such that $a^2 + b^2 + c^2 + d^2 = 1$. Topologize $\mathbb{H}$ using the topology of real numbers and give G the induced topology.

   (a) Prove that G is a Lie group.

   (b) If $H \subset G$ is a finite subgroup, show that the quotient map $p\colon G \to G/H$ is a covering map.

   (c) Identify the Lie algebra $\mathfrak{g}$ with the set of imaginary quaternions $\operatorname{Im}\mathbb{H} = \{q = bi + cj + dk : b, c, d \in \mathbb{R}\}$. Identify the map
$$\begin{aligned} \varphi\colon G &\longrightarrow \operatorname{Aut}\bigl(\operatorname{Im}\mathbb{H}\bigr) \\ g &\longmapsto (q \mapsto gqg^{-1}) \end{aligned}$$
   with the adjoint representation of G. Show that φ maps onto the group SO_3 of rotations in $\mathbb{R}^3$ and the kernel of φ is cyclic of order two.

   (d) Construct an isomorphism of G with the Lie group SU_2 of 2×2 unitary matrices of unit determinant.

   (e) A 3×3 real matrix always has a fixed line, and if the matrix is orthogonal there is a fixed line that is *pointwise* fixed: the eigenvalue is 1. Show that this is so. Except for the identity matrix I, this line is unique. Show that the map $f\colon \mathrm{SO}_3 \setminus \{I\} \to \mathbb{RP}^2$ so defined is a surjective submersion. What is the inverse image of a point? Is f a fiber bundle?

   (f) Do the conjugation classes of SO_3 form a foliation? Identify the diffeomorphism types of the conjugacy classes.

   (g) If $\overline{H} \subset \mathrm{SO}_3$ is a finite subgroup, then $H \subset \varphi^{-1}(\overline{H})$ is a finite subgroup of G of twice the order of $\overline{H}$. Let $P \subset \mathbb{R}^3$ be a finite set of n vectors whose "tips" lie on a regular polygon in $\mathbb{R}^2 \subset \mathbb{R}^3$. What is the finite subgroup of rotations which preserve P? What is its lift to G?

   (h) More interesting is to let I be the set of 12 vectors whose tips lie at the vertices of a regular icosahedron. In this case the quotient space G/H is called the *Poincaré sphere*. Can you compute the order of H?

3. Let $\omega \in \Omega^1_{\mathbb{R}}$ be a 1-form on the real line, and suppose G is a Lie group.
   (a) Prove that locally about any $t \in \mathbb{R}$ there exists a map into G such that the pullback of the Maurer-Cartan form equals ω. Write the first order ordinary differential equation this map satisfies.
   (b) Do global solutions always exist?
   (c) Construct an integrable distribution on $\mathbb{R} \times G$ that is transverse to the "vertical" fibers $\{t\} \times G$ and such that leaves of the resulting foliation of $\mathbb{R} \times G$ do *not* project surjectively onto $\mathbb{R}$.

4. Let G be a Lie group with Maurer-Cartan form θ, and suppose P is a right G-torsor.
   (a) Compute the pullback of θ under multiplication $m\colon G \times G \to G$ and under inversion $i\colon G \to G$.
   (b) Use your formulas to carry out the computation (7.14) in the notes for an arbitrary Lie group G.
   (c) Suppose $\xi \in \mathfrak{g}$ is a parallel (left-invariant) vector field. Compute $(R_g)_*\xi$. Is it parallel?
   (d) Let θ_P be the transport of the Maurer-Cartan form to a $\mathfrak{g}$-valued 1-form on P. Compute $R_g^*(\theta_P)$, $g \in G$. Compute $\varphi^*(\theta_P)$ for $\varphi \in \operatorname{Aut} P$.
   (e) An element $\xi \in \mathfrak{g}$ determines a vector field $\hat{\xi} \in \mathfrak{X}(P)$. Review the definition. Compute $(R_g)_*(\hat{\xi})$. Compute $\varphi_*(\hat{\xi})$.

5. Let G be a Lie group. Recall that a left G-torsor is a smooth manifold X equipped with a simply transitive left G action. Now we relax to a *homogeneous manifold* X, which is equipped with a transitive left G-action that is not necessarily simple. For $x \in X$ the stabilizer subgroup $H \subset G$ is a closed Lie subgroup. It is true—and you may want to prove—that G/H has a canonical smooth manifold structure such that the quotient map $G \to G/H$ is a principal H-bundle, i.e., a fiber bundle whose fibers are H-torsors. Accepting this (if you don't prove it), construct a diffeomorphisms $G/H \to X$. The diffeomorphism depends on the basepoint x. If you choose a different basepoint x' can you still construct a diffeomorphism $G/H \to X$, now one that maps the basepoint of G/H (the coset H) to x'? Is it canonical? Explore the fate of parallel vector fields and the Maurer-Cartan form, both on G/H and on X. Are there canonical vector fields? Canonical 1-forms?

6. The symplectic group Sp_1 consists of $n \times n$ quaternionic matrices P such that $P\,{}^t\overline{P}$ is the identity matrix.
   (a) Identify Sp_1 with the group G in the previous problem.
   (b) Prove that Sp_n is a compact Lie group.
   (c) Describe the Lie algebra $\mathfrak{sp}_n$ as a sub Lie algebra of $\mathfrak{gl}_n\mathbb{R}$.
   (d) Compute $\dim \mathrm{Sp}_n$.
   (e) Can you recognize the homogeneous manifolds $\mathrm{Sp}_1 / \mathrm{U}_1$ and $\mathrm{Sp}_2 / \mathrm{Sp}_1$?

Problem Set # 5

Math230a: Differential Geometry

Due: October 9

1. Let n be a positive integer, V be an n-dimensional vector space, A an affine space over V, and $U \subset A$ an open subset.

   (a) Let $n = 2$. Suppose $\theta^1, \theta^2 \in \Omega^1(U)$ are 1-forms so that for each $p \in U$ the values θ^1_p, θ^2_p form a basis of V^*. (We call this a *moving (co)frame* on U.) Prove that there exists a unique $\Theta \in \Omega^1(U)$ such that
$$d\theta^1 + \Theta \wedge \theta^2 = 0$$
$$d\theta^2 - \Theta \wedge \theta^1 = 0$$

   (b) Repeat for arbitrary n. Hence $\theta^1, \dots, \theta^n$ is a moving coframe on U and we seek unique 1-forms $\Theta^i_j \in \Omega^1(U)$ such that for all $1 \leqslant i, j \leqslant n$ we have
$$d\theta^i + \Theta^i_j \wedge \theta^j = 0$$
$$\Theta^i_j + \Theta^j_i = 0$$

2. Let X be a smooth manifold and $\Theta \in \Omega^1_X$ a nowhere vanishing 1-form. Let $H \in TX$ be the kernel of Θ, which is a codimension one distribution on X. Use Θ to construct a trivialization of the quotient bundle $TX/H \to X$. Compute the Frobenius tensor of H in terms of Θ.

3. Let $I \subset \mathbb{R}$ be an interval and suppose $(x, y)\colon I \to \mathbb{A}^2_{x,y}$ is an immersion into the standard Euclidean plane. Compute the curvature of this immersion as a function of $t \in I$.

4. Compute the curvature of the following curves in the Euclidean plane $\mathbb{A}^2_{x,y}$.

   (a) The parabola $y = kx^2$, $k \in \mathbb{R}$.

   (b) The ellipse $x^2/a^2 + y^2/b^2 = 1$, $a, b \in \mathbb{R}^{>0}$.

   (c) The cycloid that is the image of
$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{A}^2 \\ t &\longmapsto (t - \sin t,\ 1 - \cos t) \end{aligned}$$

5. Let A be a Euclidean plane and suppose $C \subset A$ is a 1-dimensional submanifold.

   (a) At each $p \in C$ prove that there is an *osculating circle*, that is, a circle $C_p \subset A$ that is tangent to C at p and among all such circles "best approximates" C near p. How is the radius of C_p related to the curvature of C at p? The center of C_p is called the *center of curvature* of C at p.

   (b) Is there a well-defined notion of a *parabola* in A? If so, can you replace circles by parabolas in (a)?

6. The *evolute* of a curve $C \subset A$ in a Euclidean plane is the locus of its centers of curvature.

   (a) Is the evolute a one-dimensional submanifold?

   (b) Compute the evolute of each of the curves in Problem 3.

7. Let $A = \mathbb{A}^3_{x,y,z}$ be standard Euclidean space (so the tangent space $\mathbb{R}^3$ has its usual inner product). I strongly encourage you to try moving frames for this problem, though you are of course free to use other techniques. The *curvature* in this problem is the real-valued function $\widetilde{K}$ on a surface $\Sigma \subset A$ introduced in Proposition 9.52 of the notes. You can compute using a local moving frame (a local section of the orthonormal frame bundle).

   (a) Compute the curvature of a sphere of radius R.

   (b) Compute the curvature of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

   where $a, b, c \in \mathbb{R}^{>0}$. (Can you use symmetry?)

   (c) Repeat for the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

   Now can you use symmetry?

8. Let A be a 3-dimensional Euclidean space, $\Sigma \in A$ a surface (2-dimensional submanifold), and $C \subset \Sigma$ a curve (1-dimensional submanifold). Suppose $\Sigma \subset A$ and $C \subset \Sigma$ are cooriented. Construct a bundle of frames adapted to $C \subset \Sigma \subset A$. Work out the restrictions of the six Maurer-Cartan forms $\theta^1, \theta^2, \theta^3, \Theta^2_1, \Theta^3_1, \Theta^3_2$ to your bundle. What curvatures do you discover? Can you see the meaning of vanishing curvature? What does your discussion reduce to if Σ is an affine plane in A?

Problem Set # 6

Math230a: Differential Geometry

Due: October 16

1. Let A be a 3-dimensional Euclidean space and $\Sigma \subset A$ a surface (2-dimensional submanifold) which is cooriented. A point $p \in \Sigma$ is *umbilic* is the second fundamental form at p is a multiple of the metric at p. Suppose e_1, e_2, e_2 is a moving frame on an open subset of Σ, and $\theta^1, \theta^2, \Theta^2_1, \Theta^3_1, \Theta^3_2$ the induced 1-forms from $\mathcal{B}_O(A)$. Express the umbilic condition in terms of these forms. Suppose now that *every* point of Σ is umbilic. Then there is a function $\lambda \colon \Sigma \to \mathbb{R}$ such that the second fundamental form is λ times the first fundamental form (metric). Prove that λ is locally constant.

2. Let M be a smooth manifold of dimension n for some $n \in \mathbb{Z}^{\geqslant 0}$.

   (a) Construct the principal $\mathrm{GL}_n\mathbb{R}$-bundle of frames $\mathcal{B}(M) \to M$. This can be done in several ways. For example, first construct a vector bundle over M whose fibers consist of ordered sets of n tangent vectors. (This is the fiber product of the tangent bundle with itself n times, so you may want to investigate fiber products of fiber bundles first.) Or you may patch $\mathcal{B}(M) \to M$ together from an cover of M by coordinate charts.

   (b) Now suppose M carries a Riemannian metric. Construct the principal O_n-bundle of orthonormal frames $\mathcal{B}_O(M) \to M$. One method is to consider the function $\mathcal{B}(M) \to \mathrm{Sym}_n\mathbb{R}$ with values in the vector space of symmetric $n \times n$ real matrices that maps a frame $e_1, \dots, e_n$ to the symmetric matrix $\big(\langle e_i, e_j \rangle\big)_{1 \leqslant i,j \leqslant n}$.

3. Does there exist an example of a Riemannian 2-manifold Σ and an embedded oriented circle $C \subset \Sigma$ such that parallel transport around C is a reflection (rather than a rotation)? If so, what does this say about the lift of C to the orientation double cover, which is identified with $\mathcal{B}_O(\Sigma)/\mathrm{SO}_2 \to \Sigma$? Does there exist an example in which Σ is a submanifold of a Euclidean 3-space?

4. Consider a surface of revolution as follows. In the standard Euclidean plane $\mathbb{A}^2_{x,z}$ consider the embedded curve C which is the graph of a positive function $x = f(z)$, $z_0 < z < z_1$, for some $z_0 < z_1$. Now revolve C about the z-axis in standard Euclidean 3-space $\mathbb{A}^3_{x,y,z}$. Use z, ϕ as coordinates, where ϕ is the angle of revolution about the z-axis. Compute the Gauss curvature in terms of f. Compute the second fundamental form. What are the principal curvatures and what are the principal directions? (These are the eigenvalues and eigenlines of the shape operator.) Check your computations against standard examples (cylinder, sphere). Can you find f so that the resulting surface has constant curvature -1? Investigate geodesics on this surface of revolution.

5. Continuing the notation of the previous problem, the *catenoid* is the surface of revolution obtained by revolving the *catenary* curve, which is the graph of $f(z) = a\cosh(z/a)$ for some $a > 0$. The *helicoid* in $\mathbb{A}^3_{x,y,z}$ is the image of the map

$$\begin{aligned} (0,\infty) \times (0,2\pi) &\longrightarrow \mathbb{A}^3 \\ (r,\theta) &\longmapsto (r\cos\theta, r\sin\theta, a\theta) \end{aligned}$$

Prove that the catenoid and the helicoid are *minimal surfaces*: their mean curvature vanishes. In addition, construct a local isometry between them.

6. Let Σ be a Riemannian 2-manifold and $\pi\colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$ the principal O_2-bundle of orthonormal frames.
   (a) Explain how to construct global horizontal vector fields ∂_1, ∂_2 on $\mathcal{B}_{\mathrm{O}}(\Sigma)$ from the global 1-forms $\theta^1, \theta^2 \in \Omega^1_{\mathcal{B}_{\mathrm{O}}(\Sigma)}$. Are ∂_1, ∂_2 π-projectible to vector fields on Σ?
   (b) Compute $[\partial_1, \partial_2]$.
   (c) Compute $[\hat{\zeta}, \partial_i]$, $i = 1, 2$, where $\hat{\zeta}$ is the vertical vector field discussed in lecture.
   (d) Prove that integral curves of ∂_1 project to geodesic motions on Σ.

7. Let Σ be a Riemannian 2-manifold and $\pi\colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$ the principal O_2-bundle of orthonormal frames.
   (a) Suppose $f\colon D^2 \to \Sigma$ is a smooth map from the (closed) 2-disk into Σ which is an immersion on ∂D^2 and has a regular point p in the interior. Factor $f\big|_{\partial D^2}$ through π using an *adapted* moving frame. What choices do you make?
   (b) Extend that lift to $D^2 \backslash B_\epsilon(p)$, where $B_\epsilon(p)$ is a ball of small radius ϵ. (Or simply a sufficiently small open set about p; we haven't discussed distance on a Riemannian manifold.) What theorem can you use to prove existence?
   (c) What can you say about the lift on $\partial B_\epsilon(p)$? Try to get a geometric picture of its image in $\mathcal{B}_{\mathrm{O}}(\Sigma)$.
   (d) Integrate the pullback of the equation $d\Theta^2_1 = -(\pi^*K)\,\theta^1 \wedge \theta^2$ under your lift. What can you conclude as $\epsilon \to 0$.

Problem Set # 7

Math230a: Differential Geometry

Due: October 23

1. Suppose

$$\begin{tikzcd} E' \arrow[r, dashed, "\tilde{f}"] \arrow[d, dashed, "\pi'"'] & E \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

is a pullback diagram of manifolds. Prove that the differentials

$$\begin{tikzcd} TE' \arrow[r, dashed, "\tilde{f}_*"] \arrow[d, dashed, "\pi'_*"'] & TE \arrow[d, "\pi_*"] \\ TX' \arrow[r, "f_*"] & TX \end{tikzcd}$$

also form a pullback.

2. Let Σ be a Riemannian 2-manifold and $\pi\colon \mathcal{B}_{\mathrm{O}}(\Sigma) \to \Sigma$ the principal O_2-bundle of orthonormal frames.

   (a) Construct an explicit identification of $\mathcal{B}_{\mathrm{O}}(\Sigma)/\mathrm{SO}_2 \to \Sigma$ with the orientation double cover of Σ.

   (b) Prove that $\theta^1 \wedge \theta^2 \in \Omega^2_{\mathcal{B}_{\mathrm{O}}(\Sigma)}$ descends to the total space of the orientation double cover.

   (c) Let $\widehat{X} \to X$ be the orientation double cover of a smooth n-manifold X. A *density* on X is $\omega \in \Omega^n_{\widehat{X}}$ such that $\sigma^*\omega = -\omega$, where σ is the deck transformation. Show that a compactly supported density can be integrated over X. (Recall the integral over an *oriented* smooth manifold of a compactly supported *n-form*. The key point is the change of variables formula.)

3. (a) Let A be a 3-dimensional Euclidean space and $c\colon (a,b) \to A$ a smooth curve parametrized by arc length. The surface M which is the union of the tangent lines to the image C of c is called the *tangent developable* of C. Find conditions on c so that M is a smooth manifold. Compute its Gauss curvature.

   (b) More generally, a *ruled surface* is a union of lines. One example is a Möbius strip. Find a parametrization of the Möbius strip. Compute the Gauss curvature. Another ruled surface is the *helicoid*, obtained by fixing perpendicular lines $\ell, \ell' \subset A$ and taking the surface swept out by ℓ' as it moves along ℓ, remaining perpendicular to ℓ and rotating at a constant speed as it moves. Find a parametrization and compute the Gauss curvature and mean curvature.

4. Let A be a 4-dimensional Euclidean space and $\Sigma \subset A$ an embedded 2-dimensional manifold. Define a submanifold $\mathcal{B}_{\mathrm{O}}(\Sigma) \subset \mathcal{B}_{\mathrm{O}}(A)$ of adapted orthonormal frames. Investigate the restrictions of the Maurer-Cartan forms and Maurer-Cartan equations. Notice that the normal bundle to Σ in A has rank 2, and it has an associated principal O_2-bundle of orthonormal frames.

5. This problem is designed to develop some intuition about horizontal distributions and parallel transport. Let $A = \mathbb{A}_{x,y}$ be the standard affine plane, and consider the product fiber bundle over A with fiber $\mathbb{R}$.

   (a) Which functions $F\colon A \times \mathbb{R}^2 \to \mathbb{R}$ define a horizontal distribution? How?

   (b) For example, we might say that the infinitesimal rate of change of a parallel section of $A \times \mathbb{R} \to A$ at $(x, y) \in A$ along a unit vector in the direction that makes angle θ with the x-axis is $\cos\theta$. Does that define a horizontal distribution? Is it integrable?

   (c) What if instead we say that the infinitesimal rate of change is $x\cos\theta$?

   (d) In both cases consider parallel transport along the path that travels around the triangle with initial vertex $(0, 0)$, second vertex $(0, 1)$, third vertex $(1, 1)$, and then back to the initial vertex $(0, 0)$.

   (e) This fiber bundle is extrinsic: a section attaches a real number to each point, and the real number is unrelated to the intrinsic geometry of A. What if instead we consider the fiber bundle whose fibers are lines in the vector space $\mathbb{R}^2$. Now the rule for infinitesimal parallel transport is to rotate at the speed specified in (b) or (c). The objects we are transporting are different—lines as opposed to real numbers—but do the answers to the questions change?

Problem Set # 8

Math230a: Differential Geometry

Due: October 30

1. Let G be a Lie group and suppose $\pi\colon P \to X$ is a principal G-bundle. In particular, π is a surjective submersion, so $\pi^*\colon \Omega^\bullet_X \to \Omega^\bullet_P$ is an injection. Characterize the image by proving Lemma 14.12 in the notes.

2. Let G be a Lie group and $H \subset G$ a closed Lie subgroup. Suppose $\pi\colon P \to X$ is a principal G-bundle.
   (a) Prove that reductions of π to H are in 1:1 correspondence with sections of $P/H \to X$.
   (b) Suppose that Θ is a connection on π. Prove that reductions of π to H such that Θ is induced from a connection on the reduction are in 1:1 correspondence with *flat* sections of $P/H \to X$.
   (c) Consider the case $G = \mathrm{GL}_n\mathbb{R}$ and $H = \mathrm{O}_n$. What do these reductions imply for the induced horizontal distribution and parallel transport on the vector bundle associated to the action of $\mathrm{GL}_n\mathbb{R}$ and O_n on $\mathbb{R}^n$?

3. (a) Prove that the action of a *compact* Lie group on a smooth manifold is proper.
   (b) Let G be a Lie group and suppose H is a *closed* Lie subgroup. Prove that the action of H on G by left multiplication is proper. What if $H \subset G$ is not closed?

4. (a) Let $Q \to [0,1] \times M$ be a principal G-bundle. Choose a connection on Q. (Why does it exist?) Use parallel transport to construct an isomorphism $Q\big|_{\{0\}\times M} \to Q\big|_{\{1\}\times M}$.
   (b) Let $f_t\colon M \to N$ be a smooth homotopy and $P \to N$ a principal bundle. Prove that $f_0^*P \cong f_1^*P$.
   (c) Prove that a principal bundle over a contractible manifold is trivializable.
   (d) Classify up to isomorphism principal $\mathbb{T}$-bundles on S^n for all n.

5. Let G be a Lie group and let $\pi\colon P \to X$ be a principal G-bundle. I showed in lecture that a connection on π is a section of a fiber bundle of affine spaces $A_\pi \to X$. Construct this bundle. Is it associated to π via the action of G on some affine space? If not, is it associated to some other principal bundle over X?

6. (a) Let P be a Riemannian manifold equipped with a free action of a Lie group H by isometries. Assume that the quotient map $\pi\colon P \to X$ is a principal H-bundle. Use the metric to construct a connection on π.

   (b) Let G be a Lie group and let H be a closed Lie subgroup. Define the notion of a *bi-invariant Riemannian metric* on G. (Bi-invariant=left and right invariant.) Give examples of a Lie group and a bi-invariant metric on it. Give an example of a Lie group that does not admit a bi-invariant metric. Can you prove the nonexistence?

   (c) Assuming a bi-invariant metric exists and is chosen, use it to construct a connection on $\pi\colon G \to G/H$. Compute the curvature of this connection.

   (d) Apply to the Hopf bundle $\mathrm{U}_1 \to S^3 \to S^2$. What about the Hopf bundle $\mathrm{Sp}_1 \to S^7 \to S^4$? The connection you construct on the latter is the basic *instanton* (self-dual connection).

7. Let G be a Lie group and let $\pi\colon P \to X$ be a principal G-bundle. A *gauge transformation* of π is a diffeomorphism $\varphi\colon P \to P$ that is G-equivariant and covers the identity map of X, i.e., the diagram

$$\begin{tikzcd} P \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & P \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

commutes.

   (a) Construct a function $g\colon P \to X$ that satisfies $\varphi(p) = p \cdot g(p)$ for all $p \in P$. How does g transform under the G-action on P?

   (b) Express g as a section of a fiber bundle associated to π. What kind of a fiber bundle is it?

   (c) Do gauge transformations always exist?

   (d) Are there any simplifications if G is abelian? If G is discrete?

   (e) Let $\mathrm{Aut}(P)$ denote the group of G-equivariant diffeomorphisms of P, and let $\mathrm{Aut}(\pi)$ denote the group of gauge transformations. Construct an exact sequence

$$1 \longrightarrow \mathrm{Aut}(\pi) \longrightarrow \mathrm{Aut}(P) \longrightarrow \mathrm{Diff}(X)$$

   where $\mathrm{Diff}(X)$ is the group of diffeomorphisms of X. Is the last map surjective? Give a proof or (counter)example to verify your answer.

   (f) Suppose Θ is a connection 1-form on P, and let $\varphi\colon P \to P$ be a gauge transformation. Show that $\varphi^*\Theta$ is a connection 1-form. Give a formula for $\varphi^*\Theta$ in terms of the function g defined in (a). How do gauge transformations act on connections as G-invariant horizontal distributions on P.

Problem Set # 9

Math230a: Differential Geometry

Due: November 6

1. Let $\pi\colon P \to X$ be a principal G-bundle with connection $\Theta \in \Omega^1_P(\mathfrak{g})$.
   (a) Denote the right G-action of G on P as $R\colon P \times G \to P$. Compute $R^*\Theta$.
   (b) Suppose $s\colon X \to P$ is a section of π and $g\colon X \to G$ is a function. Compute $\alpha' = (sg)^*(\Theta)$ in terms of $\alpha = s^*(\Theta)$.
   (c) Write your results in matrix notation if G is a matrix group.

2. For this problem you should recall Haar measure as discussed in Problem Set #2.
   (a) Let G be a Lie group (not necessarily compact) and let $H \subset G$ be a closed Lie subgroup which is compact. Prove that the homogeneous manifold G/H is reductive.
   (b) Find an H-invariant complement $\mathfrak{p}$ to $\mathfrak{h} \subset \mathfrak{g}$ for the homogeneous manifolds $\mathrm{O}_{n,1} / \mathrm{O}_n$ (hyperbolic space) and $\mathrm{Sp}_{2n}\mathbb{R} / \mathrm{U}_n$ (Siegel upper half space). The Lie group $\mathrm{O}_{n,1}$ consists of invertible $n \times n$ real matrices that preserve the inner product $\langle \xi, \eta \rangle = \sum_{i=1}^{n} \xi^i \eta^i - \xi^{n+1}\eta^{n+1}$.
   (c) Compute the curvature of the induced linear connection on hyperbolic space. Compare to the case of the sphere (in the lecture notes).
   (d) Describe the Riemannian metric on hyperbolic space. Identify the total space of the orthonormal frame bundle.

3. Let $\pi\colon P \to X$ be a principal G-bundle with connection Θ. Prove that Θ is flat iff the holonomy group $\mathrm{Hol}_p(\Theta) \subset G$ is a discrete subgroup for all $p \in P$.

4. My proof of Proposition 14.71 in the lecture notes is a complete waffle. Write a convincing proof.

5. In (**14.64**) of those same notes I indicated that parallel transport reduces to a concrete ordinary differential equation. You will flesh that out in this problem.
   (a) Let G be a Lie group and let $\pi\colon P \to [0,1]$ be a principal G-bundle with connection Θ. Let $s\colon [0,1] \to P$ be a section, and write $s^*\Theta = A(t)dt \in \Omega^1_{[0,1]}(\mathfrak{g})$. Write the ODE for a function $g\colon [0,1] \to G$ which satisfies the condition that the section sg of π is flat. Impose the initial condition $g(0) = e$.
   (b) Suppose first that $A(t) = A$ is a constant function. Write a solution to the ODE.

(c) Suppose $G = \mathrm{GL}_n\mathbb{R}$ and write g as an invertible $n \times n$ matrix M. You should have found in part (a) an equation similar to

$$\frac{dM}{dt} + M(t)B(t) = 0$$

Consider the function

$$Q(t) = 1 - \int_0^t B(u_1)\,|du_1| + \iint\limits_{0 \leqslant u_2 < u_1 \leqslant t} B(u_1)B(u_2)\,|du_1 du_2| - \cdots$$

Draw a picture. Write the rest of the series. Prove convergence. Does this give a solution to the ODE? What if $B(t) = B$ is a constant function? (Remark: This power series is sometimes denoted

$$: \exp\left(-\int_0^t B(u)\,|du|\right) :$$

and is called the *path-ordered exponential*.)

6. (a) Fix $m \in \mathbb{Z}^{>0}$. Construct a $\mathbb{C}^\times$-action on $\mathbb{CP}^m$ with precisely $m+1$ fixed points.

   (b) Let $\pi\colon P \to X$ be a principal $\mathbb{C}^\times$-bundle over a smooth manifold X. Let $L \to X$ be the associated complex line bundle (to the action of $\mathbb{C}^\times$ on $\mathbb{C}$ by multiplication). Construct the fiber bundle with fiber $\mathbb{CP}^m$ associated to π directly in terms of $L \to X$. Construct a canonical isomorphism from the associated bundle to the fiber bundle you construct.

   (c) What if the $\mathbb{C}^\times$-action has more fixed points?

   (d) Specialize to $X = S^1$ and now put a connection on π. Describe the monodromy of the induced horizontal distribution on the associated fiber bundle.

7. Let G be a Lie group and suppose $\pi\colon P \to S^1 \times S^1$ is a principal G-bundle with a *flat* connection Θ. Fix a basepoint $p \in P$. Prove that the holonomy subgroup $\mathrm{Hol}_p(\Theta) \subset G$ is abelian.

Problem Set # 10

Math230a: Differential Geometry

Due: 20

Reminder: There is an extra lecture on Friday, November 8 and there is no lecture on Wednesday, November 13. I am giving you two weeks to complete this problem set. All references to equations and theorems are to the lecture notes.

1. Carry out the computations to prove Proposition 16.67.

2. (a) Verify that (17.33) is a connection form.
   (b) Verify the claim in Remark 17.34.
   (c) Verify the claim that follows Definition 17.43.

3. Let X be a smooth manifold. Suppose $\gamma\colon (a,b) \to X$ is a smooth motion. Assume $0 \in (a,b)$ and write $x = \gamma(0)$.
   (a) Let G be a Lie group, let $\pi\colon P \to X$ be a principal G-bundle with connection. Suppose F is a smooth manifold equipped with a smooth left action $G \circlearrowright F$. Let $\sigma\colon X \to F_P$ be a smooth section of the associated fiber bundle with fiber F. Explain how to use parallel transport to define a motion $\delta\colon (a,b) \to (F_P)_x$, where $(F_P)_x$ is the fiber of the associated bundle $F_P \to X$ at $x \in X$. Prove that δ is smooth.
   (b) Now suppose X has a linear connection. Use parallel transport to define a motion $\delta\colon (a,b) \to T_xX$. Prove that δ is smooth.

4. Let X be a Riemannian 2-manifold, and suppose x^1, x^2 is a local coordinate system. Compute the Gauss curvature in terms of the Riemann curvature tensor $R^i_{jk\ell}$.

5. Let G be a Lie group. There are two canonical linear connections Θ_L, Θ_R defined using left and right multiplication.
   (a) Define them carefully.
   (b) Compute their curvature and torsion. What are the geodesic motions that begin at the identity element?
   (c) The Lie group $G \times G$ acts transitively on G by left and right multiplication. Is there a homogeneous connection? Give examples. If so, is it related to Θ_L and/or Θ_R? Compute its curvature and torsion and also the geodesic motions that begin at the identity element $e \in G$.
   (d) In (3.57) we defined an exponential map on a Lie group without using a linear connection. In (17.45) we define an exponential map using the affine connection associated to a linear connection. Compare them on G at $e \in G$ for each of the linear connections above.

6. Let $X \subset E$ be a submanifold of a Euclidean space. The dimensions of X and E are not fixed.
   (a) Use the global parallelism of E to induce a parallelism—a covariant derivative—on X. So if $\xi \in T_xX$ is a tangent vector at some point $x \in X$ and η a vector field on X defined in a neighborhood of X, use the natural covariant derivative on E to define the covariant derivative $\nabla_\xi \eta$ on X.
   (b) Prove that ∇ preserves the induced Riemannian metric on X.
   (c) Consider the example of a unit 2-sphere X in a 3-dimensional Euclidean space. Let C be the circle obtained by intersecting X with a plane whose distance from the nearest parallel tangent plane is $d < 1$. The holonomy of the parallel transport around C is rotation through some angle θ. Compute θ as a function of d. Make clear your orientations.

7. Let $S^3 \subset \mathbb{E}^4$ be a sphere of radius R in Euclidean 4-space.
   (a) Introduce local coordinates. Compute the Cristoffel symbols and the Riemann curvature tensor in that coordinate system.
   (b) Write S^3 as a homogeneous manifold for a Lie group of isometries. Is there a homogeneous connection? Does it induce the Levi-Civita connection? If so, recover the curvature you computed in (a) from the curvature of the homogeneous connection.
   (c) Compute yet another way by constructing a local orthonormal frame.

8. Read Riemann's essay ("Riemann1") that is posted on Canvas. Choose one small passage and explain it in modern terms.

9. Let X be a smooth manifold and $E \subset TX$ a distribution of rank k. Let $\mathcal{B}_E(X) \subset \mathcal{B}(X)$ be the subbundle of frames for which the first k vectors form a basis of E. Prove that $\mathcal{B}(X) \to X$ admits a torsionfree connection if and only if E is involutive.

10. Let X be a smooth manifold equipped with a linear connection. Is it possible for a geodesic to intersect itself? Example or counter-proof.

Problem Set # 11

Math230a: Differential Geometry

Due: December 4

1. Let X be a Riemannian manifold and suppose $\{x_1, \dots, x_N\} \subset X$ is a finite subset. In lecture (notes) I constructed a manifold with boundary $\hat{X}$ and a map $p\colon \hat{X} \to X$ that is a diffeomorphism on the complement of $\bigsqcup_i p^{-1}(x_i)$. Furthermore, if ξ is a vector field on X with a transverse zero at each x_i and no other zero, then the normalization $s := \xi/\|\xi\|\colon X\backslash\{x_1, \dots, x_N\} \to S(TX)$ extends over $\hat{X}$, where $S(TX) \subset TX$ is the total space of the unit sphere bundle of the tangent bundle. Explore this construction in detail when $\dim X = 1$. Draw pictures.

2. Consider Chern–Simons–Weil forms for $G = \mathbb{T}$ the circle group of unit norm complex numbers. The Lie algebra is $\mathfrak{g} \cong \sqrt{-1}\mathbb{R}$. Be sure to make orientations explicit when working this problem.
   (a) For each $k \in \mathbb{Z}^{>0}$ identify the vector space of degree k G-invariant polynomials on $\mathfrak{g}$.
   (b) If $\pi\colon P \to X$ is a principal G-bundle and h is a G-invariant polynomial of degree k, then the Chern–Simons form restricts to a closed $(2k-1$-form on each fiber of π, and this form may be identified with a biinvariant form on G. Identify this form for $k = 1$. For which h is the integral of this form (with respect to some orientation of G) an integer?
   (c) Consider the Hopf bundle $\pi\colon S^3 \to S^2$, which is a principal G-bundle. Construct a connection (perhaps one that is invariant under left translation by $\mathrm{SU}_2 \approx S^3$). Construct a section of π on the complement of a point of S^2, and use a blow-up construction as in Problem #1 to extend over a manifold with boundary. Use this to compute the integral of the Chern–Weil form over S^2. For which h is this an integer?
   (d) Now consider $k = 2$, so the Chern–Weil form has degree 4. Construct a nontrivial principal G-bundle over $S^2 \times S^2$ by first taking the Cartesian product of the Hopf bundle above with itself to form a principal $(G \times G)$-bundle over $S^2 \times S^2$, and then use the homomorphism $G \times G \to G$ to form the associated principal G-bundle. (Why is multiplication a homomorphism?) Do so with the connection you constructed above. Compute the integral of the Chern–Weil form. For which h is the answer an integer?
   (e) Continuing with $k = 2$, take the base 4-manifold to be $\mathbb{CP}^2$. For which h do you find an integer when you integrate the Chern–Weil form?

3. Now consider $G = \mathrm{SU}_2$ with Lie algebra $\mathfrak{su}_2$ the space of 2×2 traceless skew-Hermitian matrices.
   (a) Identify the space of G-invariant polynomials of degree 2 on $\mathfrak{g}$.
   (b) Write an explicit formula for the Chern–Simons 3-form of a connection. Use matrix multiplication in your formula rather than the Lie bracket.

(c) Use the homomorphism

$$\begin{aligned} \mathbb{T} &\longrightarrow \quad \mathrm{SU}_2 \\ \lambda &\longmapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \end{aligned}$$

to construct SU_2-connections from $\mathbb{T}$-connections. How are the Chern–Simons forms related?

(d) Investigate integrality of the integral of Chern–Weil forms for $k = 2$.

4. Let $\Sigma \subset \mathbb{E}^3$ be a closed surface that is a submanifold of Euclidean 3-space. Prove that Σ has a point of positive Gauss curvature.

5. Let G be a Lie group, let $H \subset G$ be a closed Lie subgroup, and assume that the homogeneous manifold G/H has a reductive structure, i.e., an H-invariant splitting $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$ as vector spaces.
   (a) Recall the definition of the canonical connection on the principal H-bundle $\pi\colon G \to G/H$.
   (b) Compute its curvature.
   (c) What is the meaning of the torsion of this connection? Compute it.
   (d) What is the meaning of geodesics of this connection? Compute them.
   (e) Consider the transitive action of $G\times G$ on G by left and right multiplication. Is this homogeneous space reductive?

6. (a) Let (Σ, g) be a Riemannian 2-manifold, and suppose $\phi\colon \Sigma \to \mathbb{R}$ is a smooth function. Compute the Gauss curvature K' of the metric $e^{2\phi}g$ in terms of ϕ and the Gauss curvature K of the metric g.
   (b) Can you do the same for the Riemann curvature tensor in arbitrary dimensions?

7. Let G be a Lie group with a left-invariant Riemannian metric, denoted $\langle -, - \rangle$. Suppose ξ, η, ν, ζ are left-invariant vector fields on G.
   (a) Prove that
   $$\nabla_\xi \eta = \frac{1}{2}\big([\xi, \eta] - (\mathrm{ad}_\xi)^*\eta - (\mathrm{ad}_\eta)^*\xi\big),$$
   where the star denotes the adjoint with respect to the inner product on the Lie algebra.
   (b) Derive a formula for $\langle R(\xi, \eta)\nu, \zeta\rangle$, where R is the Riemann curvature tensor.
   (c) Derive a formula for $\langle R(\xi, \eta)\eta, \xi\rangle$, which is the *sectional curvature.*
   (d) How do your formulas simplify for bi-invariant Riemannian metrics?