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CLASSICAL CHERN-SIMONS THEORY, PART 2

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For S. S. Chern

Introduction

Connections in fiber bundles, particularly in principal bundles, appear in many parts of differential geometry. For example, the basic invariant of a Riemannian metric—the Riemann curvature tensor—is the curvature of a canonical connection on the tangent bundle associated to the metric—the Levi-Civita connection. In recent years connections on extrinsic bundles have been used to construct topological invariants in low dimensions. Many of these mathematical constructions are inspired by quantum field theory, where connections on principal bundles are called *gauge fields* and appear as generalizations of the electromagnetic field of Maxwell.

Let θ be a connection on a principal bundle $P \to M$ with structure group a compact Lie group G. The basic local invariant of θ is its curvature, a 2-form with values in the adjoint bundle. Closed scalar differential forms, called Chern-Weil forms, then arise from invariant polynomials on the Lie algebra of G. Their dependence on θ leads to secondary geometric invariants, called Chern-Simons forms. We remark that Chern and Simons were motivated by concrete geometric questions in combinatorial and conformal geometry. Topologically, the Chern-Weil forms encode *real* characteristic classes of P. Refined secondary invariants, called Cheeger-Simons differential characters, capture *integral* characteristic classes. In §1 we quickly review these geometric invariants of connections and their relationship to characteristic classes.

Witten [W] used the secondary invariant attached to a class in $H^4(BG;\mathbb{Z})$ to construct a topological quantum field theory in three dimensions. The resulting quantum invariants include the Jones polynomials of knots as well as new invariants of 3-manifolds. There are also lower dimensional quantum invariants:

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in two dimensions they include representations of the mapping class groups of surfaces and Riemann-Roch numbers, while in dimension one quantum groups and a relationship to a twisted form of K-theory appear.[1] All of these quantum invariants are at least formally derived from classical invariants of families of G-connections. For example, to a family of G-connections over a closed oriented 3-manifold with parameter space T the Cheeger-Simons differential character assigns a function $T \to \mathbb{R}/\mathbb{Z}$. The quantum invariant is formally the integral of the exponential of this function over the space of equivalence classes of G-connections. In lower dimensions the classical invariant is a more sophisticated geometric object, as is the corresponding quantum invariant.

In [F1] the classical invariants were developed in detail assuming that the group G is connected and simply connected; finite groups were handled by other methods [F2]. We recognized that the right tool for the general case is a cochain model for Cheeger-Simons differential characters, also known as smooth Deligne cohomology. Unfortunately, our attempts to develop a cochain model which includes integration failed. Meanwhile, other geometric and physical motivations for this and related cochain models surfaced. In a forthcoming paper [HS] M. Hopkins and I. Singer develop such cochain models and apply them to quadratic forms in topology; other applications will follow. In this note we simply point out that their theory may be used to construct the classical geometric invariants of families of connections needed in the 3-dimensional field theory. Only a broad outline appears here as detailed statements and proofs depend on [HS], which has not yet appeared.

Gomi [Go] has a similar discussion of classical Chern-Simons theory for 2- and 3-manifolds which uses a different cochain model.

One can go some distance towards geometric invariants of families of connections without these cochain models, which is the subject of §2. In §3 we briefly introduce the Hopkins-Singer theory and its application to classical Chern-Simons theory.

I thank Mike Hopkins for many conversations about cochain models for differential characters and their application to connections.

It is a pleasure to dedicate this paper to Professor Chern. I am fortunate to count him as one of my teachers in Berkeley, where he presided over a golden period of activity in geometry. His mathematical guidance and personal warmth remain an inspiration.

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[1]The last remark refers to ongoing joint work of the author, C. Teleman, and M. Hopkins.

1. Geometric Invariants of Connections

We begin with connections on principal circle bundles. Everything which follows for arbitrary principal G-bundles, G a compact Lie group, is a generalization:

$$\begin{aligned} \text{curvature} &\longrightarrow \text{Chern-Weil forms} \\ \text{connection} &\longrightarrow \text{Chern-Simons forms} \\ \text{Chern class} &\longrightarrow \text{characteristic classes} \\ \text{log holonomy} &\longrightarrow \text{Cheeger-Simons differential characters} \end{aligned}$$

Our presentation is terse. The original references are [CherS] and [CheeS]. A nice account of the theory for circle bundles is [K]. Chern-Weil theory is explained in detail in [KN].

Our exposition is based mostly on the theory of differential forms and Stokes' theorem. As there is a sign at stake in the latter, we specify our sign convention. Suppose $p : \mathcal{M} \to T$ is a smooth fiber bundle whose fibers are compact manifolds of dimension n and whose relative tangent bundle is oriented. Then integration along the fibers

$$\int_{\mathcal{M}/T} : \Omega^q(\mathcal{M}) \longrightarrow \Omega^{q-n}(T)$$

is defined. For $\mathcal{M} = T \times M$ we have

$$\int_{\mathcal{M}/T} \alpha \wedge \beta = \left(\int_M \beta \right) \alpha, \qquad \alpha \in \Omega^\bullet(T), \quad \beta \in \Omega^\bullet(M).$$

Stokes' theorem asserts

$$d \int_{\mathcal{M}/T} \eta = \int_{\mathcal{M}/T} d\eta \; - \; (-1)^{q-n} \int_{\partial\mathcal{M}/T} \eta, \qquad \eta \in \Omega^q(\mathcal{M}). \tag{1.1}$$

1.1. **Circle bundles.** Let M be a smooth manifold and $\pi : P \to M$ a principal circle bundle. The canonical Maurer-Cartan 1-form θ_0 on the circle group $\mathbb{T}$ induces a family of 1-forms on the fibers of π. These forms are purely imaginary. A *connection* θ on P is a $\mathbb{T}$-invariant imaginary 1-form on P which restricts on each fiber to θ_0. Its differential is the pullback of a 2-form ω on M, the *curvature* of θ. Summarizing, if $\iota : P_m \hookrightarrow P$ is the inclusion of a fiber, then

$$\begin{aligned} \iota^*\theta &= \theta_0 \\ d\theta &= \pi^*\omega. \end{aligned} \tag{1.2}$$

We say that ω is a *transgression* of θ_0 to M. Any connection on P has the form $\theta + \pi^*\alpha$ for some imaginary 1-form α on M, and its curvature is $\omega + d\alpha$.

It follows that the curvature ω is closed and its de Rham cohomology class is independent of the connection.

There is a single topological invariant of a principal circle bundle on M, the Chern class $c(P) \in H^2(M;\mathbb{Z})$. Its image $c(P)_{\mathbb{R}}$ under the natural map

$$H^2(M;\mathbb{Z}) \longrightarrow H^2(M;\mathbb{R}) \tag{1.3}$$

is the de Rham cohomology class of $\sqrt{-1}\,\omega/2\pi$. The *integer* Chern class $c(P)$ can also be recovered from the differential geometry, but we need to use *holonomy* as well as curvature. A useful fact [MS, §2] is that an integral cohomology class is determined by its $\mathbb{Z}$-periods and its $\mathbb{Z}/\ell\mathbb{Z}$-periods, i.e., by integration over $\mathbb{Z}$-cycles and $\mathbb{Z}/\ell\mathbb{Z}$-cycles. A $\mathbb{Z}/\ell\mathbb{Z}$-cycle consists of a (smooth singular oriented) 2-chain $f:\Sigma \to M$ whose boundary is ℓ copies of a fixed 1-cycle $g:S\to M$. This determines a class $[\Sigma] \in H_2(M;\mathbb{Z}/\ell\mathbb{Z})$, and

$$c(P)[\Sigma] = \int_{\Sigma} f^*\left(\frac{\sqrt{-1}\,\omega}{2\pi}\right) \;-\; \ell\int_S \tilde{g}^*\left(\frac{\sqrt{-1}\,\theta}{2\pi}\right) \pmod{\ell},$$

where $\tilde{g}$ is a lift of g to P. The second term is proportional to the logarithm of the holonomy of θ around $g(S)$.

Finally, note that the curvature can be recovered from the holonomy around small loops. In fact, the connection θ is determined up to isomorphism by the holonomy. Let

$$\check{H}^2(M) = \text{set of equivalence classes of } \mathbb{T}\text{-connections on } M. \tag{1.4}$$

The previous remark shows that $\check{H}^2(M)$ is also the set of holonomy functions, that is maps

$$\log\mathrm{hol} : Z_1(M) \longrightarrow \mathbb{R}/\mathbb{Z}$$

such that if a 1-cycle $g:S\to M$ bounds a 2-cycle $f:\Sigma\to M$ we have

$$\log\mathrm{hol}(g) = -\int_S f^*\left(\frac{\sqrt{-1}\,\omega}{2\pi}\right) \pmod{1}.$$

There is an exact sequence

$$0 \longrightarrow H^1(M;\mathbb{R}/\mathbb{Z}) \longrightarrow \check{H}^2(M) \xrightarrow{\text{curvature}} \Omega^2_{\mathbb{Z}}(M) \longrightarrow 0, \tag{1.5}$$

where $\Omega_{\mathbb{Z}}$ denotes closed forms with integral periods. (The curvature map in (1.5) includes the factor of $\sqrt{-1}/2\pi$.) The kernel is the set of equivalence classes of flat connections. Another form of (1.5) is the exact sequence

$$0 \longrightarrow \frac{\Omega^1(M)}{\Omega^1_{\mathbb{Z}}(M)} \longrightarrow \check{H}^2(M) \xrightarrow{\text{Chern class}} H^2(M;\mathbb{Z}) \longrightarrow 0. \tag{1.6}$$

1.2. **Chern-Weil and Chern-Simons.** Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and fix an Ad-invariant polynomial $\rho \in \mathrm{Sym}^{\bullet}_{\mathrm{inv}}(\mathfrak{g}^*)$. For simplicity suppose ρ is homogeneous of degree k. Let $\pi : P \to M$ be a principal G-bundle and $\theta \in \Omega^1(P;\mathfrak{g})$ a connection on P, i.e., a G-invariant form whose restriction to each fiber is the Maurer-Cartan form. Then

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = \pi^*\Omega \tag{1.7}$$

for the curvature $\Omega \in \Omega^2(M;\mathfrak{g}_P)$, a 2-form with values in the adjoint bundle. Since ρ is Ad-invariant we obtain a scalar differential form $\rho(\Omega^{\otimes k}) \in \Omega^{2k}(M)$, which for simplicity we denote $\rho(\theta)$. The key fact in Chern-Weil theory is the following lemma.

Lemma 1.1. *$\rho(\theta)$ is closed.*

Proof. Differentiating (1.7) we have

$$\begin{aligned} \frac{1}{k} d\left[\rho(\Omega^{\otimes k})\right] &= \rho\big(d\Omega \otimes \Omega^{k-1}\big) \\ &= \rho\big(-[d\theta \wedge \theta] \otimes \Omega^{k-1}\big) \\ &= \rho\big(-[\Omega \wedge \theta] \otimes \Omega^{k-1} + \frac{1}{2}[[\theta \wedge \theta] \wedge \theta] \otimes \Omega^{k-1}\big). \end{aligned}$$

The second term vanishes by the Jacobi identity and the first by the Ad-invariance of ρ. □

The space $\mathcal{A}_P$ of connections on P is an affine space modeled on $\Omega^1(M;\mathfrak{g}_P)$. Thus $\theta_0, \theta_1 \in \mathcal{A}_P$ determine a straight line path $\theta_t = (1-t)\theta_0 + t\theta_1$, $0 \le t \le 1$, which in turn determines a connection Θ on $[0,1] \times P \to [0,1] \times M$. Define

$$\tau_\rho(\theta_1, \theta_0) = \int_{[0,1]} \rho(\Theta) \in \Omega^{2k-1}(M), \qquad \theta_0, \theta_1 \in \mathcal{A}_P. \tag{1.8}$$

Then Stokes' theorem (1.1) applied to the projection $[0,1] \times M \to M$ implies

$$d\tau_\rho(\theta_1, \theta_0) = \rho(\theta_1) - \rho(\theta_0). \tag{1.9}$$

Therefore, the de Rham cohomology class of $\rho(\theta)$ is independent of θ, so is a topological invariant of the G-bundle $\pi : P \to M$.

The form $\tau_\rho(\theta_1, \theta_0)$ is one kind of *Chern-Simons form*; it depends on two connections. There is also a Chern-Simons form $\tau_\rho(\theta) \in \Omega^{\bullet}(P)$ which depends on a single connection θ. It lives on the total space P, not on the base M. To construct $\tau_\rho(\theta)$ note that the G-bundle $\pi^*P \to P$ has a tautological section $P \to \pi^*P$, so a tautological trivialization and tautological connection $\theta_{\mathrm{taut}} \in \mathcal{A}_{\pi^*P}$. Define

$$\tau_\rho(\theta) = \tau_\rho(\pi^*\theta, \theta_{\mathrm{taut}}) \in \Omega^{2k-1}(P), \qquad \theta \in \mathcal{A}_P. \tag{1.10}$$

Since the curvature of θ_{taut} vanishes, (1.10) implies

$$d\tau_\rho(\theta) = \pi^*\rho(\theta). \tag{1.11}$$

It follows that the restriction $\iota^*\tau_\rho(\theta)$ to a fiber is closed. In fact, the restriction to the fiber $P_m = \pi^{-1}(m)$ compares two canonical connections on $P_m \times P_m \to P_m$: one derived from the section $p \mapsto (p, p_0)$ for any p_0 fixed, and the other from the section $p \mapsto (p,p)$. Since these connections are flat the associated Chern-Weil forms vanish, and so the Chern-Simons form relating these two connections determines a left-invariant closed $(2k-1)$-form α_ρ on G. Equation (1.12) asserts that $\rho(\theta)$ is a transgression of α_ρ; compare (1.2).

1.3. **A universal connection.** We generalize a basic construction of the previous section as follows. Consider the G-bundle $\mathcal{A}_P \times P \to \mathcal{A}_P \times M$. It carries a canonical connection Θ: the restriction of Θ to $\{\theta\} \times P$ is θ and its restriction to $\mathcal{A}_P \times \{m\}$ vanishes. Then $\rho(\Theta) \in \Omega^{2k}(\mathcal{A}_P \times M)$ is closed, and its restriction to $\{\theta\} \times M$ is $\rho(\theta)$. If Δ is any manifold of dimension $\le 2k$, possibly with boundary, consider the diagram

$$\begin{array}{ccc} \Delta \times \operatorname{Map}(\Delta, \mathcal{A}_P) \times M & \xrightarrow{\;e\;} & \mathcal{A}_P \times M \\ \big\downarrow & & \\ \operatorname{Map}(\Delta, \mathcal{A}_P) \times M & & \end{array}$$

The vertical map is projection and e is evaluation. Define

$$\tau_\Delta = \int_\Delta e^*\rho(\theta) \in \Omega^{2k-\dim\Delta}\big(\operatorname{Map}(\Delta, \mathcal{A}_P) \times M\big). \tag{1.12}$$

Stokes' theorem implies

$$d\tau_\Delta = -(-1)^{\dim\Delta}\,\tau_{\partial\Delta}. \tag{1.13}$$

Equation (1.8) is the application of (1.12) to $\Delta = [0,1]$ and affine maps; then (1.13) specializes to (1.9).

1.4. **Topology.** The de Rham cohomology class of $\rho(\theta)$ is best understood in terms of the *classifying space* BG. It is the base of a *universal* G-bundle $EG \to BG$, universal in the sense that any G-bundle $P \to M$ admits a G-equivariant map

$\gamma : P \to EG$ called a *classifying map*[2] for P. The space EG is characterized (in an appropriate category of spaces) as being a contractible G-space. A classifying map is a section of $P \times_G EG \to M$, a bundle with contractible fibers, so the space of classifying maps is contractible. Now a classifying map induces a quotient $\bar{\gamma} : M \to BG$, and it follows that the pullback to M of any cohomology class on BG is independent of $\bar{\gamma}$. Such pullbacks are *characteristic classes*.

Fix a smooth (infinite dimensional) model of $EG \to BG$, for example by embedding $G \hookrightarrow U(N)$ and letting EG be the Stiefel manifold of N orthonormal vectors in a complex Hilbert space. The Chern-Weil construction applied to any connection θ_{EG} on $EG \to BG$ induces a map

$$\begin{aligned} \mathrm{Sym}^{\bullet}_{\mathrm{inv}}(\mathfrak{g}^*) &\longrightarrow H^{2\bullet}(BG;\mathbb{R}) \\ \rho &\longmapsto \big[\rho(\theta_{EG})\big] \end{aligned} \tag{1.14}$$

From (1.9) we conclude that this map is independent of θ_{EG}. It is a topological fact that (1.14) is an isomorphism. In other words, any real characteristic class is realized in Chern-Weil theory.

The image of $H^{\bullet}(BG;\mathbb{Z})$ in $H^{\bullet}(BG;\mathbb{R})$ is a full lattice, so pulls back to a full lattice $\Lambda^{\bullet}(G) \subset \mathrm{Sym}^{\bullet}_{\mathrm{inv}}(\mathfrak{g}^*)$ in the space of invariant polynomials on $\mathfrak{g}$. If $\rho \in \Lambda^k(G)$ and θ is a connection on a principal bundle $P \to M$, then for any $2k$-cycle $f : \Sigma \to M$ we have

$$\int_{\Sigma} f^*\rho(\theta) \ \in \mathbb{Z}. \tag{1.15}$$

In the next section we use another basic fact about the topology of the classifying space of a compact Lie group: $H^{\mathrm{odd}}(BG;\mathbb{R}) = 0$. Equivalently, the odd homology groups $H_{\mathrm{odd}}(BG)$ are torsion.

1.5. **Cheeger-Simons.** An integral characteristic class $\lambda \in H^{2k}(BG;\mathbb{Z})$ leads to a generalization of log holonomy. Let $\rho \in \Lambda^k(G)$ be the invariant polynomial corresponding to the image of λ in real cohomology. Suppose $P \to M$ is a principal bundle with connection θ. Let $\gamma : P \to EG$ be a classifying map. Fix a connection θ_{EG} on $EG \to BG$. The *Cheeger-Simons differential character* associated to λ is a homomorphism

$$\sigma_\lambda(\theta) : Z_{2k-1}(M) \longrightarrow \mathbb{R}/\mathbb{Z}$$

[2]We remark that in this paper we never use classifying maps for connections, only classifying maps for bundles.

on $(2k-1)$-cycles $f:S\to M$ such that if $f=\partial F$ for $F:\Sigma^{2k}\to M$, then

$$\sigma_\lambda(\theta)(f)=\int_\Sigma F^*\rho(\theta)\pmod 1.$$

We construct $\sigma_\lambda(\theta)$ as follows. For any $(2k-1)$-cycle $f:S\to M$ the composition $\bar\gamma\circ f:S\to BG$ is torsion in $H_{2k-1}(BG)$, since the real odd homology of the classifying space vanishes, so there is a chain $F:\Sigma^{2k}\to BG$ so that ∂F is ℓ copies of $\bar\gamma\circ f$ for some ℓ. Let $[\frac1\ell\Sigma]\in H_{2k}(BG;\frac1\ell\mathbb{Z}/\mathbb{Z})$ be the homology class determined by this data. Then

$$\sigma_\lambda(\theta)(f)=\frac1\ell\int_\Sigma F^*\rho(\theta_{EG})\;+\;\int_S f^*\tau_\rho(\theta,\gamma^*\theta_{EG})\;-\;\lambda[\tfrac1\ell\Sigma]\pmod 1,$$

where τ_ρ is the Chern-Simons form (1.8) associated to ρ.

The set of differential characters of degree[3] $2k$ forms a group $\check H^{2k}(M)$, and there are exact sequences analogous to (1.5) and (1.6). Yet a third rewriting of these exact sequences is also useful. Define

$$A^q(M)=\bigl\{(\mu,\omega)\in H^q(M;\mathbb{Z})\times\Omega^q_{\mathbb{Z}}(M):\mu_{\mathbb{R}}=[\omega]_{\text{de Rham}}\bigr\},\tag{1.16}$$

where $\mu_{\mathbb{R}}\in H^q(M;\mathbb{R})$ is the image of μ under the degree q version of (1.3). A differential character $\sigma\in\check H^q(M)$ determines an element of $A^q(M)$. The form $\omega(\sigma)\in\Omega^q_{\mathbb{Z}}(M)$ is part of the definition of σ, and to fix the cohomology class $\mu(\sigma)\in H^q(M;\mathbb{Z})$ we need to specify its $\mathbb{Z}/\ell\mathbb{Z}$-periods. (Its $\mathbb{Z}$-periods are integrals of $\omega(\sigma)$.) Thus given a q-chain $F:\Sigma^q\to M$ with boundary ℓ copies of $f:S\to M$, we have

$$\mu(\sigma)[\Sigma]=\int_\Sigma F^*\omega(\sigma)\;-\;\ell\cdot\sigma(f)\pmod\ell.\tag{1.17}$$

The kernel of the map $\sigma\mapsto\bigl(\mu(\sigma),\omega(\sigma)\bigr)$ is a torus, as indicated in the exact sequence

$$0\longrightarrow\frac{H^{q-1}(M;\mathbb{R})}{H^{q-1}(M;\mathbb{Z})}\longrightarrow\check H^q(M)\longrightarrow A^q(M)\longrightarrow 0.\tag{1.18}$$

The Cheeger-Simons differential character $\sigma_\lambda(\theta)\in\check H^{2k}(M)$ of a connection θ is a distinguished lift of $\bigl(\lambda(P),\rho(\theta)\bigr)\in A^{2k}(M)$. Its existence is based on the vanishing of the kernel torus in (1.18) for the classifying space BG.

[3]Our indexing differs from that in [CheeS]

2. Families of Connections

The geometric invariants constructed in §1 give rise to invariants of families of connections. Let $p : \mathcal{M} \to T$ be a smooth fiber bundle whose fibers are *closed* of relative dimension n. In order to integrate we assume the relative tangent bundle is oriented. Given a G-connection ϑ over $\mathcal{M}$ and $\lambda \in H^{2k}(BG;\mathbb{Z})$ we integrate $\sigma_\lambda(\vartheta)$ over the fibers of p to obtain a differential character of degree $q = 2k - n$ on the parameter space T. When $q = 0, 1$ the differential character is suitably geometric, but for $q \geq 2$ it is an equivalence class of geometric objects. For many reasons—some of which are explained at the end of this section—we would like to refine the construction to produce geometric invariants, not simply equivalence classes. For example, for $q = 2$ the integral of the Cheeger-Simons character is an equivalence class of circle bundles with connection. With special hypotheses we recount a construction (which generalizes [RSW], [F1]) of a functorial circle bundle with connection in the given equivalence class. In general we expect that a choice of geometric representative depends on more choices, and indeed this is a feature of the general construction we present in §3.

2.1. **Integration over the fiber.** A G-connection over the fiber bundle $p : \mathcal{M} \to T$ is a principal G-bundle $\pi : \mathcal{P} \to \mathcal{M}$ with connection ϑ. It restricts to a family of connections ϑ_t on $\mathcal{P}_t \to \mathcal{M}_t$ for $t \in T$, but there is "horizontal" information as well.

Fix $\rho \in \mathrm{Sym}^k_{\mathrm{inv}}(\mathfrak{g}^*)$. The integral of the Chern-Weil form $\rho(\vartheta) \in \Omega^{2k}(\mathcal{M})$ over the fibers of p is a differential form of degree $q = 2k - n$:

$$\int_{\mathcal{M}/T} \rho(\vartheta) \ \in \Omega^q(T). \tag{2.1}$$

By Stokes' theorem it is closed.

Fix $\lambda \in H^{2k}(BG;\mathbb{Z})$. Then $\mathcal{P} \to \mathcal{M}$ has a characteristic class $\lambda(\mathcal{P}) \in H^{2k}(\mathcal{M};\mathbb{Z})$. There is integration over the fiber $H^{2k}(\mathcal{M}) \to H^q(T)$ in cohomology with any coefficients, so we obtain

$$\int_{\mathcal{M}/T} \lambda(\mathcal{P}) \ \in H^q(T;\mathbb{Z}). \tag{2.2}$$

If $\rho \in \Lambda^k(G)$ is the invariant polynomial corresponding to λ, then (2.1) and (2.2) are compatible in the sense that (c.f. (1.16))

$$\Big(\int_{\mathcal{M}/T} \lambda(\mathcal{P}), \int_{\mathcal{M}/T} \rho(\vartheta)\Big) \ \in A^q(T). \tag{2.3}$$

It is natural to ask for a lift of (2.3) to a differential character of degree q (c.f. (1.18)). Not surprisingly, such a lift is provided by integrating the differential character $\sigma_\lambda(\vartheta)$ over the fibers of p.

Proposition 2.1. *There is a functorially defined differential character $\int_{\mathcal{M}/T}\sigma_\lambda(\vartheta)\in\check H^q(T)$ compatible with $\bigl(\int_{\mathcal{M}/T}\lambda(\mathcal{P}),\int_{\mathcal{M}/T}\rho(\vartheta)\bigr)$.*

Proof. It suffices to define the sought-after differential character $\bar\sigma$ on a $(q-1)$-*simplex* $f:\Delta\to T$. Form the pullback square

$$\begin{array}{ccc} f^*\mathcal{M} & \xrightarrow{\tilde f} & \mathcal{M} \\ \Big\downarrow & & \Big\downarrow \\ \Delta & \xrightarrow{f} & T \end{array}$$

Now $f^*\mathcal{M}$ is homeomorphic to $\Delta\times M$ for a fixed closed oriented manifold M, so composing with a chain representative of the fundamental class of M we can suppose that $\tilde f$ is a $(2k-1)$-chain. Define

$$\bar\sigma(f)=\sigma_\lambda(\vartheta)(\tilde f). \tag{2.4}$$

It follows that (2.4) holds for any $(q-1)$-*chain* $f:\Delta\to T$.

We claim $\bar\sigma$ is a differential character with "curvature" $\int_{\mathcal{M}/T}\rho(\vartheta)$. For if $F:\Sigma\to T$ is a q-chain with boundary $f:S\to T$, then $\tilde F:F^*\mathcal{M}\to\mathcal{M}$ is a $2k$-chain with boundary $\tilde f:f^*\mathcal{M}\to T$ and

$$\bar\sigma(f)=\sigma_\lambda(\vartheta)(\tilde f)=\int_{F^*\mathcal{M}}\tilde F^*\rho(\vartheta)=\int_\Sigma\int_{F^*\mathcal{M}/\Sigma}\tilde F^*\rho(\vartheta)=\int_\Sigma F^*\left\{\int_{\mathcal{M}/T}\rho(\vartheta)\right\}.$$

Similarly, the integral characteristic class $\mu(\bar\sigma)$ of $\bar\sigma$ is $\int_{\mathcal{M}/T}\lambda(\mathcal{P})$. For if $F:\Sigma\to T$ is a q-chain with boundary ℓ copies of a $(q-1)$-cycle $f:S\to T$, then from (1.17) we have, computing modulo 1,

$$\begin{aligned}\mu(\bar\sigma)[\Sigma]&\equiv\int_\Sigma F^*\left\{\int_{\mathcal{M}/T}\rho(\vartheta)\right\}\;-\;\ell\cdot\bar\sigma(f)\\ &\equiv\int_{F^*\mathcal{M}}\tilde F^*\rho(\vartheta)\;-\;\ell\cdot\sigma_\lambda(\vartheta)(\tilde f)\\ &\equiv\mu\bigl(\sigma_\lambda\bigr)(\vartheta)(\tilde f)\\ &\equiv\lambda(\mathcal{P})[f^*\mathcal{M}]\\ &\equiv\left(\int_{\mathcal{M}/T}\lambda(\mathcal{P})\right)[\Sigma].\end{aligned}$$

□

2.2. **Low degrees.** The simplest invariants occur for $q = 0, 1$, in which case they take a geometric form. In fact, we have only implicitly defined $\check{H}^0$ from the exact sequences (e.g. (1.18)). Explicitly,

$$\check{H}^0(T) = \mathrm{Map}(T; \mathbb{Z})$$

is the set of (locally constant) maps $T \to \mathbb{Z}$. The integrated differential character of a connection on a principal G-bundle $\mathcal{P} \to \mathcal{M}$ over a family of closed oriented $2k$-manifolds is simply the characteristic number of $\mathcal{P}$ associated to $\lambda \in H^{2k}(BG; \mathbb{Z})$.

For $q = 1$ it follows quickly from the definitions that

$$\check{H}^1(T) = \mathrm{Map}(T; \mathbb{R}/\mathbb{Z})$$

is the space of *smooth* maps $T \to \mathbb{R}/\mathbb{Z}$. For a single connection θ on $P \to M$ the integrated differential character $\int_M \sigma_\theta(\theta) \in \check{H}^1(pt) \cong \mathbb{R}/\mathbb{Z}$ is the *Chern-Simons invariant* of θ. It varies smoothly in a smooth family of connections; its differential is the 1-form (2.1).

2.3. **Degree two.** In (1.4) we define $\check{H}^2(T)$ to be the set of equivalence classes of circle bundles with connection on T. We observed that this agrees with the general definition of differential characters, since a $\mathbb{T}$-connection is determined up to isomorphism by its holonomy. For a family of connections on an oriented family $\mathcal{M} \to T$ of $(2k-2)$-manifolds Proposition 2.1 gives an equivalence class of $\mathbb{T}$-bundles with connection over T. It is desirable to refine this to a $\mathbb{T}$-bundle with connection rather than an equivalence class. There is a special construction which works under the following hypotheses:

$$\begin{aligned} &\text{(i) } \mathcal{P}_t \to \mathcal{M}_t \text{ is trivializable for all } t \in T, \\ &\text{(ii) } \mathrm{Map}(\mathcal{M}_t, G) \text{ is connected for all } t \in T. \end{aligned} \tag{2.5}$$

For example, these hypotheses are satisfied if $k = n = 2$ and G is connected and simply connected, which is the case considered in [F1]. We explain this construction in our current setup; in §3 we give a different construction which works in general, but depends on additional "orientation" choices for $\mathcal{M} \to T$.

First, recall that a circle bundle with connection on T may be constructed by patching from the following data: (i) a cover $\mathcal{U} = \{U_i\}_{i \in I}$ of T; (ii) a 1-form $\alpha_i \in \Omega^1(U_i)$ for each $i \in I$; and (iii) a function $\alpha_{i_0 i_1} : U_{i_0} \cap U_{i_1} \to \mathbb{R}/\mathbb{Z}$ for each $i_0, i_1 \in I$. They are required to satisfy

$$\begin{aligned} d\alpha_{i_0 i_1} &= \alpha_{i_1} - \alpha_{i_0} &&\text{on } U_{i_0} \cap U_{i_1}, \\ 0 &= \alpha_{i_1 i_2} - \alpha_{i_0 i_2} + \alpha_{i_0 i_1} &&\text{on } U_{i_0} \cap U_{i_1} \cap U_{i_2}. \end{aligned}$$

We use a generalization in which I indexes a set which maps onto a cover of T.

Let ϑ be a connection on $\mathcal{P} \xrightarrow{\pi} \mathcal{M} \xrightarrow{p} T$, and assume (2.5) is satisfied. Define

$$\mathcal{U} = \bigl\{(U,s) : U \subset T \text{ open},\ s : p^{-1}U \to \pi^{-1}p^{-1}U \text{ is a section of } \pi\bigr\}.$$

Hypothesis (i) of (2.5) implies $\mathcal{U}$ maps onto a cover of T. For $(U,s) \in \mathcal{U}$ set

$$\alpha_{(U,s)} = \int_{p^{-1}U/U} s^*\tau_\rho(\vartheta) \ \in \Omega^1(U),$$

where τ_ρ is the Chern-Simons form (1.10). Now suppose $(U_{i_0}, s_{i_0}), (U_{i_1}, s_{i_1}) \in \mathcal{U}$. Set $U = U_{i_0} \cap U_{i_1}$ and $Q = \pi^{-1}p^{-1}U$ the total space of the G-bundle $\pi^{-1}p^{-1}U \to p^{-1}U$. There is a unique automorphism φ_1 of Q so that $s_{i_1} = \varphi_1 \circ s_{i_0}$, and by hypothesis (ii) of (2.5) there is a path φ_t of automorphisms of Q joining the identity φ_0 to φ_1. Let Θ be the connection on $[0,1] \times Q$ whose restriction to $\{t\} \times Q$ is $\varphi_t^*\vartheta$ and whose restriction to $[0,1] \times \{q\}$ vanishes. Let

(2.6) $$\alpha_{i_0 i_1} = \int_{[0,1]\times p^{-1}U/U} s_0^*\tau_\rho(\Theta) \pmod 1.$$

The integrality of λ, in the form (1.15), implies that

(2.7) $\alpha_{i_0 i_1} : U_{i_0} \cap U_{i_1} \to \mathbb{R}/\mathbb{Z}$ is independent of the path φ_t.

For if $\tilde{\varphi}_t$ is another path, then we have two paths $\varphi_t^*\vartheta, \varphi_t^*\vartheta$ in $\mathcal{A}_Q$ from ϑ to $\varphi_1^*\vartheta$, each contained in a fixed orbit of the gauge transformations. Since $\mathcal{A}_Q$ is contractible, there is a disk $F : \Delta \to \mathcal{A}_Q$ whose boundary is the difference of the paths. By Stokes' theorem the difference of the integrals in (2.6) along the two paths is

$$I_1 = \int_{(\Delta\times p^{-1}U)/U} (F \times \mathrm{id}_{p^{-1}U})^*\rho(\Theta) = \int_{(\Delta\times p^{-1}U)/U} \rho\bigl((F\times \mathrm{id}_Q)^*\Theta\bigr).$$

The bundle automorphisms $\varphi_t, \tilde{\varphi}_t$ construct a continuous bundle $\overline{Q}$ which fits into the diagram

(2.8) $$\begin{array}{ccc} \Delta \times Q & \xrightarrow{\tilde{q}} & \overline{Q} \\ \big\downarrow & & \big\downarrow \\ \Delta \times p^{-1}U & \xrightarrow{q} & (\Delta/\partial\Delta) \times p^{-1}U \end{array}$$

Smooth the spaces on the right and the maps in (2.8). Then for any connection $\overline{\Theta}$ on $\overline{Q}$ we have $I_2 = \int_{((\Delta/\partial\Delta)\times p^{-1}U)/U} \rho(\overline{\Theta})$ is a $\mathbb{Z}$-valued map on U, since on each fiber of $(\Delta/\partial\Delta)\times p^{-1}U \to U$ the Chern-Weil integral computes an integral characteristic number. By pullback in (2.8), the integral I_2 equals $I_3 = \int_{(\Delta\times p^{-1}U)/U} \rho\bigl(\tilde{q}^*(\overline{\Theta})\bigr)$, which is then also $\mathbb{Z}$-valued. Now from (1.9) and Stokes

$$I_1 - I_3 = \int_{\partial\Delta\times p^{-1}U/U} \tau_\rho\bigl((\partial F\times \mathrm{id}_Q)^*\Theta, \partial\tilde{q}^*\overline{\Theta}\bigr). \tag{2.9}$$

But the two connections which appear in (2.9) are equal, and so (2.9) vanishes. Therefore, I_1 is $\mathbb{Z}$-valued, which completes the verification of (2.7).

2.4. **Higher degrees.** The foregoing is not sufficient for geometric and physical applications. Most apparently, we would like a generalization of the preceding construction which does not assume (2.5). Then too we want to construct geometric representatives of the differential characters in $\check{H}^q(T)$ for $q > 2$. For example, one can regard $\check{H}^3(T)$ as the set of equivalence classes of *gerbes with connection* on T. There are many recent references, some of which also discuss integration: for example, see [B], [H], [CMW], [G], [BM].

More is needed. Consider a fiber bundle $\mathcal{M} \to T$ whose fibers are compact oriented n-manifolds *with boundary*. In that case the form (2.1) is still defined, but is no longer closed: by Stokes' theorem

$$d\int_{\mathcal{M}/T} \rho(\vartheta) = \int_{\partial\mathcal{M}/T} \rho(\vartheta). \tag{2.10}$$

In other words, $\int_{\mathcal{M}/T} \rho(\vartheta)$ is a *cochain* in de Rham theory, not a cocycle. Of course, there are cochain theories for integral cohomology, and any of them gives sense to the integral in (2.2) as a cochain. This suggests constructing $\int_{\mathcal{M}/T} \sigma_\lambda(\vartheta)$ as a cochain in a cochain theory for differential characters. We expect a suitable version of (2.10) to hold: $\int_{\mathcal{M}/T} \sigma_\lambda(\vartheta)$ should "trivialize" $\int_{\partial\mathcal{M}/T} \sigma_\lambda(\vartheta)$. For example, if $n = 2k-1$ and we refine $\int_{\partial\mathcal{M}/T} \sigma_\lambda(\vartheta)$ to a circle bundle with connection over T, we expect $\int_{\mathcal{M}/T} \sigma_\lambda(\vartheta)$ to be a trivialization[4] of the circle bundle. This construction was carried out in [F1] in a special case (which falls under (2.5)), but only for $n = 2k-1$. Notice that what are called "geometric representatives" of classes in $\check{H}^q$ in the previous paragraph are here called "cocycles".[5]

[4] In fact, we expect a *nonflat* trivialization whose "covariant derivative" is $\int_{\mathcal{M}/T} \rho(\vartheta)$.

[5] We understand "cochain" and "cocycle" in a broad sense—not just in the context of cochain complexes of abelian groups. For example, a cocycle for an element of $H^q(T;\mathbb{Z})$ can be taken to

3. Geometric Representatives

Just as closed differential forms are differential geometric representatives of real cohomology, there are well-known geometric representatives of integer cohomology in low degrees. For example a degree one class is a map to the circle and a degree two class is a circle bundle with connection. *Differential cohomology theory* is a generalization to cohomology classes of higher degree, as well as to generalized cohomology theories. After a brief introduction to these ideas, which are under development, we apply them to families of connections and in particular to classical Chern-Simons theory.

3.1. **Differential cochains and differential functions.** In a forthcoming paper [HS] M. Hopkins and I. Singer construct cochain models for Cheeger-Simons differential characters, including integration. In any model there is a bigrading, and the differential characters of degree q appear as the degree (q,q) cohomology.

Let $\bigl(C^\bullet(M),\delta\bigr)$ denote the smooth singular cochain complex on a smooth manifold M. Recall that integration over smooth singular chains defines a cochain map $\Omega^\bullet(M)\hookrightarrow C^\bullet(M;\mathbb{R})$. Define the complex of *differential cochains* by

$$\check C(q)^p(M)=C^p(M;\mathbb{Z})\times C^{p-1}(M;\mathbb{R})\times\Omega^p(M), p\ge q; C^p(M;\mathbb{Z})\times C^{p-1}(M;\mathbb{R})\times\{0\}, p<q,$$

with differential

$$d(c,h,\omega)=(\delta c,\omega-c_{\mathbb{R}}-\delta h,d\omega).$$

Then $\check H^q(M)$ is the q^{th} cohomology group of $\check C(q)^\bullet(M)$. The assignment $M\longmapsto\check C(q)^\bullet(M)$ has good functoriality properties, and moreover integration in singular theory and integration in de Rham theory combine to give a theory of integration for this cochain model. But whereas integration of differential forms requires only a topological orientation, the "orientation" required to integrate cochains in singular theory, and hence in differential cohomology as well, is more elaborate.

Other models are possible. For example, fix an Eilenberg-MacLane space $K(\mathbb{Z},q)$ and a cocycle $\iota\in Z^q\bigl(K(\mathbb{Z},q);\mathbb{R}\bigr)$ whose cohomology class is the image of the canonical class in $H^q\bigl(K(\mathbb{Z},q);\mathbb{Z}\bigr)$. Then we define a geometric representative of a class in $\check H^q(M)$ to be a triple (c,h,ω) where

$$\begin{aligned}c&:M\longrightarrow K(\mathbb{Z},q)\\ h&\in C^{q-1}(M;\mathbb{R})\\ \omega&\in\Omega^q_{\mathbb{Z}}(M)\end{aligned}\tag{3.1}$$

be a particular map $T\to K(\mathbb{Z},q)$ into a given Eilenberg-MacLane space; its cohomology class is the homotopy class of the map. A trivialization of the cocycle is a homotopy to the constant map.

satisfy

$$\delta h = \omega - c^*\iota. \tag{3.2}$$

More generally, Hopkins-Singer consider pairs (K,ι) for any space K, where now $\iota \in Z^q(K;\mathbb{R})$ is any cocycle. Then a triple (c,h,ω) as in (3.1) and (3.2) is called a *differential function* from M to (K,ι). This is the starting point of a construction of differential versions of more general topological invariants than integer cohomology.

We apply this to construct differential cocycle representatives of Cheeger-Simons differential characters. Fix a connection θ_{EG} on a smooth model $EG \to BG$ of the universal G-bundle, as in §1, and fix also an invariant polynomial $\rho \in \mathrm{Sym}^k_{\mathrm{inv}}(\mathfrak{g}^*)$. If θ is a connection on a principal G-bundle $P \to M$, and $\gamma : P \to EG$ is a classifying map, then there is a natural differential function from M to $\big(BG, \rho(\theta_{EG})\big)$, namely the triple

$$\begin{aligned} \bar\gamma\ &: M \longrightarrow BG \\ \tau_\rho(\theta, \gamma^*\theta_{EG})\ &\in \Omega^{2k-1}(M) \subset C^{2k-1}(M;\mathbb{R}) \\ \rho(\theta)\ &\in \Omega^{2k}_{\mathbb{Z}}(M) \end{aligned} \tag{3.3}$$

Note that equation (3.2) for this triple follows from (1.9). If $\lambda \in H^{2k}(BG;\mathbb{Z})$ and $\rho \in \Lambda^k(G)$ is the corresponding invariant polynomial, then once and for all we fix a differential cocycle

$$\check\lambda = (c_{BG}, h_{BG}, \omega_{BG}) \in \check C(2k)^{2k}(BG), \tag{3.4}$$

where $c_{BG} \in Z^{2k}(BG;\mathbb{Z})$ is a cocycle representative of λ, the form $\omega_{BG} = \rho(\theta_{EG})$, and $h_{BG} \in C^{2k-1}(BG;\mathbb{R})$ is a cochain which satisfies $\delta h_{BG} = \omega_{BG} - c_{BG}$. Then a connection θ and classifying map γ determine a differential cocycle

$$\check\sigma_{\check\lambda}(\theta,\gamma) = (c,h,\omega) \in \check C(2k)^{2k}(M),$$

essentially by pulling back $\check\lambda$ via (3.3):

$$\begin{aligned} c &= \bar\gamma^* c_{BG} \\ h &= \tau_\rho(\theta, \gamma^*\theta_{EG}) + \bar\gamma^* h_{BG} \\ \omega &= \rho(\theta) \end{aligned}$$

We integrate the refinement $\check\sigma_{\check\lambda}(\theta,\gamma)$ of the Cheeger-Simons class $\sigma_\lambda(\theta)$ over the fibers of a suitably oriented family $p : \mathcal{M} \to T$ to obtain geometric representatives of the invariants of §2. This is the sought-after geometric refinement of the differential character in Proposition 2.1.

As expected, the cocycle $\check{\sigma}_{\check{\lambda}}(\theta,\gamma)$ depends on more than the connection θ; it also depends on a classifying map γ. But in the sense we now explain, the collection of pairs (θ,γ) is equivalent to the collection of connections θ. Namely, over a fixed manifold M each forms a *groupoid* and the groupoids are equivalent. An object in the groupoid $\mathcal{C}_G(M)$ of G-connections is a connection θ on a G-bundle $P\to M$. A morphism $\theta'\to\theta$ is a bundle isomorphism $\varphi: P'\to P$ covering the identity on M such that $\varphi^*\theta=\theta'$. An object in the groupoid $\mathcal{C}'_G(M)$ is a pair (θ,γ) with θ as before and $\gamma: P\to EG$ a classifying map. A morphism $(\theta',\gamma')\xrightarrow{\varphi}(\theta,\gamma)$ is as before; the classifying maps play no role. There is an obvious forgetful functor $\mathcal{C}'_G(M)\to\mathcal{C}_G(M)$ which is an isomorphism of groupoids. In many geometric problems—and in the physics we consider next—there is no loss in replacing $\mathcal{C}_G$ with $\mathcal{C}'_G$.

3.2. **Classical Chern-Simons theory.** We specialize to $k=2$, which is the relevant case for the 3-dimensional topological quantum field theory known to physicists as *Chern-Simons theory*. The classical action is in the first instance a function[6] $\mathcal{C}_G(X)\to\mathbb{R}/\mathbb{Z}$ for each closed oriented 3-manifold X. It is useful to extend the notion of classical action to lower dimensions. Thus, if Y is a closed oriented 2-manifold, the classical action assigns to each G-connection on Y a $\mathbb{T}$-torsor, and more generally to a family of G-connections on Y it assigns a $\mathbb{T}$-bundle with connection over the parameter space T. The construction we use depends on a choice of classifying map. But since the categories $\mathcal{C}'_G(Y)$ and $\mathcal{C}_G(Y)$ of connections with and without classifying maps are equivalent, and since ultimately the quantum invariants are computed from the space of equivalence classes of connections, there is no loss in equipping connections with a classifying map. Similarly, there are classical actions on any compact oriented manifold—with or without boundary—of dimension ≤ 3.

The theory depends on a class $\lambda\in H^4(BG;\mathbb{Z})$ and a refinement to a differential cocycle $\check{\lambda}$ as in (3.4). Let M be a compact oriented n-manifold. A family of G-connections on M parametrized by T is a connection ϑ on a G-bundle $\mathcal{P}\to T\times M$. As emphasized we assume that $\mathcal{P}$ is equipped with a classifying map $\gamma:\mathcal{P}\to EG$. The Chern-Simons classical action is then

$$\int_{(T\times M)/T}\check{\sigma}_{\check{\lambda}}(\vartheta,\gamma)\ \in\check{C}(4-n)^{4-n}(T). \tag{3.5}$$

[6]In other words, a functor from $\mathcal{C}_G(X)$ to $\mathbb{R}/\mathbb{Z}$, where the latter is regarded as a small category with no nontrivial automorphisms.

The properties enumerated in [F1]—notably functoriality and gluing—should follow from analogous properties of integration in the Hopkins-Singer model. It is important to observe that the geometric invariants (3.5) depend on an orientation of M for integration of differential cochains, which is much more than simply a topological orientation. Of course, the equivalence class of (3.5) (see Proposition 2.1) depends only on a topological orientation.

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