FIBER BUNDLES AND VECTOR BUNDLES

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These notes, written for another class, are provided for reference. I begin with fiber bundles. Then I will discuss the particular case of vector bundles and the construction of the tangent bundle. Intuitively, the tangent bundle is the disjoint union of the tangent spaces (see (21)). What we must do is define a manifold structure on this disjoint union and then show that the projection of the base is *locally trivial*.

Fiber bundles

Definition 1. Let $\pi\colon E \to M$ be a map of sets. Then the *fiber* of π over $p \in M$ is the inverse image $\pi^{-1}(p) \in E$.

In some cases, as in the context of fiber bundles, it it convenient to denote the fiber $\pi^{-1}(p)$ as E_p. If π is surjective then each fiber is nonempty, and the map π partitions the domain E:

$$(2) \qquad E = \coprod_{p \in M} E_p$$

Recall that 'II' is the notation for *disjoint union*; that is, an ordinary union in which the sets are disjoint. (So 'disjoint' functions as an adjective; 'disjoint union' is not a compound noun.)

Definition 3. Let $\pi\colon E \to M$ be a surjective map of manifolds. Then π is a *fiber bundle* if for every $p \in M$ there exists a neighborhood $U \subset M$ of p, a manifold F, and a diffeomorphism

$$(4) \qquad \varphi\colon \pi^{-1}(U) \longrightarrow U \times F$$

such that the diagram

$$(5) \qquad
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\ \varphi\ } & U \times F \\
& {\scriptstyle \pi}\searrow \quad \swarrow {\scriptstyle \pi_1} & \\
& U &
\end{array}$$

commutes. If $\pi'\colon E' \to M$ is also a fiber bundle, then a *fiber bundle map* $\varphi\colon E \to E'$ is a smooth map of manifolds such that the diagram

$$(6) \qquad
\begin{array}{ccc}
E & \xrightarrow{\ \varphi\ } & E' \\
& {\scriptstyle \pi}\searrow \quad \swarrow {\scriptstyle \pi'} & \\
& M &
\end{array}$$

commutes. If φ has an inverse, then we say φ is an *isomorphism* of fiber bundles.

In the diagram $\pi_1 \colon U \times F \to U$ is projection onto the first factor. (We will often use the notation $\pi_k \colon X_1 \times X_2 \times \cdots \times X_n \to X_k$ for projection onto the kth factor of a Cartesian product.) The commutation of the diagram is the assertion that $\pi = \pi_1 \circ \varphi$, which means that φ maps fibers of π diffeomorphically onto F. The manifold F may vary with the local trivialization.

Remark 7. It is very important that the existence of *local trivializations* (4) in the definition of a fiber bundle is a *condition*. The local trivializations are not part of the *data* in the definition.

Definition 8. The modification of Definition 3 in which F is fixed once and for all defines a *fiber bundle with fiber F*.

You should prove that we can always take F to be fixed on each component of M.

   Terminology: E is called the *total space* of the bundle and M is called the *base*. As mentioned, F is called the *fiber*.

Example 9. The simplest example of a fiber bundle is $\pi = \pi_1 \colon M \times F \to M$, where M and F are fixed manifolds. This is called *the trivial bundle with fiber F*. A fiber bundle is *trivializable* if it is isomorphic to the trivial bundle.

The characteristic property of a fiber bundle is that it is *locally trivializable*: compare (5) and (6).

Exercise 10. Prove that every fiber bundle $\pi \colon E \to M$ is a submersion.

Remark 11. We can also define a fiber bundle of topological spaces: in Definition 3 replace 'manifold' by 'topological space' and 'diffeomorphism' by 'homeomorphism'.

Transition functions

   A local trivialization of a fiber bundle is analogous to a chart in a smooth manifold. Notice, though, that a topological manifold has no intrinsic notion of smoothness, so we must define smooth manifolds by comparing charts via transition functions and then specifying an atlas of C^∞ compatible charts. By contrast, when defining the notion of a fiber bundle we already know what a smooth manifold is and so only assert the existence of smooth local trivializations. But we can still construct fiber bundles by a procedure analogous to the construction of smooth manifolds when we don't have the total space as a manifold.

   Let $\pi \colon E \to M$ be a fiber bundle and $\varphi_1 \colon \pi^{-1}(U_1) \to U_1 \times F$ and $\varphi_2 \colon \pi^{-1}(U_2) \to U_2 \times F$ two local trivializations with the same fiber F. Then the *transition function* from φ_1 to φ_2 is

$$(12) \qquad\qquad g_{21} \colon U_1 \cap U_2 \longrightarrow \mathrm{Aut}(F)$$

defined by

$$(13) \qquad (\varphi_2 \circ \varphi_1^{-1})(p, f) = \big(p, g_{21}(p)(f)\big), \qquad p \in U_1 \cap U_2, \quad f \in F.$$

Here $\mathrm{Aut}(F)$ is the group of diffeomorphisms of F. The map g_{21} is smooth in the sense that the associated map

$$(14) \qquad \begin{aligned} \tilde{g}_{21} \colon (U_1 \cap U_2) \times F &\longrightarrow F \\ (p \quad, \quad f) &\longmapsto g_{21}(p)(f) \end{aligned}$$

is smooth. When we come to vector bundles F is a vector space and the transition functions land in the finite dimensional Lie group of linear automorphisms; then the map (12) is *smooth* if and only if (14) is smooth. Note in the formulas that $g_{21}(p)(f)$ means the diffeomorphism $g_{21}(p)$ applied to f.

Just as the overlap, or transition, functions between coordinate charts encode the smooth structure of a manifold, the transition functions between local trivializations encode the global properties of a fiber bundle.

We can use transition functions to construct a fiber bundle when we are only given the base and fiber but not the total space. For that start with the base manifold M and the fiber manifold F and suppose $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M. Now suppose given transition functions

$$(15) \qquad g_{\alpha_1 \alpha_0} \colon U_{\alpha_0} \cap U_{\alpha_1} \longrightarrow \mathrm{Aut}(F)$$

for each pair $\alpha_0, \alpha_1 \in F$, and assume these are smooth in the sense defined above using (14). We demand that $g_{\alpha\alpha}(p)$ be the identity map for all $p \in U_\alpha$, that $g_{\alpha_1 \alpha_0} = g_{\alpha_0 \alpha_1}^{-1}$ on $U_{\alpha_0} \cap U_{\alpha_1}$, and that

$$(16) \qquad (g_{\alpha_0 \alpha_2} \circ g_{\alpha_2 \alpha_1} \circ g_{\alpha_1 \alpha_0})(p) = \mathrm{id}_F, \qquad p \in U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2}.$$

Equation (16) is called the *cocycle condition*. We are going to use the transition functions (15) to construct E from the local trivial bundles $U_\alpha \times F \to U_\alpha$, and the cocycle condition (16) ensures that the gluing is consistent. So define

$$(17) \qquad E = \coprod_{\alpha \in A} (U_\alpha \times F) \big/ \sim$$

where

$$(18) \qquad (p_{\alpha_0}, f) \sim (p_{\alpha_1}, g_{\alpha_1 \alpha_0}(f)), \qquad p_{\alpha_0} = p_{\alpha_1} \in U_{\alpha_0} \cap U_{\alpha_1}, \quad f \in F.$$

The projections $\pi_1 \colon U_\alpha \times F \to U_\alpha$ fit together to define a surjective map $\pi \colon E \to M$. It is straightforward to verify that each fiber $\pi^{-1}(p)$ of π is diffeomorphic to F. Also, observe that the quotient map restricted to $U_\alpha \times F$ is injective.

Proposition 19. *The quotient* (17) *has the natural structure of a smooth manifold and $\pi \colon E \to M$ is a fiber bundle with fiber F.*

I will only sketch the proof, which I suggest you think through carefully. Once we prove E is a manifold then the fiber bundle property—the local triviality—is easy as the construction comes with local trivializations. Equip E with the quotient topology: a set $G \subset E$ is open if and only if its inverse image in $\coprod_{\alpha \in A} (U_\alpha \times F)$ is open. This topology is Hausdorff: if $q_1, q_2 \in E$ have different projections in M they can be separated by open subsets of M; if they lie in the same fiber, then we use the fact that F is Hausdorff to separate them in some $U_\alpha \times F$. The topology is also second countable: since M is second countable there is a countable subset of A for which E is the quotient (17), and then as F is second countable we can find a countable base for the topology. To construct an atlas, cover each U_α by coordinate charts of M and cover F by coordinate charts. Then the Cartesian product of these charts produces charts of $U_\alpha \times F$, and so charts of E. It remains to check that the overlap of these coordinate charts is C^∞.

Vector bundles

The notion of fiber bundle is very general: the fiber is a general manifold. In many cases the fibers have extra structure. In lecture we met a fiber bundle of affine spaces. There are also fiber bundles of Lie groups. One important special type of fiber bundle is a *vector bundle*: the fibers are vector spaces.

Definition 20. A *vector bundle* is a fiber bundle as in Definition 3 for which the fibers $\pi^{-1}(p)$, $p \in M$ are vector spaces, the manifolds F in the local trivialization are vector spaces, and for each $p \in U$ the local trivialization (4) restricts to a vector space isomorphism $\pi^{-1}(p) \to F$.

As mentioned earlier, the transition functions (15) take values in the Lie group of linear automorphisms of the vector space F. (For $F = \mathbb{R}^n$ we denote that group as $GL_n\mathbb{R}$.)

You should picture a vector bundle over M as a smoothly varying locally trivial family (2) of vector spaces parametrized by M. "Smoothly varying" means that the collection of vector spaces fit together into a smooth manifold.

The tangent bundle

Let M be a smooth manifold and assume $\dim M = n$. (If different components of M have different dimensions, then make this construction one component at a time.) One of the most important consequences of the smooth structure is the *tangent bundle*, the collection of tangent spaces

$$
(21) \qquad\qquad \pi \colon \coprod_{p \in M} T_p M \longrightarrow M
$$

made into a vector bundle. We can construct it as a vector bundle using Proposition 19 as follows. Let $\{(U_\alpha, x_\alpha)\}_{\alpha \in A}$ be a countable covering of M by coordinate charts. (As remarked earlier countability is not an issue and we can use the entire atlas.) Then we obtain local trivializations (4) for

each coordinate chart:

$$\varphi_\alpha \colon \pi^{-1}(U_\alpha) \longrightarrow U \times \mathbb{R}^n$$

(22)
$$\xi = \sum_i \xi^i \frac{\partial}{\partial x^i} \longmapsto (p; \xi^1, \xi^2, \ldots, \xi^n),$$

where $\xi \in T_p M$. This is well-defined, but so far only a map of sets as we have not even topologized the total space in (21). But we can still use (22) to compute the transition functions via (13). Namely, define $g_{\alpha_1 \alpha_0} \colon U_{\alpha_0} \cap U_{\alpha_1} \to GL_n \mathbb{R}$ by

(23)
$$g_{\alpha_1 \alpha_0}(p) = d(x_{\alpha_1} \circ x_{\alpha_0}^{-1})_p.$$

In other words, the transition functions for the tangent bundle are the differentials of the overlap functions for the charts.