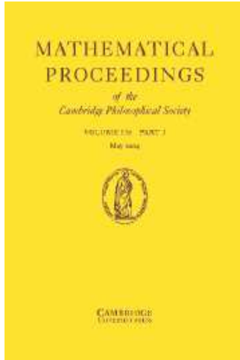

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Mathematical Proceedings of the Cambridge Philosophical Society / Volume 136 / Issue 03 / May 2004, pp 617 - 623
DOI: 10.1017/S0305004103007114, Published online: 21 April 2004

Math. Proc. Camb. Phil. Soc. (2004), **136**, 617 © 2004 *Cambridge Philosophical Society* 617
DOI: 10.1017/S0305004103007114 *Printed in the United Kingdom*

A quick proof of the rational Hurewicz theorem and a computation of the rational homotopy groups of spheres

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(*Received* 13 *June* 2002)

Abstract

We give an elementary proof of the rational Hurewicz theorem and compute the rational cohomology groups of Eilenberg–MacLane spaces and the rational homotopy groups of spheres. Instead of using the Serre spectral sequence, we only assume the classical Hurewicz theorem, and give a short proof of the rational Gysin and Wang long exact sequences, which are applied inductively to the path fibration of Eilenberg–MacLane spaces.

1. *Introduction*

We give an elementary proof of the following well-known theorem in topology:

THEOREM 1·1. *Let X be a simply connected topological space with $\pi_i(X) \otimes \mathbb{Q} = 0$ for $1 < i < r$. Then the Hurewicz map induces an isomorphism*

$$H\colon \pi_i(X) \otimes \mathbb{Q} \longrightarrow H_i(X; \mathbb{Q})$$

for $1 \leqslant i < 2r - 1$ and a surjection for $i = 2r - 1$.

This result is proved, for example, in [**2**, p. 65], as a consequence of a spectral sequence of Kahn. For a finite complex, it is a consequence of a stronger result which follows from results of Serre and which can be found in [**5**, p. 207]. This stronger version states that, if X is a simply connected finite complex with $\pi_i(X) \otimes \mathbb{Q} = 0$ for $1 < i < r$, then the kernel of $H\colon \pi_i(X) \to H_i(X; \mathbb{Z})$ is a finite abelian group for $i < 2r - 1$ and the cokernel is a finite abelian group for $i \leqslant 2r - 1$. The proof of this result uses the Serre spectral sequence, the mod $\mathcal{C}$ theory of Serre, as well as the classical Hurewicz theorem. Our proof of the above result only uses the classical Hurewicz theorem plus some elementary algebraic topological considerations.

One of the reasons for this paper is that Theorem 1·1 is needed to complete the computation of rational oriented bordism groups in the book of Milnor and Stasheff

[**5**]. Their arguments, up to page 207, are at a similar elementary level to ours, which we propose for the last step. This is particularly useful since the rational oriented bordism groups are the basic input into the proof of the important Hirzebruch signature Theorem. Of course, there are many other situations where one needs to compute rational homotopy groups. Often Theorem 1·1 is the main tool needed to carry out such computations.

We also prove some other fundamental results concerning rational cohomology and homotopy groups like the rational cohomology of $K(\mathbb{Z}, r)$ and the rational cohomology groups of spheres:

THEOREM 1·2. *The rational cohomology ring of $K(\mathbb{Z}, n)$ is given by*

$$H^*(K(\mathbb{Z}, 2r+1); \mathbb{Q}) \cong H^*(S^{2r+1}; \mathbb{Q}),$$

$$H^*(K(\mathbb{Z}, 2r); \mathbb{Q}) \cong \mathbb{Q}[\iota_{2r}],$$

where the last term denotes the rational polynomial ring in the $2r$-dimensional fundamental cohomology class ι_{2r}.

THEOREM 1·3. *The rational homotopy groups of spheres vanish with only the exception of*

$$\pi_r(S^r) \otimes \mathbb{Q} \cong \mathbb{Q},$$

$$\pi_{4r-1}(S^{2r}) \otimes \mathbb{Q} \cong \mathbb{Q}.$$

We assume that the reader is familiar with basic algebraic topology as for example presented in [**1**] or [**4**]. In particular, we will apply the classical Hurewicz theorem.

2. *Gysin and Wang sequences*

In this section we prepare the proof with some (co)homological considerations. We will work in the category of CW-complexes which is no restriction as any topological space can be replaced by a weakly equivalent CW-complex, the weak equivalence inducing isomorphisms in homotopy and homology groups, of course. In particular, this also applies if we replace a map by a weakly equivalent fibration, and if we consider the homotopy fibre of a map, see [**6**].

LEMMA 2·1. *Let $p: E \to B$ be a fibration over a connected CW-complex B with path connected fibre F over the base point $b_0 \in B$. If $\tilde{H}_i(F; \mathbb{Q}) = 0$ for $i < r$, then the (relative) induced map of the projection*

$$q_*: H_i(E, F; \mathbb{Q}) \longrightarrow H_i(B, b_0; \mathbb{Q})$$

is an isomorphism for $i \leqslant r$. Thus, the (absolute) induced map

$$p_*: H_i(E; \mathbb{Q}) \longrightarrow H_i(B; \mathbb{Q})$$

is also an isomorphism for $i < r$ and a surjection for $i = r$. If, in addition, B is simply connected, then q_* is also an isomorphism for $i = r + 1$, hence p_* is also an isomorphism for $i = r$ and a surjection for $i = r + 1$.

Proof. We prove this lemma by induction over the k-skeleton B^k of B. The case $k = 0$ is clear. Since B is connected, we can assume $B^0 = \{b_0\}$ and, if B is simply connected, we can assume $B^1 = \{b_0\}$, too. We denote the restriction of E to B^k by E^k. The result follows from the 5-lemma by comparing the exact sequences with rational coefficients

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_{i+1}(E^{k+1}, E^k) & \longrightarrow & H_i(E^k, F) & \longrightarrow & H_i(E^{k+1}, F) & \longrightarrow & \cdots \\ & & \downarrow q_* & & \downarrow q_* & & \downarrow q_* & & \\ \cdots & \longrightarrow & H_{i+1}(B^{k+1}, B^k) & \longrightarrow & H_i(B^k, b_0) & \longrightarrow & H_i(B^{k+1}, b_0) & \longrightarrow & \cdots \end{array}$$

once we know that $q_*: H_i(E^{k+1}, E^k) \to H_i(B^{k+1}, B^k)$ is an isomorphism for $i < r + 1$ and a surjection for $i = r + 1$ (if B is simply connected, an isomorphism for $i < r + 2$ and a surjection for $i = r + 2$).

To show that q_* is an isomorphism or surjective we apply excision and a trivialization of E over the $(k + 1)$-cells D^{k+1}_α of B^{k+1} to conclude:

$$H_{i+1}(E^{k+1}, E^k) \cong H_i\left(\bigsqcup_\alpha D^{k+1}_\alpha \times F, \bigsqcup_\alpha S^k_\alpha \times F\right).$$

Then the statement follows from the Künneth theorem and the fact that $\tilde{H}_j(F) = 0$ for $j < r$, since $k \geqslant 0$ and, if B is simply connected, $k \geqslant 1$. The statement on p_* follows from the exact homology sequence of the pair (E, F).

Corollary 2·2. *Let p: $E \to B$ be a fibration over a simply connected CW-complex with path-connected fibre F. If p_*: $H_i(E; \mathbb{Q}) \to H_i(B; \mathbb{Q})$ is an isomorphism for $i < r$ and a surjection for $i = r$, then $\tilde{H}_i(F; \mathbb{Q}) = 0$ for $i < r$.*

Proof. Since $\tilde{H}_0(F; \mathbb{Q}) = 0$, we know from Lemma 2·1 that $q_*: H_i(E, F; \mathbb{Q}) \to H_i(B, b_0; \mathbb{Q})$ is an isomorphism for $i = 1$ and $i = 2$. Then the exact homology sequence of (E, F) implies that $H_1(F; \mathbb{Q}) = 0$, if $H_i(E; \mathbb{Q}) \to H_i(B; \mathbb{Q})$ is an isomorphism for $i = 1$ and a surjection for $i = 2$. This proves the case $r = 2$ and inductively one obtains the general result.

Lemma 2·3. *Let G be an abelian torsion group. Then for all $r > 0$,*

$$\tilde{H}_*(K(G, r); \mathbb{Q}) = 0.$$

Proof. The result follows by induction over r using Lemma 2·1 and the path fibrations [**6**]

$$\Omega K(G, r) \longrightarrow PK(G, r) \longrightarrow K(G, r),$$

where the path space $PK(G, r)$ is contractible having fibre $\Omega K(G, r) \simeq K(G, r - 1)$.

Thus we reduce the proof to $K(G, 1)$. Here we first note that for a cyclic group $G = \mathbb{Z}/n$, $n \geqslant 1$, the space $K(\mathbb{Z}/n, 1)$ is an infinite dimensional lens space $S^\infty/(\mathbb{Z}/n)$, where $\mathbb{Z}/n$ is the group of nth roots of unity acting on $\mathbb{C}^\infty$ by scalar multiplication. Next note that $H_i(K(\mathbb{Z}/n, 1); \mathbb{Q}) \cong H_i(S^{2k+1}/(\mathbb{Z}/n); \mathbb{Q})$ as long as $2k + 1 > i$, and that this group vanishes for $i > 0$. Using the Künneth theorem one concludes $\tilde{H}_*(K(G, 1); \mathbb{Q}) = 0$ for all finite abelian groups.

If G is an arbitrary abelian torsion group and $z \in H_i(K(G, 1); \mathbb{Q})$, then there is a finite subcomplex j: $X \hookrightarrow K(G, 1)$ such that $z = j_* x$ for some $x \in H_i(X; \mathbb{Q})$. Then the image of $\pi_1(X)$ in $\pi_1(K(G,1)) = G$ is a finite subgroup H and the inclusion $j\colon X \hookrightarrow K(G,1)$ factors through $K(H,1)$ implying that x maps to 0 in $H_i(K(H,1);\mathbb{Q}) = 0$ and thus $z = 0$.

We use Lemma 2·1 and Lemma 2·3 to deduce the following:

LEMMA 2·4. *Let X be a simply connected space with $\tilde{H}_i(X;\mathbb{Q}) = 0$ for $i < r$. Then there is an $(r-1)$-connected space Y and a map $f\colon Y \to X$ inducing isomorphisms in all rational homotopy and homology groups.*

Proof. We do this inductively. If X is $(s-1)$-connected for some $1 < s < r$ we take a map $X \to K(\pi_s(X), s)$ inducing an isomorphism in π_s and consider its homotopy fibre X'. By construction, X' is s-connected, and $X' \to X$ induces an isomorphism in rational homotopy groups. The homotopy fibre of $X' \to X$ is a $K(\pi_s(X), s-1)$. Since $\pi_s(X)$ is a torsion group, Lemma 2·3 with Lemma 2·1 implies that $X' \to X$ induces an isomorphism in rational homology.

With the classical Hurewicz theorem, an immediate application of this result is the following:

COROLLARY 2·5. *Let X be a simply connected CW-complex. Then all rational homotopy groups vanish in dimension $< r$ if and only if all rational homology groups vanish in dimension $< r$, and the Hurewicz map induces an isomorphism $\pi_r(X)\otimes\mathbb{Q} \to H_r(X;\mathbb{Q})$. Moreover, assume that a map $f\colon X \to Y$ of simply connected CW-complexes has a simply connected homotopy fibre. Then f induces an isomorphism in all rational homology groups $\pi_i \otimes \mathbb{Q}$ for $i < r$ and a surjection for $i = r$ if and only if it induces an isomorphism in all rational homotopy groups $H_i(-;\mathbb{Q})$ for $i < r$ and a surjection for $i = r$.*

Here, the statement on f follows by consideration of the homotopy fibre and Lemmas 2·1 and 2·2.

Our next ingredient is a rational Gysin sequence. Let $p\colon S \to b$ be a fibration whose fibre is a rational r-homology sphere Σ^r, i.e., $H_*(\Sigma^r;\mathbb{Q}) = H_*(S^r;\mathbb{Q})$. To such a fibration we associate its *Thom space*

$$MS \coloneqq (S \times I \cup_p B)/(S \times \{1\}),$$

where we consider p as a map $S \times \{0\} \to B$. Note that if S is the sphere bundle of a vector bundle E, then MS is the Thom space of E. We say that S is $\mathbb{Q}$-*orientable* if there is a *Thom class* $\theta_S \in H^{r+1}(MS;\mathbb{Q})$, which by definition is a class whose restriction to each fibre $M\Sigma^r_x \coloneqq \Sigma^r_x \times I/(\Sigma^r_x \times \{0\} \cup \Sigma^r_x \times \{1\})$ is a non-trivial element in $H^{r+1}(M\Sigma^r_x;\mathbb{Q}) \cong \mathbb{Q}$ for all $x \in B$. The choice of a Thom class gives an *orientation* of S.

The cap product with the Thom class gives a homomorphism

$$\tilde{H}_{k+r+1}(MS;\mathbb{Q}) \longrightarrow H_k(B;\mathbb{Q})$$

$$x \mapsto p_X(x \cap \theta_S).$$

The standard induction argument over the cells of B shows that this is an isomorphism for B a finite complex [**1**, **3**]. Arguing with limits over finite subcomplexes of B the same result holds for arbitrary CW-complexes. Passing to rational cohomology by dualizing, we obtain for arbitrary CW-complexes the *Thom isomorphism*

$$H^k(B;\mathbb{Q}) \longrightarrow \tilde{H}^{k+r+1}(MS;\mathbb{Q})$$

$$x \mapsto p^*x \cup \theta_S.$$

As for vector bundles, one defines the *rational Euler class* of S as $e_S := i^*\theta_S \in H^{r+1}(B;\mathbb{Q})$, where i: $B \to MS$ is the inclusion.

From the long exact sequence in rational cohomology and the Thom isomorphism one obtains the rational *Gysin sequence*:

LEMMA 2·6. *Let p: $S \to B$ an oriented fibration over a CW-complex B with fibre a rational homology r-sphere Σ^r. Then there is a long exact sequence*

$$\cdots \longrightarrow H^i(B;\mathbb{Q}) \xrightarrow{\cup e_S} H^{i+r+1}(B;\mathbb{Q}) \xrightarrow{p^*} H^{i+r+1}(S;\mathbb{Q}) \longrightarrow H^{i+1}(B;\mathbb{Q}) \longrightarrow \cdots.$$

We finish our general study of these fibrations by noting that, if B is simply connected, then S is orientable. This follows again by induction starting from a CW-decomposition of B without 1-cells, using the same argument as in the proof of the Thom isomorphism.

Finally, we introduce the Wang sequence for a fibration p: $E \to S^{r+1}$ with fibre F. We decompose S^{r+1} into upper and lower hemispheres S^{r+1}_+ and S^{r+1}_-. Since fibrations over contractible spaces are trivial [**6**], one has homotopy equivalences $E_\pm \simeq F$ for the restrictions. On the intersection $S^{r+1}_+ \cap S^{r+1}_- = S^r$, the trivialization over the upper hemisphere gives a homotopy equivalence $E_+ \cap E_- = E|_{S^r} \simeq S^r \times F$. Applying the Mayer–Vietoris sequence to the covering $E_\pm$ and the Künneth theorem, one obtains the rational *Wang sequence*:

LEMMA 2·7. *Let p: $E \to S^{r+1}$ be a fibration with fibre F. Then there is a long exact sequence*

$$\cdots \longrightarrow H^i(E;\mathbb{Q}) \xrightarrow{i^*} H^i(F;\mathbb{Q}) \longrightarrow H^{i-r}(F;\mathbb{Q}) \xrightarrow{\delta} H^{i+1}(E;\mathbb{Q}) \longrightarrow \cdots$$

with $\delta(i^*x \cup y) = (-1)^p x \cup \delta(y)$ for $x \in H^p(E;\mathbb{Q})$ and $y \in H^q(F;\mathbb{Q})$.

The product formula is a consequence of the fact that the cross product is a derivation with respect to the boundary operator in the Mayer–Vietoris sequence.

3. *Proof of the main theorems*

In this section, all cohomology groups have rational coefficients. Now we are ready to prove Theorem 1·2 on the rational cohomology of $K(\mathbb{Z}, n)$:

Proof. The proof proceeds by induction, using the fibration

$$K_{n-1} \longrightarrow P \longrightarrow K_n$$

with P contractible, where $K_n := K(\mathbb{Z}, n)$. The case $n = 1$ is clear, since $K(\mathbb{Z}, 1) = S^1$.

We begin, for $r \geqslant 1$, with the induction step from $H^*(K_{2r-1}) \cong H^*(S^{2r-1})$ to $H^*(K_{2r})$. Since K_{2r} is simply connected and K_{2r-1} is a rational homology $(2r-1)$-sphere, the Gysin sequence, Lemma 2·6 of the above fibration with $n = 2r$ gives isomorphisms

$$H^i(K_{2r}) \xrightarrow{\cup e} H^{i+2r}(K_{2r})$$

with $e \in H^{2r}(K_{2r})$ denoting the corresponding Euler class. In particular, e and the fundamental class $\iota_{2r} \in H^{2r}(K_{2r}) \cong \mathbb{Q}$ are non-zero multiples of each other, and

$$H^*(K_{2r}) \cong \mathbb{Q}[\iota_{2r}].$$

For the other induction step from $H^*(K_{2r}) \cong \mathbb{Q}[\iota_{2r}]$ to $H^*(K_{2r+1})$, we consider again the above fibration with $n = 2r+1$. Let $f\colon S^{2r+1} \to K_{2r+1}$ represent a generator of $\pi_{2n+1}(K_{2n+1}) = \mathbb{Z}$ and consider the pullback fibration

$$K_{2r} \longrightarrow f^*P \longrightarrow S^{2r+1}.$$

By comparing the exact homotopy sequences of P and f^*P, we conclude that f^*P is $2r$-connected. Thus by the classical Hurewicz theorem, $\tilde{H}^i(f^*P;\mathbb{Z}) = 0$ for $i \leqslant 2r$, implying $\tilde{H}^i(f^*P;\mathbb{Q}) = 0$ for $i \leqslant 2r$, too. Combining this and the Wang sequence, Lemma 2·7 with the induction hypothesis $H^*(K_{2r}) \cong \mathbb{Q}[\iota_{2r}]$ we conclude

$$\tilde{H}^*(f^*P;\mathbb{Q}) = 0.$$

Since f^*P is simply connected, we can apply Lemma 2·5 and so $\pi_*(f^*P) \otimes \mathbb{Q} = 0$. By comparing the exact homotopy sequences of P and f^*P, again, we conclude that $f\colon S^{2r+1} \to K_{2r+1}$ induces isomorphisms in all rational homotopy groups. Thus by Lemma 2·5, f induces an isomorphism in all rational homology groups. This completes the proof of Theorem 1·2

Now we prove Theorem 1·3 on the rational homotopy groups of spheres.

Proof. In the last paragraph of the previous proof we showed that

$$\pi_*(S^{2r+1}) \otimes \mathbb{Q} \cong \pi_*(K_{2r+1}) \otimes \mathbb{Q},$$

giving the proof of Theorem 1·3 for odd dimensions. We finish the proof for even dimension $n=2r$ by choosing a map $f\colon S^{2r} \to K_{2r}$ representing a generator of $\pi_{2r}(K_{2r}) = \mathbb{Z}$. The pullback fibration

$$K_{2r-1} \longrightarrow f^*P \longrightarrow S^{2r}$$

satisfies that f^*P is $2r$-connected and $\pi_i(f^*P) \cong \pi_i(S^{2r})$ for $i \neq 2r$. As by Theorem 1·2, K_{2r-1} is a rational homology sphere, we can apply the Gysin sequence, Lemma 2·6

$$\cdots \longrightarrow H^i(S^{2r}) \xrightarrow{\cup e} H^{i+2r}(S^{2r}) \xrightarrow{p^*} H^{i+2r}(f^*P) \longrightarrow H^{i+1}(S^{2r}) \longrightarrow \cdots.$$

From this and $H^{2r}(f^*P) = 0$, we conclude that $\tilde{H}^i(f^*P) \cong \mathbb{Q}$ for $i = 4r-1$ and vanishes otherwise. Hence, f^*P is a rational homology $(4r-1)$-sphere. Now we apply Lemma 2·4 to replace f^*P by an $(4r-2)$-connected space Y with the same rational homology and homotopy groups as f^*P. Choose a map $g\colon S^{4r-1} \to Y$ representing a non-trivial element in $\pi_{4r-1}(Y) \cong \mathbb{Q}$. Since the rational homology of Y is that of f^*P, which is the same as S^{4r-1}, the map g induces an isomorphism on all rational homology groups. By Lemma 2·5, g induces an isomorphism on all rational homotopy groups, implying that $\pi_*(X) \otimes \mathbb{Q} \cong \pi_*(S^{4n-1}) \otimes \mathbb{Q}$.

Finally, we prove the rational Hurewicz Theorem 1·1.

Proof. First we need to know the rational homology of a $K(G, r)$ for G a rational vector space. In the case $G = \mathbb{Q}$, the fibration

$$\Omega K(\mathbb{Q}/\mathbb{Z}, r) \longrightarrow K(\mathbb{Z}, r) \longrightarrow K(\mathbb{Q}, r)$$

yields by Lemma 2·1 and Lemma 2·3 that $K(\mathbb{Z}, r) \rightarrow K(\mathbb{Q}, r)$ is a rational homology eqivalence. Hence, $\tilde{H}_i(K(G, r); \mathbb{Q}) = 0$ for $i < 2r$ with the exception of $H_r(K(G, r); \mathbb{Q}) = G$, which follows for finite dimensional G from the Künneth theorem and for general G by a direct limit argument.

Now we consider the map

$$f\colon X \longrightarrow \prod_{i=r}^{2r-1} K(H_i(X; \mathbb{Q}), i)$$

where the map $X \rightarrow K(H_i(X; \mathbb{Q}), i)$, $r \leqslant i < 2r$, corresponds by the universal coefficient theorem to the map $H_i(X; \mathbb{Z}) \rightarrow H_i(X; \mathbb{Q})$ induced from the inclusion $\mathbb{Z} \rightarrow \mathbb{Q}$. By construction, f induces an isomorphism in rational homology in dimension $i < 2r$ and has simply connected homotopy fibre. With Lemma 2·5 we conclude that f induces an isomorphism in rational homotopy groups for $i < 2r - 1$ and a surjection for $i = 2r - 1$.

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