CHAPTER IV.

Characteristic Classes

After the excursion into homotopy theory in the previous chapter, we return now to the differentiable category. Thus in this chapter, in the absence of explicit qualifications, all spaces are smooth manifolds, all maps are smooth maps, and $H^*(X)$ denotes the de Rham cohomology.

In Section 6 we first encountered the Euler class of a C^∞ oriented rank 2 vector bundle. It is but one of the many characteristic classes—that is, cohomology classes intrisically associated to a vector bundle. In its modern form the theory of characteristic classes originated with Hopf, Stiefel, Whitney, Chern, and Pontrjagin. It has since found many applications to topology, differential geometry, and algebraic geometry.

In its most rudimentary form the point of view towards the Chern classes really goes back to the old Italian algebraic geometers, but in Section 20 we recast it along the ideas of Grothendieck. We introduce in Section 21 the computational and proof technique known as the splitting principle. This is followed by the Pontrjagin classes, which may be considered the real analogue of the Chern classes. We also include an application to the embedding of manifolds.

In the final section the Chern classes are shown to be the only complex characteristic classes in the following sense: any natural transformation from the complex vector bundles to the cohomology ring is a polynomial in the Chern classes. An added dividend is a classification theorem for complex vector bundles. With its aid we fulfill an earlier promise (see the remark following Prop. 11.9) to show that the vanishing of the Euler class of an oriented sphere bundle does not imply the existence of a section.

For the Euler class of a rank 2 bundle we had in (6.38) an explicit formula in terms of the patching data on the base manifold M. Elegant as the Grothendieck approach to the Chern classes is, it is not directly linked to the geometry of M, for it gives no such patching formulas. In the concluding remarks to this chapter we describe without proof a recipe for

constructing the Chern classes of a complex vector bundle $\pi: E \to M$ out of the transition functions of E and a partition of unity on M relative to some trivializing good cover for E.

§20 Chern Classes of a Complex Vector Bundle

In this section we will study the characteristic classes of a complex vector bundle. To begin with we define the first Chern class of a complex line bundle as the Euler class of its underlying real bundle. Applying the Leray-Hirsch theorem, we then compute the cohomology ring of the projectivization $P(E)$ of a complex vector bundle E and define the Chern classes of E in terms of the ring structure of $H^*(P(E))$. We conclude with a list of the main properties of the Chern classes.

The First Chern Class of a Complex Line Bundle

Recall that a complex vector bundle of rank n is a fiber bundle with fiber $\mathbb{C}^n$ and structure group $GL(n, \mathbb{C})$. A complex vector bundle of rank 1 is also called a *complex line bundle*. Just as the structure group of a real vector bundle can be reduced to the orthogonal group $O(n)$, so by the Hermitian analogue of (6.4), the structure group of a rank n complex vector bundle can be reduced to the unitary group $U(n)$. Every complex vector bundle E of rank n has an underlying real vector bundle $E_{\mathbb{R}}$ of rank $2n$, obtained by discarding the complex structure on each fiber. By the isomorphism of $U(1)$ with $SO(2)$, this sets up a one-to-one correspondence between the complex line bundles and the oriented rank 2 real bundles. We define the *first Chern class* of a complex line bundle L over a manifold M to be the Euler class of its underlying real bundle $L_{\mathbb{R}}$: $c_1(L) = e(L_{\mathbb{R}}) \in H^2(M)$.

If L and L' are complex line bundles with transition functions $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$,

$$g_{\alpha\beta},\ g'_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*,$$

then their tensor product $L \otimes L'$ is the complex line bundle with transition functions $\{g_{\alpha\beta} \cdot g'_{\alpha\beta}\}$. By the formula (6.38) which gives the Euler class in terms of the transition functions, we have

$$c_1(L \otimes L') = c_1(L) + c_1(L'). \tag{20.1}$$

Let L^* be the dual of L. Since the line bundle $L \otimes L^* = \mathrm{Hom}(L, L)$ has a nowhere vanishing section given by the identity map, $L \otimes L^*$ is a trivial bundle, By (20.1), $c_1(L) + c_1(L^*) = c_1(L \otimes L^*) = 0$. Therefore,

$$c_1(L^*) = -c_1(L). \tag{20.2}$$

EXAMPLE 20.3 (Tautological bundles on a projective space). Let V be a complex vector space of dimension n and $P(V)$ its projectivization:

$$P(V) = \{\text{1-dimensional subspaces of } V\}.$$

On $P(V)$ there are several God-given vector bundles: the product bundle $\hat{V} = P(V) \times V$, the *universal subbundle* S, which is the subbundle of $\hat{V}$ defined by

$$S = \{(\ell, v) \in P(V) \times V \mid v \in \ell\},$$

and the *universal quotient bundle* Q, defined by the exact sequence

$$(20.4) \qquad 0 \to S \to \hat{V} \to Q \to 0.$$

The fiber of S above each point ℓ in $P(V)$ consists of all the points in ℓ, where ℓ is viewed as a line in the vector space V. The sequence (20.4) is called the *tautological exact sequence* over $P(V)$, and S^* the *hyperplane bundle*.

Consider the composition

$$\sigma : S \hookrightarrow P(V) \times V \to V$$

of the inclusion followed by the projection. The inverse image of any point v is

$$\sigma^{-1}(v) = \{(\ell, v) \mid v \in \ell\}.$$

If $v \neq 0$, $\sigma^{-1}(v)$ consists of precisely one point (ℓ, v) where ℓ is the line through the origin and v; if $v = 0$, then $\sigma^{-1}(0)$ is isomorphic to $P(V)$. Thus S may be obtained from V by separating all the lines through the origin in V. This map $\sigma : S \to V$ is called the *blow-up* or the *quadratic transformation* of of V at the origin. Over the real numbers the blow-up of a plane may be pictured as the portion of a helicoid in Figure 20.1 with its top and bottom edges identified. Indeed, we may view the (x, y)-plane as being traced out by a horizontal line rotating about the origin. In order to separate these lines at the origin, we let the generating line move with constant velocity along the z-axis while it is rotating horizontally. The resulting surface in $\mathbb{R}^3$ is a helicoid.

We now compute the cohomology of $P(V)$. Endow V with a Hermitian metric and let E be the unit sphere bundle of the universal subbundle S:

$$E = \{(\ell, v) \mid v \in \ell, \|v\| = 1\}.$$

Note that $\sigma^{-1}(0)$ is the zero section of the universal subbundle S. Since $S - \sigma^{-1}(0)$ is diffeomorphic to $V - \{0\}$, we see that E is diffeomorphic to the sphere S^{2n-1} in V and that the map $\pi : E \to P(V)$ gives a fibering

$$\begin{array}{c} S^1 \to S^{2n-1} \\ \downarrow \\ P(V). \end{array}$$

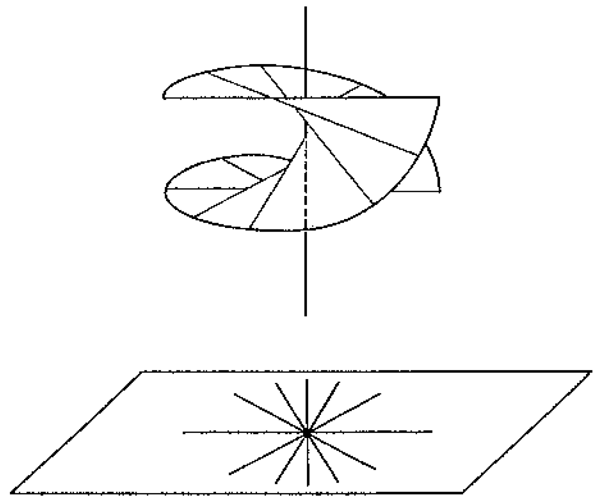

Figure 20.1

By a computation similar to (14.32), the cohomology ring $H^*(P(V))$ is seen to be generated by the Euler class of the circle bundle E, i.e., the first Chern class of the universal subbundle S. It is customary to take $x = c_1(S^*) = -c_1(S)$ to be the generator and write

$$(20.5) \qquad H^*(P(V)) = \mathbb{R}[x]/(x^n), \qquad \text{where } n = \dim_{\mathbb{C}} V.$$

We define the *Poincaré series* of a manifold M to be

$$P_t(M) = \sum_{i=0}^{\infty} \dim H^i(M)\, t^i.$$

By (20.5) the Poincaré series of the projective space $P(V)$ is

$$P_t(P(V)) = 1 + t^2 + \cdots + t^{2(n-1)} = \frac{1 - t^{2n}}{1 - t^2}.$$

The Projectivization of a Vector Bundle

Let $\rho : E \to M$ be a complex vector bundle with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{C})$. We write E_p for the fiber over p and $PGL(n, \mathbb{C})$ for the projective general linear group $GL(n, \mathbb{C})/\{\text{scalar matrices}\}$. The *projectivization* of E, $\pi : P(E) \to M$, is by definition the fiber bundle whose fiber at a point p in M is the projective space $P(E_p)$ and whose transition functions $\bar{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to PGL(n, \mathbb{C})$ are induced from $g_{\alpha\beta}$. Thus a point of $P(E)$ is a line ℓ_p in the fiber E_p.

As on the projectivization of a vector space, on $P(E)$ there are several tautological bundles: the pullback $\pi^{-1}E$, the *universal subbundle* S, and the *universal quotient bundle* Q.

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] & \pi^{-1}E \arrow[r] \arrow[d] & Q \arrow[r] & 0 & \\ & & P(E) \arrow[drrr, "\pi"'] & & & E \arrow[d, "p"] \\ & & & & & M \end{tikzcd}$$

The pullback bundle $\pi^{-1}E$ is the vector bundle over $P(E)$ whose fiber at ℓ_p is E_p. When restricted to the fiber $\pi^{-1}(p)$ it becomes the trivial bundle,

$$\pi^{-1}E\,|_{P(E)_p} = P(E)_p \times E_p,$$

since $\rho : E_p \to \{p\}$ is a trivial bundle. The universal subbundle S over $P(E)$ is defined by

$$S = \{(\ell_p, v) \in \pi^{-1}E \mid v \in \ell_p\}.$$

Its fiber at ℓ_p consists of all the points in ℓ_p. The universal quotient bundle Q is determined by the tautological exact sequence

$$0 \to S \to \pi^{-1}E \to Q \to 0.$$

Set $x = c_1(S^*)$. Then x is a cohomology class in $H^2(P(E))$. Since the restriction of the universal subbundle S on $P(E)$ to a fiber $P(E_p)$ is the universal subbundle $\tilde{S}$ of the projective space $P(E_p)$, by the naturality property of the first Chern class (6.39), it follows that $c_1(\tilde{S})$ is the restriction of x to $P(E_p)$. Hence the cohomology classes $1, x, \ldots, x^{n-1}$ are global classes on $P(E)$ whose restrictions to each fiber $P(E_p)$ freely generate the cohomology of the fiber. By the Leray-Hirsch theorem (5.11) the cohomology $H^*(P(E))$ is a free module over $H^*(M)$ with basis $\{1, x, \ldots, x^{n-1}\}$. So x^n can be written uniquely as a linear combination of $1, x, \ldots, x^{n-1}$ with coefficients in $H^*(M)$; these coefficients are by definition the *Chern classes* of the complex vector bundle E:

$$(20.6) \qquad x^n + c_1(E)x^{n-1} + \cdots + c_n(E) = 0, \qquad c_i(E) \in H^{2i}(M).$$

In this equation by $c_i(E)$ we really mean $\pi^*c_i(E)$. We call $c_i(E)$ the *ith Chern class* of E and

$$c(E) = 1 + c_1(E) + \cdots + c_n(E) \in H^*(M)$$

its *total Chern class*. With this definition of the Chern classes, we see that the ring structure of the cohomology of $P(E)$ is given by

$$(20.7) \qquad H^*(P(E)) = H^*(M)[x]/(x^n + c_1(E)x^{n-1} + \cdots + c_n(E)),$$

where $x = c_1(S^*)$ and n is the rank of E. Since additively

$$H^*(P(E)) = H^*(M) \otimes H^*(P^{n-1}),$$

the Poincaré series of $P(E)$ is

$$(20.8) \qquad P_t(P(E)) = P_t(M)\,\frac{1 - t^{2n}}{1 - t^2}.$$

We now have two definitions of the first Chern class of a line bundle L: as the Euler class of $L_{\mathbb{R}}$, and as a coefficient in (20.6). To check that these two definitions agree we will temporarily reserve the notation $c_1(\quad)$ for the second definition. What must be shown is that $e(L_{\mathbb{R}}) = c_1(L)$.

$$(20.9) \qquad \begin{tikzcd} \pi^{-1}L \arrow[d] & \\ P(L) \arrow[dr, "\pi"'] & L \arrow[d] \\ & M \end{tikzcd}$$

For a line bundle L, $P(L) = M$, $\pi^{-1}L = L$ and the universal subbundle S on $P(L)$ is L itself. Therefore, $x = e(S^*_{\mathbb{R}}) = -e(S_{\mathbb{R}}) = -e(L_{\mathbb{R}})$. So the relation (20.6) is $x + e(L_{\mathbb{R}}) = 0$, which proves that $c_1(L) = e(L_{\mathbb{R}})$.

If E is the trivial bundle $M \times V$ over M, then $P(E) = M \times P(V)$, so $x^n = 0$. Hence *all the Chern classes of a trivial bundle are zero*. In this sense the Chern classes measure the twisting of a complex vector bundle.

Main Properties of the Chern Classes

In this section we collect together some basic properties of the Chern classes.

(20.10.1) (Naturality) *If f is a map from Y to X and E is a complex vector bundle over X, then $c(f^{-1}E) = f^*c(E)$.*

$$\begin{tikzcd} f^{-1}E \arrow[d] & E \arrow[d] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

PROOF. Basically this property follows from the functoriality of all the constructions in the definition of the Chern class. To be precise, by (6.39) the first Chern class of a line bundle is functorial. Write S_E for the universal subbundle over PE. Now $f^{-1}PE = P(f^{-1}E)$ and $f^{-1}S^*_E = S^*_{f^{-1}E}$, so if $x_E = c_1(S^*_E)$, then

$$x_{f^{-1}E} = c_1(S^*_{f^{-1}E}) = c_1(f^{-1}S^*_E) = f^*x_E.$$

Applying f^* to

$$x_E^n + c_1(E)x_E^{n-1} + \cdots + c_n(E) = 0,$$

we get

$$x_{f^{-1}E}^n + f^*c_1(E)x_{f^{-1}E}^{n-1} + \cdots + f^*c_n(E) = 0.$$

Hence

$$c_1(f^{-1}E) = f^*c_1(E). \qquad \square$$

It follows from the naturality of the Chern class that if E and F are isomorphic vector bundles over X, then $c(E) = c(F)$.

(20.10.2) *Let V be a complex vector space. If S^* is the hyperplane bundle over $P(V)$, then $c_1(S^*)$ generates the algebra $H^*(P(V))$.*

This was proved earlier (20.5).

(20.10.3) (Whitney Product Formula) $c(E' \oplus E'') = c(E')c(E'')$.

The proof will be given in the next section.

In fact, these three properties uniquely characterize the Chern class (Hirzebruch [1, pp. 58–60]). For future reference we list below three more useful properties.

(20.10.4) *If E has rank n as a complex vector bundle, then $c_i(E) = 0$ for $i > n$.*

This is really a definition.

(20.10.5) *If E has a nonvanishing section, then the top Chern class $c_n(E)$ is zero.*

PROOF. Such a section s induces a section $\tilde{s}$ of $P(E)$ as follows. At a point p in X, the value of $\tilde{s}$ is the line in E_p through the origin and $s(p)$.

$$\begin{tikzcd} P(E) \arrow[d, shift left, "\pi"] \\ X \arrow[u, shift left, "\tilde{s}"] \end{tikzcd}$$

Then $\tilde{s}^{-1}S_E$ is a line bundle over X whose fiber at p is the line in E_p spanned by $s(p)$. Since every line bundle with a nonvanishing section is isomorphic to the trivial bundle, we have the tautology

$$\tilde{s}^{-1}S_E \simeq \text{the trivial line bundle.}$$

It follows from the naturality of the Chern class that

$$\tilde{s}^*c_1(S_E) = 0,$$

which implies that

$$\tilde{s}^*x = 0.$$

Applying $\tilde{s}^*$ to

$$x^n + c_1x^{n-1} + \cdots + c_n = 0,$$

we get

$$\tilde{s}^*c_n = 0.$$

By our abuse of notation this really means $\tilde{s}^*\pi^*c_n = 0$. Therefore $c_n = 0$. $\square$

(20.10.6) *The top Chern class of a complex vector bundle E is the Euler class of its realization:*

$$c_n(E) = e(E_{\mathbb{R}}), \quad \text{where } n = \text{rank } E.$$

This proposition will be proved in the next section after we have established the splitting principle.

§21 The Splitting Principle and Flag Manifolds

In this section we prove the Whitney product formula and compute a few Chern classes. The proof and the computations are based on the splitting principle, which, roughly speaking, states that if a polynomial identity in the Chern classes holds for direct sums of line bundles, then it holds for general vector bundles. In the course of establishing the splitting principle we introduce the flag manifolds. We conclude by computing the cohomology ring of a flag manifold.

The Splitting Principle

Let $\tau : E \to M$ be a C^∞ complex vector bundle of rank n over a manifold M. Our goal is to construct a space $F(E)$ and a map $\sigma : F(E) \to M$ such that:

(1) the pullback of E to $F(E)$ splits into a direct sum of line bundles: $\sigma^{-1}E = L_1 \oplus \cdots \oplus L_n$;

(2) σ^* embeds $H^*(M)$ in $H^*(F(E))$.

Such a space $F(E)$, which is in fact a manifold by construction, is called a *split manifold* of E.

If E has rank 1, there is nothing to prove.

If E has rank 2, we can take as a split manifold $F(E)$ the projective bundle $P(E)$, for on $P(E)$ there is the exact sequence

$$0 \to S_E \to \sigma^{-1}E \to Q_E \to 0;$$

by the exercise below, $\sigma^{-1}E = S_E \oplus Q_E$, which is a direct sum of line bundles.

Exercise 21.1. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of C^∞ complex vector bundles. Then B is isomorphic to $A \oplus C$ as C^∞ a bundle.

Now suppose E has rank 3. Over $P(E)$ the line bundle S_E splits off as before. The quotient bundle Q_E over $P(E)$ has rank 2 and so can be split into a direct sum of line bundles when pulled back to $P(Q_E)$.

$$\begin{tikzcd}
 & & \beta^{-1}S_E \oplus S_{Q_E} \oplus Q_{Q_E} \arrow[d] \\
 & S_E \oplus Q_E \arrow[d] & P(Q_E) \arrow[dl, "\beta"'] \\
E \arrow[d] & P(E) \arrow[dl, "\alpha"'] & \\
M & &
\end{tikzcd}$$

Thus we may take $P(Q_E)$ to be a split manifold $F(E)$. Let $x_1 = \beta^* c_1(S_E^*)$ and $x_2 = c_1(S_{Q_E}^*)$. By the result on the cohomology of a projective bundle (20.7),

$$H^*(F(E)) = H^*(M)[x_1, x_2]/(x_1^3 + c_1(E)x_1^2 + c_2(E)x_1 + c_3(E),$$
$$x_2^2 + c_1(Q_E)x_2 + c_2(Q_E)).$$

The pattern is now clear; we split off one subbundle at a time by pulling back to the projectivization of a quotient bundle.

(21.2)
$$\begin{tikzcd}
 & & & S_1 \oplus \cdots \oplus S_{n-2} \oplus S_{n-1} \oplus Q_{n-1} \arrow[d] \\
 & & S_1 \oplus S_2 \oplus Q_2 \arrow[d] & P(Q_{n-2}) = F(E) \arrow[dl] \\
 & S_1 \oplus Q_1 \arrow[d] & P(Q_1) \arrow[dl] & \\
E \arrow[d] & P(E) \arrow[dl] & & \\
M & & &
\end{tikzcd}$$

So for a bundle E of any rank n, a split manifold $F(E)$ exists and is given explicitly by (21.2). Its cohomology $H^*(F(E))$ is a free $H^*(M)$-module having as a basis all monomials of the form

(21.3)
$$x_1^{a_1}x_2^{a_2} \cdots x_{n-1}^{a_{n-1}}, \quad a_1 \le n-1,\ a_2 \le n-2, \ldots, a_{n-1} \le 1,$$
$$a_1, \ldots, a_{n-1} \text{ nonnegative},$$

where $x_i = c_1(S_i^*)$ in the notation of the diagram.

More generally, by iterating the construction above we see that given any number of vector bundles $E_1, \ldots, E_r$ over M, there is a manifold N and a map $\sigma : N \to M$ such that the pullbacks of $E_1, \ldots, E_r$ to N are all direct sums of line bundles and that $H^*(M)$ injects into $H^*(N)$ under σ^*. The manifold N is a *split manifold* for $E_1, \ldots, E_r$.

Because of the existence of the split manifolds we can formulate the following general principle.

The Splitting Principle. *To prove a polynomial identity in the Chern classes of complex vector bundles, it suffices to prove it under the assumption that the vector bundles are direct sums of line bundles.*

For example, suppose we want to prove a certain polynomial relation $P(c(E), c(F), c(E \otimes F)) = 0$ for vector bundles E and F over a manifold M. Let $\sigma : N \to M$ be a split manifold for the pair E, F. By the naturality of the Chern classes

$$\sigma^* P(c(E), c(F), c(E \otimes F)) = P(c(\sigma^{-1}E), c(\sigma^{-1}F), c((\sigma^{-1}E) \otimes (\sigma^{-1}F))),$$

where $\sigma^{-1}E$ and $\sigma^{-1}F$ are direct sums of line bundles. So if the identity holds for direct sums of line bundles, then

$$\sigma^* P(c(E), c(F), c(E \otimes F)) = 0.$$

By the injectivity of $\sigma^* : H^*(M) \to H^*(N)$,

$$P(c(E), c(F), c(E \otimes F)) = 0.$$

In the next two subsections we give some illustrations of this principle.

Proof of the Whitney Product Formula and the Equality of the Top Chern Class and the Euler Class

We consider first the case of a direct sum of line bundles:

$$E = L_1 \oplus \cdots \oplus L_n.$$

By abuse of notation we write $\pi^{-1}E = L_1 \oplus \cdots \oplus L_n$ for the pullback of E

to the projectivization $P(E)$. Over $P(E)$, the universal subbundle S splits off from $\pi^{-1}E$.

$$\begin{tikzcd} & S \subset \pi^{-1}E \arrow[d] \\ E \arrow[d] & P(E) \arrow[dl, "\pi"] \\ M & \end{tikzcd}$$

Let s_i be the projection of S onto L_i. Then s_i is a section of $\mathrm{Hom}(S, L_i) = S^* \otimes L_i$. Since at every point y of $P(E)$, the fiber S_y is a 1-dimensional subspace of $(\pi^{-1}E)_y$, the projections $s_1, \ldots, s_n$ cannot be simultaneously zero. It follows that the open sets

$$U_i = \{y \in P(E) \mid s_i(y) \neq 0\}$$

form an open cover of $P(E)$. Over each U_i the bundle $(S^* \otimes L_i)|_{U_i}$ has a nowhere-vanishing section, namely s_i; so $(S^* \otimes L_i)|_{U_i}$ is trivial. Let ξ_i be a closed global 2-form on $P(E)$ representing $c_1(S^* \otimes L_i)$. Then $\xi_{i|U_i} = d\omega_i$ for some 1-form ω_i on U_i. The crux of the proof is to find a global form on $P(E)$ that represents $c_1(S^* \otimes L_i)$ and that vanishes on U_i; because ω_i is not a global form on $P(E)$, $\xi_i - d\omega_i$ won't do. However, by shrinking the open cover $\{U_i\}$ slightly we can extend $\xi_i - d\omega_i$ to a global form. To be precise we will need the following lemmas.

Exercise 21.4 (The Shrinking Lemma). Let X be a normal topological space and $\{U_i\}_{i \in I}$ a finite open cover of X. Then there is an open cover $\{V_i\}_{i \in I}$ with

$$\bar{V}_i \subset U_i.$$

Exercise 21.5. Let M be a manifold, U an open subset, and A a closed subset contained in U. Then there is a C^∞ function f which is identically 1 on A and is 0 outside U.

It follows from these two lemmas that on $P(E)$ there exits an open cover $\{V_i\}$ and C^∞ functions ρ_i satisfying

(a) $\bar{V}_i \subset U_i$
(b) ρ_i is 1 on $\bar{V}_i$ and is 0 outside U_i.

Now $\rho_i \omega_i$ is a global form which agrees with ω_i on V_i so that

$$\xi_i - d(\rho_i \omega_i)$$

is a global form representing $c_1(S^* \otimes L_i)$ and vanishing on V_i. In summary,

there is an open cover $\{V_i\}$ of $P(E)$ such that $c_1(S^* \otimes L_i)$ may be represented by a global form which vanishes on V_i.

Since $\{V_i\}$ covers $P(E)$, $\prod_{i=1}^n c_1(S^* \otimes L_i) = 0$. Writing $x = c_1(S^*)$, this gives by (20.1)

$$\prod_{i=1}^n (x + c_1(L_i)) = x^n + \sigma_1 x^{n-1} + \cdots + \sigma_n = 0$$

where σ_i is the ith elementary symmetric polynomial of $c_1(L_1), \ldots, c_1(L_n)$. But this equation is precisely the defining equation of $c(E)$. Thus

$$\sigma_i = c_i(E)$$

and

$$c(E) = \prod (1 + c_1(L_i)) = \prod c(L_i).$$

So the Whitney product formula holds for a direct sum of line bundles. By the splitting principle it holds for any complex vector bundle. As an illustration of the splitting principle we will go through the argument in detail. Let E and E' be two complex vector bundles of rank n and m respectively and let $\pi : F(E) \to M$ and $\pi' : F(\pi^{-1}E') \to F(E)$ be the splitting constructions. Both bundles split completely when pulled back to $F(\pi^{-1}E')$ as indicated in the diagram below.

$$\begin{tikzcd} & & L_1 \oplus \cdots \oplus L_n \oplus L_1' \oplus \cdots \oplus L_m' \arrow[d] \\ & L_1 \oplus \cdots \oplus L_n \oplus \pi^{-1}E' \arrow[d] & F(\pi^{-1}E') \arrow[dl, "\pi'"'] \\ E \oplus E' \arrow[d] & F(E) \arrow[dl, "\pi"'] & \\ M & & \end{tikzcd}$$

Let $\sigma = \pi' \circ \pi$. Then

$$\begin{aligned} \sigma^* c(E \oplus E') &= c(\sigma^{-1}(E \oplus E')) = c(L_1 \oplus \cdots \oplus L_n \oplus L_1' \oplus \cdots \oplus L_m') \\ &= \prod c(L_i) c(L_i') \\ &= \sigma^* c(E) \sigma^*(E') = \sigma^* c(E) c(E'). \end{aligned}$$

Since σ^* is injective, $c(E \oplus E') = c(E)c(E')$. This concludes the proof of the Whitney product formula.

REMARK 21.6. By Exercise (21.1) and the Whitney product formula, whenever we have an exact sequence of C^∞ complex vector bundles

$$0 \to A \to B \to C \to 0,$$

then $c(B) = c(A)c(C)$.

As an application of the existence of the split manifold and the Whitney product formula, we will prove now the relation (20.10.6) between the top Chern class and the Euler class. Let E be a rank n complex vector bundle and $\sigma : F(E) \to E$ its split manifold. Write $\sigma^{-1}E = L_1 \oplus \cdots \oplus L_n$, where the L_i's are line bundles on the split manifold $F(E)$.

$\sigma^* c_n(E) = c_n(\sigma^{-1}E)$	by the naturality of c_n
$= c_1(L_1) \cdots c_1(L_n)$	by the Whitney product formula (20.10.3)
$= e((L_1)_{\mathbb{R}}) \cdots e((L_n)_{\mathbb{R}})$	by the definition of the first Chern class of a complex line bundle
$= e((L_1)_{\mathbb{R}} \oplus \cdots \oplus (L_n)_{\mathbb{R}})$	by the Whitney product formula for the Euler class (12.5)
$= e((\sigma^{-1}E)_{\mathbb{R}})$	
$= \sigma^* e(E_{\mathbb{R}})$.	

By the injectivity of σ^* on cohomology, $c_n(E) = e(E_{\mathbb{R}})$.

Computation of Some Chern Classes

Given a rank n complex vector bundle E we may write formally

$$c(E) = \prod_{i=1}^{n} (1 + x_i),$$

where the x_i's may be thought of as the first Chern class of the line bundles into which E splits when pulled back to the splitting manifold $F(E)$. Since the Chern classes $c_1(E), \ldots, c_n(E)$ are the elementary symmetric functions of $x_1, \ldots, x_n$, by the symmetric function theorem (van der Waerden [1, p. 99]) any symmetric polynomial in $x_1, \ldots, x_n$ is a polynomial in $c_1(E), \ldots, c_n(E)$; a similar result holds for power series.

Example 21.7 (Exterior powers, symmetric powers, and tensor products). Recall that if V is a vector space with basis $\{v_1, \ldots, v_n\}$, then the exterior power $\Lambda^p V$ is the vector space with basis $\{v_{i_1} \wedge \cdots \wedge v_{i_p}\}_{1 \leqslant i_1 < \cdots < i_p \leqslant n}$. So if E is the direct sum of line bundles $E = L_1 \oplus \cdots \oplus L_n$, then

$$\Lambda^p E = \bigoplus_{1 \leqslant i_1 < \cdots < i_p \leqslant n} (L_{i_1} \otimes \cdots \otimes L_{i_p}).$$

Hence

$c(\Lambda^p E) = \prod (1 + c_1(L_{i_1} \otimes \cdots \otimes L_{i_p}))$	by the Whitney product formula
$= \prod (1 + x_{i_1} + \cdots + x_{i_p})$	by (20.1), with $x_i = c_1(L_i)$,

where the product is over all multi-indices $1 \leqslant i_1 < \cdots < i_p \leqslant n$. Since the right-hand side is symmetric in $x_1, \ldots, x_n$, it is expressible as a polynomial

Q in $c_1(E), \ldots, c_n(E)$, so

$$c(\Lambda^p E) = Q(c_1(E), \ldots, c_n(E)).$$

By the splitting principle this formula holds for every rank n vector bundle, whether it is a direct sum or not. It should be pointed out that the polynomial Q depends only on n and p, not on E; for example, the Chern class of $\Lambda^2 E$, where rank $E = 3$, is given by

$$\begin{aligned} c(\Lambda^2 E) = Q(c_1, c_2, c_3) &= (1 + c_1 - x_1)(1 + c_1 - x_2)(1 + c_1 - x_3) \\ &= (1 + c_1)^3 - c_1(1 + c_1)^2 + c_2(1 + c_1) - c_3. \end{aligned}$$

Similarly, if V and W are vector spaces with bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$ respectively, then the pth symmetric power $S^p V$ of V is the vector space with basis $\{v_{i_1} \otimes \cdots \otimes v_{i_p}\}_{1 \leqslant i_1 \leqslant \cdots \leqslant i_p \leqslant n}$ and the tensor product $V \otimes W$ is the vector space with basis $\{v_i \otimes w_j\}_{1 \leqslant i \leqslant n,\, 1 \leqslant j \leqslant m}$. By the same discussion as above, if E is a rank n vector bundle with $c(E) = \prod_{i=1}^n (1 + x_i)$ and F is a rank m vector bundle with $c(F) = \prod_{j=1}^m (1 + y_j)$, then

$$c(S^p E) = \prod_{1 \leqslant i_1 \leqslant \cdots \leqslant i_p \leqslant n} (1 + x_{i_1} + \cdots + x_{i_p}) \tag{21.8}$$

and

$$c(E \otimes F) = \prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} (1 + x_i + y_j). \tag{21.9}$$

In particular if L is a complex line bundle with first Chern class y, then

$$c(E \otimes L) = \prod_{i=1}^{n} (1 + y + x_i) = \sum_{i=0}^{n} c_i(E)(1 + y)^{n-i}, \tag{21.10}$$

where by convention we set $c_0(E) = 1$.

Example 21.11 (The L-class and the Todd class). In the notation of the preceding example the power series

$$\prod_{i=1}^{n} \frac{\sqrt{x_i}}{\tanh \sqrt{x_i}}$$

is symmetric in $x_1, \ldots, x_n$, hence is some power series L in $c_1(E), \ldots, c_n(E)$. This power series $L(E) = L(c_1(E), \ldots, c_n(E))$ is called the L-class of E. By the splitting principle the L-class automatically satisfies the product formula

$$L(E \oplus F) = L(E)L(F).$$

Similarly,

$$\prod_{i=1}^{n} \frac{x_i}{1 - e^{x_i}} = \mathrm{Td}(c_1(E), \ldots, c_n(E)) = \mathrm{Td}(E)$$

defines the *Todd class* of E. By the splitting principle the Todd class also automatically satisfies the product formula. The L-class and the Todd class turn out to be of fundamental importance in the Hirzebruch signature formula (see Remark 22.9) and the Riemann–Roch theorem (see Hirzebruch [1]).

EXAMPLE 21.12 (The dual bundle). Let L be a complex line bundle. By (20.2),

$$c_1(L^*) = -c_1(L).$$

Next consider a direct sum of line bundles

$$E = L_1 \oplus \cdots \oplus L_n.$$

By the Whitney product formula

$$c(E) = c(L_1) \cdots c(L_n) = (1 + c_1(L_1)) \cdots (1 + c_1(L_n)).$$

On the other hand

$$E^* = L_1^* \oplus \cdots \oplus L_n^*$$

and

$$c(E^*) = (1 - c_1(L_1)) \cdots (1 - c_1(L_n)).$$

Therefore

$$c_q(E^*) = (-1)^q c_q(E).$$

By the splitting principle this result holds for all complex vector bundles E.

EXAMPLE 21.13 (The Chern classes of the complex projective space). By analogy with the definition of a differentiable manifold, we say that a second countable, Hausdorff space M is a *complex manifold* of dimension n if every point has a neighborhood U_α homeomorphic to some open ball in $\mathbb{C}_n$, $\phi_\alpha : U_\alpha \to \mathbb{C}^n$, such that the transition functions

$\mathbb{C}^n$

$$g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \mathbb{C}^n$$

are holomorphic. Smooth maps and smooth vector bundles have obvious analogues in the holomorphic category. If $u_1, \ldots, u_n$ are the coordinate functions on $\mathbb{C}^n$, then $z_i = u_i \circ \phi_\alpha$, $i = 1, \ldots, n$, are the coordinate functions on U_α. At each point p in U_α the vectors $\partial/\partial z_1, \ldots, \partial/\partial z_n$ span over $\mathbb{C}$ the *holomorphic tangent bundle* of M. It is a complex vector bundle of rank n. The Chern class of a complex manifold is defined to be the Chern class of its holomorphic tangent bundle.

The complex projective space $\mathbb{C}P^n$ is an example of a complex manifold, since, as in Exercise 6.44, the transition functions g_{ji} relative to the standard open cover are given by multiplication by z_i/z_j, which are holomorphic functions from $\phi_i(U_i \cap U_j)$ to $\phi_j(U_i \cap U_j)$. Recall that there is a tautological exact sequence on $\mathbb{C}P^n$

$$0 \to S \to \mathbb{C}^{n+1} \to Q \to 0,$$

where $\mathbb{C}^{n+1}$ denotes the trivial bundle of rank $n+1$ over $\mathbb{C}P^n$. A tangent

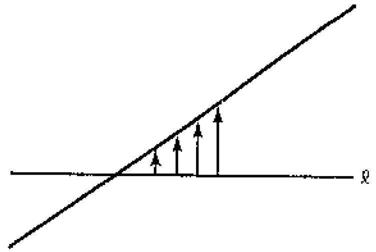

Figure 21.1

vector to $\mathbb{C}P^n$ at a line ℓ in $\mathbb{C}^{n+1}$ may be regarded as an infinitesimal motion of the line ℓ (Figure 21.1). Such a motion corresponds to a linear map from ℓ to the quotient space $\mathbb{C}^{n+1}/\ell$, which may be represented by the complementary subspace of ℓ in $\mathbb{C}^{n+1}$ (relative to some metric). Thus, denoting the holomorphic tangent bundle by T, we have

$$T \simeq \operatorname{Hom}(S, Q) = Q \otimes S^*.$$

We will compute the Chern class of T in two ways.

(1) Tensoring the tautological sequence with S^*, we get

$$0 \to \mathbb{C} \to S^* \otimes \mathbb{C}^{n+1} \to S^* \otimes Q \to 0.$$

By the Whitney product formula

$$c(T) = c(S^* \otimes Q) = c(S^* \otimes \mathbb{C}^{n+1}) = c(S^* \oplus \cdots \oplus S^*) = (1 + x)^{n+1},$$

where $x = c_1(S^*)$.

(2) From the tautological exact sequence and the Whitney product formula

$$c(Q) = \frac{1}{c(S)} = \frac{1}{1-x} = 1 + x + \cdots + x^n,$$

since $x^{n+1} = 0$ in $H^*(\mathbb{C}P^n)$. By (21.10)

$$\begin{aligned}
c(\mathbb{C}P^n) = c(Q \otimes S^*) &= \sum_{i=0}^{n} c_i(Q)(1 + x)^{n-i} = \sum_{i=0}^{n} x^i (1 + x)^{n-i} \\
&= (1 + x)^n \sum_{i=0}^{n} \left(\frac{x}{1+x}\right)^i \\
&= (1 + x)^n \left[\left(1 - \left(\frac{x}{1+x}\right)^{n+1}\right) \Big/ \left(1 - \frac{x}{1+x}\right)\right] \\
&= (1 + x)^{n+1} \left[1 - \left(\frac{x}{1+x}\right)^{n+1}\right] \\
&= (1 + x)^{n+1} - x^{n+1} \\
&= (1 + x)^{n+1}.
\end{aligned}$$

Exercise 21.14. Chern classes of a hypersurface in a complex projective space. Let H be the hyperplane bundle over the projective space $\mathbb{C}P^n$ (see (20.3)), and $H^{\otimes k}$ the tensor product of k copies of H. The line bundle H is in fact more than a C^∞ complex line bundle; because its transition functions are holomorphic, it is a *holomorphic* line bundle. The total space of a holomorphic bundle over a complex manifold is again a complex manifold, so that the notion of a *holomorphic section* makes sense. The zero locus of a holomorphic section of $H^{\otimes k}$ is called a *hypersurface of degree* k in $\mathbb{C}P^n$. If the section is transversal to the zero section, then the hypersurface is a smooth complex manifold. Compute the Chern classes of a smooth hypersurface of degree k in $\mathbb{C}P^n$. (*Hint*: apply Prop. 12.7 to get the normal bundle of the hypersurface.)

Flag Manifolds

Given a complex vector space V of dimension n, a *flag* in V is a sequence of subspaces $A_1 \subset A_2 \subset \cdots \subset A_n = V$, $\dim_{\mathbb{C}} A = i$. Let $Fl(V)$ be the collection of all flags in V. Clearly any flag can be carried into any other flag in V by an element of the general linear group $GL(n, \mathbb{C})$, and the stabilizer at a flag is the group H of the upper triangular matrices. So as a set $Fl(V)$ is isomorphic to the coset space $GL(n, \mathbb{C})/H$. Since the quotient of a Lie group by a closed subgroup is a manifold (Warner [1, p. 120]), $Fl(V)$ can be made into a manifold. It is called the *flag manifold* of V.

Given a vector bundle E, just as one can form its projectivization $P(E)$, so one can form its *associated flag bundle* $Fl(E)$. The bundle $Fl(E)$ is obtained from E by replacing each fiber E_p by the flag manifold $Fl(E_p)$; the local trivialization $\phi_\alpha : E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{C}^n$ induces a natural trivialization $Fl(E)|_{U_\alpha} \simeq U_\alpha \times Fl(\mathbb{C}^n)$. Since $GL(n, \mathbb{C})$ acts on $Fl(\mathbb{C}^n)$, we may take the transition functions of $Fl(E)$ to be those of E, but note that $Fl(E)$ is not a vector bundle.

Proposition 21.15. *The associated flag bundle $Fl(E)$ of a vector bundle is the split manifold $F(E)$ constructed earlier.*

PROOF. We first show this for $E = V$ a vector space of dimension 3, viewed as a rank 3 vector bundle over a point.

$$\begin{tikzcd}
 & & S_V \oplus S_{Q_V} \oplus Q_{Q_V} \arrow[d] \\
 & S_V \oplus Q_V \arrow[d] & P(Q_V) = F(V) \arrow[dl] \\
V \arrow[d] & P(V) \arrow[dl] & \\
\text{point} & &
\end{tikzcd}$$

In what follows all lines and planes go through the origin. A point in $P(V)$ is a line L in V. A point of $P(Q_V)$ is a line L in V and a line L' in V/L. L' may be regarded as a 2-plane in V containing L. Thus $Fl(V) = P(Q_V) = \{A_1 \subset A_2 \subset V, \dim A_i = i\} = F(V)$.

Now let E be a vector bundle of rank n over M. The split manifold $F(E)$ is obtained by a sequence of $n-1$ projectivizations as in (21.2). A point of $P(E)$ is a pair (p, ℓ), where p is in M and ℓ is a line in E_p. By introducing a Hermitian metric on E, we may regard all the quotient bundles $Q_1, \ldots, Q_{n-1}$ in (21.2) as subbundles of E. Then a point of $P(Q_1)$ over (p, ℓ_1) in $P(E)$ is a triple (p, ℓ_1, ℓ_2) where ℓ_2 is a line in the orthogonal complement of ℓ_1 in E_p. A point of $P(Q_2)$ over (p, ℓ_1, ℓ_2) in $P(Q_1)$ is a 4-tuple $(p, \ell_1, \ell_2, \ell_3)$ where ℓ_3 is a line in the orthogonal complement of ℓ_1 and ℓ_2 in E_p. Thus, more generally, a point in the split manifold $F(E) = P(Q_{n-1})$ may be identified with the flag

$$(p, \ell_1 \subset \{\ell_1, \ell_2\} \subset \{\ell_1, \ell_2, \ell_3\} \subset \cdots \subset E_p).$$

This proves the equality of the split manifold $F(E)$ and the flag bundle $Fl(E)$. $\square$

From now on the notations $F(E)$ and $Fl(E)$ will be used interchangeably.

The formula (21.3) gives one description of the vector space structure of the cohomology of a flag bundle. To compute its ring structure we first recall from (20.7) that if E is a rank n complex vector bundle over M, then the cohomology ring of its projectivization is

$$H^*(P(E)) = H^*(M)[x]/(x^n + c_1(E)x^{n-1} + \cdots + c_n(E)), \text{ where } x = c_1(S^*).$$

NOTATION. If A is a polynomial ring, and $a, b, c, f \in A$, then (a, b, c) denotes the ideal generated by a, b, and c, while $(f = 0)$ the ideal generated by the homogeneous components of f.

There is an alternate description of the ring structure which is sometimes very useful. We write $H^*(M)[c(S), c(Q)]$ for $H^*(M)[c_1(S), c_1(Q), \ldots, c_{n-1}(Q)]$, where S and Q are the universal subbundle and quotient bundle on $P(E)$.

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] & \pi^*E \arrow[r] \arrow[d] & Q \arrow[r] & 0 \\
 & & P(E) \arrow[dr, "\pi"'] & E \arrow[d] & \\
 & & & M &
\end{tikzcd}$$

Proposition 21.16. $H^*(P(E)) = H^*(M)[c(S), c(Q)]/(c(S)c(Q) = \pi^*c(E))$.

PROOF. The idea is to eliminate the generators $c_1(Q), \ldots, c_{n-1}(Q)$ by using the relation $c(S)c(Q) = \pi^*c(E)$. Let $x = c_1(S^*)$, $y_i = c_i(Q)$, and $c_i = \pi^*c_i(E)$. Equating the terms of equal degrees in

$$(1 - x)(1 + y_1 + \cdots + y_{n-1}) = 1 + c_1 + \cdots + c_n,$$

we get

$$\begin{aligned} y_1 - x &= c_1, \\ y_2 - xy_1 &= c_2, \\ y_3 - xy_2 &= c_3, \\ &\vdots \\ y_{n-1} - xy_{n-2} &= c_{n-1}, \\ - xy_{n-1} &= c_n. \end{aligned}$$

By the first $n-1$ equations, $y_1, \ldots, y_{n-1}$ can be expressed in terms of x and elements of $H^*(M)$, and so can be eliminated as generators of $H^*(M)[c(S), c(Q)]/(c(S)c(Q) = \pi^*c(E))$. The last equation $-xy_{n-1} = c_n$ translates into

$$(*) \qquad x^n + c_1x^{n-1} + \cdots + c_n = 0.$$

Hence $H^*(M)[c(S), c(Q)]/(c(S)c(Q) = \pi^*c(E))$ is isomorphic to the polynomial ring over $H^*(M)$ with the single generator x and the single relation $(*)$. □

By (21.2) and (21.15) the flag bundle $Fl(E)$ is obtained from a sequence of $n-1$ projectivizations. Applying Proposition 21.16 to (21.2), we have

$$\begin{aligned} &H^*(P(Q_1)) \\ &= H^*(P(E))[c(S_2), c(Q_2)]/(c(S_2)c(Q_2) = c(Q_1)) \\ &= H^*(M)[c(S_1), c(Q_1), c(S_2), c(Q_2)]/(c(S_1)c(Q_1) = c(E), c(S_2)c(Q_2) = c(Q_1)) \\ &= H^*(M)[c(S_1), c(S_2), c(Q_2)]/(c(S_1)c(S_2)c(Q_2) = c(E)). \end{aligned}$$

By induction

$$\begin{aligned} &H^*(P(Q_{n-2})) \\ &\quad = H^*(M)[c(S_1), \ldots, c(S_{n-1}), c(Q_{n-1})]/(c(S_1)\cdots c(S_{n-1})c(Q_{n-1}) = c(E)). \end{aligned}$$

Writing $x_i = c_1(S_i)$, $i = 1, \ldots, n-1$, and $x_n = c(Q_{n-1})$, the cohomology ring of the flag bundle $Fl(E)$ is

$$H^*((Fl(E)) = H^*(M)[x_1, \ldots, x_n]\Big/\left(\prod_{i=1}^{n}(1 + x_i) = c(E)\right).$$

Specializing this theorem to a complex vector space V, considered as the trivial bundle over a point, we obtain the cohomology ring of the flag manifold

$$H^*((Fl(V)) = \mathbb{R}[x_1, \ldots, x_n]\Big/\left(\prod_{i=1}^{n}(1 + x_i) = 1\right).$$

As for the Poincaré polynomial of the flag manifold we note again that the flag manifold is obtained by a sequence of $n-1$ projectivizations (21.2).

By (20.8) each time we projectivize a rank k vector bundle, the Poincaré polynomial is multiplied by $(1 - t^{2k})/(1 - t^2)$. So the Poincaré polynomial of the flag manifold $Fl(V)$ is

$$P_t(Fl(V)) = \frac{1 - t^{2n}}{1 - t^2} \cdot \frac{1 - t^{2n-2}}{1 - t^2} \cdot \cdots \cdot \frac{1 - t^2}{1 - t^2}.$$

This discussion may be summarized in the following proposition.

Proposition 21.17. *Let V be a complex vector space of dimension n. The cohomology ring of the flag manifold $Fl(V)$ is*

$$H^*(Fl(V)) = \mathbb{R}[x_1, \ldots, x_n]\Big/\left(\prod_{i=1}^{n}(1 + x_i) = 1\right).$$

It has Poincaré polynomial

$$P_t(Fl(V)) = \frac{(1 - t^2)(1 - t^4)\cdots(1 - t^{2n})}{(1 - t^2)(1 - t^2)\cdots(1 - t^2)}.$$

REMARK 21.18. Similarly, if E is a rank n complex vector bundle over a manifold M, then the cohomology ring of the flag bundle $Fl(E)$ is

$$H^*(Fl(E)) = H^*(M)[x_1, \ldots, x_n]\Big/\left(\prod_{i=1}^{n}(1 + x_i) = c(E)\right),$$

and the Poincaré series is

$$P_t(Fl(E)) = P_t(M)\,\frac{(1 - t^2)(1 - t^4)\cdots(1 - t^{2n})}{(1 - t^2)(1 - t^2)\cdots(1 - t^2)}.$$

REMARK 21.19. Since projectivization does not introduce any torsion element in integer cohomology, the integer cohomology ring of the flag manifold $Fl(V)$ is torsion-free and is given by the same formula as (21.17) with $\mathbb{Z}$ in place of $\mathbb{R}$. The integer cohomology ring of a flag bundle is given by the same formula as (21.18). In fact, with a little care, the entire discussion can be translated into the Čech theory.

§22 Pontrjagin Classes

Although the Chern classes are invariants of a complex bundle, they can be used to define invariants of a real vector bundle, called the *Pontrjagin classes*. In this section we define the Pontrjagin classes, compute a few examples, and as an application obtain an embedding criterion for differentiable manifolds.

Conjugate Bundles

Let V be a complex vector space. If $z \in \mathbb{C}$ and $v \in V$, the formula

$$z * v = \bar{z}v$$

defines an action of $\mathbb{C}$ on V. The underlying additive group of V with this action as scalar multiplication is called the *conjugate vector space* of V, denoted $\bar{V}$. The conjugate space $\bar{V}$ may be thought of as V with the opposite complex structure; as a vector space, $\bar{V}$ is anti-isomorphic to V. A linear map $f\colon V \to W$ of two complex vector spaces V and W is also a linear map of the conjugate vector spaces $f\colon \bar{V} \to \bar{W}$; we denote both by f as they are represented by the same matrix.

Given a complex vector bundle E with trivialization

$$\phi_\alpha : E|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^n,$$

we construct the *conjugate vector bundle* $\bar{E}$ by replacing each fiber of E by its conjugate. The trivialization of $\bar{E}$ is given by

$$\bar{\phi}_\alpha : \bar{E}|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^n,$$

which is the composition

$$\bar{E}|_{U_\alpha} \overset{\phi_\alpha}{\simeq} U_\alpha \times \bar{\mathbb{C}}^n \xrightarrow{\text{conjugation}} U_\alpha \times \mathbb{C}^n.$$

In terms of transition functions, if the cocycle $\{g_{\alpha\beta}\}$ defines E, then its conjugate $\{\bar{g}_{\alpha\beta}\}$ defines $\bar{E}$.

As in (6.4), by endowing a complex vector bundle on a manifold with a Hermitian metric, we can reduce its structure group to the unitary group. Since unitary matrices $g_{\alpha\beta}$ satisfy $\bar{g}_{\alpha\beta} = (g_{\alpha\beta}^t)^{-1}$, we see that the conjugate bundle $\bar{E}$ and the dual bundle E^* have the same transition functions and hence are isomorphic. So by Example 21.12, if $c(E) = \prod (1 + x_i)$, then $c(\bar{E}) = \prod (1 - x_i)$.

Realization and Complexification

By simply forgetting the complex structure, we can regard a linear map of complex vector spaces $L : \mathbb{C}^n \to \mathbb{C}^n$ with coordinates $z_1, \dots, z_n$ as a linear map of the underlying real vector spaces $L_{\mathbb{R}} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with coordinates $x_1, \dots, x_{2n}$ where $z_k = x_{2k-1} + ix_{2k}$. Conversely, via the natural embedding of $\mathbb{R}^n$ in $\mathbb{C}^n$, a linear map of real vector spaces $L : \mathbb{R}^n \to \mathbb{R}^n$ gives rise to a map $L \otimes \mathbb{C} : \mathbb{C}^n \to \mathbb{C}^n$. The first operation is called *realization* and the second, *complexification*. The complexification of a real matrix is the matrix itself, but with the entries viewed as complex numbers. The realization of a complex matrix is described in Examples 22.2 and 22.3 below. In terms of

matrices these two operations give a sequence of embeddings

$$\begin{array}{ccccc} U(n) & \hookrightarrow & O(2n) & \hookrightarrow & U(2n) \\ \cap & & \cap & & \cap \\ GL(n, \mathbb{C}) & \hookrightarrow & GL(2n, \mathbb{R}) & \hookrightarrow & GL(2n, \mathbb{C}) \\ A & \mapsto & A_{\mathbb{R}} & \mapsto & A_{\mathbb{R}} \otimes \mathbb{C}. \end{array} \tag{22.1}$$

EXAMPLE 22.2. Let $L : \mathbb{C} \to \mathbb{C}$ be given by multiplication by the complex number $\lambda = \alpha + i\beta$. Since

$$(\alpha + i\beta)(x_1 + ix_2) = (\alpha x_1 - \beta x_2) + i(\beta x_1 + \alpha x_2),$$

as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, $L_{\mathbb{R}}$ is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus

$$(\alpha + i\beta)_{\mathbb{R}} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

EXAMPLE 22.3 Let $L : \mathbb{C}^2 \to \mathbb{C}^2$ be given by the complex matrix $\left(\begin{smallmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{smallmatrix}\right)$ where $\lambda_k = \alpha_k + i\beta_k$. A little computation shows that $L_{\mathbb{R}} : \mathbb{R}^4 \to \mathbb{R}^4$ is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} \alpha_1 & -\beta_1 & \alpha_2 & -\beta_2 \\ \beta_1 & \alpha_1 & \beta_2 & \alpha_2 \\ \alpha_3 & -\beta_3 & \alpha_4 & -\beta_4 \\ \beta_3 & \alpha_3 & \beta_4 & \alpha_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Thus

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}_{\mathbb{R}} = \begin{pmatrix} (\lambda_1)_{\mathbb{R}} & (\lambda_2)_{\mathbb{R}} \\ (\lambda_3)_{\mathbb{R}} & (\lambda_4)_{\mathbb{R}} \end{pmatrix}$$

It is clear from these two examples what the realization of an n by n complex matrix should be.

Lemma 22.4. *Let A be an n by n complex matrix. There is a $2n$ by $2n$ matrix B, independent of A, such that $A_{\mathbb{R}} \otimes \mathbb{C}$ is similar to $\left(\begin{smallmatrix} A & 0 \\ 0 & \bar{A} \end{smallmatrix}\right)$ via B.*

PROOF. In the 1 by 1 case, this is a matter of diagonalizing

$$(\alpha + i\beta)_{\mathbb{R}} \otimes \mathbb{C} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

Corresponding to the eigenvalues $\alpha + i\beta$ and $\alpha - i\beta$ are the eigenvectors $\binom{1}{-i}$ and $\binom{1}{i}$. Therefore, $B = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$.

Now consider the 2 by 2 case:

$$A = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}, \quad \lambda_k = \alpha_k + i\beta_k$$

$$A_{\mathbb{R}} \otimes \mathbb{C} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \quad \text{where} \quad A_k = \begin{pmatrix} \alpha_k & -\beta_k \\ \beta_k & \alpha_k \end{pmatrix}.$$

Note that

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ -i\lambda_1 \\ \lambda_3 \\ -i\lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & * & * \\ -i & 0 & * & * \\ 0 & 1 & * & * \\ 0 & -i & * & * \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_3 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ -i & 0 & i & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -i & 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -i & 0 & i & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -i & 0 & i \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 & & \\ \lambda_3 & \lambda_4 & & \\ & & \bar{\lambda}_1 & \bar{\lambda}_2 \\ & & \bar{\lambda}_3 & \bar{\lambda}_4 \end{pmatrix}.$$

So

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -i & 0 & i & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -i & 0 & i \end{pmatrix}.$$

For the n by n case, we can take B to be

$$\begin{pmatrix} 1 & & & & 1 & & & \\ -i & & & & i & & & \\ & 1 & & & & 1 & & \\ & -i & & & & i & & \\ & & 1 & & & & 1 & \\ & & -i & & & & i & \\ & & & \ddots & & & & \ddots \end{pmatrix}$$

□

If E is a complex vector bundle of rank n with transition functions $\{g_{\alpha\beta}\}$, then $E_{\mathbb{R}} \otimes \mathbb{C}$ is the complex vector bundle of rank $2n$ with transition functions $\{(g_{\alpha\beta})_{\mathbb{R}} \otimes \mathbb{C}\}$. By Lemma 22.4,

$$E_{\mathbb{R}} \otimes \mathbb{C} \simeq E \oplus \bar{E}. \tag{22.5}$$

This result may be seen alternatively as follows. On the complex vector space $E_{\mathbb{R}} \otimes \mathbb{C}$, multiplication by i is a linear transformation J satisfying $J^2 = -1$. Therefore, the eigenvalues of J are $\pm i$ and $E_{\mathbb{R}} \otimes \mathbb{C}$ accordingly decomposes into a direct sum

$$E_{\mathbb{R}} \otimes \mathbb{C} = (i\text{-eigenspace}) \oplus ((-i)\text{-eigenspace}).$$

On the i-eigenspace, J acts as multiplication by i, hence

$$(i\text{-eigenspace}) \simeq E.$$

Similarly,

$$((-i)\text{-eigenspace}) \simeq \bar{E}.$$

It follows by reasons of dimension that

$$E_{\mathbb{R}} \otimes \mathbb{C} = E \oplus \bar{E}.$$

The Pontrjagin Classes of a Real Vector Bundle

By their naturality property the Chern classes of a C^∞ complex vector bundle are C^∞ invariants of the bundle. For a real vector bundle E similar invariants may be obtained by considering the Chern classes of its complexification $E \otimes_{\mathbb{R}} \mathbb{C}$; these are the *Pontrjagin classes* of E. More precisely, if E is a rank n real vector bundle over M, then its *total Pontrjagin class* is

$$\begin{aligned} p(E) &= 1 + p_1(E) + \cdots + p_n(E) \\ &= 1 + c_1(E \otimes \mathbb{C}) + \cdots + c_n(E \otimes \mathbb{C}) \in H^*(M). \end{aligned}$$

It follows from the corresponding properties of the total Chern class that the Pontrjagin class is functorial and satisfies the Whitney product formula

$$p(E \oplus E') = p(E)p(E').$$

The Pontrjagin class of a manifold is defined to be that of its tangent bundle.

REMARK 22.6. Let E be a real vector bundle. Because the transition functions of $E \otimes \mathbb{C}$ are the same as those of E, they are real-valued, and therefore $E \otimes \mathbb{C}$ is isomorphic to its conjugate $\overline{E \otimes \mathbb{C}}$. It follows that $c_i(E \otimes \mathbb{C}) = c_i(\overline{E \otimes \mathbb{C}}) = (-1)^i c_i(E \otimes \mathbb{C})$. For an odd i, then, $2c_i(E \otimes \mathbb{C}) = 0$. Thus the odd Pontrjagin classes, as we have defined them, are zero in the de Rham cohomology, and torsion of order 2 in the integral cohomology. The usual definition of the Pontrjagin classes in the literature (see, for instance, Milnor and Stasheff [1, p. 174]) ignores these odd Chern classes and defines $p_i(E)$ to be

$$(-1)^i c_{2i}(E \otimes \mathbb{C}).$$

EXAMPLE 22.7. (The Pontrjagin class of the sphere). Since the sphere S^n is orientable, its normal bundle N in $\mathbb{R}^{n+1}$ is trivial. From the exact sequence

$$0 \to T_{S^n} \to T_{\mathbb{R}^{n+1}}|_{S^n} \to N \to 0,$$

we see by the Whitney product formula that

$$p(S^n)p(N) = p(T_{\mathbb{R}^{n+1}}|_{S^n}).$$

Therefore,

$$p(S^n) = 1.$$

EXAMPLE 22.8 (The Pontrjagin class of a complex manifold). The Pontrjagin class of a complex manifold M is defined to be that of the underlying real manifold $M_{\mathbb{R}}$. Let T be the holomorphic tangent bundle to M. Then the tangent bundle to $M_{\mathbb{R}}$ is the realization of T and

$$p(M) = p(T_{\mathbb{R}}) = c(T_{\mathbb{R}} \otimes \mathbb{C}) = c(T \oplus \bar{T}) = c(T)c(\bar{T}).$$

So if the total Chern class of the complex manifold M is $c(M) = \prod (1 + x_i)$, then the Pontrjagin class is $p(M) = \prod (1 - x_i^2)$.

REMARK 22.8.1. If we had followed the usual sign convention for the Pontrjagin classes (see Remark 22.6), the Pontrjagin class of a complex manifold would be $p(M) = \prod (1 + x_i^2)$, where the x_i's are defined as above. To have only positive terms in this formula is one of the reasons for the sign in $(-1)^i c_{2i}(E \otimes \mathbb{C})$ in the usual definition of the Pontrjagin class.

REMARK 22.9. Let M be a compact oriented manifold of dimension $4n$. By Poincaré duality the wedge product $\wedge : H^{2n}(M) \otimes H^{2n}(M) \to \mathbb{R}$ is a nondegenerate symmetric bilinear form and hence has a signature; this is called the *signature* of M. Recall that the signature of a symmetric matrix is the number of positive eigenvalues minus the number of negative eigenvalues. Hirzebruch proved that the signature is expressible in terms of the Pontrjagin classes.

Hirzebruch signature formula:

$$\text{signature of } M = \int_M L(p_1(M), \ldots, p_{4n}(M)),$$

where L is the polynomial defined in Example 21.11. For a proof of the signature formula, see Milnor and Stasheff [1, p. 224].

Application to the Embedding of a Manifold in a Euclidean Space

Using the Pontrjagin class one can sometimes decide if a conjectured embedding is possible. We illustrate this with the following example.

EXAMPLE 22.10. *Decide if $\mathbb{C}P^4$ can be differentiably embedded in $\mathbb{R}^9$.*

By (22.8) and (21.13) the Pontrjagin class of $\mathbb{C}P^4$ is

$$p(\mathbb{C}P^4) = c(T_{\mathbb{C}P^4})c(\bar{T}_{\mathbb{C}P^4}) = (1 + x)^5(1 - x)^5 = (1 - x^2)^5.$$

If $\mathbb{C}P^4$ can be differentiably embedded in $\mathbb{R}^9$, then there is an exact sequence

$$0 \to (T_{\mathbb{C}P^4})_{\mathbb{R}} \to T_{\mathbb{R}^9}|_{\mathbb{C}P^4} \to N \to 0,$$

where $(T_{\mathbb{C}P^4})_{\mathbb{R}}$ is the realization of the holomorphic tangent bundle $T_{\mathbb{C}P^4}$ and N is the normal bundle of $\mathbb{C}P^4$ in $\mathbb{R}^9$. By the Whitney product formula

$$p(T_{\mathbb{R}^9}|_{\mathbb{C}P^4}) = p((T_{\mathbb{C}P^4})_{\mathbb{R}})p(N). \tag{22.11}$$

Since the restriction $T_{\mathbb{R}^9}|_{\mathbb{C}P^4}$ is the pullback of $T_{\mathbb{R}^9}$ to $\mathbb{C}P^4$ under the embedding $i : \mathbb{C}P^4 \to \mathbb{R}^9$, by the functoriality of the Pontrjagin class

$$p(T_{\mathbb{R}^9}|_{\mathbb{C}P^4}) = i^*p(T_{\mathbb{R}^9}) = 1.$$

Therefore, by (22.11)

$$p(N) = \frac{1}{p((T_{\mathbb{C}P^4})_{\mathbb{R}})} = \frac{1}{(1 - x^2)^5} = 1 + 5x^2 + 15x^4. \tag{22.12}$$

Since N is a real line bundle, the top component of $p(N)$ should be in $H^2(\mathbb{C}P^4)$. This contradicts the fact that $5x^2$ and $15x^4$ are nonzero classes in $H^4(\mathbb{C}P^4)$ and $H^8(\mathbb{C}P^4)$. Thus $\mathbb{C}P^4$ cannot be embedded in $\mathbb{R}^9$.

From (22.12), if $\mathbb{C}P^4$ can be embedded in $\mathbb{R}^n$, then the normal bundle has rank at least 4, since the top-degree term of the Pontrjagin class of a rank k real bundle is in dimension $2k$. It follows that $\mathbb{C}P^4$ cannot be embedded in a Euclidean space of dimension 11 or less.

§23 The Search for the Universal Bundle

Let $f : M \to N$ be a complex map between two manifolds and E a complex bundle over N. The pullback $f^{-1}E$ is a bundle over M. If the Chern classes of E vanish, by the naturality property (20.10.1), so do those of $f^{-1}E$. Taking the Chern classes to be a measure of the twisting of a bundle, we may assert that pulling back "dilutes" a bundle, i.e., makes it less twisted. One extreme example is when f is constant; in this case $f^{-1}E$ is trivial. Another example is the flag construction of Section 21; pulling E back to the split manifold $F(E)$ splits E into a direct sum of line bundles. One may wonder if there exists a bundle which is so twisted that every bundle is a pullback of this universal bundle. Such a bundle indeed exists, at least for manifolds of finite type; it is the universal quotient bundle on the Grassmannian $G_k(\mathbb{C}^n)$ for n sufficiently large. We will prove this result and conclude from it that every natural transformation from the complex vector bundles to the cohomology classes is expressible in terms of the Chern classes, all for manifolds of finite type. We also indicate how the theorems generalize to an arbitrary manifold.

The Grassmannian

Let V be a complex vector space of dimension n. The *complex Grassmannian* $G_k(V)$ is the set of all subspaces of complex codimension k in V. We sometimes call such a subspace an $(n-k)$-plane in V. Given a Hermitian metric on V, the unitary group $U(n)$ is the group of all metric-preserving endomorphisms of V. Clearly $U(n)$ acts transitively on the collection of all $(n-k)$-planes in V. Since a unitary matrix which sends an $(n-k)$-plane to itself must also fix the complementary orthogonal k-plane, the stabilizer of an $(n-k)$-plane in V is $U(n-k) \times U(k)$. Thus the Grassmannian can be represented as a homogeneous space

$$G_k(V) = \frac{U(n)}{U(k) \times U(n-k)}.$$

As the coset space of a Lie group by a closed subgroup, $G_k(V)$ is a differentiable manifold (Warner [1, p. 120]). Note that $G_{n-1}(V)$ is the projective space $P(V)$.

Just as in the case of the projective space, over the Grassmannian $G_k(V)$ there are three tautological bundles: the *universal subbundle* S, whose fiber at each point Λ of $G_k(V)$ is the $(n-k)$-plane Λ itself; the *product bundle* $\hat{V} = G_k(V) \times V$; and the *universal quotient bundle* Q defined by

$$0 \to S \to \hat{V} \to Q \to 0.$$

This exact sequence is called the *tautological sequence* on $G_k(V)$. Over $G_k(V)$ the universal subbundle S has rank $n-k$ and the universal quotient bundle has rank k.

Similarly, if V is a real vector space, one can define the *real Grassmannian* $G_k(V)$ of codimension k real subspaces of V, and the analogous real universal bundles. The real Grassmannian can also be represented as a homogeneous space

$$G_k(\mathbb{R}^n) = \frac{O(n)}{O(k) \times O(n-k)}.$$

Proposition 23.1. *The cohomology of the complex Grassmannian $G_k(V)$ has Poincaré polynomial*

$$P_t(G_k(V)) = \frac{(1-t^2)\cdots(1-t^{2n})}{(1-t^2)\cdots(1-t^{2k})(1-t^2)\cdots(1-t^{2(n-k)})}.$$

PROOF. The flag manifold $F(V)$ may be obtained from the Grassmannian $G_k(V)$ by a series of flag constructions as follows. Let $\hat{Q}$ be the pullback of Q to the flag bundle $F(S)$.

$$\begin{tikzcd}
 & \hat{Q} \arrow[d] & \\
S \oplus Q \arrow[d] & F(S) \arrow[dl] & F(\hat{Q}) \arrow[l] \\
G_k(V) & &
\end{tikzcd}$$

A point of $F(S)$ is a pair $(\Lambda, L_1 \subset \cdots \subset \Lambda)$ consisting of an $(n-k)$-plane Λ in V together with a flag in Λ. Therefore a point in $F(\hat{Q})$ consists of a point in $F(S)$, $(\Lambda, L_1 \subset \cdots \subset \Lambda)$, together with a flag in V/Λ, i.e., a point in $F(\hat{Q})$ is given by $(\Lambda, L_1 \subset \cdots \subset L_{n-k-1} \subset \Lambda \subset L_{n-k+1} \subset \cdots \subset V)$. So $F(\hat{Q})$ is the flag manifold $F(V)$, and $F(V)$ is obtained from the Grassmannian $G_k(V)$ by two flag constructions. By (21.18), the Poincaré polynomials of $F(V)$ and $G_k(V)$ satisfy the relation

$$P_t(F(V)) = P_t(G_k(V)) \frac{(1-t^2)\cdots(1-t^{2(n-k)})(1-t^2)\cdots(1-t^{2k})}{(1-t^2)\cdots(1-t^2)(1-t^2)\cdots(1-t^2)}.$$

From (21.17) it follows that

$$P_t(G_k(V)) = \frac{(1-t^2)\cdots(1-t^{2n})}{(1-t^2)\cdots(1-t^{2(n-k)})(1-t^2)\cdots(1-t^{2k})}. \qquad \square$$

As for the ring structure of the cohomology of the Grassmannian $G_k(V)$, we have the following.

Proposition 23.2. *Let V be a complex vector space of dimension n.*

(a) *As a ring*

$$H^*(G_k(V)) = \frac{\mathbb{R}[c_1(S), \ldots, c_{n-k}(S), c_1(Q), \ldots, c_k(Q)]}{(c(S)c(Q) = 1)}$$

(b) *The Chern classes $c_1(Q), \ldots, c_k(Q)$ of the quotient bundle generate the cohomology ring $H^*(G_k(V))$.*

(c) *For a fixed k and a fixed i there are no polynomial relations of degree i among $c_1(Q), \ldots, c_k(Q)$ if the dimension of V is large enough.*

PROOF. In the proof of Proposition 23.1, we saw that the flag manifold $F(V)$ is obtained from the Grassmannian by two flag constructions

$$\begin{tikzcd}
 & \hat{Q} \arrow[d] & \\
S \oplus Q \arrow[d] & F(S) \arrow[dl] & F(\hat{Q}) = F(V) \arrow[l] \\
G_k(V) & &
\end{tikzcd}$$

By (21.18) the cohomology ring of the flag manifold is

$$H^*(F(V)) = \frac{H^*(G_k(V))[x_1, \ldots, x_{n-k}, y_1, \ldots, y_k]}{(\prod (1 + x_i) = c(S), \prod (1 + y_j) = c(Q))}.$$

On the other hand, we've computed the cohomology of $F(V)$ in (21.17) to be

$$(*) \quad H^*(F(V)) = \mathbb{R}[x_1, \ldots, x_{n-k}, y_1, \ldots, y_k]/(\prod (1 + x_i) \prod (1 + y_j) = 1).$$

Thus in $H^*(G_k(V))$ the Chern classes of S and Q can satisfy no relation other than $c(S)c(Q) = 1$, for any relation among them would appear as a relation among the x_i's and y_j's in $(*)$. It follows that there is an injection of algebras

$$(23.2.1) \qquad \frac{\mathbb{R}[c(S), c(Q)]}{(c(S)c(Q) = 1)} \hookrightarrow H^*(G_k(V)).$$

From the digression following this proof, the Poincaré series of $\mathbb{R}[c_1(S), \ldots, c_{n-k}(S), c_1(Q), \ldots, c_k(Q)]/(c(S)c(Q) = 1)$ is

$$P_t\left(\frac{\mathbb{R}[c(S), c(Q)]}{(c(S)c(Q) = 1)}\right) = \frac{(1 - t^2) \cdots (1 - t^{2n})}{(1 - t^2) \cdots (1 - t^{2(n-k)})(1 - t^2) \cdots (1 - t^{2k})}.$$

But this is also the Poincaré series of $H^*(G_k(V))$. Thus the injection (23.2.1) is an isomorphism. This proves (a).

Writing $c(S) = 1/c(Q)$, we see from the description of the ring structure in (a) that $c_1(Q), \ldots, c_k(Q)$ generate the cohomology ring of $G_k(V)$.

The equation $c(S) = 1/c(Q)$ not only allows one to eliminate $c_1(S), \ldots, c_{n-k}(S)$ in terms of $c_1(Q), \ldots, c_k(Q)$, but also gives polynomial relations of degrees $2(n - k + 1), \ldots, 2n$ among $c_1(Q), \ldots, c_k(Q)$. Thus for a given degree i, if the dimension n of the vector space V is so large that $2(n - k + 1) > i$, then there are no polynomial relations of degree i among the Chern classes of Q. □

Digression on the Poincaré Series of a Graded Algebra

Let k be a field and $A = \bigoplus_{i=1}^{\infty} A_i$ a graded algebra over k. The *Poincaré series* of A is defined to be

$$P_t(A) = \sum_{i=0}^{\infty} (\dim_k A_i) t^i.$$

If A is a graded $\mathbb{Z}$-module, its Poincaré series is defined to be that of the $\mathbb{Q}$-algebra $A \otimes_{\mathbb{Z}} \mathbb{Q}$.

Example. Let A be the polynomial ring $\mathbb{R}[x]$, where x is an element of degree n. Then

$$P_t(A) = 1 + t^n + t^{2n} + \cdots = \frac{1}{1 - t^n}.$$

Example. Let A and B be two graded algebras. Suppose a basis for A as a vector space is $\{x_i\}_{i \in I}$ and a basis for B is $\{y_j\}_{j \in J}$. Then a vector space basis for $A \otimes B$ is $\{x_i \otimes y_j\}_{i \in I, j \in J}$. Therefore

$$P_t(A \otimes B) = P_t(A)P_t(B).$$

Example. Let $A = \mathbb{R}[x, y]$, with $\deg x = m$ and $\deg y = n$. Then since $\mathbb{R}[x, y] = \mathbb{R}[x] \otimes \mathbb{R}[y]$,

$$P_t(A) = P_t(\mathbb{R}[x])P_t(\mathbb{R}[y]) = \frac{1}{1 - t^m} \cdot \frac{1}{1 - t^n}.$$

We next investigate the effect of a relation on the Poincaré series of a graded algebra.

Proposition 23.3. *Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded algebra over a field k, and x a homogeneous element of degree n in A. If x is not a zero-divisor, then*

$$P_t(A/xA) = P_t(A)(1 - t^n).$$

Proof. Because x is not a zero-divisor, multiplication by x is an injection. Hence for each integer i there is an exact sequence of vector spaces

$$0 \to A_i \xrightarrow{x} A_{i+n} \to (A/xA)_{i+n} \to 0.$$

By the additivity of the dimension,

$$\dim A_{i+n} = \dim A_i + \dim(A/xA)_{i+n}.$$

Summing over all i,

$$\sum_{i=-n}^{\infty} (\dim A_{i+n}) t^{i+n} = \sum_{i=-n}^{\infty} (\dim A_i) t^{i+n} + \sum_{i=-n}^{\infty} \dim(A/xA)_{i+n} t^{i+n},$$

where we set $A_i = \{0\}$ if i is negative. Hence

$$P_t(A) = P_t(A)t^n + P_t(A/xA).$$ □

Example. If x, y, and z are elements of degree 1, then the Poincaré series of $A = \mathbb{R}[x, y, z]/(x^3y + y^2z^2 + xy^2z)$ is

$$P_t(A) = P_t(\mathbb{R}[x, y, z])(1 - t^4)$$
$$= (1 - t^4)/(1 - t^3).$$

To generalize Proposition 23.3, we will need the notion of a *regular sequence*.

Definition. Let A be a ring. A sequence of elements $a_1, \ldots, a_r$ in A is a *regular sequence* if a_1 is not a zero-divisor in A and for each $i \geq 2$, the image of a_i in $A/(a_1, \ldots, a_{i-1})$ is not a zero-divisor.

Proposition 23.4. *Let A be a graded algebra over a field k and $a_1, \ldots, a_r$ a regular sequence of homogeneous elements of degrees $n_1, \ldots, n_r$. Then*

$$P_t(A/(a_1, \ldots, a_r)) = P_t(A)(1 - t^{n_1}) \cdots (1 - t^{n_r}).$$

PROOF. This is an immediate consequence of Proposition 23.3 and induction on r. $\square$

Let I be the ideal in $\mathbb{R}[x_1, \ldots, x_j, y_1, \ldots, y_k]$ generated by the homogeneous terms of $(1 + x_1 + \cdots + x_j)(1 + y_1 + \cdots + y_k) - 1$. We will now compute the Poincaré series of $\mathbb{R}[x_1, \ldots, x_j, y_1, \ldots, y_k]/I$.

Lemma 23.5. *Let A be a graded algebra over a field k. If $a_1, \ldots, a_r$ is a regular sequence of homogeneous elements of positive degrees in A, so is any permutation of $a_1, \ldots, a_r$.*

PROOF. Since any permutation is a product of transpositions of adjacent elements, it suffices to show that $a_1, \ldots, a_{i-1}, a_{i+1}, a_i, \ldots, a_r$ is a regular sequence. For this it is enough to show that in the ring $A/(a_1, \ldots, a_{i-1})$, the images of a_{i+1}, a_i form a regular sequence. In this way the lemma is reduced to the case of two elements: if a, b is a regular sequence of elements of positive degrees in the graded algebra A, so is b, a.

If x is an element of A, we write $\bar{x}$ for the image of x in whatever quotient ring of A being discussed. Assume that a, b is a regular sequence in A.

(1) Suppose $bx = 0$ in A. Then $\bar{b}\bar{x} = 0$ in $A/(a)$. Since $\bar{b}$ is not a zero-divisor in $A/(a)$, $x = ax_1$ for some x_1 in A. Therefore, $abx_1 = 0$ in A. Since a is not a zero divisor, $bx_1 = 0$. Repeating the argument, we get $x_1 = ax_2$, $x_2 = ax_3$, and so on. Thus $x = ax_1 = a^2x_2 = a^3x_3 = \ldots$, showing that x is divisible by all the powers of a. Since a has positive degree, this is possible only if $x = 0$. Therefore b is not a zero-divisor in A.

(2) Next we show that $\bar{a}$ is not a zero-divisor in $A/(b)$. Suppose $\bar{a}\bar{x} = 0$ in $A/(b)$. Then $ax = by$ for some y in A. It follows that $\bar{b}\bar{y} = 0$ in $A/(a)$. Since $\bar{b}$ is not a zero-divisor in $A/(a)$, $y = az$ for some z. Therefore, $ax = abz$. Since a is not a zero-divisor in A, $x = bz$; hence, $\bar{x} = 0$ in $A/(b)$. $\square$

Lemma 23.6. *If $a_1, \ldots, a_r, b$ and $a_1, \ldots, a_r, c$ are regular sequences in a ring A, then so is $a_1, \ldots, a_r, bc$.*

PROOF. It suffices to check that bc is not a zero-divisor in $A/(a_1, \ldots, a_r)$. This is clear since by hypothesis neither b nor c is a zero-divisor in $A/(a_1, \ldots, a_r)$. $\square$

Proposition 23.7. *The homogeneous terms of*

$$(1 + x_1 + \cdots + x_j)(1 + y_1 + \cdots + y_k) - 1$$

form a regular sequence in $A = \mathbb{R}[x_1, \ldots, x_j, y_1, \ldots, y_k]$.

PROOF. The proof proceeds by induction on j and k. Suppose $j = 1$ and $k = 1$. Then $\mathbb{R}[x_1, y_1]/(x_1 + y_1) = \mathbb{R}[x_1]$ and the image of x_1y_1 in $\mathbb{R}[x_1, y_1]/(x_1 + y_1)$ is $-x_1^2$, which is not a zero divisor. So $x_1 + y_1$, x_1y_1 is a regular sequence in $\mathbb{R}[x_1, y_1]$. For a general j and k, let f_i be the homogeneous term of degree i in $(1 + x_1 + \cdots + x_j)(1 + y_1 + \cdots + y_k) - 1$. We first show that $f_1, \ldots, f_{j+k-1}, x_j$ and $f_1, \ldots, f_{j+k-1}, y_k$ are regular sequences. By Lemma 23.5, $f_1, \ldots, f_{j+k-1}, x_j$ is a regular sequence if and only if $x_j, f_1, \ldots, f_{j+k-1}$ is. Let $\bar{f}_i$ be the image of f_i in $A/(x_j)$. Since x_j is not a zero-divisor in A, it suffices to show that $\bar{f}_1, \ldots, \bar{f}_{j+k-1}$ is a regular sequence in $A/(x_j)$. This is true by the induction hypothesis, since

$$A/(x_j) = \mathbb{R}[x_1, \ldots, x_{j-1}, y_1, \ldots, y_k]$$

and

$$1 + \bar{f}_1 + \cdots + \bar{f}_{j+k-1} = (1 + x_1 + \cdots + x_{j-1})(1 + y_1 + \cdots + y_k).$$

Therefore, $f_1, \ldots, f_{j+k-1}, x_j$ is a regular sequence in A. Similarly, $f_1, \ldots, f_{j+k-1}, y_k$ is also a regular sequence in A. By Lemma 23.6, so is $f_1, \ldots, f_{j+k-1}, x_jy_k$. $\square$

By Propositions 23.4 and 23.7, if I is the ideal in

$$A = \mathbb{R}[x_1, \ldots, x_{n-k}, y_1, \ldots, y_k]$$

generated by the homogeneous terms of

$$(1 + x_1 + \cdots + x_{n-k})(1 + y_1 + \cdots + y_k) - 1,$$

where $\deg x_i = 2i$ and $\deg y_i = 2i$, then the Poincaré series of A/I is

$$P_t(A/I) = \frac{(1 - t^2) \cdots (1 - t^{2n})}{(1 - t^2) \cdots (1 - t^{2(n-k)})(1 - t^2) \cdots (1 - t^{2k})}$$

The Classification of Vector Bundles

We now show that the vector bundles over a manifold M may be classified up to isomorphism by the homotopy classes of maps from M into a Grassmannian. We will first discuss this in the complex case and then extend it by analogy to the real case.

Lemma 23.8. *Let E be a rank k complex vector bundle over a differentiable manifold M of finite type. There exist on M finitely many smooth sections of E which span the fiber at every point.*

Proof. Let $\{U_i\}_{i\in I}$ be a finite good cover for M. Since U_i is contractible, $E|_{U_i}$ is trivial and so we can find k sections $s_{i,1}, \ldots, s_{i,k}$ over U_i which form a basis of the fiber above any point in U_i. By the Shrinking Lemma (see (21.4) and (21.5)), there is an open cover $\{V_i\}_{i\in I}$ with $\bar{V}_i \subset U_i$ and smooth functions f_i such that f_i is identically 1 on V_i and identically 0 outside U_i. Then $\{f_i s_{i,1}, \ldots, f_i s_{i,k}\}_{i\in I}$ are global sections of E which span the fiber at every point. □

Proposition 23.9. *Let E be a rank k complex vector bundle over a differentiable manifold M of finite type. Suppose there are n global sections of E which span the fiber at every point. Then there is a map f from M to some Grassmannian $G_k(\mathbb{C}^n)$ such that E is the pullback under f of the universal quotient bundle Q; that is, $E = f^{-1}Q$.*

Proof. Let $s_1, \ldots, s_n$ be n spanning sections of E and let V be the complex vector space with basis $s_1, \ldots, s_n$. Since $s_1, \ldots, s_n$ are spanning sections, for each point p in M the evaluation map

$$\mathrm{ev}_p : V \to E_p \to 0$$

is surjective. Hence ker ev_p is a codimension k subspace of V, and the fiber of the universal quotient bundle Q at the point $\ker \mathrm{ev}_p$ of the Grassmannian $G_k(V)$ is $V/\ker \mathrm{ev}_p = E_p$. If the map $f : M \to G_k(V)$ is defined by

$$f : p \mapsto \ker \mathrm{ev}_p,$$

then the quotient bundle Q pulls back to E. We can identify V with $\mathbb{C}^n$, and $G_k(V)$ with $G_k(\mathbb{C}^n)$. □

This map $f : M \to G_k(\mathbb{C}^n)$ is called a *classifying map* for the bundle E.

In the proposition above, if $\{s'_1, \ldots, s'_n\}$ is another choice of global sections which span the fiber at every point, and V' the vector space with basis $s'_1, \ldots, s'_n$, then there is a natural isomorphism of V with V', and of $G_k(V)$ with $G_k(V')$. Therefore the classifying map $f : M \to G_k(\mathbb{C}^n)$ is well-defined up to the identification of V with $\mathbb{C}^n$, independent of the choice of the n spanning sections. More precisely, any two such maps f and f' from M to $G_k(\mathbb{C}^n)$ differ by the action of an element B of $GL(n, \mathbb{C})$ on $G_k(\mathbb{C}^n)$; that is,

$$f' = B \circ f, \quad \text{where} \quad B : G_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n).$$

Since $GL(n, \mathbb{C})$ is connected, there is a path $\gamma(t)$ joining the identity element I and B in $GL(n, \mathbb{C})$. Then $f_t = \gamma(t) \circ f$ is a homotopy between f and f'. Therefore we may refine Proposition 23.9 to include the following statement.

(23.9.1) *The homotopy class of the classifying map $f : M \to G_k(\mathbb{C}^n)$ of the vector bundle E is uniquely determined by E.*

In fact, if $g : E \to E'$ is an isomorphism of vector bundles over M, then g gives a one-to-one correspondence between the sections of E and the sections of E'. So by the same reasoning as in (23.9.1), we see that E and E' determine homotopic classifying maps f and f' from M to the Grassmannian $G_k(\mathbb{C}^n)$. Writing $\mathrm{Vect}_k(M; \mathbb{C})$ for the isomorphism classes of the rank k complex vector bundles over M and $[X, Y]$ for the set of all homotopy classes of maps from X to Y, we have the following.

(23.9.2) *For n sufficiently large, there is a well-defined map*

$$\beta : \mathrm{Vect}_k(M; \mathbb{C}) \to [M, G_k(\mathbb{C}^n)]$$

given by the classifying map of a vector bundle.

Remark 23.9.3. From the proof of Lemma 23.8, if M has a good cover with r open sets, then every rank k bundle over M has a set of rk spanning sections. Thus in (23.9) given the manifold M of finite type and the integer k, if n is an integer $\geq rk$, then the Grassmannian $G_k(\mathbb{C}^n)$ classifies all vector bundles of rank k over M.

Theorem 23.10. *Let M be a manifold having a good cover consisting of r open sets and let k be a positive integer. For every integer $n \geq kr$, there is a one-to-one correspondence*

$$\mathrm{Vect}_k(M; \mathbb{C}) \simeq [M, G_k(\mathbb{C}^n)]$$

between the isomorphism classes of rank k complex vector bundles over M and the homotopy classes of maps from M into the complex Grassmannian $G_k(\mathbb{C}^n)$.

Proof. By the homotopy property of vector bundles (Theorem 6.8), there is a map

$$\alpha : [M, G_k(\mathbb{C}^n)] \to \mathrm{Vect}_k(M; \mathbb{C})$$

given by the pullback of the universal quotient bundle over $G_k(\mathbb{C}^n)$:

$$f \mapsto f^{-1}Q.$$

By (23.9), (23.9.2), and (23.9.3), for any integer $n \geq rk$, the map

$$\beta : \mathrm{Vect}_k(M; \mathbb{C}) \to [M, G_k(\mathbb{C}^n)],$$

given by the homotopy class of the classifying map of a vector bundle, is inverse to α. □

As a corollary of the existence of the universal bundle (23.9), we now show that in a precise sense the Chern classes are the only cohomological invariants of a smooth complex vector bundle.

Proposition 23.11. *Every natural transformation from the isomorphism classes of complex vector bundles over a manifold of finite type to the de Rham cohomology can be given as a polynomial in the Chern classes.*

PROOF. Let T be a natural transformation from the functor $\mathrm{Vect}_k(\ ;\mathbb{C})$ to the functor $H^*(\ )$ in the category of manifolds of finite type. By Proposition 23.9 and the naturality of T, if E is any rank k complex vector bundle over M and $f: M \to G_k(\mathbb{C}^n)$ a classifying map for E, then

$$T(E) = T(f^{-1}Q) = f^*T(Q).$$

Because the cohomology of the Grassmannian $G_k(\mathbb{C}^n)$ is generated by the Chern classes of Q (Prop. 23.2(b)), $T(Q)$ can be written as

$$T(Q) = P_T(c_1(Q), \ldots, c_k(Q))$$

for some polynomial P_T depending on T. Therefore

$$T(E) = f^*T(Q) = P_T(f^*c_1(Q), \ldots, f^*c_k(Q)) = P_T(c_1(E), \ldots, c_k(E)). \qquad \square$$

Of course there is an analogue of Theorem 23.10 for real vector bundles. Recall that we write $\mathrm{Vect}_k(M)$ for the isomorphism classes of the rank k real vector bundles over M.

Theorem 23.12. *Let M be a manifold having a good cover consisting of r open sets and k a positive integer. For every integer $n \geq rk$, there is a one-to-one correspondence*

$$\mathrm{Vect}_k(M) \simeq [M, G_k(\mathbb{R}^n)].$$

The proof is completely analogous to that of Theorem 23.10.

We now classify the vector bundles over spheres and relate them to the homotopy groups of the orthogonal and unitary groups.

Exercise 23.13. (a) Use Exercise 17.24 and the homotopy exact sequence of the fibration

$$\begin{array}{c} O(k) \to O(n)/O(n-k) \\ \downarrow \\ G_k(\mathbb{R}^n) \end{array}$$

to show that

$$\pi_q(G_k(\mathbb{R}^n)) = \pi_{q-1}(O(k)) \quad \text{if} \quad n \geq k + q + 2.$$

(b) Similarly show that

$$\pi_q(G_k(\mathbb{C}^n)) = \pi_{q-1}(U(k)) \quad \text{if} \quad n \geq (2k + q + 1)/2.$$

Combining these formulas with Proposition 17.6.1 concerning the relation of free versus base-point perserving homotopies we find that for n sufficiently large,

$$\begin{aligned} \mathrm{Vect}_k(S^q) &= [S^q, G_k(\mathbb{R}^n)] \\ &= \pi_q(G_k(\mathbb{R}^n))/\pi_1(G_k(\mathbb{R}^n)) \\ &= \pi_{q-1}(O(k))/\pi_0(O(k)). \end{aligned}$$

Exactly the same computation works for the complex vector bundles over S^q. We summarize the results in the following.

Proposition 23.14. *The isomorphism classes of the differentiable rank k real vector bundles over the sphere S^q are given by*

$$\mathrm{Vect}_k(S^q) \simeq \pi_{q-1}(O(k))/\mathbb{Z}_2\,;$$

the isomorphism classes of the complex vector bundles are given by

$$\mathrm{Vect}_k(S^q;\mathbb{C}) \simeq \pi_{q-1}(U(k)).$$

REMARK 23.14.1 If G is a Lie group and $a \in G$, then conjugation by a defines an automorphism h_a of G:

$$h_a(g) = aga^{-1}.$$

Let m be any integer. The map h_a induces a map of homotopy groups:

$$(h_a)_*: \pi_m(G) \to \pi_m(G).$$

If two elements a and b in G can be joined by a path $\gamma(t)$ in G, then h_a is homotopic to h_b via the homotopy $h_{\gamma(t)}$. Consequently $(h_a)_* = (h_b)_*$. In this way conjugation induces an action of $\pi_0(G)$ on $\pi_m(G)$, called the *adjoint action.*

We know from (17.6) that for any space X with base point x, conjugation on the loop space $\Omega_x X$ induces an action of $\pi_1(X)$ on $\pi_q(X)$. With a little more classifying space theory, it can be shown that the action of $\pi_0(O(k))$ on $\pi_{q-1}(O(k))$ corresponding to the action of $\pi_1(G_k(\mathbb{R}^n))$ on $\pi_q(G_k(\mathbb{R}^n))$ under the identification of $\pi_{q-1}(O(k))$ with $\pi_q(G_k(\mathbb{R}^n))$ is precisely the adjoint action.

REMARK 23.14.2. It is in fact possible to explain the correspondence (23.14) directly. Let E be a rank k vector bundle over S^q with structure group $O(k)$, and let U_0 and U_1 be small open neighborhoods of the upper and lower hemispheres. Because U_0 and U_1 are contractible, E is trivial over them. Hence E is completely determined by the transition function

$$g_{01}: U_0 \cap U_1 \to O(k).$$

g_{01} is called a *clutching function* for E. Then Proposition 23.14 may be interpreted as a correspondence between the isomorphism classes of vector bundles over a sphere and the free homotopy classes of the clutching functions.

Exercise 23.15. Compute $\operatorname{Vect}_k(S^1)$, $\operatorname{Vect}_k(S^2)$, and $\operatorname{Vect}_k(S^3)$.

EXAMPLE 23.16 (An orientable sphere bundle with zero Euler class but no section). Because S^4 is simply connected, every vector bundle over S^4 is orientable (Proposition 11.5). For a line bundle orientability implies triviality. Therefore,

$$\operatorname{Vect}_1(S^4) = 0.$$

By (23.14),

$$\begin{aligned}\operatorname{Vect}_2(S^4) &= \pi_3(SO(2))/\mathbb{Z}_2 = \pi_3(S^1)/\mathbb{Z}_2 = 0,\\ \operatorname{Vect}_3(S^4) &= \pi_3(SO(3))/\mathbb{Z}_2 = \pi_3(\mathbb{R}P^3)/\mathbb{Z}_2\\ &= \pi_3(S^3)/\mathbb{Z}_2 = \mathbb{Z}/\mathbb{Z}_2.\end{aligned}$$

Consequently there is a nontrivial rank 3 vector bundle E over S^4. The Euler class of E vanishes trivially, since $e(E)$ is in $H^3(S^4) = 0$. If E has a nonzero global section, it would split into a direct sum $E = L \oplus F$ of a line bundle and a rank 2 bundle. Since $\operatorname{Vect}_1(S^4) = \operatorname{Vect}_2(S^4) = 0$, this would imply that E is trivial, a contradiction. Therefore the unit sphere bundle of E relative to some Riemannian metric is an orientable S^2-bundle over S^4 with zero Euler class but no section. This example shows that the converse of Proposition 11.9 is not true.

REMARK 23.16.1 Actually $\operatorname{Vect}_3(S^4) \simeq \mathbb{Z}$, because the action of $\mathbb{Z}_2$ on $\pi_3(SO(3))$ is trivial. Indeed, by Remark 23.14.1 this action is induced by the action of $-1 \in O(3)$ under conjugation on $SO(3)$. But conjugating by -1 clearly gives the identity map.

In general, by the same reasoning, if k is odd, then the action of $\pi_0(O(k))$ on $\pi_q(O(k))$ is trivial for all q.

The Infinite Grassmannian

We will now say a few words about vector bundles over manifolds not having a finite good cover. For Theorem 23.10 to hold here the analogue of the finite Grassmannian is the *infinite Grassmannian*. Given a sequence of complex vector spaces

$$\cdots \subset V_r \subset V_{r+1} \subset V_{r+2} \subset \cdots \qquad . \dim_{\mathbb{C}} V_i = i,$$

there is a naturally induced sequence of Grassmannians

$$\cdots \subset G_k(V_r) \subset G_k(V_{r+1}) \subset G_k(V_{r+2}) \subset \cdots.$$

The infinite Grassmannian $G_k(V_\infty)$ is the telescope constructed from this sequence. Over each $G_k(V_r)$ there are the universal quotient bundles Q_r and there are maps

$$\cdots \subset Q_r \subset Q_{r+1} \subset Q_{r+2} \subset \cdots.$$

By the telescoping construction again there is a bundle Q of rank k over $G_k(V_\infty)$. A point of $G_k(V_\infty)$ is a subspace Λ of codimension k in V and the fiber of Q over Λ is the k-dimensional quotient space V_∞/Λ.

Unfortunately the infinite Grassmannian is infinite-dimensional and so is not a manifold in our sense of the word. Since to discuss infinite-dimensional manifolds would take us too far afield, we will merely indicate how our theorems may be extended. By the countable analogue of the Shrinking Lemma (Ex. 21.4), with the finite cover replaced by a countable locally finite cover, one can show just as in Lemma 23.8 that every vector bundle over an arbitrary manifold M has a collection of countably many spanning sections $s_1, s_2, \ldots$. If V_∞ is the infinite-dimensional vector space with basis $s_1, s_2, \ldots,$ there is again a surjective evaluation map at each point p in M:

$$\operatorname{ev}_p : V_\infty \to E_p \to 0.$$

The kernel of ev_p is a codimension k subspace of V_∞. So the function $f(p) = \ker \operatorname{ev}_p$ sends M into the infinite Grassmannian $G_k(V_\infty)$. This map f is a classifying map for the vector bundle E and there is again a one-to-one correspondence

$$\operatorname{Vect}_k(M; \mathbb{C}) \simeq [M, G_k(\mathbb{C}^\infty)].$$

All this can be proved in the same way as for manifolds of finite type. From Proposition 23.2, it is reasonable to conjecture that the cohomology ring of the infinite Grassmannian $G_k(\mathbb{C}^\infty)$ is the free polynomial algebra

$$\mathbb{R}[c_1(Q), \ldots, c_k(Q)].$$

This is indeed the case. (For a proof see Milnor and Stasheff [1, p. 161] or Husemoller [1, Ch. 18, Th. 3.2, p. 269].) Hence Proposition 23.11 extends to a general manifold.

Exercise 23.17. Let V be a vector space over $\mathbb{R}$ and $V^* = \operatorname{Hom}(V, \mathbb{R})$ its dual.

(a) Show that $P(V^*)$ may be interpreted as the set of all hyperplanes in V.
(b) Let $Y \subset P(V) \times P(V^*)$ be defined by

$$Y = \{([v], [H]) \mid H(v) = 0,\ v \in V,\ H \in V^*\}.$$

In other words, Y is the incidence correspondence of pairs (line in V, hyperplane in V) such that the line is contained in the hyperplane. Compute $H^*(Y)$.

Concluding Remarks

In the preceding sections the Chern classes of a vector bundle E over M were first defined by studying the relations in the cohomology ring $H^*(PE)$ of the projective bundle, where the ring was considered as an algebra over

$H^*(M)$. This somewhat ad hoc procedure turned out to yield all characteristic classes of E only after we learned that all bundles of a given rank were pullbacks of a universal bundle and that the cohomology ring of the universal base space (the classifying space) was generated by the Chern classes of the universal bundle.

From a purely topological point of view one could therefore dispense with the original definition, for by designating a set of generators of the cohomology ring of the classifying space as the universal Chern classes, one can define the Chern classes of any vector bundle simply as the pullbacks via the classifying map of the universal Chern classes. On the other hand, from the differential-geometric point of view the projective-bundle definition is more appealing, starting as it does, with $c_1(S^*)$, a class that we understand rather thoroughly and that furnishes us with a canonical generator for $H^*(PE)$ over $H^*(M)$. However, this c_1 is taken on the space $P(E)$ rather than on M and is therefore not directly linked to the geometry of M. The question arises whether one can write down a form representing $c_k(E)$ in terms of the following data:

(1) a good cover $\mathfrak{U} = \{U_\alpha\}$ of M which trivializes E;
(2) the transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{C})$$

for E relative to such a trivialization;
(3) a partition of unity subordinate to the open cover $\mathfrak{U}$.

The answer to this question is yes and the reader is referred to Bott [2] for a thoroughgoing discussion. Here we will describe only the final recipe, for to understand it properly, we would have to explore the concepts of connections and curvature, which are beyond the scope of this book.

Observe first that we are already in possession of the desired formula for the first Chern class of a complex line bundle L (see (6.38)). Indeed, if $g_{\alpha\beta}$ is the transition function for L, the element

$$c^{1,1} = \frac{i}{2\pi} d \log g_{\alpha\beta}$$

in the Čech–de Rham complex $C^*(\mathfrak{U}, \Omega^*)$ is both d- and δ-closed. By the collating formula (9.5), once a partition of unity is selected, this cocycle yields a global form. The cohomology class of this global form is $c_1(L)$.

In the general case one can construct a cocyle $\sum_{q=0}^{k-1} c^{k-q,k+q}$, with $c^{k-q,k+q}$ in $C^{k-q}(\mathfrak{U}, \Omega^{k+q})$, that represents the k-th Chern class $c_k(E)$ by the following unfortunately rather formidable "averaging" procedure.

Let $I = (i_0, \cdots, i_q)$ correspond to a nonvacuous intersection, set

$$U_I = U_{i_0} \cap \cdots \cap U_{i_q},$$

and let

$$g_{0j} : U_{i_0} \cap U_{i_j} \to GL(n, \mathbb{C})$$

be the pertinent transition matrix function for E. Consider the expression

$$\theta_I = \sum_{j=0}^{q} t_j\, g_{0j}^{-1}\, dg_{0j}$$

as a matrix of 1-forms on $U_I \times \mathbb{R}^{q+1}$, the t's being linear coordinates in $\mathbb{R}^{q+1}$. From θ one can construct the matrix of 2-forms

$$K_I = d\theta_I + \tfrac{1}{2}\theta_I^2$$

on $U_I \times \mathbb{R}^{q+1}$ and set

$$c_I(E) = \det\left(1 + \frac{i}{2\pi} K_I\right).$$

Our recipe is now completed by the following ansatz. Let

$$\Delta_q = \{(t_1, \cdots, t_{q+1}) \mid t_j \geq 0, \textstyle\sum t_j = 1\}$$

be the standard q-simplex in $\mathbb{R}^{q+1}$. The $2k$-form $c_I^k(E)$ restricted to $U_I \times \Delta_q$, and integrated over the "fiber Δ_q" yields the desired form on U_I:

$$c_I^{k-q,k+q}(E) = \int_{\Delta_q} c_I^k(E).$$

In other words, $c_k(E)$ is represented by the chain

$$\sum_{q=0}^{k-1} c^{k-q,k+q} \in C^*(\mathfrak{U}, \Omega^*).$$

Note that for dimensional reasons this chain has no component below the diagonal and also no component in the zero-th column. This fact has interesting applications in foliation theory (Bott [1]). In any case, the collating procedure (9.5) now completes the construction of the forms $c_k(E)$ in terms of the specified data.