

QUANTUM THEORY FROM A GEOMETRIC VIEWPOINT

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These notes are based on lectures given at UT Austin in Spring 2023 and Harvard University in Fall 2023. The course is designed to provide inroads for mathematicians to engage with quantum physics. In Lectures¹ 1–23, which made up the first semester, the setting is close to traditional physics: real time quantum mechanics and quantum field theory. It ends with axioms for Wick-rotated (imaginary time) quantum field theory. The second semester Lectures 24–44 are mostly in the Wick-rotated setting and focus on parts of the theory that are most mathematically developed: topological field theory, invertible field theory, supersymmetric field theory, and applications to symplectic geometry and gauge theory. The notes are rough in parts, and certainly my point of view is constantly evolving. Nonetheless, I offer this preliminary version in the hope that (1) it is useful, and (2) you will provide feedback and suggestions.

There are many topics missing in these notes, some of which I hope to fill in in a future version. The most important of these is the relationships between classical and quantum theories, including quantization.

I appended homework problems to these notes.

My views on this material, both general and specific, are heavily influenced by many mathematicians and physicists over many years. I give many references for further reading and investigation. Nonetheless, some insights I’ve learned from others have sneaked into these notes without attribution.

I warmly welcome your comments/corrections/suggestions: email me at dafr@math.harvard.edu.

Here is some text I wrote for students in the course:

In this course you will see the application of mathematical ideas, techniques, and viewpoints to a subject outside of mathematics. Such applications draw freely on whatever mathematics is needed with no walls between subdisciplines (analysis, algebra, geometry, representation theory, etc.). As such the course places large demands on your mathematical knowledge and sophistication. I assume exposure to fundamental material at least at the level of basic graduate courses, and often beyond. While I review many mathematical ideas—and a quirky selection at that—I do not develop them in detail in this course. I give references for you to learn more, and I hope that you will delve in at some point, if not immediately. Most likely you will not emerge from this course with a *working* knowledge of *all* of this mathematics, but you will emerge with a deeper perspective on how different parts of mathematics work together. Similar remarks apply to the physics. My focus is the geometric structure of quantum theory, not on physics applications *per se*, though there are many of those as well. Bringing quantum field theory into mathematics is an ongoing longterm project; this course is a snapshot of aspects of my current understanding.

Recall the classic Three-Step Procedure of applications of mathematics to problems external to mathematics: (1) build a mathematical model, (2) solve the mathematics questions which arise in the model, and (3) apply the solution back to the external problems. Recall too that

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¹The “lectures” here do not strictly correspond to what was done in the classes.

within physics theories refine over time: relativistic mechanics replaces Newtonian mechanics at large scales, just as quantum mechanics replaces Newtonian mechanics at small scales. So too the models (axiom systems) in these lectures can be expected to apply to some range of problems, but certainly not to all. Furthermore, as we penetrate further into problems external to mathematics, we refine and amend our models. In any case, a major goal of the interaction between mathematics and quantum theory is to articulate and develop new mathematical structures, and these too are dynamic until they settle. (This is true of any notion in mathematics that we now think of as *stare decisis*: think of the early history of the notion of a topological space, or a smooth manifold, or almost any of our current mathematical objects.) All this is to reassure you that the unsettled feeling you may experience in this course is expected. Embrace it! Unsettledness is the steady state of a researcher.

Finally, mathematics is very powerful in many ways. One of them is its conceptualization of structure. The edifice of definitions, axioms, and theorems is a foundation for all of our investigations. If we can build this for a field outside of mathematics, then we have a solid basis from which to proceed, even if that edifice evolves over time. It is in this spirit that we begin this course in the next section with an abstract conceptualization of states, observables, and mechanical systems.

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Lecture 1: States and observables

We begin with very general axioms that capture the notions of ‘state’ and ‘observable’. These axioms apply to classical mechanics, statistical mechanics, and—our main foci—quantum mechanics and quantum field theory. As with all axiom systems in this course, they are not meant to be in final form, but rather serve as a guide for engagement with physics. The intuition of many mathematicians and physicists are summarized here, among them Dirac [D], von Neumann [vN1], Irving Segal [S] and Mackey [Ma1, §2-2]. I highly recommend the leisurely account in [FY].

In mechanics one has motion, which is a flow of states and observables: a representation of the additive group $\mathbb{R}$ of time translation. Later in the course we discuss systems equipped with a representation of a larger symmetry group, in particular relativistic systems.

We illustrate the axioms in the familiar case of a particle or finite set of particles moving in Euclidean space. Then we describe states and observables in abstract quantum mechanical systems. Here we summarize the structure quickly; in the next several lectures we spell it out more thoroughly and give many concrete examples of classical and quantum mechanical systems. What is a glaring omission in these notes is a systematic discussion of the relation between classical and quantum systems, though we do discuss quantization of free theories in later lectures.

Sometimes background mathematical material is included at the end of a written lecture so as not to interrupt the main exposition. You should consult it for unfamiliar terms and concepts. In this lecture we include background on affine and convex spaces §1.4.1, symplectic manifolds §1.4.2, and on bounded operators in Hilbert space §1.4.3. The spectral theorem is stated and used in this lecture; in [Lecture 4](#) we discuss it and several of its cousins in more detail.

1.1 States, observables, and motion

A physical system consists of *states* and *observables* and a pairing between them. For example, fix a positive integer k and consider a system of k particles moving in a Euclidean space. A state is the collection of trajectories of the particles, or less canonically, the collection of positions and velocities at some fixed time. An observable is a real-valued function on collections of trajectories. Example:

the distance between the 2nd and 13th particles at 4:45 AM on January 10, 2023. The following axiomatization of states and observables is due to many people. The original references [D, vN1] are the origin. A thorough mathematical account is in [Ma1, §2-2], and the first lectures of [FY] are a helpful exposition.

Axiom System 1.1. *Data of states and observables* (DOSO) consists of:

- (1) A real convex space $\mathcal{S}$. Elements of $\mathcal{S}$ are *states*, extreme points of $\mathcal{S}$ are *pure states*, and non-extreme points are *mixed states*. Let $\mathcal{PS} \subset \mathcal{S}$ denote the subspace of pure states.
- (2) A complex topological vector space $\mathcal{O}$ equipped with a real structure² $A \mapsto A^*$, a dense subspace $\mathcal{O}^\infty \subset \mathcal{O}$, a complex Lie algebra structure on $\mathcal{O}^\infty$, and a map³

$$(1.2) \quad \begin{aligned} \text{Borel}(\mathbb{R}; \mathbb{R}) \times \mathcal{O}_{\mathbb{R}} &\longrightarrow \mathcal{O}_{\mathbb{R}} \\ f, A &\longmapsto f(A) \end{aligned}$$

A real point of $\mathcal{O}$ —an element of $\mathcal{O}_{\mathbb{R}}$ —is called an *observable*.

- (3) A *measurement* pairing

$$(1.3) \quad \begin{aligned} \mathcal{O}_{\mathbb{R}} \times \mathcal{S} &\longrightarrow \text{Prob}(\mathbb{R}) \\ A, \sigma &\longmapsto \sigma_A \end{aligned}$$

where $\text{Prob}(\mathbb{R})$ is the space of probability measures on $\mathbb{R}$.

- (4) A map from $\mathcal{O}_{\mathbb{R}}^\infty$ to a one-parameter group of automorphisms of $\mathcal{S}$ and a map from $\mathcal{O}_{\mathbb{R}}^\infty$ to a one-parameter group of automorphisms of $\mathcal{O}$.

There are many compatibilities among this data, for example the following. For $f \in \text{Borel}(\mathbb{R}; \mathbb{R})$, $A, A_1, A_2 \in \mathcal{O}$, $\sigma, \sigma_i \in \mathcal{S}$, and positive real numbers x^i summing to 1 we require

$$(1.4) \quad \begin{aligned} (x^i \sigma_i)_A &= x^i (\sigma_i)_A \\ \sigma_{A_1+A_2} &= \sigma_{A_1} * \sigma_{A_2} \\ \sigma_{f(A)} &= f_* \sigma_A \\ [A_1, A_2]^* &= [A_1^*, A_2^*], \quad A_1, A_2 \in \mathcal{O}^\infty, \end{aligned}$$

where ‘ $*$ ’ in the second equation is the convolution product of measures. For $H \in \mathcal{O}_{\mathbb{R}}^\infty$, let

$$(1.5) \quad \begin{aligned} \sigma &\longmapsto \sigma^{(t)} \\ A &\longmapsto A^{(t)} \end{aligned}$$

denote the generated one-parameter groups of automorphisms of $\mathcal{S}$ and $\mathcal{O}$, respectively. Then

$$(1.6) \quad \frac{dA^{(t)}}{dt} + [H, A^{(t)}] = 0, \quad A \in \mathcal{O}^\infty, \quad t \in \mathbb{R},$$

$$(1.7) \quad \sigma^{(t)}_A = \sigma_{A^{(t)}}, \quad \sigma \in \mathcal{S}, \quad A \in \mathcal{O}, \quad t \in \mathbb{R}.$$

²A real structure on a complex vector space is an antilinear involution. Its fixed points are called *real points*.

³ $\text{Borel}(X; \mathbb{R})$ denotes the vector space of real-valued Borel functions on the topological space X .

Finally, the *expectation value*

$$(1.8) \quad \langle A \rangle_\sigma = \int_{\mathbb{R}} \lambda d\sigma_A(\lambda)$$

defines a real-valued pairing $\mathcal{S} \times \mathcal{O}_{\mathbb{R}} \rightarrow \mathbb{R}$ that separates points.

The last statement means: (1) if $\sigma_1, \sigma_2 \in \mathcal{S}$ satisfy $\langle A \rangle_{\sigma_1} = \langle A \rangle_{\sigma_2}$ for all $A \in \mathcal{O}_{\mathbb{R}}$, then $\sigma_1 = \sigma_2$; and (2) if $A_1, A_2 \in \mathcal{O}_{\mathbb{R}}$ satisfy $\langle A_1 \rangle_\sigma = \langle A_2 \rangle_\sigma$ for all $\sigma \in \mathcal{S}$, then $A_1 = A_2$.

We make several remarks about **Axiom System 1.1**.

Remark 1.9.

- (1) Equation (1.7) expresses the compatibility between (1) the *Schrödinger picture*, in which states evolve, and (2) the *Heisenberg picture*, in which operators evolve.
- (2) In this general setting, the measurement process is probabilistic. In classical mechanics, in a pure state σ , which describes a definite trajectory, the result σ_A of measuring an observable A is a point measure, so no probability is necessary. In a mixed state, which describes a statistical mechanical state, the measure σ_A has positive variance. In quantum mechanics, even in a pure state, the result of measurement usually has positive variance.
- (3) There is a map

$$(1.10) \quad \text{Prob}(\mathcal{S}_0) \longrightarrow \mathcal{S}$$

that averages pure states over a probability measure to produce a general state.

- (4) A symmetry of DOSO is the data of automorphisms of $\mathcal{S}$ and of $\mathcal{O}$ that preserve the structural data. This is typically a very large group.

Definition 1.11. Suppose given DOSO as in **Axiom System 1.1**. A *motion* is the data of compatible 1-parameter groups of automorphisms of $\mathcal{S}$ and of $\mathcal{O}$. A *mechanical system* is DOSO together with a fixed motion.

Denote the chosen one-parameter groups as

$$(1.12) \quad \begin{aligned} \sigma &\longmapsto \sigma^{(t)} \\ A &\longmapsto A^{(t)} \end{aligned}$$

Then the compatibility condition (1.7) is that these flows are adjoint under the measurement pairing (1.3), as in (1.7):

$$(1.13) \quad \sigma_A^{(t)} = \sigma_{A^{(t)}}$$

Remark 1.14.

- (1) In many cases the flows are generated by an observable via the map (4) in **Axiom System 1.1**. If so, the *negative* of that observable is called the *Hamiltonian* of the system. Our sign convention for this and other choices are elaborated in [DF1].

- (2) The flows (1.12) are actions of the *group* $\mathbb{R}$ of time translations.
- (3) Time is an affine line and time translation is its group of translations. Here, for simplicity, we use the standard affine line $\mathbb{A}^1$ for time (it should also have the standard Euclidean structure).
- (4) The symmetry group of a mechanical system is a subgroup of the symmetry group of the DOSO. We require that symmetries preserve the flows (1.12) up to reversal of time; see §11.1 for the case of time-reversing symmetries in quantum mechanics.

Often a state $\sigma \in \mathcal{S}$ is fixed as part of the data of a mechanical system, and it may be special.

Definition 1.15. Fix a mechanical system. A state $\sigma \in \mathcal{S}$ that satisfies $\sigma^{(t)} = \sigma$ for all $t \in \mathbb{R}$ is called a *stationary state*.

1.2 Classical mechanics

We illustrate how **Axiom System 1.1** and **Definition 1.11** encode a classical or statistical mechanical system. For concreteness consider a single particle moving in standard Euclidean space $\mathbb{E}^d$ of some dimension $d \in \mathbb{Z}^{>0}$. (See **Definition 1.33**. We revisit this example in more detail below in **Example 3.3**.) We also assume a potential energy function $V: \mathbb{E}^d \rightarrow \mathbb{R}$. For convenience, write $M = \mathbb{E}^d$. Let $m \in \mathbb{R}^{>0}$ be the mass of the particle.

Let $\mathcal{F} = \text{Map}(\mathbb{R}, M)$ be the space of all smooth motions $x: \mathbb{R} \rightarrow \mathbb{E}^d$ in $M = \mathbb{E}^d$; it is the space of *possible* trajectories of a particle.⁴ *Physical* or *classical* trajectories satisfy Newton's law:

$$(1.16) \quad m\ddot{x} = -\nabla V.$$

(This is an equality of functions $\mathbb{R} \rightarrow \mathbb{R}^d$.) We assume that solutions to (1.16) exist for all time, given an initial position and velocity. Then the set $N \subset \mathcal{F}$ of physical trajectories forms a finite dimensional manifold. Note that for any time $t_0 \in \mathbb{R}$, the map

$$(1.17) \quad \begin{aligned} N &\longrightarrow \mathbb{E}^d \times \mathbb{R}^d \\ x &\longmapsto (x(t_0), \dot{x}(t_0)) \end{aligned}$$

is a diffeomorphism. (This description breaks time-translation symmetry.)

Now we proceed to the data (1)–(4) in **Axiom System 1.1**. Define the space $\mathcal{PS} = N$ of *pure* states to be the manifold of classical trajectories. Then $\mathcal{S} = \text{Prob}(N)$ is the space of probability measures on N ; see **Example 1.34**. A mixed state $\sigma \in \mathcal{S} \setminus \mathcal{PS}$ describes a statistical state in which the particle does not undergo a fixed classical motion, but rather its motion is probabilistic.

Define an observable to be a function on N . It may be the restriction of a function on $\mathcal{F}$, but a possible extension beyond classical trajectories is not relevant to the data (2) in **Axiom System 1.1**. We must specify what kinds of functions we allow:

$$(1.18) \quad \begin{aligned} \mathcal{O} &= \text{Borel}(N; \mathbb{C}) \\ \mathcal{O}^\infty &= C^\infty(N; \mathbb{C}) \end{aligned}$$

⁴We can endow $\mathcal{F}$ with the structure of an infinite dimensional manifold, but we do not elaborate here. We could also work with finite dimensional families of smooth motions, which suffices for all of our considerations.

The real structure $A \mapsto A^*$ is complex conjugation, hence $\mathcal{O}_{\mathbb{R}} = \text{Borel}(N; \mathbb{R})$ consists of real-valued Borel functions.

The measurement pairing (1.3) is simply the pushforward measure: if $A \in \text{Borel}(N; \mathbb{R})$ and $\sigma \in \text{Prob}(N)$, then

$$(1.19) \quad \sigma_A = A_* \sigma$$

is a probability measure on $\mathbb{R}$. Note that if $\sigma \in \mathcal{PS}$ is a pure state supported at $x \in N$, then σ_A is a point measure supported at $A(x)$. In that case, the measurement is deterministic.

It remains to explain how smooth observables—that is, smooth functions on N —generate motion. The motion we need in [Axiom System 1.1\(4\)](#) is that of states and observables, but in fact those motions are induced from a motion on N . And the latter comes from a *symplectic structure* on N . See §1.4.2 below for some basics about symplectic manifolds, and [Lecture 2](#) for a more leisurely account. In fact, a pair (N, H) consisting of a symplectic manifold N and a function $H \in C^\infty(N; \mathbb{R})$ determines the data of a mechanical system. For the case of a particle moving in Euclidean space $\mathbb{E}^d$, the symplectic structure on the space N of classical trajectories is *derived* from a more fundamental structure: a *variational principle* or *action principle* for the classical equation of motion (1.16). This is a *Lagrangian* formulation of the classical mechanical system; the general case of a pair (N, H) is the *Hamiltonian* formulation. All of this, including the derivation of the symplectic structure in the Lagrangian framework, is explained in detail in [Lecture 3](#). Here we simply exhibit the symplectic 2-form on N .

Remark 1.20. More generally, a pair (N, φ_t) of a symplectic manifold and a flow of symplectic diffeomorphisms (symplectomorphisms) determines the data of a mechanical system. Not every such system admits a Hamiltonian function: see [Example 3.12](#).

In fact, we define a 2-form ω on the entire mapping space $\mathcal{F}$, implicitly assuming it is a smooth manifold. The restriction of ω to N is closed and nondegenerate, i.e., is a symplectic form. Write δ for the de Rham differential on $\mathcal{F}$, which is also a (partial) differential on the product manifold $\mathcal{F} \times \mathbb{R}$. Let

$$(1.21) \quad \begin{aligned} e: \mathcal{F} \times \mathbb{R} &\longrightarrow M \\ (x, t) &\longmapsto x(t) \end{aligned}$$

be the evaluation map. Then

$$(1.22) \quad \omega = m \langle \delta \dot{e} \wedge \delta e \rangle$$

is a 2-form on $\mathcal{F} \times \mathbb{R}$. The restriction of ω to $N \times \{t_0\} \subset N \times \mathbb{R}$ is closed, nondegenerate, and is independent of $t_0 \in \mathbb{R}$. We write the pullback of ω under the identification (1.17). Let $x^1, \dots, x^d$ be the standard coordinates on $\mathbb{E}^d$, and let $\dot{x}^1, \dots, \dot{x}^d$ be the standard coordinates on $\mathbb{R}^d$. Then the symplectic form is

$$(1.23) \quad \omega = \sum_{i=1}^d m \, dx^i \wedge d\dot{x}^i.$$

Remark 1.24. A symplectic manifold N together with a smooth function $H: N \rightarrow \mathbb{R}$ is a *dynamical system*. The theory of *ordinary* differential equations provides a local flow on N that integrates the symplectic gradient vector field. Existence and uniqueness of the flow are the main theorems, and they follow from fixed point theorems used to solve the flow equations. That is relatively easy. What is more interesting are the *dynamical* questions: Are there periodic orbits? Are there limit cycles? Is the system chaotic? Observe that the basic existence and uniqueness concerns small times, whereas the dynamical questions are at large times.

Classical field theory (as opposed to classical particle theory) has similar small and large time questions. In field theory there is space as well as time, and the local theory involves small time differences and small distances, said simply as *short range*. (In a relativistic field theory, a universal constant—the speed of light—relates time differences and lengths.) The short range questions involve *partial* differential equations, such as wave equations or equations from fluid dynamics. Then small time existence and uniqueness is already a delicate issue. And again the long range questions are the most interesting. Example: Is there existence for all time, or do singularities develop?

Quantum field theory has a similar dichotomy of questions, which is sometimes called kinematics vs. dynamics. The dynamical questions at long range, i.e., at large times and large distances—the two are the same in relativistic systems—are most interesting and have the most powerful mathematical ramifications.

1.3 Quantum mechanics

We turn now to quantum mechanics in the framework of **Axiom System 1.1**. Our exposition here is quite formal, presented with no history or motivation. I strongly suggest reading elsewhere, e.g., [Wei1, §1], [FY, §4]. The basics are in the classic texts of Dirac [D] and von Neumann [vN1]. Other mathematical axiom systems for *quantum* mechanics may be found in [Str, §1.3] and [Ta, §2]. In subsequent lectures we flesh this out and give many examples of quantum mechanical systems.

In our approach here we restrict to *bounded* observables. This is the approach taken in many treatments; see [Ha], for example. On the other hand, the self-adjoint operator that generates the flow in states and observables may be an unbounded operator.

Our presentation relies on the background material in §1.4.3.

Quantum theory has a universal constant of nature, Planck’s constant $\hbar$, which has units

$$(1.25) \quad [\hbar] = \frac{\text{mass} \cdot \text{length}^2}{\text{time}}$$

Recall that energy has units $\text{mass} \cdot \text{length}^2 / \text{time}^2$, so the units of $\hbar$ are those of energy $\cdot$ time.

The DOSO of a quantum mechanical system is derived from a separable complex Hilbert space $\mathcal{H}$, which may be finite or countably infinite dimensional. The space $\mathcal{PS}$ of pure states is the projective space $\mathbb{P}\mathcal{H}$; a pure state is a line in $\mathcal{H}$. The space $\mathcal{S}$ of all states is the space of nonnegative self-adjoint trace class operators S with $\text{trace}(S) = 1$. **Remark 1.59** tells how $\mathcal{PS}$ is a subset of $\mathcal{S}$, and in fact $\mathcal{PS} \subset \mathcal{S}$ is the subspace of extreme points, as it must be according to **Axiom System 1.1(1)**.

Define $\mathcal{O} = \mathcal{O}^\infty = \text{End } \mathcal{H}$ as the complex vector space of *bounded* linear operators on $\mathcal{H}$. The adjoint map $A \mapsto A^*$ is a real structure on $\mathcal{O}$. Hence $\mathcal{O}_\mathbb{R}$ is the space of bounded self-adjoint

operators on $\mathcal{H}$. Set $i = \sqrt{-1}$ and define the Lie algebra structure on $\mathcal{O}$ as

$$(1.26) \quad [A_1, A_2]_{\hbar} = \frac{-i}{\hbar}(A_1 \circ A_2 - A_2 \circ A_1),$$

Functions of observables (1.2) are defined via the functional calculus (Definition 1.52).

We turn now to measurement (1.3). It is most familiar for pure states. Thus suppose $L \subset \mathcal{H}$ is a line, i.e., a one-dimensional subspace. Fix a unit vector $\psi \in L$. Let $A \in \mathcal{O}_{\mathbb{R}}$ be a bounded self-adjoint operator, and let π_A be the associated self-adjoint projection-valued measure on $\mathbb{R}$. Then the result of pairing the pure state L and the observable A is the probability measure on $\mathbb{R}$

$$(1.27) \quad E \longmapsto \langle \psi, \pi_A(E) \psi \rangle, \quad E \text{ a Borel subset of } \mathbb{R}.$$

This is independent of the choice of unit vector $\psi \in L$. A general state is a nonnegative self-adjoint trace class operator S with unit trace, and then (1.27) generalizes to

$$(1.28) \quad E \longmapsto \text{trace}(\pi_A(E) \circ S), \quad E \text{ a Borel subset of } \mathbb{R}.$$

Finally, the motion generated by an observable $H \in \mathcal{O}_{\mathbb{R}}^{\infty}$ is defined by an application of functional calculus, as in (1.54). Namely, let

$$(1.29) \quad U_t = e^{-itH/\hbar}, \quad t \in \mathbb{R},$$

be the 1-parameter group of unitary transformations of $\mathcal{H}$ generated by H . Then the motions (1.5) are conjugation by U_t : for $S \in \mathcal{S}$ and $A \in \mathcal{O}$ set

$$(1.30) \quad \begin{aligned} S^{(t)} &:= U_t \circ S \circ U_{-t} \\ A^{(t)} &:= U_{-t} \circ A \circ U_t \end{aligned}$$

1.4 Mathematical background

1.4.1 Affine and convex spaces. A *vector space* is a group under vector addition. Infinite dimensional vector spaces typically come with a topology—the compatible topology on a finite dimensional vector space is unique—and underlying a *topological* vector space is a *topological* group. Examples include Hilbert spaces, Banach spaces, Fréchet spaces, nuclear spaces, etc. Standard textbooks on functional analysis, such as [RS], develop the theory of topological vector spaces. Here we need only the very basic definitions, and we defer to the textbooks for a detailed development of the theory.

Affine space is the arena of flat geometry. We allow both finite and infinite dimensional spaces. Global parallelism in these geometries is expressed in terms of the translation action of a topological vector space.

Definition 1.31. Let V be a real topological vector space.

- (1) An *affine space* A over V is a topological space A equipped with a continuous simply transitive V -action. We write the result of acting $\xi \in V$ on $a \in A$ as $a + \xi \in A$.
- (2) A subset $\mathcal{S} \subset A$ is *convex* if for all finite ordered tuples $\sigma_0, \dots, \sigma_k \in \mathcal{S}$ and $x^0, \dots, x^k \in \mathbb{R}^{\geq 0}$ with $x^0 + \dots + x^k = 1$ we have $x^i \sigma_i = x^0 \sigma_0 + \dots + x^k \sigma_k \in \mathcal{S}$.
- (3) A point $\sigma \in \mathcal{S}$ is *extreme* if whenever $\sigma = x\sigma' + (1-x)\sigma''$ for $x \in [0, 1]$, $\sigma', \sigma'' \in \mathcal{S}$, then $\sigma = \sigma'$ or $\sigma = \sigma''$.

A vector space is an affine space (over itself). We cannot add points in an affine space, but we can average them. See [F1, Lectures 1–2] for an elementary exposition of some affine geometry, with a focus on finite dimensions. For every $n \in \mathbb{R}^{\geq 0}$ there is a standard real affine space $\mathbb{A}^n$ whose points are n -tuples $(a^1, \dots, a^n)$ of real numbers; the standard real vector space $\mathbb{R}^n$ acts on $\mathbb{A}^n$ as usual by component-wise addition:

$$(1.32) \quad (a^1, \dots, a^n) + (\xi^1, \dots, \xi^n) = (a^1 + \xi^1, \dots, a^n + \xi^n), \quad (a^1, \dots, a^n) \in \mathbb{A}^n, \quad (\xi^1, \dots, \xi^n) \in \mathbb{R}^n.$$

Definition 1.33. A *Euclidean space* is an affine space over a real inner product space.

For an infinite dimensional Euclidean space, the underlying inner product space might be assumed complete (a real Hilbert space). For each finite dimension n there is a standard Euclidean space $\mathbb{E}^n$. Its underlying affine space is $\mathbb{A}^n$; the group of translations is $\mathbb{R}^n$ with the standard dot product.

Example 1.34. Here is an important example of an infinite dimensional affine space. Let X be a set equipped with a σ -algebra of measurable sets. The space $\text{Meas}(X)$ of (finite signed) real measures is a topological vector space. The subspace $\text{Prob}(X) \subset \text{Meas}(X)$ of probability measures is a convex subspace. The point measure μ_x at $x \in X$, defined by

$$(1.35) \quad \mu_x(E) = \begin{cases} 1, & x \in E; \\ 0, & x \notin E, \end{cases}$$

is an extreme point, and all extreme points have this form.

1.4.2 Symplectic manifolds. Informally, a symplectic manifold is a smooth manifold N equipped with a smoothly varying nondegenerate skew-symmetric bilinear form ω on its tangent spaces, and ω is required to satisfy an *integrability condition*. Thus $\omega \in \Omega_N^2$ is a differential 2-form with the property that for all $p \in N$ the skew-symmetric form $\omega_p: T_p N \times T_p N \rightarrow \mathbb{R}$ is nondegenerate. The integrability condition is that ω be closed:

$$(1.36) \quad d\omega = 0.$$

Riemannian vs. symplectic: (symmetric positive-definite) vs. (skew-symmetric nondegenerate); (no integrability condition) vs. (integrability condition). We develop this analogy in [Lecture 2](#). See [GS, MS] for much more on symplectic manifolds. (There are many more textbooks to recommend.)

The skew form ω induces an isomorphism

$$(1.37) \quad \begin{aligned} \mathcal{X}_N &\xrightarrow{\cong} \Omega_N^1 \\ \xi &\longmapsto -\iota_\xi \omega \end{aligned}$$

of the topological vector space of vector fields with the topological vector space of differential 1-forms; the notation indicates the contraction of the vector field ξ into the symplectic 2-form ω . (Note that because of the skew-symmetry there is a sign at stake; it is fixed by the formula). Also, there is a *symplectic gradient* map

$$(1.38) \quad \begin{aligned} \Omega_N^0 &\xrightarrow{d} \Omega_N^1 \xrightarrow{\cong} \mathcal{X}_N \\ f &\longmapsto df \longmapsto \xi_f \end{aligned}$$

Recall that vector fields generate flows, at least locally. Hence a function on a symplectic manifold generates a 1-parameter group of diffeomorphisms, at least locally (in time and space).

Functions on a symplectic manifold form a Lie algebra under the *Poisson bracket*:

$$(1.39) \quad \{f_1, f_2\} = \xi_{f_1} \cdot f_2 = \omega(\xi_{f_1}, \xi_{f_2}).$$

1.4.3 Bounded operators on Hilbert space. The reference [RS] is an excellent functional analysis text, among many others.

The hermitian inner product on $\mathcal{H}$ is complex antilinear in the first variable and complex linear in the second variable:

$$(1.40) \quad \langle \lambda_1 \psi_1, \lambda_2 \psi_2 \rangle = \overline{\lambda_1} \lambda_2 \langle \psi_1, \psi_2 \rangle, \quad \lambda_1, \lambda_2 \in \mathbb{C}, \quad \psi_1, \psi_2 \in \mathcal{H}.$$

See [DF1] for a justification of this choice.

Definition 1.41. Let $\mathcal{H}$ be a separable complex Hilbert space.

- (1) The *projective space* $\mathbb{P}\mathcal{H}$ is the space of lines (one-dimensional subspaces) in $\mathcal{H}$.
- (2) $\text{End } \mathcal{H}$ is the algebra of *bounded linear operators* $A: \mathcal{H} \rightarrow \mathcal{H}$. The *adjoint* of a bounded operator A is the bounded linear operator A^* characterized by

$$(1.42) \quad \langle A^* \psi_1, \psi_2 \rangle = \langle \psi_1, A \psi_2 \rangle, \quad \psi_1, \psi_2 \in \mathcal{H}.$$

Adjunction is an involution

$$(1.43) \quad \begin{aligned} \text{End } \mathcal{H} &\longrightarrow \text{End } \mathcal{H} \\ A &\longmapsto A^* \end{aligned}$$

and it satisfies

$$(1.44) \quad \begin{aligned} (\lambda A)^* &= \bar{\lambda} A^* \\ (A_1 \circ A_2)^* &= A_2^* \circ A_1^* \end{aligned}$$

for all $\lambda \in \mathbb{C}$, $A, A_1, A_2 \in \text{End } \mathcal{H}$. The map (1.43) is a real structure on the vector space underlying $\text{End } \mathcal{H}$.

- (3) A *self-adjoint operator* is a real point of $\text{End } \mathcal{H}$, i.e., $A \in \text{End } \mathcal{H}$ such that $A^* = A$.
- (4) A *projection* is an operator $P \in \text{End } \mathcal{H}$ that satisfies $P^2 = P$. A *self-adjoint projection* additionally satisfies $P^* = P$.
- (5) The *rank* of a projection P is $\dim P\mathcal{H}$, which may be finite or infinite.

The dimension of $\mathcal{H}$ is either finite or countably infinite. I recommend thinking through this material first in the finite dimensional case. The vector space $\text{End } \mathcal{H}$ is topologized using the operator norm. With that topology⁵ it is a *topological $*$ -algebra*. If P is a projection, then so too is $1 - P$ (where ‘1’ denotes the identity map $\text{id}_{\mathcal{H}}$), and there is a direct sum decomposition

$$(1.45) \quad \mathcal{H} = P\mathcal{H} \oplus (1 - P)\mathcal{H}$$

in which each subspace is closed in $\mathcal{H}$. If P is a self-adjoint projection, then (1.45) is an orthogonal decomposition. Identify $\mathbb{P}\mathcal{H}$ with the space of rank one self-adjoint projections. It may be topologized as a subspace of $\text{End } \mathcal{H}$.

The following spectral theorem, due to von Neumann, is the basic structure theorem for bounded self-adjoint operators.

Theorem 1.46. *Let $\mathcal{H}$ be a separable complex Hilbert space, and suppose $A \in \text{End } \mathcal{H}$ is a bounded self-adjoint operator. Then there is a self-adjoint projection-valued measure π_A on $\mathbb{R}$ such that*

$$(1.47) \quad A = \int_{\mathbb{R}} \lambda d\pi_A(\lambda).$$

The *spectral measure*⁶ π_A assigns to each Borel subset $E \subset \mathbb{R}$ a self-adjoint projection $\pi_A(E): \mathcal{H} \rightarrow \mathcal{H}$, and this assignment satisfies $\pi_A(\emptyset) = 0$, $\pi_A(\mathbb{R}) = \text{id}_{\mathcal{H}}$. Furthermore, if $E' \cap E'' = \emptyset$, then $\pi_A(E')$ commutes with $\pi_A(E'')$.

Example 1.48. Set $\mathcal{H} = L^2(\mathbb{R}; \mathbb{C})$ (with respect to Lebesgue measure), and suppose $f \in L^\infty(\mathbb{R}; \mathbb{R})$ is a real-valued function. Let $M_f: \mathcal{H} \rightarrow \mathcal{H}$ be the linear operator that multiplies an L^2 function by f . Then for a Borel subset $E \subset \mathbb{R}$, the spectral measure of M_f evaluated on E is the composition

$$(1.49) \quad \pi_{M_f}(E): L^2(\mathbb{R}; \mathbb{C}) \longrightarrow L^2(f^{-1}E; \mathbb{C}) \longrightarrow L^2(\mathbb{R}; \mathbb{C})$$

of restriction followed by extension by zero.

Example 1.50. Let $\Lambda \subset \mathbb{R}$ be a bounded discrete subset, and suppose given an *orthogonal* direct sum decomposition

$$(1.51) \quad \mathcal{H} = \bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda$$

in which each $\mathcal{H}_\lambda \subset \mathcal{H}$ is a closed subspace. Define a bounded self-adjoint operator $A_\Lambda: \mathcal{H} \rightarrow \mathcal{H}$ by requiring $A|_{\mathcal{H}_\lambda} = \lambda$. I leave the description of the spectral measure to the reader.

⁵The same holds in other topologies as well; of course, this must be checked in each case.

⁶The term ‘spectral measure’ seems to be used instead for the real-valued measures $\langle \psi, \pi_A(\psi) \rangle$ associated to each $\psi \in \mathcal{H}$. *Tant pis!*

The next definition goes by the name ‘functional calculus’.

Definition 1.52. Let A be a bounded self-adjoint operator on a separable complex Hilbert space $\mathcal{H}$, and suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is a bounded Borel function. Then $f(A) \in \text{End } \mathcal{H}$ is defined by specifying its spectral measure to be the pushforward $f_*\pi_A$. Thus

$$(1.53) \quad f(A) = \int_{\mathbb{R}} \lambda \, d(f_*\pi_A)(\lambda) = \int_{\mathbb{R}} f(\lambda) \, d\pi_A(\lambda).$$

An important application of functional calculus constructs a 1-parameter group of unitary automorphisms of $\mathcal{H}$ from a bounded self-adjoint operator H on $\mathcal{H}$. Namely, fix⁷ $\hbar \in \mathbb{R}$ and define

$$(1.54) \quad U_t = e^{-itH/\hbar}, \quad t \in \mathbb{R},$$

which is the functional calculus for the 1-parameter group of functions $f_t: \lambda \mapsto e^{-it\lambda/\hbar}$.

We use the spectral measure to define the spectrum as well as the notion of positivity. There are more standard alternative definitions.

Definition 1.55. Let A be a bounded self-adjoint operator on a separable complex Hilbert space $\mathcal{H}$.

- (1) The *spectrum* of A is the support⁸ of π_A .
- (2) The operator A is *nonnegative* if its spectrum is a subset of $\mathbb{R}^{\geq 0}$.

The operator A_Λ in [Example 1.50](#) has spectrum Λ . In this case we say A has *pure point spectrum*, and each $\lambda \in \Lambda$ is an *eigenvalue* of A_Λ of multiplicity $\dim \mathcal{H}_\lambda$. The subspace $\mathcal{H}_\lambda \subset \mathcal{H}$ is the corresponding *eigenspace*. If the function f in [Example 1.48](#) is continuous, then the operator M_f has no eigenvalues; its spectrum is *continuous*.

Finally, we introduce trace class operators, restricting to self-adjoint operators as a simplification.

Definition 1.56. Let A be a bounded self-adjoint operator on a separable complex Hilbert space $\mathcal{H}$. Then A is a *self-adjoint trace class operator* if it has the form A_Λ as in [Example 1.50](#), each $\mathcal{H}_\lambda$ is finite dimensional, and $\sum_{\lambda \in \Lambda} |\lambda| \dim \mathcal{H}_\lambda < \infty$. The *trace* of A is then

$$(1.57) \quad \text{trace}(A) = \sum_{\lambda \in \Lambda} (\dim \mathcal{H}_\lambda) \lambda.$$

Example 1.58. Let $\mathcal{H}' \subset \mathcal{H}$ be a finite dimensional subspace, and let $P: \mathcal{H} \rightarrow \mathcal{H}$ be the orthogonal projection with image $\mathcal{H}'$. Then P is self-adjoint trace class, and $\text{trace}(P) = \dim \mathcal{H}'$.

Remark 1.59. Building on the remark following [\(1.45\)](#), we identify the projective space $\mathbb{P}\mathcal{H}$ as the space of self-adjoint trace class operators of trace one.

⁷In quantum theory, $\hbar$ is *Planck’s constant*. It has units (mass)(length)²/(time), which are the units of tH if H has units of energy. The constant is put in so that the exponent in [\(1.54\)](#) is dimensionless. Of course, in a mathematical context we usually omit $\hbar$ and suppose H is dimensionless.

⁸The support is the complement of the union of open subsets of $\mathbb{R}$ with π_A -measure zero.

Lecture 2: Symplectic geometry

This lecture is taken up with mathematical background on symplectic manifolds. A symplectic structure is a nondegenerate *skew-symmetric* bilinear form, and we contrast its basic theory with that of a positive definite *symmetric* form. We begin with linear algebra, where the salient fact is that whereas the space of inner products on a finite dimensional real vector space is a (contractible) convex cone, the space of symplectic forms has interesting topology. We move to affine geometry, where on both a Euclidean space and a symplectic affine space functions give rise to vector fields. In turn, vector fields generate motion, and it is that motion—in the symplectic case—which is needed for [Axiom System 1.1](#)(4).

New phenomena appear on smooth manifolds. Any smooth manifold admits a Riemannian metric, in fact a contractible space of them, whereas there are topological obstructions to the existence of a smoothly varying nondegenerate skew-symmetric bilinear form on the tangent spaces of an even dimensional manifold. Furthermore, there is an *integrability condition* in the symplectic case which is not present in the Riemannian case: we ask that a symplectic 2-form be closed. Functions on a symplectic manifold have a symplectic gradient, and there is a Lie algebra structure on the vector space of functions.

In the next lecture we apply these preliminaries to tell how a symplectic manifold equipped with a one-parameter family of symplectic diffeomorphisms gives rise to a mechanical system in the sense of [Definition 1.11](#).

2.1 Bilinear forms in linear algebra

Let V be a finite dimensional real vector space. A bilinear form $B: V \times V \rightarrow \mathbb{R}$ induces two maps $V \rightarrow V^*$. We say that B is *nondegenerate* if these maps are isomorphisms.

Definition 2.1.

- (1) An *inner product* $\langle -, - \rangle$ is a positive definite symmetric bilinear form.
- (2) A *symplectic form* ω is a nondegenerate skew-symmetric bilinear form.

The positive definiteness is the condition that $\langle \xi, \xi \rangle > 0$ for all nonzero $\xi \in V$.

Example 2.2. Let $V = \mathbb{R}^n$ be the model vector space of dimension n . The model inner product is

$$(2.3) \quad \langle (\xi^i), (\eta^i) \rangle = \sum_i \xi^i \eta^i.$$

Let $n = 2m$ be even. Then the model symplectic form is

$$(2.4) \quad \omega((\xi^i), (\eta^i)) = (\xi^{m+1} \eta^1 - \xi^1 \eta^{m+1}) + \dots$$

Remark 2.5.

- (1) An inner product induces a norm—a length function on vectors—as well as a measurement of the angle between two nonzero vectors. From these one deduces a volume function on parallelepipeds of all dimensions. A symplectic form induces a *signed* area function on 2-dimensional parallelograms in V . Induced is a signed volume function on even dimensional parallelepipeds, in particular a signed volume function (on top dimensional parallelepipeds).

- (2) A skew-symmetric bilinear form on V is an element of the exterior algebra of the dual space: $\omega \in \bigwedge^2 V^*$. The form is nondegenerate iff the top exterior product

$$(2.6) \quad \frac{\omega^{\wedge m}}{m!} = \frac{\omega \wedge \cdots \wedge \omega}{m!} \in \bigwedge^{\text{top}} V^* = \bigwedge^{2m} V^* = \text{Det } V^*$$

is nonzero. In that case (2.6) is a signed volume form on V , so in particular determines an orientation of V .

The following basic result asserts the existence of normal forms.

Theorem 2.7. *Let V be a finite dimensional real vector space equipped with an inner product or symplectic form. Set $n = \dim V$. Then there exists an isomorphism $\mathbb{R}^n \rightarrow V$ that pulls back the inner product or symplectic form to the model (2.3) or (2.4).*

In the symplectic case this implies that n is even.

Corollary 2.8. *The group $\text{GL}_n \mathbb{R}$ acts transitively on the space $\text{Met}(\mathbb{R}^n)$ of inner products and on the space $\text{Symp}(\mathbb{R}^n)$ of symplectic forms.*

Sketch proof of Theorem 2.7. Consider first the case of an inner product space $(V, \langle -, - \rangle)$. Assuming $V \neq 0$, let $e_1 \in V$ be a unit length vector. Set $V_1 = (\mathbb{R} \cdot e_1)^\perp$ and apply the argument to the inner product space $(V_1, \langle -, - \rangle|_{V_1})$.

For (V, ω) a symplectic vector space, if $V \neq 0$ choose $e_1 \neq 0$ arbitrary and then, by nondegeneracy, choose⁹ $e_{m+1} \in V$ such that $\omega(e_{m+1}, e_1) = 1$. Set $V_1 = \mathbb{R} \cdot \{e_1, e_{m+1}\}^\perp$ and apply the argument to the symplectic vector space $(V_1, \omega|_{V_1})$. $\square$

The proof demonstrates that $n = \dim V = 2m$ is even in the symplectic case.

Corollary 2.8 asserts that the functions

$$(2.9) \quad \begin{aligned} \text{GL}_n \mathbb{R} &\longrightarrow \text{Met}(\mathbb{R}^n) \\ \text{GL}_n \mathbb{R} &\longrightarrow \text{Symp}(\mathbb{R}^n) \end{aligned}$$

that map an automorphism of $\mathbb{R}^n$ to the pullback of the model form are surjective, i.e., the Lie group $\text{GL}_n \mathbb{R}$ acts transitively on the respective spaces of forms. The stabilizer subgroups of the model forms are the orthogonal group O_n and the symplectic group $\text{Sp}_{2m} \mathbb{R}$, respectively. The disparate natures of these stabilizer groups are behind the distinct natures of the associated linear and nonlinear geometries. For example, O_n is compact and the inclusion $\text{O}_n \hookrightarrow \text{GL}_n \mathbb{R}$ is a homotopy equivalence: in fact, $\text{GL}_n \mathbb{R}$ deformation retracts onto the subgroup O_n . By contrast, $\text{Sp}_{2m} \mathbb{R}$ is not compact and the inclusion into $\text{GL}_n \mathbb{R}$ is not a homotopy equivalence. In the lowest dimensional case $m = 1$, we have $\text{Sp}_2 \mathbb{R} = \text{SL}_2 \mathbb{R}$. (The symplectic form is an area form and $\text{SL}_2 \mathbb{R} \subset \text{GL}_2 \mathbb{R}$ is the subgroup of invertible matrices that preserves the area form=determinant.)

The surjectivity of (2.9) and the identification of the stabilizer subgroups imply that

$$(2.10) \quad \begin{aligned} \text{GL}_n \mathbb{R} / \text{O}_n &\xrightarrow{\approx} \text{Met}(\mathbb{R}^n) \\ \text{GL}_n \mathbb{R} / \text{Sp}_{2m} \mathbb{R} &\xrightarrow{\approx} \text{Symp}(\mathbb{R}^{2m}) \end{aligned}$$

are diffeomorphisms of homogeneous $\text{GL}_n \mathbb{R}$ -manifolds.

⁹The indices can be rearranged later: ‘ m ’ need not be defined.

Remark 2.11.

- (1) $\text{Met}(\mathbb{R}^n) \subset \text{SymBil}(\mathbb{R}^n; \mathbb{R})$ is a convex subset, in fact a convex cone: positivity is a convex condition. (Here $\text{SymBil}(\mathbb{R}^n; \mathbb{R})$ is the vector space of real-valued symmetric bilinear forms on the model vector space $\mathbb{R}^n$.) In particular, $\text{Met}(\mathbb{R}^n)$ is contractible.
- (2) By contrast, for $m = 1$ we have $\text{GL}_2\mathbb{R}/\text{SL}_2\mathbb{R} \approx \mathbb{R}^\times$ is not contractible. (The diffeomorphism is realized by the determinant homomorphism of Lie groups.) The space $\text{Symp}(\mathbb{R}^{2m})$ has more complicated topology in higher dimensions.
- (3) These assertions all transport from the model space $\mathbb{R}^n$ to arbitrary vector spaces. We discuss the details of that transportation in a future lecture, though we use it implicitly at the beginning of §2.3.

2.2 Euclidean spaces and affine symplectic spaces

We pass now from linear geometry to affine geometry, which is the arena for flat geometry of global parallelism and the model for geometry on smooth manifolds. Let V be a finite dimensional real vector space, and let A be an affine space over V . We say A is a *Euclidean space* if V is equipped with an inner product, and A is an *affine symplectic space* if V is equipped with a symplectic form. A Euclidean space carries a constant/parallel/flat metric, and an affine symplectic space carries a constant/parallel/flat symplectic form. The normal form [Theorem 2.7](#) implies the existence of distinguished affine coordinate systems. In the Euclidean case, write the coordinates as¹⁰ $x^1, \dots, x^n$; the metric takes the form

$$(2.12) \quad g = dx^1 \otimes dx^1 + \dots + dx^n \otimes dx^n.$$

In the symplectic case, write the coordinates as $q^1, \dots, q^m, p_1, \dots, p_m$; the symplectic form is¹¹

$$(2.13) \quad \omega = dp_i \wedge dq^i.$$

Let $U \subset A$ be an open subset. A vector field on U is a smooth function $U \rightarrow V$. A 1-form on U is a smooth function $U \rightarrow V^*$. The metric or symplectic form induces an isomorphism between the space of vector fields and the space of 1-forms. In the symplectic case we write

$$(2.14) \quad \xi \longleftrightarrow -\iota_\xi \omega,$$

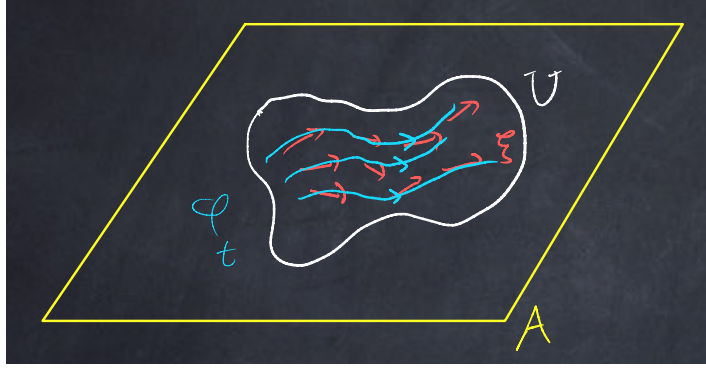
where ξ is a vector field and ι is the contraction of a vector field into a differential form; see [\(1.37\)](#).

Recall that vector fields induce local flows. The flows may patch to a global flow, which is an action of the Lie group $\mathbb{R}$ by diffeomorphisms φ_t of A . In general, these diffeomorphisms are not affine, nor do they preserve the metric or symplectic form.

The differential of a function $f \in \Omega_U^0$ is a 1-form $df \in \Omega_U^1$. This is true on any affine space. In the presence of a Euclidean or affine structure we can convert the 1-form df to a vector field $\xi_f \in \mathcal{X}_U$ on U . In the Euclidean case, we use the notation $\nabla f = \xi_f$ and call this the *gradient* vector field of f . Observe that the *gradient flow* it induces is typically not an isometry.

¹⁰Each coordinate $x^i: V \rightarrow \mathbb{R}$ is an affine function.

¹¹The summation convention tells that there is a sum over i in [\(2.13\)](#).

FIGURE 1. The flow φ_t generated by a vector field ξ

Example 2.15. Consider the function $f = \frac{1}{2}x^2$ on the Euclidean line $\mathbb{E}_x^1$ with coordinate x . Then $\nabla f = x\partial/\partial x$, and distances grow exponentially under the gradient flow.

In the symplectic case, if $f \in \Omega_U^0$ is a smooth function, then its symplectic gradient ξ_f satisfies:

$$(2.16) \quad df = -\iota_{\xi_f} \omega.$$

By contrast to the Euclidean case, the flow generated by a symplectic gradient vector field preserves the symplectic form.

Lemma 2.17. *Let A be an affine symplectic space, suppose $U \subset A$ is an open subset, and let $f: U \rightarrow \mathbb{R}$ be a smooth function with symplectic gradient $\xi_f \in \mathfrak{X}_U$. Then $\mathcal{L}_{\xi_f} \omega = 0$.*

The conclusion is that the Lie derivative $\mathcal{L}_{\xi_f} \omega$ of ω along the vector field ξ_f vanishes.

Proof. Apply Cartan's formula for the Lie derivative of a differential form:

$$(2.18) \quad \mathcal{L}_{\xi_f} \omega = (d\iota_{\xi_f} + \iota_{\xi_f} d) \omega = d\iota_{\xi_f} \omega = -d(df) = 0. \quad \square$$

Definition 2.19. A vector field ξ is *symplectic* if $\mathcal{L}_{\xi} \omega = 0$. It is *Hamiltonian* if it is the symplectic gradient of a function.

Note that a Hamiltonian vector field determines a function only up to locally constant functions.

2.3 Riemannian and symplectic manifolds

The vector space V of translations of an affine space A plays two roles. On the one hand, it acts as the Lie group of translations of A . On the other hand, it acts infinitesimally as the Lie algebra of infinitesimal translations of A : each element of V gives rise to a constant/parallel/flat vector field on A . Equivalently, V is the tangent space to A at each point of A . It is this second, infinitesimal, role of V that carries over to smooth manifolds in the form of the tangent bundle.

Let M be a smooth manifold and let $TM \rightarrow M$ be its tangent bundle. There are two canonically associated fiber bundles:

$$(2.20) \quad \begin{array}{ccc} \text{Met}(M) & & \text{Symp}(M) \\ \downarrow \rho_1 & & \downarrow \rho_2 \\ M & & M \end{array}$$

At a point $m \in M$, the fiber of ρ_1 is the space $\text{Met}(T_m M)$ of inner products on the tangent space, and the fiber of ρ_2 is the space $\text{Symp}(T_m M)$ of symplectic forms on the tangent space.

Definition 2.21.

- (1) A section of ρ_1 is a *Riemannian metric*.
- (2) A section of ρ_2 is an *almost symplectic form*.

Recall from [Remark 2.11](#)(1) that the fibers of ρ_1 are convex subsets of a vector space.¹²

Theorem 2.22. *There exist Riemannian metrics on M .*

In fact, the space of Riemannian metrics (in an appropriate topology) is contractible.

Proof. A smooth manifold is locally modeled on affine space, and on affine space inner products exist. Use a partition of unity to average these local Riemannian metrics. Such averages are Riemannian metrics by the aforementioned convexity. $\square$

By contrast, not every smooth manifold admits an almost symplectic form.

Example 2.23.

- (1) For $M = \mathbb{RP}^2$, the fiber bundle ρ_2 in (2.20) has typical fiber $\mathbb{R}^\times$; see [Remark 2.11](#)(2). As we traverse a circle $\mathbb{RP}^1 \subset \mathbb{RP}^2$, the two components of the fiber interchange, hence there is no section over that circle, much less over $\mathbb{RP}^2$.
- (2) Since the components of $\mathbb{R}^\times$ are contractible, this is the only obstruction to a section of ρ_2 in dimension two. Therefore, every orientable 2-manifold admits an almost symplectic structure.
- (3) The 4-sphere S^4 does not admit an almost symplectic structure; we do not give the proof here.

There is another crucial distinction between the Riemannian and symplectic cases. In the Riemannian case, there is no condition on a section of ρ_1 . In the symplectic case, there is an *integrability condition*.

Definition 2.24. Let N be a smooth manifold, and suppose $\omega \in \Omega_N^2$ is an almost symplectic form on N . Then ω is *symplectic* if $d\omega = 0$.

Thus an almost symplectic structure is a smoothly varying *nondegenerate* skew-symmetric bilinear form on the tangent spaces: an algebraic structure. [Definition 2.24](#) imposes a differential constraint.

¹²Those vector spaces fit together to a vector bundle $\text{SymBil}(T^*M; \mathbb{R}) \rightarrow M$.

Remark 2.25. There is a uniform theory of integrability conditions for structures on smooth manifolds. The triviality in the Riemannian case and the closure condition in the symplectic cases can be deduced from this general theory.

Example 2.26. The 4-sphere S^4 does not admit a symplectic structure. For if $\omega \in \Omega_{S^4}^2$ is a symplectic form, then its de Rham cohomology class

$$(2.27) \quad [\omega] \in H_{\text{dR}}^2 \cong H^2(S^4; \mathbb{R}) = 0$$

vanishes. But by [Remark 2.5\(2\)](#), the square of ω is a volume form and hence has a nonzero de Rham cohomology class

$$(2.28) \quad \left[\frac{\omega \wedge \omega}{2} \right] \in H_{\text{dR}}^4(S^4) \cong H^4(S^4; \mathbb{R}) \xrightarrow[\int_{S^4}]{\cong} \mathbb{R}$$

The same argument gives a cohomological obstruction to a symplectic structure on any closed manifold.

In the positive direction, we have the following.

Example 2.29.

- (1) Any orientable 2-manifold admits a symplectic structure, which is a nonzero 2-form: all 2-forms are closed. In fact, as remarked above ([2.10](#)), a symplectic form in 2-dimensions is an area form, and in general the only obstruction to the existence of a volume form on a smooth manifold is orientability.

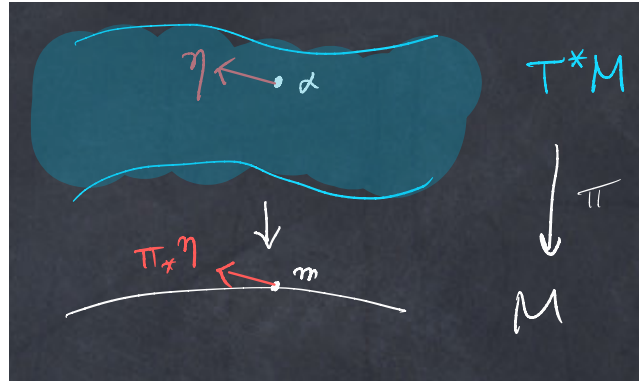


FIGURE 2. The tautological 1-form on T^*M

- (2) Let M be a smooth manifold. We construct a symplectic form on the total space T^*M of its cotangent bundle $\pi: T^*M \rightarrow M$. In fact, it is an *exact* symplectic form: the de Rham differential of a 1-form $\theta \in \Omega_{T^*M}^1$. To construct this tautological 1-form, suppose $m \in M$ and $\alpha \in T_m^*M$. Let $\eta \in T_\alpha(T^*M)$ be a tangent vector to T^*M at α . Then $\pi_*\eta \in T_mM$, and we define

$$(2.30) \quad \theta_\alpha(\eta) = \langle \alpha, \pi_*\eta \rangle.$$

The right hand side is the pairing between cotangent and tangent vectors at $m \in M$. Define the symplectic form $\omega = d\theta$. The nondegeneracy can be checked locally on M , where it reduces to a computation in affine space.

The construction of gradients carries over to Riemannian and symplectic manifolds. We only discuss the symplectic case. Let (N, ω) be a symplectic manifold. Define $\mathcal{X}_N^\omega \subset \mathcal{X}_N$ to be the subspace of vector fields ξ that (infinitesimally) preserve the symplectic form in the sense that $\mathcal{L}_\xi \omega = 0$. These vector fields generate *symplectic diffeomorphisms*, i.e., diffeomorphisms φ that preserve the symplectic form: $\varphi^* \omega = \omega$. The proof of [Lemma 2.17](#) shows that under the isomorphism $\mathcal{X}_N \xrightarrow{\cong} \Omega_N^1$ this subspace maps to the subspace of closed 1-forms. Then de Rham theory leads to the exact sequence

$$(2.31) \quad 0 \longrightarrow H^0(N; \mathbb{R}) \longrightarrow \Omega_N^0 \xrightarrow[\text{gradient}]{\text{symplectic}} \mathcal{X}_N^\omega \longrightarrow H^1(N; \mathbb{R}) \longrightarrow 0$$

In particular, there is an obstruction to expressing a symplectic vector field as the symplectic gradient of a function.

Example 2.32. Let $N = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ be the 2-torus with coordinates θ^1, θ^2 and symplectic form $\theta^1 \wedge \theta^2$. The 1-parameter group $\varphi_t(\theta^1, \theta^2) = (\theta^1 + t, \theta^2)$ is generated by the symplectic vector field $\partial/\partial\theta^1$, but this vector field is not Hamiltonian, i.e., is not a symplectic gradient. (The corresponding 1-form $\pm d\theta^2$ is not exact, despite the notation.)

Recall that vector fields on a smooth manifold have a Lie algebra structure given by the Lie bracket. On a symplectic manifold, the subspace $\mathcal{X}_N^\omega \subset \mathcal{X}_N$ is closed under Lie bracket. Then the symplectic gradient map in [\(2.31\)](#) induces a Lie algebra structure on the vector space Ω_N^0 of functions. This *Poisson bracket* is given by the formula

$$(2.33) \quad \{f_1, f_2\} = \xi_{f_1} \cdot f_2 = \omega(\xi_{f_1}, \xi_{f_2}), \quad f_1, f_2 \in \Omega_N^0.$$

2.4 Symmetries and moment maps

Let (N, ω) be a symplectic manifold. A *symmetry* of N is a diffeomorphism $\varphi: N \rightarrow N$ that preserves the symplectic form: $\varphi^* \omega = \omega$. Then φ is called a *symplectic diffeomorphism* or *symplectomorphism* for short. An *infinitesimal symmetry* is a symplectic vector field ξ ([Definition 2.19](#)): $\mathcal{L}_\xi \omega = 0$. If ξ is Hamiltonian, so $\xi = \xi_f$ for some $f \in \Omega_N^0$, then we say that f is a *momentum* of the infinitesimal symmetry ξ . Note that f is determined up to additive shifts by a locally constant function. Suppose $H \in \Omega_N^0$ is a (Hamiltonian) function and φ_t is the flow generated by $-\xi_H$. Assume that an infinitesimal symmetry ξ preserves H : $\mathcal{L}_\xi H = 0$. Then if $\xi = \xi_f$ for some $f \in \Omega_N^0$,

$$(2.34) \quad -\xi_H \cdot f = -\{H, f\} = \{f, H\} = \xi_f \cdot H = 0.$$

Therefore, $\varphi_t^* f = f$: the momentum f is conserved by the Hamiltonian evolution. This relationship between infinitesimal symmetries and conserved momenta is due to Noether.

Now consider Lie groups of symmetries and Lie algebras of infinitesimal symmetries. For the latter, suppose $\mathfrak{g}$ is a Lie algebra and

$$(2.35) \quad \alpha: \mathfrak{g} \longrightarrow \mathcal{X}_N^\omega$$

is an *antihomomorphism* of Lie algebras. (Antihomomorphisms are the infinitesimal version of *left* actions of Lie groups.) Recall the exact sequence (2.31).

Definition 2.36. Suppose the action α admits a lift to an antihomomorphism $\tilde{\alpha}$ in

$$(2.37) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^0(N; \mathbb{R}) & \longrightarrow & \Omega_N^0 & \longrightarrow & \mathcal{X}_N^\omega \longrightarrow H^1(N; \mathbb{R}) \longrightarrow 0 \\ & & & & \nwarrow \tilde{\alpha} & & \uparrow \alpha \\ & & & & & & \mathfrak{g} \end{array}$$

Then the transpose $\mu: N \rightarrow \mathfrak{g}^*$ of $\tilde{\alpha}$ is a *moment map* of α .

The composition $\mathfrak{g} \xrightarrow{\alpha} \mathcal{X}_N^\omega \rightarrow H^1(N; \mathbb{R})$ is an obstruction to the existence of $\tilde{\alpha}$. If this composition vanishes, then there is an induced central extension of Lie algebras

$$(2.38) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^0(N; \mathbb{R}) & \longrightarrow & \Omega_N^0 & \longrightarrow & \mathcal{X}_N^\omega \longrightarrow H^1(N; \mathbb{R}) \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \alpha \\ 0 & \longrightarrow & H^0(N; \mathbb{R}) & \longrightarrow & \tilde{\mathfrak{g}} & \dashrightarrow & \mathfrak{g} \longrightarrow 0 \end{array}$$

together with an antihomomorphism $\tilde{\mathfrak{g}} \rightarrow \Omega_N^0$. If the extension in the bottom row of (2.38) splits, then there exist lifts $\tilde{\alpha}: \mathfrak{g} \rightarrow \Omega_N^0$ and corresponding moment maps $\mu: N \rightarrow \mathfrak{g}^*$. A moment map is $\mathfrak{g}$ -equivariant in the sense that

$$(2.39) \quad \mu_*[\alpha(\xi)_N] = \xi \cdot \mu(n), \quad \xi \in \mathfrak{g}, \quad n \in N,$$

where on the right hand side the dot indicates the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$.¹³ Unraveling the definitions, a moment map is characterized by the equations

$$(2.41) \quad d\mu_\xi = -\iota_{\alpha(\xi)}\omega, \quad \xi \in \mathfrak{g},$$

where $\mu_\xi \in \Omega_N^0$ is the contraction of the $\mathfrak{g}^*$ -valued function μ with $\xi \in \mathfrak{g}$.

If a Lie group G acts on N on the left by symplectic diffeomorphisms, then there is an induced infinitesimal action (2.35) of its Lie algebra $\mathfrak{g}$, and we ask that a moment map $\mathfrak{g}: N \rightarrow \mathfrak{g}^*$ be G -invariant.

¹³There is a sign in the coadjoint action

$$(2.40) \quad (\xi \cdot \mu)(\eta) = -\mu([\xi, \eta]), \quad \xi, \eta \in \mathfrak{g}, \quad \mu \in \mathfrak{g}^*,$$

that cancels the sign in the *antihomomorphism* property of α and leads to the $\mathfrak{g}$ -equivariance of the moment map.

We conclude with a geometric interpretation of the moment map. Suppose $\pi: T \rightarrow N$ is a principal $\mathbb{R}$ -bundle with connection $\Theta \in \Omega_T^1$ whose curvature is

$$(2.42) \quad d\Theta = \pi^*\omega.$$

If a Lie group G acts on N by symplectic diffeomorphisms, a lift to an action $G \curvearrowright T$ by automorphisms of the principal bundle with connection leads to a moment map as follows. For $\xi \in \mathfrak{g}$, let $\tilde{\xi}$ be the vector field on T induced by the action $G \curvearrowright T$. Define the moment map $\mu_\xi \in \Omega_N^0$ by

$$(2.43) \quad \pi^*\mu(\xi) = \iota_{\tilde{\xi}}\Theta.$$

(The map π^* is injective, hence (2.43) determines the function $\mu_\xi \in \Omega_N^0$.) Then

$$(2.44) \quad 0 = \mathcal{L}_{\tilde{\xi}}\Theta = (d\iota_{\tilde{\xi}} + \iota_{\tilde{\xi}}d)\Theta = \pi^*d\mu_\xi + \iota_{\tilde{\xi}}\pi^*\omega = \pi^*\left[d\mu_\xi + \iota_{\alpha(\xi)}\omega\right],$$

from which (2.41) follows.

Lecture 3: Hamiltonian and Lagrangian mechanics

This lecture is a very condensed account of classical mechanics. The richness of the subject should be experienced through other sources, such as the classic text [Ar].

We begin with Hamiltonian systems, based on symplectic geometry. We illustrate with particle motion and also give more exotic examples, though without a particular physical model. A Hamiltonian system is the basic model of classical mechanics.

Often a Hamiltonian system can be derived from a Lagrangian system, and if so one has more tools and more control. We begin here with a discussion of variational principles in geometry, illustrating with the variational principle in Riemannian geometry that gives rise to the equation for geodesics. From here it is a short step to the Lagrangian system for particle motion. This is the *principle of least action*, for which I highly recommend the delightful lecture [Fey]. This lecture concludes with an extra structure one obtains in a Lagrangian system that is not present in a general Hamiltonian system: a connection whose curvature is the symplectic form. That connection plays a crucial role in quantization. We defer a discussion of Noether's theorem, which tells how symmetries in a Lagrangian theory give rise to observables that are preserved by the Hamiltonian flow, to a future lecture. We refer to [DF2] for many details, elaborations, and examples of Lagrangian mechanics as well as Lagrangian field theories.

3.1 Hamiltonian systems

Definition 3.1.

- (1) The data of a *Hamiltonian system* is a symplectic manifold (N, ω) and a 1-parameter group φ_t of symplectic automorphisms of N .
- (2) A Hamiltonian system is *free* if (N, ω) is an affine symplectic space and φ_t is a 1-parameter group of affine symplectic automorphisms.

The symplectic manifold N is often called the *phase space*.

A Hamiltonian system gives rise to a mechanical system in the sense of [Definition 1.11](#). Given a symplectic manifold (N, ω) and a 1-parameter group φ_t of symplectic automorphisms, define (see also [Section 1.2](#)):

- (1) The space of pure states is $\mathcal{PS} = N$. The space of all states is the space $\text{Prob}(N)$ of probability distributions on N .
- (2) The complex vector space $\mathcal{O} = \text{Borel}(N; \mathbb{C})$ is the space of complex-valued Borel functions on N . The involution of complex conjugation is the real structure. The dense smooth subspace $\mathcal{O}^\infty = C^\infty(N; \mathbb{C})$ consists of smooth functions. It carries a Lie algebra structure given by Poisson bracket [\(2.33\)](#). Observables form the real vector space $\mathcal{O}_\mathbb{R} = \text{Borel}(N; \mathbb{R})$ of Borel real-valued functions on N , and there is a dense subspace $\mathcal{O}_\mathbb{R}^\infty = C^\infty(N; \mathbb{R}) = \Omega_N^0$ of smooth observables.
- (3) The measurement pairing $\mathcal{O}_\mathbb{R} \times \mathcal{S} \rightarrow \text{Prob}(\mathbb{R})$ pairs an observable $A: N \rightarrow \mathbb{R}$ and a state $\sigma \in \text{Prob}(N)$ to the pushforward probability measure $A_*\sigma \in \text{Prob}(\mathbb{R})$.
- (4) A smooth observable $f \in \Omega_N^0$ has a symplectic gradient vector field $\xi_f \in \mathcal{X}_N$. Under appropriate completeness assumptions it generates a 1-parameter flow ψ_t of symplectic diffeomorphisms. There is an induced flow on states (measures push forward) and complex observables (functions pull back), and these induced flows are compatible [\(1.7\)](#).
- (5) The chosen φ_t generates the desired flows [\(1.12\)](#) on states and observables.

Remark 3.2. Note that not all smooth observables generate motion: vector fields always generate local flows, but a global flow may not exist. So [Axiom System 1.1\(4\)](#) is only aspirational in Hamiltonian systems.

Example 3.3. We revisit particle motion in Euclidean space ([§1.2](#)), and for simplicity consider only the Euclidean line $\mathbb{E}^1$ as target. Let $V: \mathbb{E}^1 \rightarrow \mathbb{R}$ be a smooth function (potential energy). Then Newton's law for a motion $x: \mathbb{R} \rightarrow \mathbb{E}^1$ of a particle of mass $m \in \mathbb{R}^{>0}$ is

$$(3.4) \quad m\ddot{x} = -V'(x).$$

Let N be the space of *classical trajectories*, i.e., the space of functions $x: \mathbb{R} \rightarrow \mathbb{E}^1$ that satisfy [\(3.4\)](#). Given a position and velocity—a point of the Cartesian product $\mathbb{E}^1 \times \mathbb{R}$ —there exists a *local* solution whose position and velocity at some specified time $t_0 \in \mathbb{R}$ is as given. (The solution is local in the sense that its domain is an open interval about t_0 .) Assume the potential V is sufficiently well-behaved that solutions exist for all time. Then for any $t_0 \in \mathbb{R}$, the evaluation map

$$(3.5) \quad N \longrightarrow \mathbb{E}^1 \times \mathbb{R}$$

that maps a trajectory to its position and velocity at time t_0 is a diffeomorphism. Use ‘ x ’ for the coordinate on $\mathbb{E}^1$ and ‘ y ’ for the coordinate on $\mathbb{R}$. Then the symplectic form is

$$(3.6) \quad \omega = m \, dy \wedge dx.$$

The flow $\varphi_t: N \rightarrow N$ is generated by the Hamiltonian function

$$(3.7) \quad H = \frac{my^2}{2} + V(x),$$

which is the sum of kinetic and potential energy. The symplectic gradient (2.16) of H is

$$(3.8) \quad \xi_H = y \frac{\partial}{\partial x} - \frac{1}{m} V'(x) \frac{\partial}{\partial y},$$

hence the equations for the flow φ_t are

$$(3.9) \quad \begin{cases} \dot{x} = y \\ \dot{y} = -\frac{1}{m} V'(x) \end{cases}$$

Remark 3.10.

- (1) The passage from Newton's Law (3.4) to the Hamilton equations (3.9) is the traditional move that converts a second-order ODE to a system of first-order ODEs by introducing the first derivative as a new variable.
- (2) Observe that N is an affine space. For a free particle ($V = 0$) and for a harmonic oscillator ($V = kx^2/2$, $k > 0$), the flow φ_t is by affine symplectic diffeomorphisms, hence according to Definition 3.1(2) the system is free.
- (3) Example 3.3 generalizes to particle motion on a Riemannian manifold (M, g) in the presence of a potential $V: M \rightarrow \mathbb{R}$. In this case the acceleration $\ddot{x}$ is computed using the Levi-Civita covariant derivative. Also, (3.5) is replaced by a diffeomorphism onto the total space of the tangent bundle TM , again assuming solutions to the equations of motion are defined for all time. The Riemannian metric induces an isomorphism $TM \xrightarrow{\cong} T^*M$, and the symplectic form is m times the pullback of the tautological symplectic form defined in Example 2.29(2). The Hamiltonian function (3.7) generalizes to

$$(3.11) \quad H(\xi) = \frac{m}{2} |\xi|^2 + V(\pi(\xi)), \quad \xi \in TM,$$

where $\pi: TM \rightarrow M$ is the tangent bundle.

The Riemannian manifold M is often called the *configuration space*. (Recall that $N = TM$ is called the *phase space*.)

- (4) Later in the lecture we indicate how to derive (3.6) and (3.7) (and their generalizations for particle motion on a Riemannian manifold) from a more basic starting point: a Lagrangian.

Example 3.12. The phase space N of a Hamiltonian system need not be the tangent bundle to a manifold, and the 1-parameter group φ_t need not be generated by a Hamiltonian function. For example, as in Example 2.32 take

$$(3.13) \quad N = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$$

and

$$(3.14) \quad \varphi_t(\theta^1, \theta^2) = (\theta^1 + t, \theta^2)$$

This is the quotient of a free system by the action of a discrete group $(2\pi\mathbb{Z} \times 2\pi\mathbb{Z})$ of automorphisms.

3.2 Variational principles

Variational principles are important in many parts of geometry, for example in global analysis on manifolds. Here we review the variational principle for free particle motion in Euclidean space. (The generalization to geodesic motion on a Riemannian manifold uses the Levi–Civita covariant derivative, but otherwise is the same.) We do not prove the assertions here, which can be proved using standard ideas in analysis; see [F1, Lecture 10], for example.

Let V be a finite dimensional real inner product space, and suppose A is an affine space over V (a Euclidean space). Fix real numbers $a < b$, and consider smooth motions $x: [a, b] \rightarrow A$. The length of the image path is

$$(3.15) \quad \int_{[a,b]} |\dot{x}(t)| |dt|$$

and it does not depend on the parametrization of the image. In other terms, there is an infinite dimensional symmetry group of the function that maps x to (3.15), and that symmetry makes (3.15) more difficult to use as the function(al) for a variation problem. Instead, consider the *energy*

$$(3.16) \quad E(x) = \int_{[a,b]} \frac{1}{2} |\dot{x}(t)|^2 |dt| = \int_{[a,b]} \frac{1}{2} \langle \dot{x}, \dot{x} \rangle |dt|.$$

To compute the *variational* formula, introduce a finite dimensional family of “variations”. That is, let S be a smooth manifold and consider a smooth function

$$(3.17) \quad x: S \times [a, b] \longrightarrow A.$$

For each $s \in S$ the function $x(s, -): [a, b] \rightarrow A$ is a motion in A . Let δ be the de Rham differential along S , and write the de Rham differential on $[a, b]$ as $d = dt \frac{d}{dt}$. Then $E: S \rightarrow \mathbb{R}$ is a smooth function, and its differential is

$$(3.18) \quad \begin{aligned} \delta E &= \int_{[a,b]} \langle \delta \dot{x}, \dot{x} \rangle \wedge dt \\ &= \int_{[a,b]} \langle \delta dx, \dot{x} \rangle \\ &= \int_{[a,b]} -\langle d\delta x, \dot{x} \rangle \\ &= \int_{[a,b]} -d\langle \delta x, \dot{x} \rangle - \langle \delta x \wedge d\dot{x} \rangle \\ &= \int_{[a,b]} -\langle \delta x, \ddot{x} \rangle \wedge dt - d\gamma, \end{aligned}$$

where

$$(3.19) \quad \gamma = \langle \delta x, \dot{x} \rangle.$$

$$(3.20) \quad \ddot{x}(t) = 0$$

Remark 3.21. The last term in (3.18) is the integral of an exact 1-form:

$$(3.26) \quad \begin{array}{c|cc} & 0 & 1 \\ \hline 1 & L & \rightarrow \\ & & \uparrow \\ 0 & & \gamma \end{array} \quad \mathcal{F}$$

where the horizontal differential is δ and the vertical differential is d . The crucial point is that if we restrict to the finite dimensional manifold N of solutions to the equations of motion (3.20), then the solutions extend to all time and fit into the diagram

$$(3.27) \quad \begin{array}{ccc|c} & 0 & 1 & \\ \hline 1 & L & \rightarrow & 0 \\ & & \uparrow & \\ 0 & & \gamma & \\ \hline & \mathbb{R} & & \end{array} \quad N$$

In other words, on $N \times \mathbb{R}$ we have

$$(3.28) \quad D(L + \gamma) = \delta L + d\gamma = 0.$$

Remark 3.29.

- (1) If we begin with the density L in (3.23), then we should ask: where does the *variational 1-form* γ in (3.19) come from? In other words, can we codify the integration by parts in (3.18) without integration? Yes we can, but we do not do so here: see Definition 2.37 and §2.4 in [DF2].
- (2) Is there an *a priori* argument why the critical point equation of the total energy (3.16) is an equation (3.20) that is local in time? Yes there is. Namely, consider instead of the condition for a critical point the condition for a *minimizer*. Then for each time t_0 , a minimizer must minimize the integral over the interval $[t_0, t_0 + \delta]$ for all $\delta > 0$. The limit $\lim_{\delta \rightarrow 0}$ is then a condition localized at t_0 . It ends up being the *Euler–Lagrange equation* (3.20).
- (3) Variational principles are extremely useful in theory and in practice. If an ODE or a PDE is the Euler–Lagrange equation of some functional, then that functional can be used to set up and control a minimization procedure to find solutions. Specifically, the functional leads to estimates that are often crucial in the study of the PDE.

3.3 Lagrangian systems

The Lagrangian approach dates from the *principle of least action* or the *stationary action principle* is usually ascribed to Maupertuis, though perhaps Euler and Leibniz should also receive credit. Again, I defer to Feynman [Fey]. Here I simply focus on the formal aspects, which are developed in great detail in [DF2].

Remark 3.30.

- (1) Not every Hamiltonian system has a Lagrangian formulation.
- (2) A Lagrangian formulation of a Hamiltonian system leads to estimates and greater control, as indicated in Feynman’s lecture. Compare with Remark 3.29(3).
- (3) A Lagrangian formulation leads to more geometric structure, as we indicate in Section 3.4.
- (4) A Lagrangian formulation of a mechanical theory is the input into the *path integral*, which is an approach to quantum mechanics developed by Feynman.

We do not attempt any generality here, but simply consider a particle motion on a Riemannian manifold (M, g) equipped with potential energy $V: M \rightarrow \mathbb{R}$. Then for $x: \mathbb{R} \rightarrow M$, set

$$(3.31) \quad S(x) = \underbrace{\int_{[a,b]} \left[\underbrace{\frac{m}{2} |\dot{x}(t)|^2}_{\text{kinetic energy}} - \underbrace{V(x(t))}_{\text{potential energy}} \right] dt}_{\text{Lagrangian}} \quad \underbrace{\hspace{10em}}_{\text{action}}$$

Observe that both the integrand (Lagrangian or Lagrangian *density*) and the integral (action) have units/dimensions

$$(3.32) \quad \frac{ML^2}{T} = \frac{(\text{mass})(\text{length})^2}{(\text{time})},$$

which we call the *units of action*.

Remark 3.33. The Lagrangian is a somewhat magical quantity that seemingly does not have as direct a physical interpretation as, say, quantities like energy, force, momentum, mass, etc.

As in (3.26) one does local calculus on $\mathcal{F} \times \mathbb{R}$, where now $\mathcal{F}$ is the space of smooth maps $\mathbb{R} \rightarrow M$. One computes γ , roughly using integration by parts as in (3.18), so that the equation of motion (Euler–Lagrange equation) is

$$(3.34) \quad D(L + \gamma) = \delta L + d\gamma = 0.$$

For the particle on M we find

$$(3.35) \quad \gamma = m \langle \delta x, \dot{x} \rangle,$$

and the equations of motion are Newton’s Law

$$(3.36) \quad \nabla_{\dot{x}} \dot{x} + V'(x) = 0,$$

where ∇ is the Levi–Civita covariant derivative. Let N be the space of solutions.

We briefly indicate how to derive a Hamiltonian system (Definition 3.1) from the Lagrangian. Set

$$(3.37) \quad \omega = \delta\gamma \quad \in \Omega_{N \times \mathbb{R}}^{0,2},$$

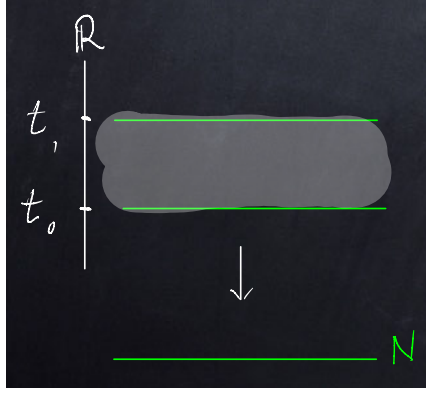
as depicted in the diagram

$$(3.38) \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 1 & L & \rightarrow & 0 \\ & & \uparrow & \\ 0 & & \gamma & \rightarrow \omega \end{array} \quad N$$

$\mathbb{R}$

Lemma 3.39.

- (1) $D\omega = 0$, from which $\delta\omega = 0$ and $d\omega = 0$.
- (2) The restriction of ω to $N \times \{t\}$ is independent of $t \in \mathbb{R}$.
- (3) ω is nondegenerate.

FIGURE 3. Proof that ω is static

Proof. For (1), observe that $\omega = D(\gamma - L)$; now apply D . For (2), choose times $t_0 < t_1$ and apply Stokes' theorem¹⁴ to the projection map $N \times [t_0, t_1] \rightarrow N$ depicted in Figure 3:

$$(3.40) \quad \delta \int_{[t_0, t_1]} \omega = \pm \int_{[t_0, t_1]} D\omega \pm \int_{\{t_1\} - \{t_0\}} \omega.$$

The left hand side vanishes: the differential δ and the integral $\int_{[t_0, t_1]}$ commute. (This is the usual differentiation under the integral sign.) The first term on the right hand side vanishes by (1). The vanishing of the remaining term is the desired statement. For (3) compute that

$$(3.41) \quad \omega = m \langle \delta \dot{x} \wedge \delta x \rangle,$$

from which the description in Remark 3.10(3) follows. Now use the fact that the tautological 2-form on the cotangent bundle is nondegenerate. $\square$

Corollary 3.42. *The restriction of ω to $N \times \{t\}$ is a symplectic form on N , and it is independent of t .*

Next, we give a formula for a (Hamiltonian) function on N derived from the Lagrangian. The flow it generates is the 1-parameter group needed to complete the data in Definition 3.1(1). Let ξ be the vector field on $\mathcal{F} \times \mathbb{R}$ that generates the *negative* of time translation. It is characterized by the contractions

$$(3.43) \quad \begin{aligned} \iota_\xi dx &= -1 \\ \iota_\xi \delta x &= \dot{x} \end{aligned}$$

¹⁴The version we need is Stokes' theorem in families. Fortunately, we do not need to pin down signs.

Then set

$$(3.44) \quad H = \iota_\xi(L + \gamma) \in \Omega_{\mathcal{F} \times \mathbb{R}}^{0,0}.$$

Remark 3.45. The motivation for (3.44) comes from Noether's theorem: the Hamiltonian is the conserved quantity derived from the negative of infinitesimal time translation. Compare (3.47) below to (2.41); see [DF2, §2.6] for further details.

Lemma 3.46. $H|_{N \times \{t\}}$ is independent of $t \in \mathbb{R}$ and

$$(3.47) \quad \delta H = -\iota_\xi \omega.$$

Proof. On $N \times \mathbb{R}$ we use $D(L + \gamma) = \omega$ to compute

$$(3.48) \quad DH = D\iota_\xi(L + \gamma) = \mathcal{L}_\xi(L + \gamma) - \iota_\xi D(L + \gamma) = -\iota_\xi \omega.$$

The first assertion now follows from the *conservation law* $dH = 0$, which is the $(0,1)$ -component of (3.48). $\square$

Example 3.49. Consider particle motion on $\mathbb{E}^1$, as in Example 3.3. Then

$$(3.50) \quad L + \gamma = \left\{ \frac{m}{2} \dot{x}^2 - V(x) \right\} dt + m \dot{x} dx.$$

Hence

$$(3.51) \quad \begin{aligned} H = \iota_\xi(L + \gamma) &= - \left\{ \frac{m}{2} \dot{x}^2 - V(x) \right\} + m \dot{x}^2 \\ &= \frac{m}{2} \dot{x}^2 + V(x) \end{aligned}$$

which is the usual formula for the total kinetic energy. Notice the sign change on the potential from the Lagrangian (negative) to the Hamiltonian (positive).

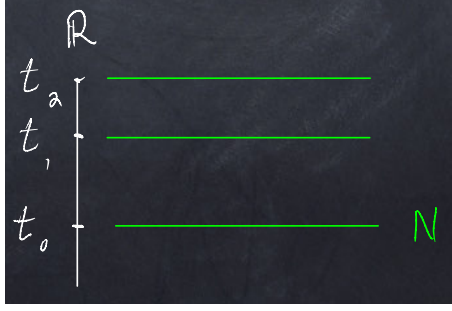
3.4 Geometric meaning of γ

On the space N of classical trajectories the equation $\delta L = -d\gamma$ holds, from which we deduce

$$(3.52) \quad \delta \int_{[t_0, t_1]} L = -\{\gamma(t_1) - \gamma(t_0)\}$$

for all times $t_0 < t_1$. We now interpret (3.52) geometrically; see [DF2, §1.1] for further discussion.

View $\mathbb{R}$ as a Lie group under addition. A connection on the trivial principal $\mathbb{R}$ -bundle over N is a (real-valued) 1-form on N . Then for each $t \in \mathbb{R}$, the 1-form $\gamma(t)$ is an $\mathbb{R}$ -connection on $N \times \{t\}$. Equation (3.52) implies that addition by $\int_{[t_0, t_1]} L$ is an isomorphism of trivial principal

FIGURE 4. Construction of an $\mathbb{R}$ -connection over N

$\mathbb{R}$ -bundles with connection; it maps the connection $\gamma(0)$ to the connection $\gamma(1)$. Furthermore, these isomorphisms are compatible for three times $t_0 < t_1 < t_2$, as in Figure 4. Then taking invariant sections in the $\mathbb{R}$ -direction, we obtain a (not trivialized) principal $\mathbb{R}$ -bundle

$$(3.53) \quad T \longrightarrow N$$

equipped with a connection whose curvature is the symplectic form. (This last assertion follows from $\delta\gamma = \omega$. This interpretation gives a new construction of the symplectic form on N as the curvature of the connection on (3.53).)

Remark 3.54. The first step in the path integral formulation of the quantum theory is to exponentiate the bundle (3.53) with its connection. (Intuitively, quantization is an exponentiation process.) Observe that the units of γ are the units ML^2/T of action. Therefore, to carry out the exponentiation we need a numerical constant with units of action. *Planck's constant* $\hbar$ is precisely such a number. It makes sense, then, to apply the function

$$(3.55) \quad \exp\left(\frac{i}{\hbar} -\right) : \mathbb{R} \longrightarrow \mathbb{C}^\times$$

to form a principal $\mathbb{C}^\times$ -bundle with connection associated to (3.53). The associated complex line bundle with covariant derivative is sometimes called the *prequantum line bundle*. Observe that it is constructed canonically from a Lagrangian formulation. This is the extra piece of structure alluded to earlier, and it is one of the powerful features of a Lagrangian formulation, when it is available.

Example 3.56. This illustrative example is called the *particle on a ring*. The ring is the Euclidean circle $\mathbb{E}^1/2\pi L\mathbb{Z}$ of length $2\pi L$ for some $L \in \mathbb{R}^{>0}$. Let $m \in \mathbb{R}^{>0}$ be the mass of the particle. We define a 1-parameter family of theories parametrized by $\theta \in \mathbb{R}$. The Lagrangian is

$$(3.57) \quad L = \left(\frac{m}{2} \dot{x}^2 + \frac{\theta}{2\pi} \dot{x} \right) dt.$$

The first term is the usual kinetic energy; the potential energy vanishes. The second term is locally exact but not globally exact, since there is no global function x on the circle. We leave the reader to identify the space N of classical solutions and show that

- the Hamiltonian system derived from (3.57) is independent of θ ; and

- the principal bundle (3.53) with connection changes with θ .

The dependence on θ shows up in the corresponding quantum systems, as we will see in §7.3.

Remark 3.58. This example illustrates that different Lagrangian systems can give rise to the same Hamiltonian system. It also illustrates that Lagrangian systems can contain more information than Hamiltonian systems.

Lecture 4: Spectral theorems

We begin with a geometric take on diagonalization of linear operators on finite dimensional vector spaces. The general operator A has a Jordan normal form, which we interpret in terms of a sheaf over the complex line. The support of the sheaf is the spectrum of the operator. If the operator is diagonalizable, then the sheaf can be re-expressed as a projection-valued measure. This is essentially the expression of A as a multiplication operator. On inner product spaces there are two special classes of operators that are diagonalizable: self-adjoint operators and unitary operators.

Then we turn to bounded (or continuous) operators on Hilbert spaces. For self-adjoint operators the main result is von Neumann's spectral theorem. We refer to §1.4.3 for background material.

In the next lecture we use these spectral ideas to introduce quantum mechanical systems.

4.1 Linear operators in finite dimensions

Let W be a finite dimensional complex vector space of positive dimension. Suppose $A: W \rightarrow W$ is a linear operator. A subspace $W' \subset W$ is *A-invariant* if $A(W') \subset W'$. The operator A is *decomposable* if there exists a direct sum decomposition $W = W' \oplus W''$ into proper¹⁵ A -invariant subspaces. An *eigenspace* of A is a subspace on which A acts as (multiplication by) a scalar; that scalar is the corresponding *eigenvalue*. The algebraic completeness of the complex numbers implies that every operator has an eigenline. For $\lambda \in \mathbb{C}$ define

$$(4.1) \quad W_\lambda = \{\psi \in W : A\psi = \lambda\psi\}.$$

The previous assertion is that $W_\lambda \neq 0$ for some $\lambda \in \mathbb{C}$. There is an inclusion

$$(4.2) \quad \bigoplus_{\lambda \in \mathbb{C}} W_\lambda \longrightarrow W$$

The operator A is *diagonalizable* if this inclusion is an isomorphism.

Example 4.3. Let $W = \mathbb{C}^3$, fix $\lambda \in \mathbb{C}$, and define $A: W \rightarrow W$ to be multiplication by the matrix

$$(4.4) \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

¹⁵A subspace $W' \subset W$ is *proper* if $W' \neq 0$ and $W' \neq W$.

Then with respect to the standard basis e_1, e_2, e_3 , the only eigenline is $\mathbb{C} \cdot \lambda$. This subspace has no A -invariant complement: the operator A is *indecomposable*.

Definition 4.5. A *Jordan block* is an indecomposable linear operator.

It follows (not immediately!) that a Jordan block has the form (4.4) in some basis, where we allow the matrix to shrink or expand. The number λ is called the *generalized eigenvalue* of the Jordan block. In case $\dim W = 1$, a Jordan block is an eigenline. For a Jordan block, the domain of (4.2) is 1-dimensional.

Let $A: W \rightarrow W$ be a Jordan block with generalized eigenvalue λ . Then $N = A - \lambda \text{id}_W$ is a *nilpotent* operator: $A^{n-1} = 0$, where $n = \dim W$. The kernels of successive powers of N form a *full flag* in W , i.e., a *filtration*

$$(4.6) \quad 0 \subset W^{(1)} \subset W^{(2)} \subset \dots \subset W^{(n)} = W$$

with $\dim W^{(i)} = i$. The line $W^{(1)}$ is a 1-dimensional eigenspace with eigenvalue λ .

Theorem 4.7. Let W be a finite dimensional complex vector space, and suppose $A: W \rightarrow W$ is a linear operator. Then there exists an A -invariant decomposition

$$(4.8) \quad W = \bigoplus_{i=1}^k W_i$$

into Jordan blocks. The decomposition is unique up to ordering.

The corresponding $\lambda_1, \dots, \lambda_k$ are the eigenvalues of A . The λ_i may be repeated. The operator A is diagonalizable iff $\dim W_i = 1$ for all i .

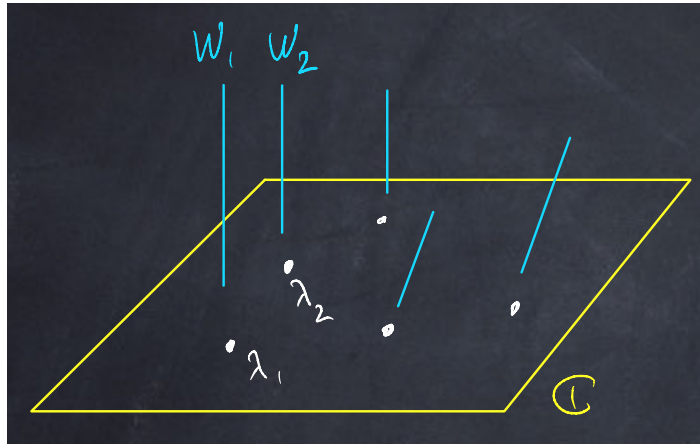


FIGURE 5. The sheaf $\mathcal{F}_A \rightarrow \mathbb{C}$ of the linear operator $A: W \rightarrow W$

For a geometric view, let $R = \mathbb{C}[\lambda]$ be the polynomial ring on one variable. Then a linear operator $A: W \rightarrow W$ determines an R -module structure on W : the indeterminate λ acts as A . In algebraic geometry, to a commutative ring one attaches a space on which the ring is realized as a ring of

functions. For the polynomial ring, that space is the affine line $\text{Spec } R = \mathbb{C}$. A module over the commutative ring determines a *sheaf*¹⁶ over its spectrum. For the linear operator A , denote the sheaf by $\mathcal{F}_A \rightarrow \mathbb{C}$. The support of this sheaf—the *spectrum* of A —is the finite set of its eigenvalues; see [Figure 5](#). For any open $E \subset \mathbb{C}$, the sections of this sheaf over E form the vector space

$$(4.9) \quad \Gamma_E(\mathcal{F}_A) = \bigoplus_{\lambda_i \in E} W_i.$$

Remark 4.10. The decomposition (4.8) depicted in [Figure 5](#) does not determine the operator A in general. If there is a Jordan block of dimension > 1 , then the structure of the sheaf $\mathcal{F}_A \rightarrow \mathbb{C}$ in a formal neighborhood of the corresponding eigenvalue encodes the flag (4.6).

4.2 Projection-valued measures

Henceforth, we restrict to operators $A: W \rightarrow W$ that are diagonalizable. By definition, the eigenspace decomposition (4.2) is an isomorphism:

$$(4.11) \quad W = \bigoplus_{\lambda \in \mathbb{C}} W_\lambda.$$

The operator A is determined by this decomposition and by the eigenvalues: A acts as multiplication by λ on W_λ . This is nicely depicted in the skyscraper sheaf of [Figure 5](#). The operator is multiplication by the coordinate λ in the base $\mathbb{C}$, and it acts on the vector space W which has been “spread out” or “sheaffied” over the base.

There is an equivalent encoding of a multiplication operator in terms of projections. Recall from [Definition 1.41\(4\)](#) that a *projection* $P: W \rightarrow W$ is a linear operator that satisfies $P^2 = P$. A projection gives rise to a direct sum decomposition

$$(4.12) \quad W = P(W) \oplus (1 - P)(W)$$

which in other terms is the direct sum of the image of P and the kernel of P . Conversely, a decomposition of $W = W'' \oplus W'$ as the ordered direct sum of *two* subspaces determines a projection

$$(4.13) \quad \text{proj}[W'](W''): W \longrightarrow W$$

with image W'' and kernel W' . Let $\text{Projections}(W)$ denote the space of all projection operators on W .

Returning to a diagonalizable $A: W \rightarrow W$ with eigenspace decomposition (4.11), define

$$(4.14) \quad \begin{aligned} \pi_A: \text{Open}(\mathbb{C}) &\longrightarrow \text{Projections}(W) \\ E &\longmapsto \text{proj} \left[\bigoplus_{\lambda \notin E} W_\lambda \right] \left(\bigoplus_{\lambda \in E} W_\lambda \right) \end{aligned}$$

Observe that although the domain is written as the set of open subsets of $\mathbb{C}$, in fact the map is well-defined on *all* subsets of $\mathbb{C}$. The map π_A satisfies:

¹⁶We discuss sheaves in a later lecture; they are only used in passing here.

- (i) $\pi_A(\emptyset) = 0$,
- (ii) $\pi_A(\mathbb{C}) = \text{id}_W$,
- (iii) If $E_1 \cap E_2 = \emptyset$, then $\pi_A(E_1 \cup E_2) = \pi_A(E_1) + \pi_A(E_2)$ and $\pi_A(E_1)$ commutes with $\pi_A(E_2)$.

These are similar to the properties of a probability measure; a projection has (eigen)values 0 and 1, and its value on a typical vector is a sort of average of 0 and 1. Property (i) says that the value of the empty set is the 0-map, and property (ii) says that the value of the entire space is the “1-map”. But there are the following differences from a probability measure:

- A probability measure is real-valued.
- A measure is defined on a σ -algebra of subsets, which for a topological space is the σ -algebra generated by the open subsets: the σ -algebra of Borel subsets.
- A measure satisfies countable additivity, not just the finite additivity in (iii).

The map (4.14) is called a *projection-valued measure* or *spectral measure*. The expression of A as a multiplication operator—its diagonalization—is

$$(4.15) \quad A = \int_{\mathbb{C}} \lambda d\pi_A(\lambda).$$

As usual, once an operator A is diagonalized we can define functions of the operator, such as its exponential e^A . If $f: \mathbb{C} \rightarrow \mathbb{C}$ is a function, set

$$(4.16) \quad f(A) = \int_{\mathbb{C}} f(\lambda) d\pi_A(\lambda).$$

This formula is the essence of *functional calculus*. When we come to Hilbert spaces, the function f is required to be Borel measurable.

4.3 Generalizations and special cases

Later we encounter, in the infinite dimensional setting, some variations that are worth pointing out now in passing. We continue with a finite dimensional complex vector space W .

Suppose $A^1, \dots, A^n: W \rightarrow W$ form a commuting set of diagonalizable operators: $A_i A_j = A_j A_i$ for all $1 \leq i, j \leq n$. Then these operators are *simultaneously* diagonalizable: there is a direct sum decomposition

$$(4.17) \quad W = \bigoplus_{j=1}^{\ell} W^{(j)}$$

and

$$(4.18) \quad A_i|_{W^{(j)}} = \lambda_{ij}$$

for some $\lambda_{ij} \in \mathbb{C}$.

A more geometric picture emerges if we consider all linear combinations of the operators $A_1, \dots, A_n$. Let V be a finite dimensional complex vector space, and suppose $A: V \rightarrow \text{End}(W)$ is a homomorphism of additive groups: it maps vector sums to operator sums. Then the eigenvalues of the operator $A(\xi)$, $\xi \in V$, are linear functions of ξ . There is one such linear function for each subspace $W^{(j)}$ in (4.17). This diagonalization can be expressed either as a skyscraper sheaf over V^* supported at these linear functions, or, as in §4.2, as a projection-valued measure on V^* with the same support. The previous paragraph is the model case $V = \mathbb{C}^n$.

Now suppose that W is a finite dimensional complex *inner product space*. (Our convention for the inner product, which is a positive definite hermitian form, is (1.40).) There are two special classes of diagonalizable linear operators $A: W \rightarrow W$, namely *self-adjoint* operators and *unitary* operators. The eigenvalues of a self-adjoint operator are real, and the associated projection-valued measure is supported on $\mathbb{R} \subset \mathbb{C}$. The eigenvalues of a unitary operator lie in the unit circle

$$(4.19) \quad \mathbb{T} = \{\lambda \in \mathbb{C} : \bar{\lambda}\lambda = 1\},$$

and the associated projection-valued measure is supported on $\mathbb{T} \subset \mathbb{C}$. Observe that $\mathbb{R} \subset \mathbb{C}$ is an (additive) subgroup, and $\mathbb{T} \subset \mathbb{C}^\times$ is a (multiplicative) subgroup. The exponential map

$$(4.20) \quad \exp(\sqrt{-1} \cdot): \mathbb{R} \longrightarrow \mathbb{T}$$

is a covering map, and using functional calculus we can exponentiate a self-adjoint operator to a unitary operator.

Example 4.21. Let V be a finite dimensional *real* vector space, and suppose

$$(4.22) \quad A: V \longrightarrow U(W)$$

is a unitary representation of the additive Lie group V . Then diagonalization produces a projection-valued measure on (the *real* dual space) V^* , but at a linear functional $\theta \in V^*$ the operator acts via the exponential of θ :

$$(4.23) \quad A = \int_{V^*} e^{i\theta} d\pi_A(\theta).$$

Remark 4.24. **Example 4.21** generalizes to unitary representations of abelian (Lie) groups; diagonalization occurs over the Pontrjagin dual group.

4.4 Spectral theorem on Hilbert space

We begin with a formal definition of a spectral measure. The books [Fo2, Nee, RS] have more detail and proofs of various spectral theorems.

Definition 4.25. Let (X, Σ) be a measure space (so Σ is a σ -algebra on X), and suppose $\mathcal{H}$ is a complex separable Hilbert space. Let $\text{Proj}^\perp(\mathcal{H})$ denote the space of self-adjoint projections in $\mathcal{H}$.

Then a map $\pi: \Sigma \rightarrow \text{Proj}^\perp(\mathcal{H})$ is a *spectral measure* if:

- (1) $\pi(\emptyset) = 0_{\mathcal{H}}$ and $\pi(X) = \text{id}_{\mathcal{H}}$;
- (2) if $E_1, E_2, \dots \in \Sigma$ is a finite or countable sequence of disjoint sets, then

$$(4.26) \quad \pi \left(\bigcup_i E_i \right) \xi = \sum_i \pi(E_i) \xi, \quad \xi \in \mathcal{H};$$

- (3) $\pi(E_1 \cap E_2) = \pi(E_1)\pi(E_2)$ for all $E_1, E_2 \in \Sigma$.

Recall now the material in §1.4.3. Since only bounded operators are discussed there, we supplement here with the following crucial result.

Definition 4.27. Let $\mathcal{H}$ be a separable Hilbert space. A *strongly continuous 1-parameter unitary group* is a homomorphism

$$(4.28) \quad U: \mathbb{R} \longrightarrow U(\mathcal{H})$$

such that for every $\psi \in \mathcal{H}$, the map $t \mapsto U_t(\psi)$ is a continuous map $\mathbb{R} \rightarrow \mathcal{H}$.

A theorem of Stone [Stone] and von Neumann [vN2] tells that such 1-parameter groups can be differentiated at $t = 0$, and in fact the derivative generates the 1-parameter group.

Theorem 4.29. *There is a one-to-one correspondence between self-adjoint operators and strongly continuous 1-parameter unitary groups.*

Crucially, the self-adjoint operator may be *unbounded*. The key ingredient in the proof is the theory of unbounded self-adjoint operators in Hilbert space and the spectral theorem for such operators.

Lecture 5: Quantum mechanics; finite dimensional systems

We introduced quantum mechanics (QM systems) quickly at the end of [Lecture 1](#), and we quickly revisit that here. Then we discuss some general features of QM systems. First, there is an important composition law, which combines two QM systems without introducing any interaction between them. We introduce the crucial property of *positive energy* as well as the basic dichotomy of QM systems: gapped vs. gapless.

In the remainder of the lecture we illustrate with finite dimensional examples. We begin with a general discussion of systems with a 2-dimensional state space, now called *qubits*. Here we encounter an important piece of geometric structure in quantum mechanics: the transition probability function on pairs of pure states. Then we introduce the general class of *lattice models*, and describe one very accessible example in detail: the *toric code*.

5.1 QM systems

The following is parallel to [Definition 3.1](#), which tells that the data of a classical Hamiltonian system is a pair (N, φ) consisting of a symplectic manifold and a 1-parameter group φ of symplectic diffeomorphisms, i.e., a homomorphism $\varphi: \mathbb{R} \rightarrow \text{SympDiff}(N)$.

Definition 5.1. The linear data of a *quantum mechanical (QM) system* is a pair $(\mathcal{H}, U)$ consisting of a complex separable Hilbert space $\mathcal{H}$ and a strongly continuous 1-parameter group $U_t: \mathcal{H} \rightarrow \mathcal{H}$, $t \in \mathbb{R}$, of unitary automorphisms.

Put differently, $U: \mathbb{R} \rightarrow U(\mathcal{H})$ is a strongly continuous unitary representation of the additive group $\mathbb{R}$ of time translation. By Stone's [Theorem 4.29](#) this unitary representation is equivalent to a (possibly unbounded) self-adjoint operator H , which is called the *Hamiltonian*.

Remark 5.2. We call this *linear* data since a quantum system is *projective* and can be specified by projective data instead, as we discuss in [Lecture 8](#).

In [§1.3](#) we explain how to obtain the data ([Axiom System 1.1](#)) of a mechanical system from the pair $(\mathcal{H}, U)$. We do not repeat here, but only emphasize the following points:

- Let $\mathcal{O} = \mathcal{O}^\infty = \text{End } \mathcal{H}$ be the algebra of *continuous* (bounded) operators on $\mathcal{H}$. That observables can be restricted to be bounded operators has been much discussed in the literature; for example, see [\[Seg, Ha\]](#). This leads to *C*-algebras* of observables.
- The formulas [\(1.30\)](#) for the flow on states and observables hold. They are defined directly in terms of the 1-parameter group of unitaries.

5.2 General remarks

5.2.1 Composition laws. There is a composition law on classical Hamiltonian systems and on QM systems: the composition of two systems is their conjunction with no interaction. In terms of the data in [Definition 3.1](#) and [Definition 5.1](#), the composition laws are

$$(5.3) \quad (N^{(1)}, \varphi^{(1)}) \times (N^{(2)}, \varphi^{(2)}) = (N^{(1)} \times N^{(2)}, \varphi^{(1)} \times \varphi^{(2)})$$

and

$$(5.4) \quad (\mathcal{H}^{(1)}, U^{(1)}) \otimes (\mathcal{H}^{(2)}, U^{(2)}) = (\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}, U^{(1)} \otimes U^{(2)}).$$

These composition laws are commutative and associative, but in a categorical sense: the structure is the categorification of a commutative monoid. There is a unit for the compositions: in the classical case it has $N = \text{pt}$ and in the quantum case $\mathcal{H} = \mathbb{C}$; for both units the 1-parameter groups are identity maps. These mechanical systems are the objects of a category (what are the morphisms?), and the composition law determines a *symmetric monoidal* structure.

Remark 5.5. The associative composition law and its unit determine the notion of an *invertible* quantum mechanical system: $(\mathcal{H}, U)$ is invertible if there exists $(\mathcal{H}', U')$ such that $(\mathcal{H}, U) \otimes (\mathcal{H}', U') \cong (\mathbb{C}, 1)$. If so, then $\dim \mathcal{H} = 1$, and so the system only has a single state (but the evolution can be by any unitary character of $\mathbb{R}$). This is not a particularly interesting system. Its analogs in quantum *field* theory turn out to be surprisingly useful.

5.2.2 Positive energy; gapped vs. gapless. Recall the spectral decomposition of a unitary representation of a vector space ([Example 4.21](#)). Applied to the unitary time-evolution $U: \mathbb{R} \rightarrow U(\mathcal{H})$ of a QM system, we obtain a self-adjoint projection-valued measure π_U such that

$$(5.6) \quad U_t = \int_{\mathbb{R}} e^{-itE/\hbar} d\pi_U(E).$$

This is the diagonalization of U in terms of unitary characters $t \mapsto e^{-itE/\hbar}$, $E \in \mathbb{R}$, of the additive group $\mathbb{R}$. The support of the measure π_U is the *spectrum* $\text{spec}(U)$ of the unitary representation U .

The following definition is critical.

Definition 5.7. Let $(\mathcal{H}, U)$ be the data of a quantum mechanical system.

- (1) $(\mathcal{H}, U)$ has *positive energy* if $\text{spec}(U)$ is bounded below.
- (2) $(\mathcal{H}, U)$ is *gapped* if it has positive energy and the minimum of $\text{spec}(U)$ is an isolated point of $\text{spec}(U)$ which is an eigenvalue of finite multiplicity. Any eigenline for the minimum eigenvalue is a *vacuum state*. If $(\mathcal{H}, U)$ is not gapped, it is called *gapless*.

Many systems are normalized so that the minimum of the spectrum is 0, i.e., the Hamiltonian is a nonnegative self-adjoint operator. Also, in many systems there is a unique vacuum state; we will see an example below (toric code) with nonunique vacua. Finally, if $\mathcal{H}$ is finite dimensional, then every Hamiltonian is gapped. For these discrete systems there is a more refined notion, at least heuristically; see [Remark 5.28\(2\)](#) below.

5.2.3 Unitarity. The evolution in quantum mechanics, as defined in [\(1.30\)](#), is unitary. This is a signature feature of quantum theory. The other pillar of quantum field theory is *locality*. In the general setup of quantum mechanics there is no notion of space—only time—so no notion of locality. Nonetheless, we will get a glimpse of locality in our description of spin systems and of the toric code, each of which takes place on a discrete space.

5.3 States and observables for two-dimensional Hilbert spaces

Let $\mathcal{H} = W$ be a 2-dimensional complex inner product space. We can fix a basis, so in essence take $\mathcal{H} = \mathbb{C}^2$, and we can give the basis elements descriptive names, such as ‘spin up’ $\uparrow$ and ‘spin down’ $\downarrow$, but to keep the geometry and all of the symmetry in the foreground we do not do so. Also, we need not fix a Hamiltonian to discuss states, observables, and the pairing to $\text{Prob}(\mathbb{R})$. A QM system with a 2-dimensional state space is often called a *qubit*.

The pure states $\mathcal{S}_0$ comprise the complex projective line $\mathbb{P}W$, the space of 1-dimensional subspaces in W . The space of all states is

$$(5.8) \quad \mathcal{S} = \{S \in \text{End } W : S^* = S, S \geq 0, \text{trace}(S) = 1\}.$$

Our goal is to make explicit the geometric structure of $\mathcal{S}$. First, $\text{End } W$ is a 4-dimensional complex vector space. The real points ($S^* = S$) lie in a 4-dimensional real subspace, depicted in [Figure 6](#). For any self-adjoint S there is an orthogonal decomposition $L \oplus L^\perp$ with respect to which the matrix of S is diagonal. Either S is a multiple of id_W or there are distinct real numbers $\lambda > \lambda^\perp$ for which

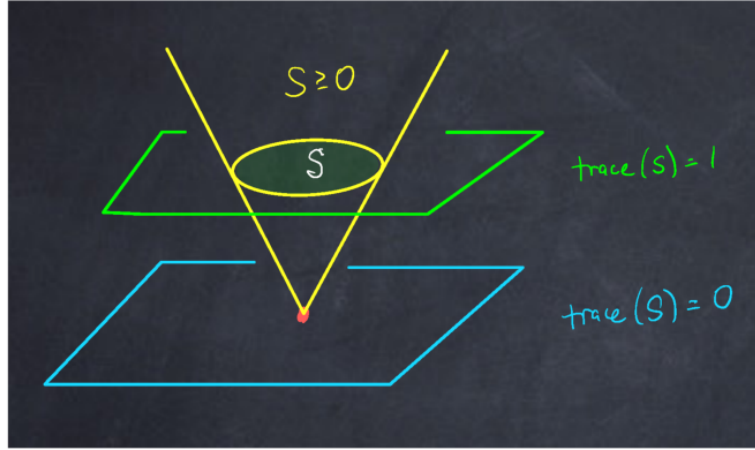


FIGURE 6. The 4-dimensional real space of self-adjoint operators, the cone of nonnegative operators, and the subspace of unit trace operators

S acts as multiplication by λ on L and multiplication by $\lambda^\perp$ on $L^\perp$. There is a closed convex cone of nonnegative self-adjoint operators, those for which $\lambda, \lambda^\perp \geq 0$; its interior is nonempty and consists of strictly positive operators ($\lambda, \lambda^\perp > 0$). The three-dimensional subspace of tracefree operators ($\lambda + \lambda^\perp = 0$) has an affine translate of trace one operators ($\lambda + \lambda^\perp = 1$). The intersection with nonnegative operators is (5.8). It is a convex set whose extreme points satisfy $\lambda = 1$ and $\lambda^\perp = 0$, in which case S is orthogonal projection onto the line $L \in \mathbb{P}W$. Identify $\mathcal{S}$ as a 3-dimensional disk (closed ball) in the 3-dimensional affine space of self-adjoint S with $\text{trace}(S) = 1$; see Figure 7. The center of the disk is the map $\frac{1}{2}\text{id}_W$, and the boundary 2-sphere is identified with the projective line $\mathbb{P}W$: a point L on the boundary 2-sphere is the self-adjoint projection onto the line $L \subset W$. Endow the affine space with a Euclidean metric so that the sphere has radius 1. Then the point at distance $0 < r \leq 1$ on the radius with endpoint L represents the self-adjoint operator with eigenspaces $L, L^\perp$ and eigenvalues $(1 \pm r)/2$.

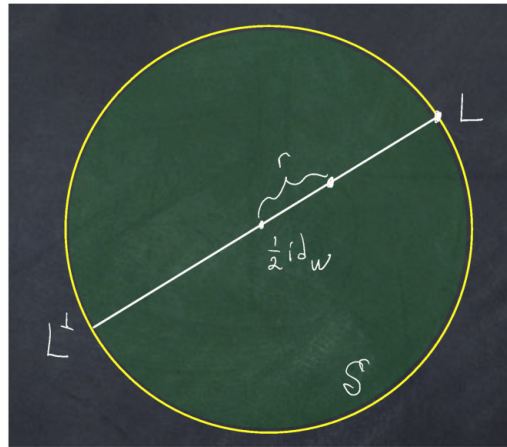


FIGURE 7. The space $\mathcal{S}$ of states as a 3-disk

Remark 5.9. It is easy to see in this example that the map $\text{Prob}(\mathcal{S}_0) \rightarrow \mathcal{S}$, which averages pure states to construct a state, while surjective is far from injective: any mixed state can be expressed in

uncountably infinitely many ways as a convex combination of pure states, even of two pure states. Contrast with the case of classical/statistical mechanics in which that map is an isomorphism.

The space $\mathcal{O}_{\mathbb{R}}$ of observables is the real 4-dimensional vector space of self-adjoint operators on W . There is a distinguished line: real multiples of id_W . For $\lambda \in \mathbb{R}$ the spectral measure of $\lambda \cdot \text{id}_W$ is depicted in Figure 8(i). It is supported at $\lambda \in \mathbb{R}$. The generic self-adjoint operator has distinct real eigenvalues $\lambda_1 < \lambda_2$ and there is an orthogonal decomposition $W = K_1 \oplus K_2$ as a sum of eigenlines; see Figure 8(ii) for its spectral measure.

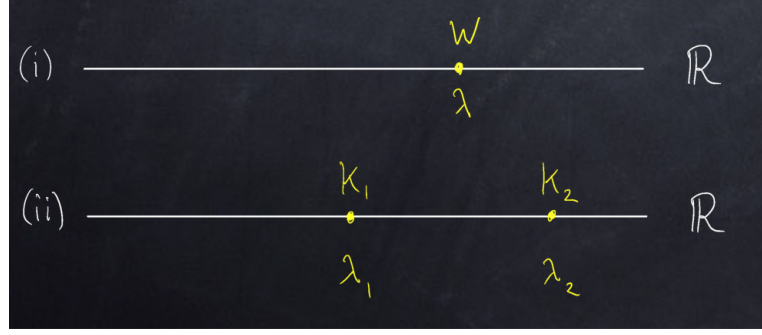


FIGURE 8. The spectral measures of (i) $\lambda \cdot \text{id}_W$ and (ii) a self-adjoint operator with eigenvalues $\lambda_1 < \lambda_2$

Next, we work out the real-valued probability measure obtained by pairing an observable and a state, as defined in (1.27) and (1.28). Begin with the pure state S that is projection onto a line $L \in \mathbb{P}W$. Consider first the observable $A = \lambda \cdot \text{id}_W$. Then the pairing works out to be the point probability measure on $\mathbb{R}$ supported at λ . In other words, the result of the pairing is λ with probability one: there is no uncertainty. For a self-adjoint operator A with spectral measure as in Figure 8(ii), the pairing is a probability measure supported on the 2-element set $\{\lambda_1, \lambda_2\}$. Let's work out the probabilities. Let $\psi \in L$ and $\xi_i \in K_i$ be unit norm vectors. Then orthogonal projection onto K_i is the linear operator

$$(5.10) \quad P_i(\psi) = \langle \xi_i, \psi \rangle \xi_i.$$

Note that (5.10) is independent of the choice of unit vector ξ_i . The probability at λ_i is

$$(5.11) \quad \langle \psi, P_i(\psi) \rangle = \langle \psi, \langle \xi_i, \psi \rangle \xi_i \rangle = \langle \xi_i, \psi \rangle \langle \psi, \xi_i \rangle = |\langle \psi, \xi_i \rangle|^2.$$

Observe that this is independent of the choice of unit vectors ψ, ξ_i . The expression for these probabilities is a fundamental piece of structure. Therefore, we introduce the function¹⁷

$$(5.12) \quad \begin{aligned} p: \mathbb{P}W \times \mathbb{P}W &\longrightarrow [0, 1] \\ L, K &\longmapsto |\langle \psi, \xi \rangle|^2 \end{aligned}$$

in which $\psi \in L, \xi \in K$ are unit vectors. The probability measures for the pairing of the operators in Figure 8 with the pure state $L \in \mathbb{P}W$ are depicted in Figure 9.

¹⁷The transition probability (5.12) extends to all states as the function $S_0, S_1 \mapsto \text{tr}(S_1 \circ S_0)$.

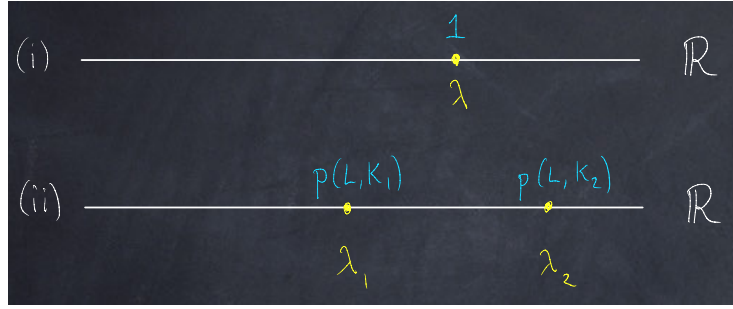


FIGURE 9. The probability measures obtained by measuring the observables of [Figure 8](#) in the pure state $L \in \mathbb{PW}$.

Remark 5.13.

- (1) Note that in [Figure 9\(ii\)](#) the result of a measurement in a pure state is a probability measure with positive variance. In other words, in quantum mechanics, as opposed to in classical mechanics, even in a pure state measurement is probabilistic. (In both theories we expect measurement to be probabilistic in mixed states.)
- (2) The function p encodes the *transition probability* between pure states. It is a basic structure in quantum mechanics, which we explore in [Lecture 8](#) in the general context of projective geometry. We will prove a generalization of the following formula for p in terms of the geodesic distance d on the 2-sphere of unit radius:

$$(5.14) \quad p = \cos^2(d/2).$$

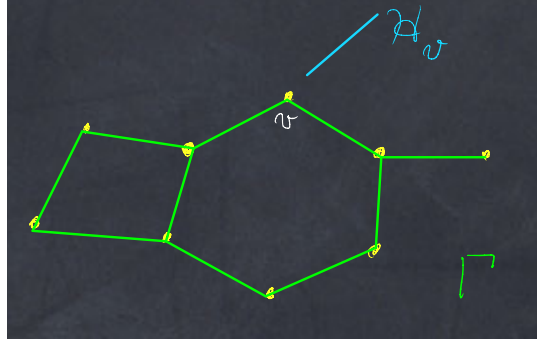
Try now to prove (5.14)—it is a fun problem in elementary geometry. You should also use it to verify that the sum of the weights in [Figure 9\(ii\)](#) is 1, a fact you can verify directly.

- (3) The entire discussion extends easily to mixed states: apply the first equation of (1.4).
- (4) There is a huge literature on measurement in quantum mechanics, and we simply make a brief comment. If the value of a measurement of an observable A in a pure state σ is a number in the point spectrum of σ_A , i.e., is an eigenvalue λ of A , then after measurement the state of the system projects to the pure state obtained by orthogonal projection of σ onto the λ -eigenspace. There is a similar rule when σ is a mixed state. The story is more complicated for values in the continuous spectrum; see [\[DP\]](#) and the references therein.

5.4 Lattice models

We give only a brief introduction to spin systems; see [\[Sa\]](#) for a more extensive introduction. These systems introduce a *discrete* notion of space while retaining continuous time. Systems of this type are a focus of theoretical condensed matter physics. There are also systems with discrete space and discrete time, so-called “stat-mech systems”.

Let Γ be a finite graph, as in [Figure 10](#). For each vertex $v \in \text{Vert}(\Gamma)$ suppose given a finite dimensional complex Hilbert space $\mathcal{H}_v$. (Alternatively, one can attach Hilbert spaces to the edges.)

FIGURE 10. A spin system on a finite graph Γ

Then define the state space of the model to be

$$(5.15) \quad \mathcal{H} = \bigotimes_{v \in \text{Vert}(\Gamma)} \mathcal{H}_v.$$

The observables are the real points in the tensor product

$$(5.16) \quad \mathcal{O} = \bigotimes_{v \in \text{Vert}(\Gamma)} \text{End } \mathcal{H}_v.$$

One can now speak of the support of an observable. For example, for a fixed vertex $v_0 \in \text{Vert}(\Gamma)$ observables of the form

$$(5.17) \quad A \otimes \bigotimes_{v \neq v_0} \text{id}_{\mathcal{H}_v}, \quad A \in \text{End } \mathcal{H}_{v_0},$$

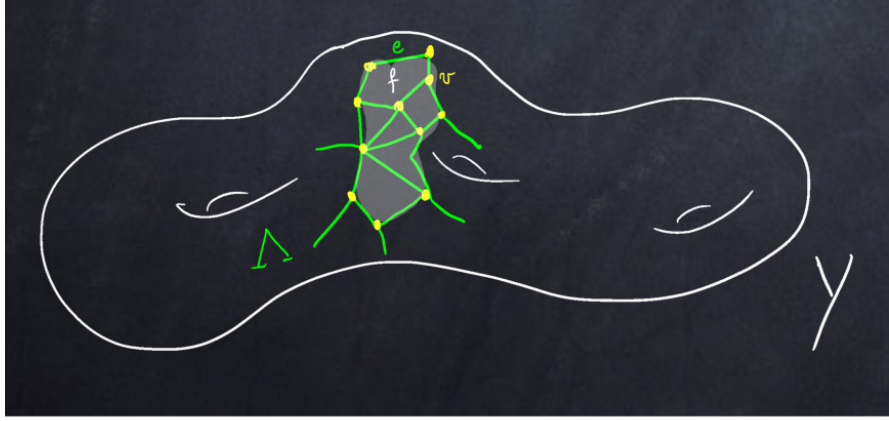
are said to be supported at v_0 . There is a similar definition of observables supported on any subset of $\text{Vert}(\Gamma)$. One imagines Γ to be very large, and then one has *local* observables that are supported on a small number of neighboring vertices. The Hamiltonian of such a system is taken to be a sum of local observables with uniformly small support.

Remark 5.18. There is no corresponding notion of the support of a state: there is no distinguished nonzero vector in $\mathcal{H}_v$, whereas in (5.17) we use the distinguished identity operator in $\text{End } \mathcal{H}_v$.

This is only a small hint of this large field of inquiry. Next, we introduce a beautiful geometric example of a system of this type.

5.5 Toric code

The toric code was introduced by Alexei Kitaev [Ki]. It can be formulated on any finite CW complex, in which case the vertices, edges, and faces in our description below are the 0-cells, 1-cells, and 2-cells of the CW structure. The model also has a generalization for any finite group G ; here G is the cyclic group of order 2.


 FIGURE 11. A closed surface Y with an embedded graph Λ

Let Y be a closed smooth manifold, which you can assume is 2-dimensional. Let $\Lambda \subset Y$ be a “nice” finite graph that is “nicely embedded” in Y , as depicted in [Figure 11](#). (Treating the model with CW structures is a way to make this precise.) Let $\Lambda^0 \subset \Lambda$ denote the set of vertices. Consider the finite set

$$(5.19) \quad \mathcal{D}(\Lambda, \Lambda^0) = \left\{ (\tilde{\Lambda} \longrightarrow \Lambda, s) : \tilde{\Lambda} \rightarrow \Lambda \text{ is a double cover, } s \text{ is a section of } \tilde{\Lambda}|_{\Lambda^0} \longrightarrow \Lambda^0 \right\} / \cong$$

of isomorphism classes of double covers of Λ equipped with a section over the vertices. Over each edge e the double cover is trivializable, and so by parallel transport we can compare the trivializations at the two endpoints: they agree (+1) or disagree (−1). This leads to an isomorphism

$$(5.20) \quad \mathcal{D}(\Lambda, \Lambda^0) \xrightarrow{\cong} \text{Map}(\text{Edge}(\Lambda), \{\pm 1\}).$$

The Hilbert space of the toric code is the complex vector space

$$(5.21) \quad \mathcal{H} = \text{Map}(\mathcal{D}(\Lambda, \Lambda^0), \mathbb{C})$$

of functions on the finite set [\(5.19\)](#). It carries the hermitian inner product which makes the basis of δ -functions orthonormal.

Introduce observables H_v, H_f associated to each vertex v and each face f . A vertex $v \in \Lambda^0$ determines an automorphism

$$(5.22) \quad \varphi_v : \mathcal{D}(\Lambda, \Lambda^0) \longrightarrow \mathcal{D}(\Lambda, \Lambda^0)$$

that flips the section at the vertex v . A face f has a boundary which is homeomorphic to a circle, and a double cover of a circle is either trivial (0) or nontrivial (1). Let

$$(5.23) \quad h_f : \mathcal{D}(\Lambda, \Lambda^0) \longrightarrow \{0, 1\}$$

be the function that returns the isomorphism type of the restriction of the double cover to ∂f . Set

$$(5.24) \quad \begin{aligned} H_v \psi &= \frac{1}{2}(\psi - \varphi_v^* \psi) \\ H_f \psi &= h_f \psi \end{aligned}$$

Each operator is self-adjoint and has spectrum $\{0, 1\}$, i.e. is a projection. The kernel of H_v consists of functions invariant under flip of the section at v , and the kernel of H_f consists of functions supported at double covers that extend over the face f . It is a nice exercise to verify that any two operators of the form (5.24) commute. Therefore, these operators are simultaneously diagonalizable.

The Hamiltonian of the toric code is the sum these of commuting projection operators:

$$(5.25) \quad H = \sum_v H_v + \sum_f H_f.$$

It follows that the spectrum of H is contained in the set $\mathbb{Z}^{\geq 0} \subset \mathbb{R}$ of nonnegative integers. Furthermore, the kernel consists of functions supported on double covers that extend over Y . More precisely, in the diagram

$$(5.26) \quad \mathcal{D}(\Lambda, \Lambda^0) \xrightarrow{\pi} \mathcal{D}(\Lambda) \xleftarrow{r} \mathcal{D}(Y)$$

the map π forgets the section s , and r restricts a double cover over Y to $\Lambda \subset Y$. (These maps pass to sets $\mathcal{D}(-)$ of isomorphism classes of double covers.) The map r is injective. Define an embedding

$$(5.27) \quad \text{Map}(\mathcal{D}(Y), \mathbb{C}) \hookrightarrow \mathcal{H}$$

that takes a function on $\mathcal{D}(Y)$, extends it by zero to a function on $\mathcal{D}(\Lambda)$, and then pulls back via π to a function on $\mathcal{D}(\Lambda, \Lambda^0)$. The descriptions of $\ker H_v$ and $\ker H_f$ show that the image of (5.27) is the vacuum space $\ker H$.

Remark 5.28.

- (1) The vacuum space is independent of the graph $\Lambda \subset Y$. Also, it has *topological* significance: it can be identified with functions on the finite set $H^1(Y; \{\pm 1\})$ of isomorphism classes of double covers of Y . We will see other examples of systems—quantum mechanical and quantum field theoretic—whose vacuum “sector” can be described in topological terms.
- (2) To say a lattice system is gapped, one wants a local formulation of a family of systems over a family of $\Lambda \subset Y$, and the systems should have a *uniform* spectral gap independent of Λ . This data and property are present here: the toric code is defined for all suitable $\Lambda \subset Y$, and the uniform spectral gap is 1 since the spectrum for any Λ is contained in $\mathbb{Z}^{\geq 0} \subset \mathbb{R}$.

Lecture 6: Infinite dimensional quantum mechanical systems

In this lecture we discuss examples of quantum mechanical systems with an infinite dimensional state space $\mathcal{H}$. We begin with a particle on a Riemannian manifold with zero potential energy. We then move to the special case of a particle in Euclidean space with a quadratic potential energy: the *harmonic oscillator*. We do this in several forms, at one point emphasizing the essentially algebraic nature of the system, and then telling the crucial interpretation as a representation of the *Heisenberg group*. The corresponding Lie algebra representation is a realization of the *canonical commutation relations*. We state some general theorems about the one-dimensional Schrödinger equation, but we leave proofs to the homework. We conclude with some heuristics about the double well potential and tunneling.

6.1 Particle on a Riemannian manifold

Let (M, g) be a Riemannian manifold. There is an associated quantum mechanical system $(\mathcal{H}, U_t)$ in which $\mathcal{H} = L^2(M; \mathbb{C})$ is the Hilbert space of L^2 complex functions on M and U_t is the strongly continuous 1-parameter group of unitaries on $\mathcal{H}$ generated by the Laplace operator $H = \Delta$, where $\Delta f = d^*df$ (on the dense subspace of smooth functions $f \in \Omega_M^0(\mathbb{C})$). This is a system of positive energy (Definition 5.7(1)): $\text{spec } H \subset \mathbb{R}^{\geq 0}$. This follows since Δ is a nonnegative operator: for any smooth function f on M with compact support, integration by parts shows

$$(6.1) \quad \langle f, \Delta f \rangle = \langle f, d^*df \rangle = \langle df, df \rangle \geq 0,$$

where $\langle -, - \rangle$ is the L^2 hermitian inner product

$$(6.2) \quad \langle f, g \rangle = \int_M \bar{f}g \, d\mu_g.$$

(Here $d\mu_g$ is the Riemannian measure on M .) If M has components of positive dimension, then the spectrum of H is unbounded above: Δ is an unbounded self-adjoint operator.

The corresponding classical Hamiltonian system is described in Remark 3.10(3). In that case—assuming M has components of positive dimension—the spectrum of the classical Hamiltonian function, which is the closure of its image in $\mathbb{R}$ —equals $\mathbb{R}^{\geq 0}$. This is not always the case for the quantum theory.

Remark 6.3. One should not skip lightly over the problematic phrase ‘corresponding classical Hamiltonian system’. There is no easy correspondence between classical and quantum systems.

If M is closed (compact without boundary) and positive dimensional, then the quantum Hamiltonian $H = \Delta$ has pure point spectrum: its spectrum is a countable sequence $0 < \lambda_1 < \lambda_2 < \dots$ of nonnegative real eigenvalues, each with finite multiplicity. The discreteness of the spectrum and the finiteness of the multiplicities is a basic result in the theory of elliptic partial differential equations. Hence this is a gapped system (Definition 5.7(2)). Furthermore, (6.1) proves that for $f \in \Omega_M^0$ we have $\Delta f = 0$ iff $df = 0$ iff f is a locally constant function. Therefore, the vacuum space $\ker H$ is finite dimensional and consists of locally constant functions, so it has topological significance:

it is the space of functions on the finite set $\pi_0 M$ of (path) components of M . A generalization for $q \in \mathbb{Z}^{\geq 0}$: let $\mathcal{H}$ be the Hilbert space of complex differential q -forms on M , and let H be the Hodge Laplace operator $\Delta = dd^* + d^*d$. The spectrum is pure point, as for $q = 0$, and the Hodge theorem tells that the vacuum space $\ker H$ of complex harmonic q -forms is isomorphic to the complex cohomology group $H^q(M; \mathbb{C})$. Notice that this is independent of the Riemannian metric, and as in [Remark 5.28\(1\)](#) the vacuum space has topological significance. This is an important feature of many gapped systems.

Remark 6.4. On a compact manifold, then, the classical spectrum is continuous whereas the quantum spectrum is discrete. This is one meaning of ‘quantization’: the quantization of energy.

If M is noncompact, then the Laplace operator may have continuous spectrum. For example, if $M = \mathbb{E}^d$ is standard Euclidean space then the spectrum is $\mathbb{R}^{\geq 0}$ and it is all continuous: there are no eigenvalues. There are certainly harmonic functions on $\mathbb{E}^d$ —constant functions in any dimension d , polynomial functions such as $(x^1)^2 - (x^2)^2$, say, in dimension $d = 2$ on $\mathbb{E}_{(x^1, x^2)}^2$ —but no nonzero harmonic function lies in L^2 .

We can allow a potential energy function $V: M \rightarrow \mathbb{R}$ in each of these systems. Even on Euclidean space a potential function with suitable growth leads to a gapped quantum system with pure point spectrum, as we illustrate next.

6.2 Harmonic oscillator I: simple harmonic oscillator on the Euclidean line

We treat a unit mass particle moving on $\mathbb{E}_x^1$ with potential function

$$(6.5) \quad V(x) = \frac{1}{2}x^2.$$

Consider first the *classical* particle moving in this potential. The classical trajectories $x: \mathbb{R} \rightarrow \mathbb{E}^1$ obey Newton’s law [\(1.16\)](#), which works out to be

$$(6.6) \quad \ddot{x} = -x.$$

The classical trajectories are oscillatory and form a real affine space N , isomorphic to $\mathbb{E}^1 \times \mathbb{R}^1$ by fixing a time t_0 :

$$(6.7) \quad \begin{aligned} N &\longrightarrow \mathbb{A}^2 \\ x &\longmapsto (x(t_0), \dot{x}(t_0)) \end{aligned}$$

The Hamiltonian on N is the function

$$(6.8) \quad H = \frac{1}{2}(\dot{x}^2 + x^2),$$

which is evaluated at any time t_0 ; on a classical trajectory it is independent of t_0 . The classical “spectrum” of the Hamiltonian function H , the closure of its range, is $\mathbb{R}^{\geq 0}$. The trajectories are illustrated in [Figure 12](#).

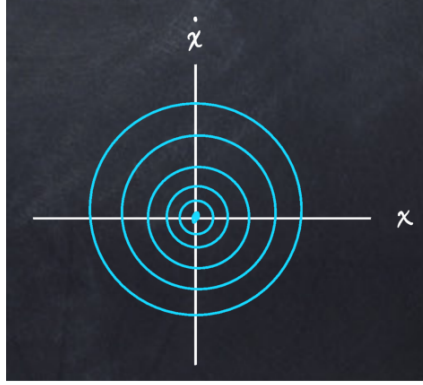


FIGURE 12. Classical trajectories of a harmonic oscillator

We should, of course, include constants to get the units correct. Let $m \in \mathbb{R}^{>0}$ have units of (mass) and $k \in \mathbb{R}^{>0}$ have units of (mass)/(time)². Then the classical Hamiltonian

$$(6.9) \quad H = \frac{1}{2}(m\dot{x}^2 + kx^2),$$

has units of energy. We leave the reader to put these constants into the following treatment of the corresponding quantum system.

Let $\mathcal{H} = L^2(\mathbb{E}^1; \mathbb{C})$ and define the self-adjoint Hamiltonian operator

$$(6.10) \quad H = \frac{1}{2} \left(-\frac{d^2}{dx^2} + x^2 \right).$$

This is an unbounded operator, and it is nonnegative: the quantum harmonic oscillator has positive energy. Note that H is the sum of two unbounded nonnegative operators, each of which has continuous spectrum $[0, \infty)$. We now show that the sum H has pure point spectrum. In particular, the quantum harmonic oscillator is a gapped system.

Set

$$(6.11) \quad \begin{aligned} A &= \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + x \right) \\ A^* &= \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + x \right) \end{aligned}$$

Then A^* is the (formal) adjoint of A and the commutator is $[A, A^*] = 1$ (meaning $\text{id}_{\mathcal{H}}$). Also,

$$(6.12) \quad H = A^*A + \frac{1}{2},$$

which implies that $\text{spec}(H) \subset [\frac{1}{2}, \infty)$. We claim there exists a unique line $\mathbb{C} \cdot \Omega \subset \mathcal{H}$ such that $A\Omega = 0$. For if Ω is represented by a function $f: \mathbb{E}^1 \rightarrow \mathbb{C}$, then

$$(6.13) \quad \frac{df}{dx} + xf(x) = 0,$$

from which

$$(6.14) \quad f(x) = Ce^{-x^2/2}$$

for $C \in \mathbb{C}$. It follows from (6.12) that Ω is an eigenfunction of H with eigenvalue $1/2$. Furthermore, any eigenfunction of H with eigenvalue $1/2$ is annihilated by A ; argue analogously to (6.1). Compute the Hamiltonian on $A^*\Omega$:

$$(6.15) \quad \begin{aligned} HA^*\Omega &= \frac{1}{2}A^*AA^*\Omega + \frac{1}{2}A^*\Omega \\ &= \frac{1}{2}A^*A^*A\Omega + A^*\Omega + \frac{1}{2}A^*\Omega \\ &= \frac{3}{2}A^*\Omega, \end{aligned}$$

so $A^*\Omega$ is an eigenfunction with eigenvalue $3/2$. Also

$$(6.16) \quad \langle A^*\Omega, \Omega \rangle = \langle \Omega, A\Omega \rangle = 0,$$

thus $\mathbb{C}\langle A^*\Omega \rangle$ is orthogonal to $\mathbb{C}\langle \Omega \rangle$, as it must be. Continue by induction to prove that for $n \geq 0$ the vector $(A^*)^n\Omega$ is an eigenfunction of H with eigenvalue $\frac{1}{2} + n$. One can prove that $\{(A^*)^n\Omega\}_{n \in \mathbb{Z}_{\geq 0}}$ is an orthonormal basis of $\mathcal{H} = L^2(\mathbb{E}^1; \mathbb{C})$. Hence the spectrum of the Hamiltonian is $\text{spec}(H) = \frac{1}{2} + \mathbb{Z}_{\geq 0}$ (Figure 13) and every eigenvalue is simple: the eigenspaces are one-dimensional.

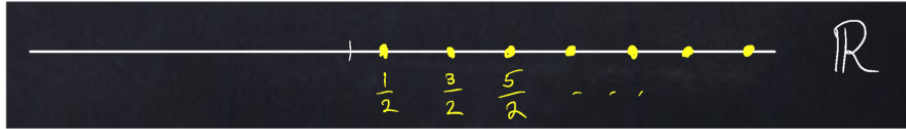


FIGURE 13. The spectrum of the harmonic oscillator Hamiltonian

6.3 Harmonic oscillator II: algebraic treatment

Now we consider what amounts to a finite number of simple harmonic oscillators, but we consider the collection as a whole—no decomposition into simple oscillators—which amounts to not choosing a basis for the vector space W below. Our treatment focuses on operators rather than states, and we give a purely algebraic description, which applies quite generally, for example to vector spaces over arbitrary fields.

Let W be a finite dimensional real vector space. Set

$$(6.17) \quad F(W) = \text{Sym}^\bullet W_{\mathbb{C}}^*,$$

where $W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of W . This symmetric algebra, which is $\mathbb{Z}$ -graded, is the *Fock space* of W . It is the algebra of complex-valued polynomials on W . For $w \in W$ and $\ell \in W^*$ define linear operators A_w and M_ℓ of degrees -1 and $+1$, respectively, as follows:

$$(6.18) \quad \begin{aligned} A_w(\theta) &= \theta(w), & \theta &\in \text{Sym}^1 W_{\mathbb{C}}^* \\ M_\ell(p) &= \ell \cdot p, & p &\in F(W), \end{aligned}$$

and extend A_w to be a derivation: $A_w(p_1 \cdot p_2) = A_w(p_1) \cdot p_2 + p_1 \cdot A_w(p_2)$ for all $p_1, p_2 \in F(W)$. Then compute the commutator

$$(6.19) \quad [A_w, M_\ell] = \ell(w)$$

to be scalar multiplication by $\ell(w)$. Other commutators vanish: for $w_1, w_2 \in W$, $\ell_1, \ell_2 \in W^*$,

$$(6.20) \quad [A_{w_1}, A_{w_2}] = [M_{\ell_1}, M_{\ell_2}] = 0.$$

Equations (6.19) and (6.20) are the *canonical commutation relations*.

Remark 6.21.

- (1) The real vector space of operators A_w, M_ℓ is the quotient in a central extension

$$(6.22) \quad 0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{h} \longrightarrow W \oplus W^* \longrightarrow 0$$

of real Lie algebras. $\mathfrak{h}$ is called a *Heisenberg algebra*. The commutator (6.19) induces a symplectic structure on $W \oplus W^*$. In the next section we replace $W \oplus W^*$ with finite dimensional real symplectic vector space. Here the symplectic vector space is *polarized*.

- (2) There are operators A_w, M_ℓ for w, ℓ in the complexifications $W_{\mathbb{C}}, W_{\mathbb{C}}^*$, and they generate an infinite dimensional complex algebra: the *Weyl algebra*.
- (3) Observe that nonzero polynomials on W , which are the nonzero vectors in the Fock space $F(W)$, cannot be L^2 with respect to any measure, since they do not tend to zero at infinity.

There is a canonical element $C \in W^* \otimes W$. It has the form $C = w^i \otimes w_i$ for any basis $w_1, \dots, w_N$ of W and dual basis $w^1, \dots, w^N$ of W^* . It corresponds to the degree 0 linear operator

$$(6.23) \quad \hat{C} = \sum_i M_{w^i} \circ A_{w_i}$$

on $F(W)$. The harmonic oscillator Hamiltonian is $H = \hat{C} + 1/2$.

Remark 6.24.

- (1) The ordering in (6.23) is important since the commutator (6.19) is nonzero.
- (2) If W has a real inner product, then there is an induced hermitian inner product on $F(W)$ and one can complete it to a Hilbert space. We have not done so here, in part to emphasize the algebraic nature of the harmonic oscillator.

6.4 Harmonic oscillator III: Heisenberg group

Suppose (V, ω) is a finite dimensional real symplectic vector space. The *Heisenberg Lie algebra*

$$(6.25) \quad \mathfrak{h}(V) = V \oplus \mathbb{R}$$

is the direct sum in which $\mathbb{R}$ is central and $[v_1, v_2] = \omega(v_1, v_2) \in \mathbb{R}$ for $v_1, v_2 \in V$. This is a *2-step nilpotent Lie algebra*:

$$(6.26) \quad [v_1, [v_2, v_3]] = 0, \quad v_1, v_2, v_3 \in V.$$

A more geometric picture emerges for a general affine symplectic space A over V . Then the *Heisenberg Lie algebra* $\mathfrak{h}(A)$ is defined from (2.31) by restriction to constant vector fields:

$$(6.27) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega_A^0 & \longrightarrow & \mathcal{X}_A^\omega \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathfrak{h}(A) & \longrightarrow & V \longrightarrow 0 \end{array}$$

In the first line we use $H^0(A; \mathbb{R}) = \mathbb{R}$ and $H^1(A; \mathbb{R}) = 0$. The subspace $\mathfrak{h}(A) \subset \Omega_A^0$ is the Lie algebra of affine functions on A : the Lie bracket is the Poisson bracket.

We can exponentiate $\mathfrak{h}(A)$ to obtain a Lie group $H(A)$, the *Heisenberg group*. In doing so replace the center $\mathbb{R}$ of (6.27) with $\sqrt{-1}\mathbb{R}$, and so upon exponentiation we obtain $\mathbb{T} \subset \mathbb{C}$ in place of $\mathbb{R}^{>0} \subset \mathbb{R}$. The Heisenberg group is a *group extension*¹⁸

$$(6.28) \quad 1 \longrightarrow \mathbb{T} \longrightarrow H(A) \longrightarrow \exp V \longrightarrow 1$$

The main result about harmonic oscillators is the following.

Theorem 6.29 (Stone-von Neumann). *There exists a unique isomorphism class of irreducible unitary representations of $H(A)$ on which $\mathbb{T} \subset H(A)$ acts as scalar multiplication.*

In §6.3 we indicated how to construct an algebraic representation of a *polarized* Heisenberg Lie algebra. Recall that there is an increasing filtration on the underlying vector space. One can introduce inner products so that the exponentiated representation is unitary, and then one can complete to a Hilbert space representation. Then one must prove that the representation is independent of the polarization, up to isomorphism. There is a rich story associated to this Heisenberg representation; see [LV, Fo1] for example.

Remark 6.30.

- (1) This unitary representation does *not* contain the 1-parameter group U_t needed to define a QM system. In fact, the Heisenberg representation extends to a larger group that includes the *metaplectic group*, and it is in this larger unitary representation that one finds U_t . This too is a rich story recounted in many places, among them [GV, Sel1].
- (2) There is a generalization [Sh] of Theorem 6.29 for infinite dimensional V , but now a new geometric structure is required: a *polarization* (class) on V . See [Sel1] for an exposition of this result.

¹⁸We discuss group extensions more in §8.7.4.

6.5 One-dimensional Schrödinger operators

An important general class of QM systems is a single particle of mass $m \in \mathbb{R}^{>0}$ on $\mathbb{E}^1$ with some potential energy $V: \mathbb{E}^1 \rightarrow \mathbb{A}$, which we assume is a smooth function. The Hamiltonian is

$$(6.31) \quad H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x).$$

An eigenfunction of H with eigenvalue $E \in \mathbb{R}$ is an L^2 function $\psi: \mathbb{E}^1 \rightarrow \mathbb{C}$ that satisfies the *Schrödinger equation*

$$(6.32) \quad -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + (V(x) - E)\psi(x) = 0.$$

A nonzero eigenfunction represents a pure state called a *bound state*. Depending on V there may be no bound states, the bound states may span (after closure) the Hilbert space $\mathcal{H} = L^2(\mathbb{E}^1; \mathbb{C})$, or they may span a proper subspace. For $V = 0$ the first occurs; for $V = x^2/2$ the second occurs. Schrödinger equations have been the object of much mathematical study.

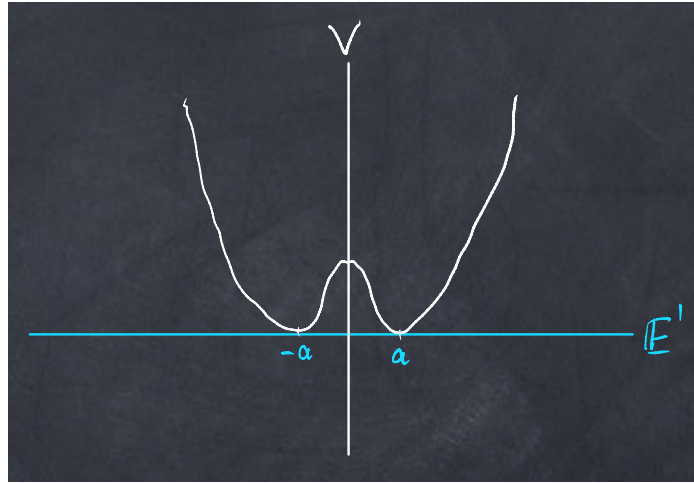


FIGURE 14. Double well potential

6.6 The double well potential

This is heuristic; rigorous and even exact solutions for particular potentials can be found in the literature. Here $V: \mathbb{E}^1 \rightarrow \mathbb{R}$ has two distinct minima and tends to $+\infty$ as the argument tends to $\pm\infty$, as depicted in [Figure 14](#). For example, we can take the quartic $V(x) = (x^2 - a^2)^2$ for some $a \in \mathbb{R}^{>0}$. The minimal energy classical trajectories $x: \mathbb{R} \rightarrow \mathbb{E}^1$ are stationary sitting at a minima of V , so there are two: $x = \pm a$. We might therefore anticipate two quantum vacua, each localized at a minimum. If we approximate V near a minimum by a quadratic potential, then the resulting harmonic oscillator has a unique vacuum ([6.14](#)). These two smooth L^2 functions are only *approximate* vacua for the double well potential. In fact, the exact vacuum has energy less than the energy of each approximate vacuum, and there is a second eigenvalue that is slightly more. A detailed treatment of this problem leads to instantons, tunneling, and many more fun topics; see [\[C\]](#) for an entrée.

Lecture 7: More QM systems; Klein’s *Erlangen Program*

In the first part of this lecture we consider two more QM systems. Each of them comes in a parametrized family. The first depends on a real-valued function on the Euclidean line; the second depends on a number $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. The first is a family of systems discussed by Hori [HKKPTVVZ, Chapter 10]; it is also described in the lecture notes [To]. There are approximate vacua from quadratic (harmonic oscillator) approximations, and—at first glance surprisingly—one can solve explicitly for the exact vacua. This is due to a hidden symmetry—a *supersymmetry*—that we do not discuss in this lecture, though we do derive some consequences. We work on the Euclidean line; the entire discussion has a beautiful generalization to Riemannian manifolds, as explored in Witten’s influential paper [W6]. The second family of QM systems is the instructive *particle on a ring*; see [GKKS, Appendix D]. We focus on a symmetry group, which we show only acts *projectively*. This is motivation for Lecture 8, in which we explore the projective nature of quantum mechanics.

As preparation for that discussion, the second part of this lecture is a modern interpretation of Felix Klein’s *Erlangen Program* [K, BB]. We define a “type” of geometry as a model space F together with the action of a group G . The pair $G \curvearrowright F$ is a specification; an instantiation is a right G -torsor, from which we construct an instance of the geometry using the *mixing* or *sandwich* construction. This can also be carried out in families. The lecture concludes with some mathematical background on torsors, etc.

7.1 A supersymmetric QM system

Although we use ‘supersymmetric’ in the section title, we do not elaborate on that term now.

The Hilbert space of the system is a space of L^2 vector-valued functions on the Euclidean line:

$$(7.1) \quad \mathcal{H} = L^2(\mathbb{E}^1; \mathbb{C}^2) = \left\{ \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} : \psi^i \in L^2(\mathbb{E}^1; \mathbb{C}) \right\} = \mathcal{H}^0 \oplus \mathcal{H}^1$$

Vectors in $\mathcal{H}^0$ have the form $\begin{pmatrix} \psi^0 \\ 0 \end{pmatrix}$ and are called *even*; vectors in $\mathcal{H}^1$ have the form $\begin{pmatrix} 0 \\ \psi^1 \end{pmatrix}$ and are called *odd*. Let $h: \mathbb{E}^1 \rightarrow \mathbb{R}$ be a smooth function. The (unbounded) Hamiltonian operator is

$$(7.2) \quad H = H_h = \frac{1}{2} \left(-\frac{d^2}{dx^2} + h'(x)^2 \right) \otimes \begin{pmatrix} +1 & \\ & +1 \end{pmatrix} - \underbrace{\frac{1}{2} h''(x) \otimes \begin{pmatrix} +1 & \\ & -1 \end{pmatrix}}_T$$

The first term is the standard Schrödinger operator (6.31), with $\hbar = 1$, $m = 1$, and potential energy

$$(7.3) \quad V = \frac{1}{2}(h')^2.$$

Observe that $V \geq 0$ and $V(x) = 0$ iff $x \in \text{Crit}(h)$: the potential vanishes exactly at critical points of h . The key to the system is the second term T , which is also algebraic—no differentiation. We view (7.2) as a family $(\mathcal{H}, H_h)$ of QM systems parametrized by a function $h: \mathbb{E}^1 \rightarrow \mathbb{R}$.

7.1.1 Semiclassical vacua. We first ask about vacua of (7.2), that is, pure states of minimal energy. If $x = x_0$ is an isolated critical point of h , then we can approximate h near $x = x_0$ by a quadratic function. Then V is also approximated by a quadratic function. The first term of (7.2) is then approximated by a harmonic oscillator with minimal energy $|h''(x_0)|/2$, and there are approximate eigenfunctions in both $\mathcal{H}^0$ and $\mathcal{H}^1$. The second term T now alters the minimal energy to 0, and the corresponding eigenfunction is even if $h''(x_0) > 0$ and is odd if $h''(x_0) < 0$. (Assume *nondegeneracy* of the critical point: $h''(x_0) \neq 0$.) Therefore, the prediction from a quadratic approximation is that, if each critical point is nondegenerate, then

- (i) the set of (pure) vacuum states is isomorphic to $\text{Crit}(h)$, and
- (ii) the vacuum at $x_0 \in \text{Crit}(h)$ lies in $\mathbb{P}\mathcal{H}^0$ if $h''(x_0) > 0$; it lies in $\mathbb{P}\mathcal{H}^1$ if $h''(x_0) < 0$.

We will see that these semiclassical predictions are usually *false* in the full theory.

7.1.2 Quantum vacua and first order operators. We analyze the quantum theory using the following observation. Introduce the (formally) adjoint¹⁹ first-order differential operators

$$(7.4) \quad \begin{aligned} Q &= \left(-i\frac{d}{dx} - ih'(x)\right) \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ Q^* &= \left(-i\frac{d}{dx} + ih'(x)\right) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Observe that we can view these as differential operators

$$(7.5) \quad \mathcal{H}^0 \begin{matrix} \xrightarrow{Q} \\ \xleftarrow{Q^*} \end{matrix} \mathcal{H}^1$$

Then an easy computation shows that

$$(7.6) \quad H = \frac{1}{2}(QQ^* + Q^*Q).$$

Lemma 7.7. $H \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} = 0$ iff $Q\psi^0 = 0$ and $Q\psi^1 = 0$.

Proof. The implication $\Leftarrow$ is clear from (7.6). In the converse direction, if $\Psi = \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix}$, then

$$(7.8) \quad \langle \Psi, H\Psi \rangle = \frac{1}{2}\|Q\psi^0\|^2 + \frac{1}{2}\|Q^*\psi^1\|^2,$$

and the implication $\Rightarrow$ follows. (Compare (6.1).) □

¹⁹We have used usual conventions for adjoints. When we view these as *odd* in the world of $\mathbb{Z}/2\mathbb{Z}$ -graded algebra, then there are additional factors of i and signs to made the adjoints conform to the Koszul sign rule.

Of course, we are assuming that $\Psi \in \mathcal{H}$ is L^2 , which is used in the integration by parts implicit in (7.8).

The first-order ordinary differential equations $Q\psi^0 = 0$ and $Q^*\psi^1 = 0$ have solutions

$$(7.9) \quad \begin{aligned} \psi^0 &= e^{-h} \\ \psi^1 &= e^h \end{aligned}$$

and multiples thereof. So there are at most two vacua. However, these functions are not necessarily L^2 , and in fact they cannot be simultaneously L^2 . So there is at most one vacuum, and it lies in either $\mathbb{P}\mathcal{H}^0$ or in $\mathbb{P}\mathcal{H}^1$.

Remark 7.10. There is a variation of this system in which we replace $\mathbb{E}^1$ by the Euclidean circle $\mathbb{E}^1/2\pi\mathbb{Z}$. In this case the function h is periodic, as are the functions (7.9). Then there are two vacua for every choice of h : one vacuum lies in $\mathcal{H}^0$ and the other lies in $\mathcal{H}^1$.

We make one more observation about the system (7.5) and the Hamiltonian (7.6). First, observe

$$(7.11) \quad H = \begin{cases} \frac{1}{2}Q^*Q, & \text{on } \mathcal{H}^0; \\ \frac{1}{2}QQ^*, & \text{on } \mathcal{H}^1. \end{cases}$$

Lemma 7.12. *For $\lambda \in \mathbb{R}^{>0}$ let $\mathcal{H}_\lambda^i$, $i = 0, 1$, be the eigenspace of H with eigenvalue λ . Then*

$$(7.13) \quad Q: \mathcal{H}_\lambda^0 \longrightarrow \mathcal{H}_\lambda^1$$

is an isomorphism.

Proof. The inverse is $Q^*/2\lambda$. □

Thus eigenspaces with *positive* eigenvalues are “paired up”. However, as we have already seen, the zero eigenspaces may have different dimensions.

We now illustrate various phenomena that occur for different h .

7.1.3 Quadratic h : the harmonic oscillator. Consider the one parameter family

$$(7.14) \quad h_s(x) = \frac{1}{2}sx^2, \quad s \in \mathbb{R}.$$

Then

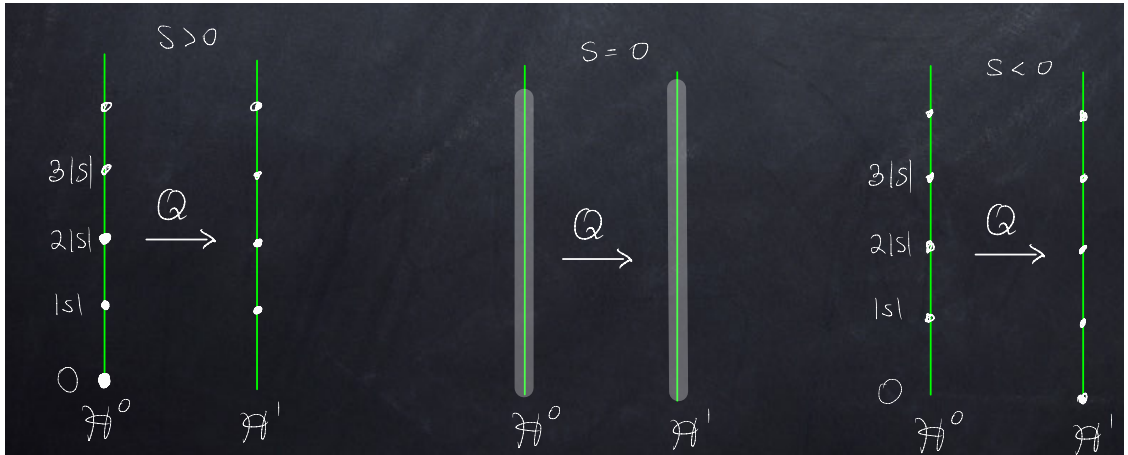
$$(7.15) \quad V_s(x) = \frac{1}{2}s^2x^2,$$

and the system is a harmonic oscillator if $s \neq 0$. The functions h_s, V_s are depicted in [Figure 15](#) for $s > 0$. The second term in the Hamiltonian (7.2) is the constant matrix

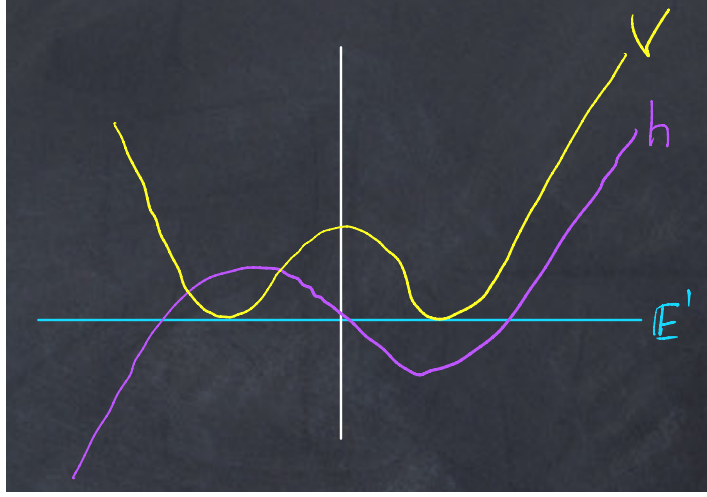
$$(7.16) \quad T = \begin{pmatrix} -s/2 & \\ & s/2 \end{pmatrix}$$

FIGURE 15. The harmonic oscillator ($s > 0$)

From our analysis (§6.2) of the harmonic oscillator, one can easily deduce the spectrum of H . It is plotted in Figure 16 for $s > 0$, $s = 0$, and $s < 0$. Recall the fundamental bifurcation between gapped and gapless systems. Here H_s , $s \neq 0$ is gapped and H_0 is gapless. That is reflected in Figure 16: the spectrum is continuous down to 0 for $s = 0$, and there is a spectral gap at 0 for $s \neq 0$. (In fact, the entire spectrum is discrete for $s \neq 0$.) Finally, observe that in this case the semiclassical prediction of vacua (§7.1.1) is true, as expected since the semiclassical approximation is exact.

FIGURE 16. The spectrum of H as s varies

Remark 7.17. At first glance the following is strange: as $s \in \mathbb{R}$ increases through 0 from negative values to positive values, the unique vacuum for $s \neq 0$ jumps from $\mathbb{P}\mathcal{H}^0$ to $\mathbb{P}\mathcal{H}^1$. One might think it is due to the *continuous* spectrum down to 0 at $s = 0$. In fact, there are other examples in which it is only *discrete* spectrum that comes down to 0. These are examples of *phase transitions* at $s = 0$. One passes from a connected component of *gapped* systems to another connected component of gapped systems through a codimension one “wall” in the parameter space, and on that wall the gap in the spectrum is breached.

FIGURE 17. A cubic function h and the corresponding potential energy V

7.1.4 Cubic h . Next, consider a typical cubic function h , as depicted in Figure 17. Notice that neither function in (7.9) lies in L^2 , so there are no zero energy states. The potential $V(x)$ still tends to $+\infty$ as $|x| \rightarrow \infty$; more precisely $V(x) = O(x^4)$ as $|x| \rightarrow \infty$. It follows (not by arguments we give here) that the spectrum of H is discrete, and as we have argued all eigenvalues are positive. In this case $Q: \mathcal{H}^0 \rightarrow \mathcal{H}^1$ is an isomorphism. Observe that the semiclassical picture predicts two vacua, one in $\mathbb{P}\mathcal{H}^0$ and one in $\mathbb{P}\mathcal{H}^1$. This is in a sense correct, but the equal energies of these vacua is strictly positive. Also, we may have thought that the quartic potential V would exhibit tunneling between the two minimal energy states, as described in §6.6. However, these states lie in different projective Hilbert spaces and there is no tunneling between them.

FIGURE 18. A quartic function h and the corresponding potential energy V

7.1.5 Quartic h . We use the quartic depicted in Figure 18: $h(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. It follows that ψ^0 in (7.9) lies in L^2 , and so there is a unique vacuum in $\mathbb{P}\mathcal{H}^0$. On the other hand, the semiclassical analysis predicts three vacua: two in $\mathbb{P}\mathcal{H}^0$ and one in $\mathbb{P}\mathcal{H}^1$. In this case there is tunneling between

the two semiclassical vacua in $\mathcal{H}^0$ to produce a single true quantum vacuum; the energy of the minimal energy state in $\mathbb{P}\mathcal{H}^1$ is strictly positive and matches the energy of the positive energy state from the splitting of the two semiclassical vacua.

It is illuminating to consider what happens in a family of quartics. Consider a family h_s , $s \in \mathbb{R}$, in which

- h is a quartic polynomial for $x < 0$,
- h is a quadratic polynomial for $x > 0$, and
- $h(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$.

In this family there is a unique vacuum in $\mathbb{P}\mathcal{H}^0$ for *all* $s \in \mathbb{R}$. But think about semiclassical vacua and can convince yourself—at least heuristically—that as s passes from negative to positive values, there are some eigenvalues that come close to 0. In fact, one can split $\mathcal{H}^i = \mathcal{H}_\downarrow^i \oplus \mathcal{H}_\uparrow^i$ into a direct sum of a finite dimensional space of “low lying eigenvectors” and an infinite dimensional space on which the Hamiltonian H is bounded below. Such spectral splittings are useful in the geometric study of index theory, for example.

7.2 A quick word about symmetry

Suppose $(\mathcal{H}, U_t)$ is the data of a QM system. One might define a Lie group G of symmetries to be a unitary representation $\rho: G \rightarrow \mathrm{U}(\mathcal{H})$ such that

$$(7.18) \quad \rho(g) U_t = U_t \rho(g), \quad g \in G, \quad t \in \mathbb{R}.$$

These are symmetries that preserve the direction of time translation, and that is what we need for this lecture.

Remark 7.19. One should also allow *time-orientation reversing* symmetries, as well as symmetries that act antilinearly. We will discuss these in [Section 11.1](#).

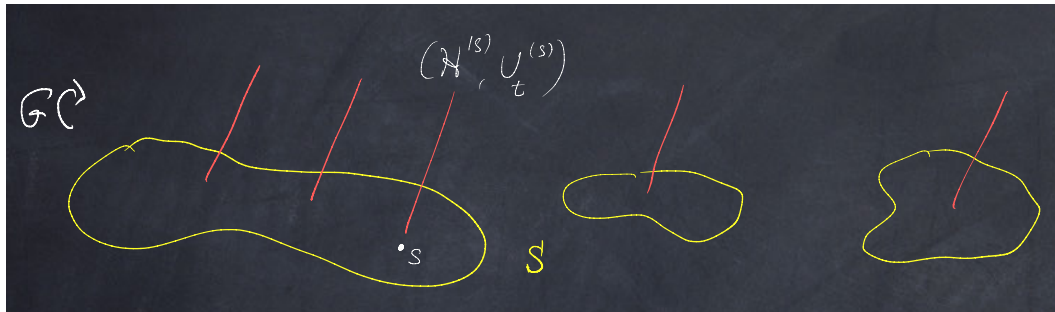


FIGURE 19. A symmetry group G acting on a family of QM systems

More generally, consider a family $(\mathcal{H}^{(s)}, U_t^{(s)})$ of QM data parametrized by $s \in S$ for a smooth manifold S . Then a symmetry group G can also act nontrivially on the parameter space S ; it can permute the QM systems, as depicted in [Figure 19](#). Alternatively, one can quotient by G and say that we have a family of systems parametrized by the *groupoid* or *stack* $S//G$.

7.3 Particle on a ring

The classical version²⁰ of this system was introduced in [Example 3.56](#). Set $M = \mathbb{E}^1/2\pi\mathbb{Z}$ and $\mathcal{H} = L^2(M; \mathbb{C})$. Consider the family of Hamiltonian operators

$$(7.20) \quad H_\theta = \frac{1}{2} \left(i \frac{d}{dx} + \frac{\theta}{2\pi} \right)^2, \quad \theta \in \mathbb{R}.$$

Then $H_\theta = T_\theta^2$ for the self-adjoint operator

$$(7.21) \quad T_\theta = \frac{1}{\sqrt{2}} \left(-i \frac{d}{dx} - \frac{\theta}{2\pi} \right).$$

The spectrum is discrete, a basis of eigenfunctions is

$$(7.22) \quad \psi_n(x) = e^{inx}, \quad x \in M, \quad n \in \mathbb{Z},$$

and the corresponding eigenvalues of T_θ and H_θ are

$$(7.23) \quad \begin{aligned} \mu_n(\theta) &= \frac{1}{\sqrt{2}} \left(n - \frac{\theta}{2\pi} \right) \\ E_n(\theta) &= \frac{1}{2} \left(n - \frac{\theta}{2\pi} \right)^2 \end{aligned}$$

respectively. Observe that the spectrum is periodic in θ ; it is unchanged by $\theta \rightarrow \theta + 2\pi$. In fact, the entire family of quantum systems is periodic: there is an automorphism of $\mathcal{H}$ that sends H_θ to $H_{\theta+2\pi}$. So the parameter space of this family of systems is $S = \mathbb{R}/2\pi\mathbb{Z}$.

Remark 7.24.

- (1) One can put in a parameter $L \in \mathbb{R}^{>0}$ by setting $M_L = \mathbb{E}^1/2\pi L\mathbb{Z}$ and modifying [\(7.20\)](#) to

$$(7.25) \quad H_{\theta,L} = \frac{1}{2} \left(i \frac{d}{dx} + \frac{\theta}{2\pi L} \right)^2.$$

For each θ, L the QM system is gapped, but the gap is not uniform in L ; the gap tends to zero as $L \rightarrow \infty$.

- (2) At $\theta = 0$ and $\theta = \pi$ the eigenspaces are not all lines. In fact, other than the 0-eigenspace at $\theta = 0$ all eigenspaces are 2-dimensional. At $\theta = \pi$ all eigenspaces are 2-dimensional.

It is illuminating to visualize the flow of eigenvalues of H_θ as θ varies over $S = \mathbb{R}/2\pi\mathbb{Z}$. In [Figure 20](#) we indicate the direction of flow of the eigenvalues $E_n(\theta)$ as θ increases. We have depicted snapshots at $\theta = 0$ and $\theta = \pi$. Note the doubling observed in [Remark 7.24\(2\)](#). One can also contemplate the *spectral flow* of the eigenvalues of the first-order differential operators T_θ ;


 FIGURE 20. The flow of eigenvalues of $\{H_\theta\}$ as θ increases

 FIGURE 21. Spectral flow of the family of operators $\{T_\theta\}$

this is depicted in **Figure 21**. The eigenvalues fit together into a single helix embedded in the cylinder $S \times \mathbb{R}$.

Recall the classical Lagrangian (3.57) for a particle motion $x: \mathbb{R} \rightarrow M$:

$$(7.26) \quad \left(\frac{m}{2} \dot{x}^2 + \frac{\theta}{2\pi} \dot{x} \right) dt.$$

There is an evident translation action by $c \in G_0 = \mathbb{R}/2\pi\mathbb{Z}$ that acts trivially on $\theta \in S$ and sends

$$(7.27) \quad x \mapsto x + c.$$

²⁰The phrase ‘classical version’ is problematic: there is no isomorphism between classical and quantum systems. But in this simple system it is less controversial.

Then G_0 acts on $\mathcal{H}$ and commutes with the Hamiltonian (as in (7.18)), so $\mathcal{H}$ decomposes according to the irreducible representations of G_0 . Since G_0 is abelian, these representations are 1-dimensional; they are *characters* of G_0 . In fact, (7.22) shows that each character occurs with multiplicity one, independently of $\theta \in S$.

There is also a reflection symmetry which has order 2 classically and acts nontrivially on the parameter space S :

$$(7.28) \quad \begin{aligned} x &\longrightarrow -x \\ \theta &\longrightarrow -\theta \end{aligned}$$

Note that the fixed points on the parameter space are at $\theta = 0$ and $\theta = \pi$. Reflections reverse the direction of translation under conjugation, so altogether we have the classical symmetry group

$$(7.29) \quad G = \mu_2 \ltimes \mathbb{R}/2\pi\mathbb{Z} = \mu_2 \ltimes G_0,$$

where recall $\mu_2 = \{\pm 1\}$. The Lie group G is isomorphic to the orthogonal group O_2 .

In the quantum system, we can implement (7.27) and (7.28) by using the formula (7.22) as a guide. Restrict to $\theta = \pi$, where the energy eigenvalues (7.23) are $E_n = \frac{1}{2}(n - \frac{1}{2})^2$. (It is a good exercise to write the symmetry on the entire family of QM theories.) We can implement translation by c and reflection by the unitary operators

$$(7.30) \quad \begin{aligned} T_c(\psi_n) &= e^{inc}\psi_n \\ R(\psi_n) &= \psi_{1-n} \end{aligned}$$

Then $R^2 = \text{id}_{\mathcal{H}}$ and $\{T_c\}_{c \in \mathbb{R}/2\pi\mathbb{Z}}$ satisfies the group law of G_0 : we obtain separate unitary representations of μ_2 and G_0 . However, we do not obtain a unitary representation of the semidirect product; rather,

$$(7.31) \quad RT_cR^{-1} = e^{-ic}T_{-c}.$$

The homothety that is multiplication by e^{-ic} ruins the group law of the semidirect product.

However, and this is the key point, the group G does act (unitarily) on the *projective* space $\mathbb{P}\mathcal{H}$. In other words, it acts on the pure states of the QM system. Further analysis shows that G acts on the convex space of all states, G acts on the vector space of observables, and that these actions preserve the measurement pairing and all other data (Axiom System 1.1) of the QM system. There is nothing “sick” about this quantum symmetry; it is just as healthy as the symmetry of the corresponding classical system. This example is one indication of the projective nature of quantum mechanics—indeed, of quantum theory in general—as we pursue in Lecture 8.

Remark 7.32. One often hears that a classical symmetry has an *anomaly* when acting on the corresponding quantum system. This anomaly is not a sickness. It is simply a measure of the *projectivity* of the symmetry, an obstruction to its *linearization*.

As preparation for our discussion in the next lecture, we conclude this lecture with a rather different and general topic: geometric structures à la Felix Klein.

7.4 The Erlangen Program

Klein's *Erlangen Program* [K, BB] states that “a geometry is determined by its symmetries”. We interpret this as follows, vaguely at first:

- (1) One specifies a *geometric type* as a group action $G \curvearrowright F$.
- (2) Then one gives data related to G to construct an instance of this geometric type, or even better a parametrized family of these geometries.

Here F is a *model* of the geometry, G is the *model* symmetry group, and the pair $G \curvearrowright F$ completely determines the geometric type. We can carry this out in different contexts (categories). For our applications to physics, F is a smooth manifold and G is a Lie group.

Example 7.33. The geometric type ‘linear geometry of dimension n over a field $\mathbb{F}$ ’ has model

$$(7.34) \quad \mathrm{GL}_n \mathbb{F} \curvearrowright \mathbb{F}^n.$$

The geometric type ‘spherical geometry of dimension n ’ has a model

$$(7.35) \quad \mathrm{O}_{n+1} \curvearrowright \mathbb{S}^n,$$

where $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the sphere of unit norm vectors.

To explain instantiation of geometric types, we begin with a digression on *torsors*. In the following definition the group can be discrete, topological, or Lie. The torsor has the corresponding structure of a set, topological space, or smooth manifold, and the group action is unconstrained, continuous, or smooth.

Definition 7.36. Let G be a group. A *right G -torsor* is a set T equipped with a simply transitive right G -action $T \times G \rightarrow T$. A *left G -torsor* is a set T equipped with a simply transitive left G -action $G \times T \rightarrow T$.

Simple transitivity for a right G -torsor is the condition that the map

$$(7.37) \quad \begin{aligned} T \times G &\longrightarrow T \times T \\ (t, g) &\longmapsto (t, t \cdot g) \end{aligned}$$

be an isomorphism.

Remark 7.38. We use a strict convention for left vs. right: *left* group actions are *geometric* and *right* group actions are *structural*:

$$(7.39) \quad \text{left} = \text{geometric} \quad \text{right} = \text{structural}$$

There is a category of right G -torsors. Namely, a morphism $T' \rightarrow T$ of right G -torsors is a map that commutes with the right G -actions. Every morphism is invertible, so the category of right G -torsors is a groupoid. The automorphism group $\mathrm{Aut}(T)$ of a right G -torsor T acts on the left:

$$(7.40) \quad \mathrm{Aut}(T) \curvearrowright T \curvearrowright G.$$

These left and right actions commute. Each point $t_0 \in T$ determines an isomorphism $G \rightarrow T$ under which the left $\text{Aut}(T)$ -action pulls back to the action of G on itself by left multiplication. In particular, the automorphism group $\text{Aut}(T)$ is isomorphic to G , but the isomorphism is not canonical: it requires a choice of point in T .

For topological or Lie groups, the set F in the following definition is a topological space or a smooth manifold, respectively.

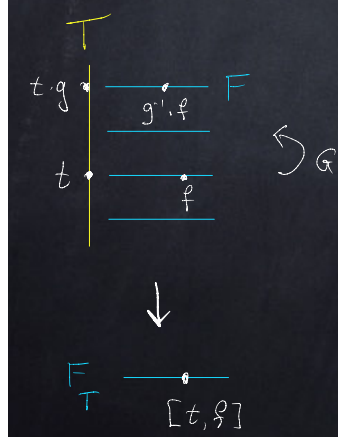


FIGURE 22. The mixing construction

Definition 7.41. Let G be a group and suppose F is a set equipped with a left G -action. Then for every right G -torsor T , there is an associated *mixing construction*

$$(7.42) \quad F_T = T \times_G F = (T \times F) / G$$

in which the right G -action on $T \times F$ is $(t, f) \cdot g = (t \cdot g, g^{-1} \cdot f)$.

Each point of T induces an isomorphism $F \xrightarrow{\cong} F_T$. The mixing construction (7.42) can also be called a *sandwich*: the group G is sandwiched between right and left G -torsors to quotient out the G -action. The linear analog is the tensor product of right and left modules over a ring. Observe that $\text{Aut}(T)$ acts as symmetries of F_T :

$$(7.43) \quad \text{Aut}(T) \curvearrowright F_T.$$

The group action (7.43) is an instantiation of the model $G \curvearrowright F$.

As a special case of Definition 7.41, suppose $\phi: G \rightarrow H$ is a homomorphism of groups, and let T be a right G -torsor. Then there is a left G -action on H ($g \cdot h = \phi(g)h$), and

$$(7.44) \quad \phi(T) = T \times_G H$$

is a right H -torsor. This is analogous to a change of ring (base change) in the case of rings and modules.

In these terms we have the following.

Definition 7.45 (Klein’s Erlangen Program).

- (1) A *geometric type* is a group action $G \curvearrowright F$.
- (2) A *geometry of type* $G \curvearrowright F$ is a right G -torsor T .
- (3) A *family of geometries of type* $G \curvearrowright F$ *parametrized by* S is a fiber bundle of right G -torsors $P \rightarrow S$.

In (2) the corresponding geometric space is the mixing construction $F_T = T \times_G F$. A ‘fiber bundle of right G -torsors’ is also called a ‘principal G -bundle’.

Example 7.46. We usually specify a linear geometry (7.34) as an n -dimensional vector space V over the field $\mathbb{F}$. Given V define

$$(7.47) \quad T(V) = \{b: \mathbb{F}^n \xrightarrow{\cong} V\}$$

to be the set of bases of V . Then $T(V)$ is a right $\mathrm{GL}_n\mathbb{F}$ -torsor by precomposition, and there is a canonical isomorphism

$$(7.48) \quad \begin{aligned} T(V) \times_G \mathbb{F}^n &\longrightarrow V \\ (b, v) &\longmapsto b(v) \end{aligned}$$

In other words, a vector space may be specified as an instantiation of linear geometry, i.e., via a right $\mathrm{GL}_n\mathbb{F}$ -torsor. In other geometries there is typically not a direct definition of an instantiation.

Lecture 8: Quantum mechanics and projective geometry

In Definition 5.1 we defined the data of a quantum mechanical system to be a pair $(\mathcal{H}, U)$ of a complex Hilbert space $\mathcal{H}$ and a strongly continuous unitary representation of the additive group $\mathbb{R}$ of time translations. But the derived data of states and observables is projective rather than linear. For example, the space of pure states is the projective space $\mathbb{P}\mathcal{H}$, which comes equipped with the *transition probability* function (5.12). And in §7.3 we gave an example—the particle on a ring—of a symmetry group of a QM system that only acts projectively, not linearly.

In this lecture we examine the projective nature of quantum mechanics and determine the precise data needed to define a quantum mechanical system. We find that it is not a linear space $\mathcal{H}$ that we need, but rather it is a complex projective space. Recall (§7.4) that a model geometry is not simply a model space F , which for quantum mechanics is a complex projective space, but also needed is a model group action $G \curvearrowright F$ of a symmetry group G . It is the symmetry group that determines the precise nature of the geometry—that is Klein’s insight—and for quantum mechanics a fundamental result of Wigner helps determine G , as we explain in §8.2. The model geometry of quantum mechanics is determined in §8.4: see (8.47). A choice of linear space $\mathcal{H}$ is additional data, and we end the lecture (§8.6) by examining the nature of the choice of a *linearization* $\mathcal{H}$.

In fact, if we consider a family of QM systems, or a QM system with symmetry, then there is an obstruction to the existence of a linearization.

The considerations in this lecture extend to *any* quantum system: quantum field theory and string theory included. The obstruction to linearization is called an *anomaly*, and it is a crucial and accessible piece of structure in a quantum theory.²¹ We like to summarize the basic idea underlying this lecture as a slogan

$$(8.1) \quad \text{QUANTUM THEORY IS PROJECTIVE. QUANTIZATION IS LINEAR.}$$

with a corollary

$$(8.2) \quad \text{THE ANOMALY OF A QUANTUM THEORY EXPRESSES ITS PROJECTIVITY.}$$

8.1 What is a projective space?

Three of the classical geometries are: linear geometry, affine geometry, and projective geometry. As a prelude to the question posed above, let us briefly consider: What is a linear space? What is an affine space? Well, a linear space V (also called a vector space) is defined by well-known data and axioms. It is an algebraic object, and it can be defined over any field $\mathbb{F}$. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$ one often introduces a compatible topology, which is unique in finite dimensions. **Definition 1.31(1)** tells what is an affine space A over a linear space V . Affine spaces are the arena of flat geometry with a global parallelism. They can be defined over any field $\mathbb{F}$, and for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ one can introduce topologies as in linear geometry. The latter is the arena for calculus.

There are *direct* definitions of both linear spaces and affine spaces. The definition of a projective space, which can be given over any field $\mathbb{F}$, is simplest if we begin with an *auxiliary* linear space W . Then there is a free action of the units $\mathbb{F}^\times$ on $W \setminus \{0\}$ by scalar multiplication. Projective space is the quotient

$$(8.3) \quad \mathbb{P}W = (W \setminus \{0\}) / \mathbb{F}^\times.$$

The set $\mathbb{P}W$ parametrizes lines in W . If $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and W has a topology, then $\mathbb{P}W$ has the quotient topology. In fact, it can be given a compatible structure of a smooth manifold—a complex manifold if $\mathbb{F} = \mathbb{C}$. If $\dim W = 2$, then $\mathbb{P}W$ is a *projective line*. Over $\mathbb{R}$ a projective line is diffeomorphic to a circle S^1 ; over $\mathbb{C}$ a projective line is diffeomorphic to a 2-sphere S^2 . We can work over the division ring of quaternions as well, and a quaternionic projective line is diffeomorphic to a 4-sphere S^4 .

Can we construct a projective space $\mathbb{P}$ without first choosing a linear space W ? Here we turn to the notion of geometric type, discussed in §7.4, which we illustrate in linear and affine geometry first. Fix a nonnegative integer n . The model n -dimensional linear geometry over a field $\mathbb{F}$ is

$$(8.4) \quad \mathrm{GL}_n \mathbb{F} \curvearrowright \mathbb{F}^n,$$

²¹Although the example in §7.3 is the anomaly of a symmetry, we emphasize that the projective nature of quantum theory is more fundamental and is not necessarily tied to symmetry; see [F14].

as in (7.34). The model n -dimensional affine geometry over $\mathbb{F}$ is

$$(8.5) \quad \text{Aff}_n \mathbb{F} \curvearrowright \mathbb{A}_{\mathbb{F}}^n,$$

where $\text{Aff}_n \mathbb{F}$ is the group of affine transformations of the model n -dimensional affine space $\mathbb{A}_{\mathbb{F}}^n$; it fits into the group extension

$$(8.6) \quad 1 \longrightarrow \mathbb{F}^n \longrightarrow \text{Aff}_n \mathbb{F} \longrightarrow \text{GL}_n \mathbb{F} \longrightarrow 1.$$

(As a set $\mathbb{A}_{\mathbb{F}}^n = \mathbb{F}^n$, but it has a different structure than the model linear space.)

Remark 8.7. We give basic definitions about group extensions in §8.7.4. Group extensions appear often in subsequent lectures, as do variations of the particular group extension (8.6).

For n -dimensional projective geometry over $\mathbb{F}$ a natural model geometry is

$$(8.8) \quad \text{PGL}_{n+1} \mathbb{F} \curvearrowright \mathbb{FP}^n,$$

where the group of projective transformations $\text{PGL}_{n+1} \mathbb{F} = \text{GL}_{n+1} \mathbb{F} / \mathbb{F}^\times$ is the quotient of the general linear group by the homotheties. It acts effectively on the set $\mathbb{FP}^n = (\mathbb{F}^{n+1} \setminus \{0\}) / \mathbb{F}^\times$. The following is the special case of Definition 7.45 for the model geometry (8.8).

Definition 8.9. Fix a field $\mathbb{F}$ and a nonnegative integer n . The data of an n -dimensional projective space over $\mathbb{F}$ is a right $\text{PGL}_{n+1} \mathbb{F}$ -torsor T . The associated projective space is

$$(8.10) \quad \mathbb{P} = T \times_{\text{PGL}_{n+1} \mathbb{F}} \mathbb{FP}^n.$$

A point of T induces an isomorphism $\mathbb{FP}^n \xrightarrow{\cong} \mathbb{P}$ of the model projective space with the abstract projective space, a kind of “basis” in projective geometry. Such “bases” are determined up to the symmetry group of the geometry.

Matrix algebras are also “projective” in the following sense.

Remark 8.11. Let $M_{n+1} \mathbb{F}$ denote the algebra of $(n+1) \times (n+1)$ matrices over $\mathbb{F}$. The conjugation action $\text{GL}_{n+1} \mathbb{F} \curvearrowright M_{n+1} \mathbb{F}$ factors through the quotient group $\text{PGL}_{n+1} \mathbb{F}$. So associated to a projective space $\mathbb{P}$ as in Definition 8.9 is an algebra

$$(8.12) \quad E = T \times_{\text{PGL}_{n+1} \mathbb{F}} M_{n+1} \mathbb{F}.$$

An element $t \in T$ gives rise to an identification of E as a matrix algebra.

The idea that “a matrix algebra is projective” can be expressed in the following terms. Let W be a vector space (over any field), and let K be a line. Then there is a canonical isomorphism

$$(8.13) \quad \begin{aligned} \text{End}(W) &\longrightarrow \text{End}(W \otimes K) \\ T &\longmapsto T \otimes \text{id}_K \end{aligned}$$

Similarly, projective space is projective: there is a canonical isomorphism

$$(8.14) \quad \begin{aligned} \mathbb{P}W &\longrightarrow \mathbb{P}(W \otimes K) \\ L &\longmapsto L \otimes K \end{aligned}$$

Specialize to $\mathbb{F} = \mathbb{C}$. The general model projective geometry (8.8) specializes to

$$(8.15) \quad \mathrm{PGL}_{n+1}\mathbb{C} \curvearrowright \mathbb{CP}^n,$$

which is the model for holomorphic projective geometry. Restrict to the projective unitary group

$$(8.16) \quad \mathrm{PU}_{n+1}\mathbb{C} \curvearrowright \mathbb{CP}^n,$$

to obtain the model for Fubini–Study geometry. The smaller symmetry group in (8.16) has more invariants, in this case the Fubini–Study metric on $\mathbb{CP}^n$. (See §8.7.3 for a brief introduction to this geometry.) We will see that (8.16) is the correct model geometry for quantum mechanics.

8.2 The model geometry of finite dimensional quantum mechanical DOSO

In quantum mechanics the base field is $\mathbb{F} = \mathbb{C}$: complex numbers encode interference phenomena. We might at first think the appropriate symmetry group for a quantum mechanical system with an n -dimensional space of pure states is the complex Lie group $\mathrm{PGL}_{n+1}\mathbb{C}$. Then the model for the space $\mathcal{S}_0$ of pure states would be $\mathrm{PGL}_{n+1}\mathbb{C} \curvearrowright \mathbb{CP}^n$, and the model for the complex vector space $\mathcal{O}$ would be $\mathrm{PGL}_{n+1}\mathbb{C} \curvearrowright M_{n+1}\mathbb{C}$. But there is more data in a DOSO (Axiom System 1.1): there is an entire convex space $\mathcal{S}$ of states, the vector space $\mathcal{O}$ has a real structure and a Lie algebra structure, and there is a pairing of states and observables to probability measures on $\mathbb{R}$. This additional data depends on the adjoint map, which is the conjugate transpose map $A \mapsto A^*$ on complex matrices. If $U \in \mathrm{GL}_{n+1}\mathbb{C}$ is to preserve this map, then for all $A \in M_{n+1}\mathbb{C}$ we must have

$$(8.17) \quad UA^*U^{-1} = (UAU^{-1})^* = (U^{-1})^*A^*U^* = (U^*)^{-1}A^*U^*,$$

from which it follows that $(U^*U)A^*(U^*U)^{-1} = A^*$ for all A . Hence U^*U is a scalar multiple of the identity. Therefore, the image of U in the projective linear group $\mathrm{PGL}_{n+1}\mathbb{C}$ equals the image of a unitary matrix. In other words, it is the projective unitary subgroup

$$(8.18) \quad \mathrm{PU}_{n+1} \subset \mathrm{PGL}_{n+1}\mathbb{C}$$

that preserves the map $A \mapsto A^*$. This subgroup also preserves the subspace of positive matrices and the trace one condition on states. Hence Fubini–Study geometry

$$(8.19) \quad \mathrm{PU}_{n+1} \curvearrowright \mathbb{CP}^n$$

seems to be a reasonable model geometry for quantum mechanics. But we need to look deeper, and also to incorporate motion, as in Definition 1.11 and Definition 5.1.

Remark 8.20. For $n = 1$ the model geometry is $\mathbb{CP}^1 \approx S^2$ with the round metric. The Lie group $\mathrm{PU}_2 \cong \mathrm{SO}_3$ is not the full group of isometries—that group is O_3 . It has SO_3 as its identity component, and the second component $\mathrm{O}_3 \setminus \mathrm{SO}_3$ acts as antiholomorphic transformations. Elements in this second component—reflections—do preserve the DOSO structure in quantum mechanics, as we now argue.

Recall from (5.12) that the space of pure states in quantum mechanics is equipped with a function

$$(8.21) \quad p: \mathbb{CP}^n \times \mathbb{CP}^n \longrightarrow [0, 1]$$

that expresses transition probabilities. The pairing of states and observables is expressed in terms of p ; see (5.11). This leads to the following

Hypothesis 8.22. *The symmetry group of the model pure state space $\mathbb{CP}^n$ in quantum mechanics is the group of self-maps of $\mathbb{CP}^n$ that preserves the function p .*

We emphasize that this is not the model we seek for quantum mechanics; for that we must incorporate motion as well as DOSO.

The following theorem is the first step toward determining this group.

Theorem 8.23. *Let*

$$(8.24) \quad d: \mathbb{CP}^n \times \mathbb{CP}^n \longrightarrow \mathbb{R}^{\geq 0}$$

be the distance function of the Fubini–Study Riemannian metric. Then

$$(8.25) \quad \cos(d) = 2p - 1.$$

In [Lecture 5](#) we stated (8.25) in the equivalent form

$$(8.26) \quad p = \cos^2(d/2).$$

We prove (8.25) in the next section.

Next, we apply the Myers–Steenrod theorem in Riemannian geometry [[MySt](#), [Pa](#)]. The first part of the theorem is the following: A Riemannian manifold (M, g) has an underlying metric space (M, d) with distance function d , and if $\varphi: M \rightarrow M$ is a metric space isometry, then φ is smooth and is a Riemannian isometry. The second part of the theorem asserts that the group of Riemannian isometries of M has a natural Lie group structure. Putting this together with [Theorem 8.23](#) we are led to the following consequence of [Hypothesis 8.22](#).

Definition 8.27. For a nonnegative integer n , let PQ_{n+1} be the Lie group of isometries of $\mathbb{CP}^n$. Then *the model geometry for quantum mechanical DOSO with an n -dimensional space of pure states is*

$$(8.28) \quad \mathrm{PQ}_{n+1} \curvearrowright \mathbb{CP}^n.$$

A right PQ_{n+1} -torsor T has an associated projective space

$$(8.29) \quad \mathbb{P} = T \times_{\mathrm{PQ}_{n+1}} \mathbb{CP}^n,$$

and the claim is that the rest of the data that defines quantum mechanical DOSO is also associated to T . We leave that verification to a homework problem.

Remark 8.30.

- (1) We remind that [Definition 8.27](#) is the model geometry for DOSO; the model for quantum mechanics that we arrive at below takes into account time evolution as well.
- (2) We reiterate the logic at work and the power of the Myers–Steenrod theorem. The model of pure states in quantum mechanics is the *set* $\mathbb{CP}^n$ equipped with the function p in [\(8.21\)](#), and so what comes to us from physics is the group of permutations of $\mathbb{CP}^n$ that preserve p . Note that such permutations are simply maps of an uncountable set to itself; there is no *a priori* topology, so no continuity. We apply [Theorem 8.23](#) and the Myers–Steenrod theorem to prove that this group has a natural *Lie group* structure, and that it is precisely the group of isometries of $\mathbb{CP}^n$ with the Riemannian metric defined by Fubini–Study.

Next, we determine the group PQ_{n+1} . Note that the projective unitary group PU_{n+1} is a subgroup, since it acts by isometries on $\mathbb{CP}^n$. It turns out that PQ_{n+1} has two components, if $n > 0$, and PU_{n+1} is the identity component. As an example, recall from [Remark 8.20](#) that $\mathrm{PQ}_2 \cong \mathrm{O}_3$; the identity component is SO_3 , which fits into the group extension

$$(8.31) \quad 1 \longrightarrow \mathbb{T} \longrightarrow \mathrm{U}_2 \longrightarrow \mathrm{SO}_3 \longrightarrow 1.$$

The Kähler manifold $\mathbb{CP}^n$ is a *symmetric space*, in fact, it is a symmetric space of a special type: a hermitian symmetric space. Groups of isometries of symmetric spaces have been determined by general theory [[Loo](#)]. Here we outline an argument for $\mathbb{CP}^n$.

The key result is called Wigner’s theorem, though it seems [[Bo](#)] that the statement of the theorem first appeared in a joint paper [[vNW](#)] of von Neumann and Wigner. To state it we need to introduce *antilinear* or *conjugate linear* transformations (see §[8.7.2](#) for definitions) of a Hilbert space that preserve the underlying real inner product.

Definition 8.32.

- (1) Let W be a complex Hilbert space. A real linear map $A: W \rightarrow W$ is *antiunitary* if it is conjugate linear and

$$(8.33) \quad \langle A\psi_1, A\psi_2 \rangle = \overline{\langle \psi_1, \psi_2 \rangle}, \quad \psi_1, \psi_2 \in W.$$

- (2) For $W = \mathbb{C}^{n+1}$, let Q_{n+1} be the 2-component Lie group of unitary and antiunitary transformations of $\mathbb{C}^{n+1}$.

Note that the composition of two antiunitary transformations is a unitary transformation. The structure of PQ_{n+1} is encapsulated in the following. (See §[8.7.4](#) for the definition of a group extension.) Let $\mathbb{T} \subset \mathbb{C}$ denote the group complex numbers of modulus one.

Theorem 8.34. *There is a group extension*

$$(8.35) \quad 1 \longrightarrow \mathbb{T} \longrightarrow Q_{n+1} \xrightarrow{\pi} PQ_{n+1} \longrightarrow 1$$

It is straightforward that elements of Q_{n+1} act on $\mathbb{CP}^n$ by isometries, and that homotheties in $\mathbb{T}$ act as the identity. The import of [Theorem 8.34](#) is the surjectivity of the map $Q_{n+1} \rightarrow PQ_{n+1}$: every isometry of $\mathbb{CP}^n$ lifts to a unitary or antiunitary transformation of $\mathbb{C}^{n+1}$. Wigner's theorem holds for infinite dimensional Hilbert spaces as well.

Remark 8.36. For $n = 1$ the Lie group Q_2 sits inside a Clifford algebra: $Q_2 \cong \text{Pin}_3^c$. (Spin and pin groups and their position inside Clifford algebras are discussed in [\[ABS\]](#). We give an introduction in the next lecture.)

Remark 8.37. The preceding discussion leads to the model geometry for quantum mechanical DOSO with a choice of state space:

$$(8.38) \quad Q_{n+1} \curvearrowright \mathbb{C}^{n+1}.$$

Elements in the identity component $U_{n+1} \subset Q_{n+1}$ act unitarily; elements in the off-component act antiunitarily.

Remark 8.39. Symmetries in quantum theory that reverse the direction of time are typically antiunitary; see [Lemma 11.3](#).

8.3 Sketch proofs

Here we only give some ideas toward proofs of [Theorem 8.23](#) and [Theorem 8.34](#). Complete proofs, including the infinite dimensional case, can be found in [\[F3\]](#).

Sketch proof of Theorem 8.23. The first step is to reduce to the case of $\mathbb{CP}^1 \approx S^2$. Observe that [\(8.25\)](#) holds on the diagonal of $\mathbb{CP}^n \times \mathbb{CP}^n$, so fix $(L_1, L_2) \in \mathbb{CP}^n \times \mathbb{CP}^n$ with $L_1 \neq L_2$ and let $V = L_1 + L_2 \subset \mathbb{C}^{n+1}$ be the 2-dimensional subspace they span. Then the unitary automorphism φ of $\mathbb{CP}^n$ induced from $\text{id}_V \oplus -\text{id}_{V^\perp}$ on $\mathbb{C}^{n+1} = V \oplus V^\perp$ has $\mathbb{P}V \subset \text{Fix}(\varphi)$ as a component of its fixed point set. It follows that $\mathbb{P}V \subset \mathbb{CP}^n$ is *totally geodesic*: a geodesic from L_1 to L_2 in $\mathbb{P}V$ is also a geodesic in $\mathbb{CP}^n$. This means that $d(L_1, L_2)$ can be computed in the complex projective line $\mathbb{P}V$.

At this point the proof is elementary geometry in the 2-sphere. Let $e_1 \in L_1$ be a unit norm vector, and choose e_2 so that $\{e_1, e_2\}$ is an orthonormal basis of V . Then $\lambda e_1 + e_2 \in L_2$ for a unique $\lambda \in \mathbb{C}$. After normalizing to obtain a unit vector, we compute

$$(8.40) \quad p(L_1, L_2) = \frac{|\lambda|^2}{|\lambda|^2 + 1}.$$

In case $\lambda = 0$ the lines L_1, L_2 are orthogonal, and so $d = \pi$ and $p = 0$: [\(8.25\)](#) holds. It suffices to assume $\lambda \neq 0$, and then identify

$$(8.41) \quad \mathbb{P}V \setminus \mathbb{C}\langle e_2 \rangle \longleftrightarrow \mathbb{C} \cong \{0\} \times \mathbb{C} \longleftrightarrow S^2 \setminus \{(1, 0)\} \subset \mathbb{R} \times \mathbb{C},$$

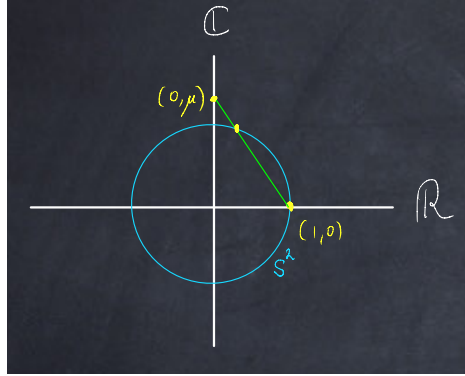


FIGURE 23. Stereographic projection

where the first identification is $\mathbb{C}\langle e_1 + \mu e_2 \rangle \leftrightarrow \mu$, $\mu \in \mathbb{C}$, and the second is stereographic projection as depicted in [Figure 23](#). Compute

$$(8.42) \quad \begin{aligned} L_1 &\longleftrightarrow (-1, 0) \\ L_2 &\longleftrightarrow \left(-\frac{|\lambda|^2 - 1}{|\lambda|^2 + 1}, \frac{2|\lambda|^2}{|\lambda|^2 + 1} \frac{1}{\lambda} \right) \end{aligned}$$

Finally, the cosine of the distance between two points on the unit sphere in a real inner product space is the inner product of the unit vectors from the origin to the points. $\square$

Two proofs of [Theorem 8.34](#) are given in [\[F3\]](#), one leaning on Riemannian geometry and the other on complex geometry. For the latter, the first step is the following.

Lemma 8.43. *An isometry of $\mathbb{CP}^n$ is either holomorphic or antiholomorphic.*

The argument is as follows. Let $I: T\mathbb{CP}^n \rightarrow T\mathbb{CP}^n$ be the complex structure. Then I is parallel with respect to the Levi-Civita covariant derivative. So too is its pullback by an isometry. We use the fact that a parallel almost complex structure commutes with the Riemann curvature tensor to prove that the only parallel complex structures are $\pm I$.

The second step is standard.

Lemma 8.44. *Let W be a finite dimensional complex vector space. Then any holomorphic map $\phi: \mathbb{PW} \rightarrow \mathbb{PW}$ lifts to a linear map $W \rightarrow W$.*

Proof. To prove this let $\mathcal{L} \rightarrow \mathbb{PW}$ be the tautological holomorphic line bundle whose fiber at $L \subset W$ is the line L . Then $\phi^*\mathcal{L} \rightarrow \mathbb{PW}$ is isomorphic to $\mathcal{L} \rightarrow W$, since a holomorphic line bundle over $\mathbb{PW}$ is determined by its topology (its Chern class) and the group of holomorphic automorphisms of $\mathbb{PW}$ is connected. Choose an isomorphism. The pullback on global sections of the dual hyperplane bundle

$$(8.45) \quad \phi^*: H^0(\mathbb{PW}; \mathcal{L}^*) \longrightarrow H^0(\mathbb{PW}; \phi^*\mathcal{L}^*) \cong H^0(\mathbb{PW}; \mathcal{L}^*)$$

is the automorphism of W^* dual to a linear automorphism of W that lifts ϕ . $\square$

8.4 The model geometry of quantum mechanics

We continue the discussion of §8.2. There we determined that the structure group of quantum mechanical DOSO is PQ_{n+1} , if the pure states form a complex projective space of dimension n . A quantum mechanical system also has a time evolution, and the structure group of quantum mechanics must be compatible with it as well. In a linear presentation of quantum mechanics, time evolution is a *unitary* representation of the group $\mathbb{R}$ of time translations. In fact, there is a potentially larger symmetry group $\mathcal{G}$ that is represented. This group is a semidirect product

$$(8.46) \quad \mathcal{G} = \mathbb{R} \ltimes K$$

where K consists of symmetries that do not move time: *internal* symmetries of the system. In a projective presentation of quantum mechanics, which is the more fundamental one, time evolution is a projective unitary representation of $\mathcal{G}$. Define two systems to be *similar* if these projective unitary representations are equivalent. In other words, $U: \mathcal{G} \rightarrow \text{PU}(\mathbb{P})$ and $U': \mathcal{G} \rightarrow \text{PU}(\mathbb{P}')$ are similar if there is a projective unitary isomorphism $\mathbb{P} \rightarrow \mathbb{P}'$ that intertwines them.

The model geometry we seek is of similarity classes of quantum mechanical systems of a fixed dimension. It is now clear that the structure group is the projective unitary group. Namely, fix $n \in \mathbb{Z}^{>0} \cup \{\infty\}$ the dimension of the projective space of pure states. Then

$$(8.47) \quad \text{THE MODEL GEOMETRY FOR QUANTUM MECHANICS IS } \text{PU}_{n+1} \curvearrowright \mathbb{CP}^n$$

This is precisely the same as the model of complex projective space as a Kähler manifold with the Fubini–Study metric.

8.5 The model geometry in infinite dimensions

The main issue here is to use appropriate topologies on spaces of operators and on projective spaces. I hope to return to that issue in a future version of these notes. For now I refer to [F3], where proofs are given for infinite dimensional versions of the theorems in the previous section, and to [AS, Appendix 1], [FM1, Appendix D], where operator topologies are discussed.

As for a model geometry, fix a separable complex Hilbert space, say ℓ^2 the L^2 sequence space. Let Q be the group of unitary and antiunitary transformations of ℓ^2 , topologized with the compact-open topology as in the references. There is a projective quotient group PQ which fits into the group extension

$$(8.48) \quad 1 \longrightarrow \mathbb{T} \longrightarrow Q \xrightarrow{\pi} PQ \longrightarrow 1$$

as in (8.35). The identity component of PQ is the projective unitary group PU , and the model geometry for an infinite dimensional quantum system is

$$(8.49) \quad PU \curvearrowright \mathbb{P}\ell^2.$$

8.6 Linearization of projective space

The considerations here apply to any of the three projective geometries we have discussed, both in finite and infinite dimensions, but the situation for quantum projective geometry has an extra twist. We begin with a generic definition, which ultimately is too vague (why?).

Definition 8.50. Let $\mathbb{P}$ be a projective space.

- (1) A *linearization* of $\mathbb{P}$ is a pair (W, θ) consisting of a vector space W and an isomorphism

$$(8.51) \quad \theta: \mathbb{P} \xrightarrow{\cong} \mathbb{P}W.$$

- (2) An *isomorphism* $(W', \theta') \rightarrow (W, \theta)$ of linearizations is a linear isomorphism $W' \rightarrow W$ whose projectivization fits into the commutative diagram

$$(8.52) \quad \begin{array}{ccc} & & \mathbb{P}W' \\ & \nearrow^{\theta'} & \downarrow \\ \mathbb{P} & & \mathbb{P}W \\ & \searrow_{\theta} & \end{array}$$

Linearizations and their isomorphisms comprise a groupoid $\mathcal{L}(\mathbb{P})$.

A better version of [Definition 8.50](#) uses the symmetry/structure groups to pin down the precise projective geometry in question. Recall that these groups fit into group extensions, written here for the infinite dimensional case:

$$(8.53) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathrm{GL} & \xrightarrow{\pi} & \mathrm{PGL} \longrightarrow 1 \\ & & & & 1 & \longrightarrow & \mathbb{T} \longrightarrow \mathrm{U} \xrightarrow{\pi} \mathrm{PU} \longrightarrow 1 \\ & & & & & & 1 \longrightarrow \mathbb{T} \longrightarrow \mathrm{Q} \xrightarrow{\pi} \mathrm{PQ} \longrightarrow 1 \end{array}$$

Crucially, in the first two the kernel $\mathbb{T}$ is *central*; in the third it is not: scalars anticommute with antiunitary transformations. For the following definition, we choose one of the three geometries and write the corresponding group extension [\(8.53\)](#) as

$$(8.54) \quad 1 \longrightarrow \mathbb{T} \longrightarrow \mathrm{G} \xrightarrow{\pi} \mathrm{PG} \longrightarrow 1$$

Recall the mixing construction, and especially [\(7.44\)](#).

Definition 8.55. Let T be a right PG-torsor.

- (1) A *linearization* of T is a pair $(\tilde{T}, \theta)$ consisting of a right G -torsor $\tilde{T}$ and an isomorphism

$$(8.56) \quad \theta: T \xrightarrow{\cong} \pi(\tilde{T})$$

of PG-torsors, where the associated torsor $\pi(\tilde{T})$ is defined in [\(7.44\)](#) below.

- (2) An *isomorphism* $(\tilde{T}', \theta') \rightarrow (\tilde{T}, \theta)$ of linearizations is an isomorphism $\tilde{T}' \rightarrow \tilde{T}$ of G -torsors whose induced isomorphism of PG -torsors fits into the commutative diagram

$$(8.57) \quad \begin{array}{ccc} & & \pi(\tilde{T}') \\ & \nearrow^{\theta'} & \downarrow \\ T & & \pi(\tilde{T}) \\ & \searrow_{\theta} & \end{array}$$

Linearizations and their isomorphisms comprise a groupoid $\mathcal{L}(\mathbb{P})$, where here $\mathbb{P} = T \times_{PG} \mathbb{P}^2$ is the projective space associated to T (with its induced “quantum projective geometry”).

For the two geometries in which the kernel $\mathbb{T}$ is central, the groupoid $\mathcal{L}(\mathbb{P})$ can be given the structure of a *gerbe*. (See [Definition 8.60](#) in §8.7.1.) To see this, observe that automorphisms of an object (T, θ) are given by the action of some $\lambda \in \mathbb{T}$ on T : since $\mathbb{T} \subset G$ is central, the action of λ commutes with the G -action, so it is an automorphism of the G -torsor. It follows that every morphism space is naturally a $\mathbb{T}$ -torsor. Observe too that given a $\mathbb{T}$ -torsor S and an object $(\tilde{T}, \theta)$ of $\mathcal{L}(\mathbb{P})$, then we can construct a new object by tensoring $\tilde{T}$ with S .

In the third geometry—quantum projective geometry, the case of most interest in these lectures—the groupoid $\mathcal{L}(\mathbb{P})$ is a *twisted* version of a gerbe. We do not spell out that structure now, but encourage the reader to do so.

A single projective space $\mathbb{P}$ always admits linearizations: simply choose an object of $\mathcal{L}(\mathbb{P})$. However, there are two closely related situations in which linearizations may not exist: parametrized families of projective spaces and projective spaces equipped with the action of a Lie group of symmetries. A motivating example is the particle on a ring with its symmetry group, as explained in §7.3.

Remark 8.58. In algebraic geometry one can express linearization in the terms of the proof of [Lemma 8.44](#). Namely, a linearization of a projective space $\mathbb{P}$ is a choice of tautological holomorphic (or algebraic) line bundle $\mathcal{L} \rightarrow \mathbb{P}$. For a single projective space it exists, but in a family there is an obstruction, as there is to lifting a group action on $\mathbb{P}$ to $\mathcal{L} \rightarrow \mathbb{P}$.

Remark 8.59. In quantum theory, the twisted gerbe $\mathcal{L}(\mathbb{P})$ of linearizations is sometimes called the *anomaly*. It is an accessible invariant of a theory, and it has strong implications in quantum field theory and string theory. When one wants to quantize, the anomaly is an obstruction—see the guiding slogans (8.1) and (8.2).

8.7 Mathematical background

8.7.1 Gerbes. For an *abelian* group A , the category Tor_A of right A -torsors is a *Picard groupoid*: there is a multiplication operation $\text{Tor}_A \times \text{Tor}_A \rightarrow \text{Tor}_A$ which tensors A -torsors. The following is probably nonstandard, but sufficient.

Definition 8.60. Let A be an abelian group. An A -gerbe is a groupoid $\mathcal{C}$ and, for each pair of objects $x, y \in \mathcal{C}$, the structure of a right A -torsor $T_{x,y}$ on the morphism set $\mathcal{C}(x, y)$. Furthermore, for $x, y, z \in \mathcal{C}$ there are isomorphisms

$$(8.61) \quad T_{x,y} \otimes T_{y,z} \xrightarrow{\cong} T_{x,z}$$

which fit into a commutative diagram with composition:

$$(8.62) \quad \begin{array}{ccc} \mathcal{C}(x, y) \times \mathcal{C}(y, z) & \xrightarrow{\circ} & \mathcal{C}(x, z) \\ \cong \downarrow & & \downarrow \cong \\ T_{x,y} \times T_{y,z} & \xrightarrow{\otimes} & T_{x,z} \end{array}$$

A *trivialization* of $\mathcal{C}$ is a choice of object.

The isomorphisms (8.61) are required to be coherent; I leave you to construct the appropriate commutative diagram for four objects in $\mathcal{C}$.

8.7.2 Conjugate linear vector spaces and transformations.

Definition 8.63. Let W be a complex vector space.

- (1) The *complex conjugate vector space* $\overline{W}$ is the same underlying real vector space with the complex conjugate scalar multiplication:

$$(8.64) \quad \bar{\lambda} \cdot \bar{w} = \overline{\lambda w}, \quad w \in W, \quad \lambda \in \mathbb{C}.$$

Here $\bar{w} \in \overline{W}$ corresponds to $w \in W$ under the identification of underlying real vector spaces.

- (2) The *complex conjugate* of a linear map $T: W' \rightarrow W$ between complex vector spaces is the linear map

$$(8.65) \quad \begin{array}{l} \overline{T}: \overline{W'} \longrightarrow \overline{W} \\ \overline{w'} \longmapsto \overline{T w'} \end{array}$$

- (3) An *antilinear* transformation of W is a linear map $A: W \rightarrow \overline{W}$.

Alternatively, an antilinear transformation of W is a real linear transformation of the underlying real vector space $W_{\mathbb{R}}$ such that $A(\lambda w) = \bar{\lambda} A(w)$ for all $w \in W$, $\lambda \in \mathbb{C}$.

8.7.3 The Fubini–Study metric on complex projective space. Let W be a finite dimensional complex inner product space and let $\mathbb{P}W$ be the associated complex projective space, which is a complex manifold. Then at $L \in \mathbb{P}W$, the tangent space to $\mathbb{P}W$ is canonically the complex vector space

$$(8.66) \quad T_L \mathbb{P}W = \text{Hom}(L, L^\perp).$$

(The graph of a linear map $L \rightarrow L^\perp$ is a point of $\mathbb{P}W$. So a smooth motion in $\mathbb{P}W$ with initial position L can be expressed, for small times, as a motion in $\text{Hom}(L, L^\perp)$, hence the initial velocity lies in that vector space.) The Fubini–Study metric is a hermitian metric on each tangent space $T_L \mathbb{P}W$. In terms of the identification (8.66) it is

$$(8.67) \quad \langle f_1, f_2 \rangle \longmapsto \text{trace}(f_1^* \circ f_2)$$

The composition $f_1^* \circ f_2$ is an endomorphism of the line L , so is scalar multiplication by a complex number; the trace extracts the complex number. The formula makes evident that the projectivization of unitary and antiunitary transformations of W are isometries of the Fubini–Study metric.

8.7.4 Group extensions. We do not comment further on topology, but say once and for all that for topological groups all group homomorphisms are assumed continuous, and for Lie groups all group homomorphisms are assumed smooth.

Definition 8.68. A *group extension* is a sequence of group homomorphisms

$$(8.69) \quad 1 \longrightarrow G' \xrightarrow{i} G \xrightarrow{\pi} G'' \longrightarrow 1$$

that is exact in the sense that the kernel of any homomorphism equals the image of the preceding homomorphism. We call G' the *kernel* and G'' the *quotient*.

Exactness at G' implies that i is injective. The inclusion i identifies G' with its image, which is a subgroup of G . At G the exactness implies that π factors through an injective map of the quotient group G/G' into G'' , and exactness at G'' implies that this is an isomorphism. We use it to identify G'' as this quotient.

There is a category of group extensions with fixed kernel and quotient.

Definition 8.70. Let G', G'' be groups and G^{τ_1}, G^{τ_2} group extensions with kernel G' and quotient G'' . A *morphism of group extensions* G^{τ_1}, G^{τ_2} is a group homomorphism $\varphi: G^{\tau_1} \rightarrow G^{\tau_2}$ which fits into the commutative diagram

$$(8.71) \quad \begin{array}{ccccccc} & & & G^{\tau_1} & & & \\ & & \nearrow & \downarrow \varphi & \searrow & & \\ 1 & \longrightarrow & G' & & G'' & \longrightarrow & 1 \\ & & \searrow & \downarrow & \nearrow & & \\ & & & G^{\tau_2} & & & \end{array}$$

As usual, φ is an *isomorphism* if there exists a homomorphism $\psi: G^{\tau_2} \rightarrow G^{\tau_1}$ of group extensions so that the compositions $\psi \circ \varphi$ and $\varphi \circ \psi$ are identity maps, which is simply equivalent to the condition that φ be an isomorphism of groups..

There is a notion of a trivialization of a group extension.

Definition 8.72. A *splitting* of the group extension (8.69) is a homomorphism $j: G'' \rightarrow G$ such that $\pi \circ j = \text{id}_{G''}$.

Not every group extension is split: for example, the cyclic group of order 4 is a nonsplit extension of the cyclic group of order 2 by the cyclic group of order 2:

$$(8.73) \quad 1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

Lecture 9: Introduction to superalgebra

So far our discussion has been for mechanical systems which are *bosonic*. But in the real world there are two species of particles: *bosons* and *fermions*. This statement depends on features of space, and so far our axioms only have incorporated time. Furthermore, in two spatial dimensions more exotic possibilities abound: *anyons* [Wil]. Hence in our current context we cannot give a convincing argument for a boson/fermion dichotomy. Nonetheless, we begin the lecture with a heuristic discussion.

The main part of the lecture is an introduction to *superalgebra*, the algebraic formalism which neatly encodes the boson/fermion dichotomy, and which appears in many other mathematical contexts as well. The word ‘super’ is synonymous with $\mathbb{Z}/2\mathbb{Z}$ -graded. The key point is the *Koszul sign rule* (9.11): there is a minus sign when commuting two odd elements. For example, this sign dictates the proper definition of commutativity in super associative algebras, of skew-symmetry and the Jacobi identity in super Lie algebras, and of the hermitian structure and adjoints in the context of a super Hilbert space.

In the last part of the lecture we give an introduction to Clifford algebras and the spin group. The venerable paper [ABS] has a more comprehensive treatment, and there are many other accounts.

9.1 Quantum statistics

Suppose $(\mathcal{H}, U)$ is a complex Hilbert space and unitary strongly continuous 1-parameter group, data that defines a QM system. We use language as if the system models a single particle: $\mathcal{H}$ is the Hilbert space of *1-particle states*. Consider the conjunction of two copies of this system with no interaction. The data for this composition follows from (5.4):

$$(9.1) \quad (\mathcal{H} \otimes \mathcal{H}, U \otimes U).$$

Then if $\psi_1, \psi_2 \in \mathcal{H}$ are unit norm vectors, $\psi_1 \otimes \psi_2$ represents a decomposable pure state in the combined system. To encode that (9.1) is a QM system of two *identical* particles, we ask that $\psi_1 \otimes \psi_2$ and $\psi_2 \otimes \psi_1$ represent the same state, that is, they are scalar multiples of each other. In other words, the exchange map

$$(9.2) \quad \begin{aligned} \mathcal{H} \otimes \mathcal{H} &\longrightarrow \mathcal{H} \otimes \mathcal{H} \\ \psi_1 \otimes \psi_2 &\longmapsto \psi_2 \otimes \psi_1 \end{aligned}$$

should reduce to scalar multiplication:

$$(9.3) \quad \psi_2 \otimes \psi_1 = \lambda \psi_1 \otimes \psi_2.$$

Furthermore—this is the unmotivated step—we ask that applying (9.3) twice we get back to $\psi_1 \otimes \psi_2$ on the nose. Therefore, we posit $\lambda = \pm 1$:

$$(9.4) \quad \psi_2 \otimes \psi_1 = \begin{cases} \psi_1 \otimes \psi_2, & (\text{boson}); \\ -\psi_1 \otimes \psi_2, & (\text{fermion}), \end{cases}$$

and so factor the putative 2-particle state space $\mathcal{H} \otimes \mathcal{H}$ to the quotient

$$(9.5) \quad \mathcal{H} \otimes \mathcal{H} \longrightarrow \begin{cases} \text{Sym}^2 \mathcal{H}, & (\text{boson}); \\ \wedge^2 \mathcal{H}, & (\text{fermion}). \end{cases}$$

Remark 9.6.

- (1) Exterior algebra encodes the *Pauli exclusion principle*.
- (2) A typical QM system has both bosonic and fermionic states, so we want a formalism that incorporates both simultaneously. That is provided by superalgebra. There are also states that are mixtures; they are neither bosonic nor fermionic.
- (3) We also use superalgebra to define bosonic and fermionic observables, though ‘observable’ is inappropriate for fermionic observables: only bosonic observables can be measured.
- (4) The presence of both bosonic and fermionic states and observables does not imply that the system is *supersymmetric*. Supersymmetries exchange bosonic states and fermionic states, and those may or may not be present.

We expound on these points in the next lecture.

9.2 Superalgebra

The word ‘super’ is synonymous with ‘ $\mathbb{Z}/2\mathbb{Z}$ -graded’.

The following discussion applies over any field $\mathbb{F}$ not of characteristic 2, but for our application we only use $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$.

Definition 9.7.

- (1) A *super vector space* $V = V^0 \oplus V^1$ is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space. A homogeneous element $v \in V^i$ has *parity* $i \in \mathbb{Z}/2\mathbb{Z}$. A *morphism* of super vector spaces is a parity-preserving linear map. The *grading automorphism* of V is

$$(9.8) \quad \epsilon_V = \text{id}_{V^0} \oplus -\text{id}_{V^1}.$$

If V is finite dimensional, then its *dimension* is $\dim V = n^0 |n^1|$, where $n^i = \dim V^i$, $i \in \mathbb{Z}/2\mathbb{Z}$.

- (2) Let V, W be super vector spaces. The *tensor product* $V \otimes W$ is the usual tensor product with $\mathbb{Z}/2\mathbb{Z}$ -grading

$$(9.9) \quad (V \otimes W)^k = \bigoplus_{i+j=k} (V^i \otimes W^j), \quad k \in \mathbb{Z}/2\mathbb{Z}.$$

The associativity isomorphism for the tensor product of super vector spaces U, V, W is

$$(9.10) \quad \begin{aligned} (U \otimes V) \otimes W &\longrightarrow U \otimes (V \otimes W) \\ (u \otimes v) \otimes w &\longmapsto u \otimes (v \otimes w) \end{aligned}$$

The commutativity isomorphism for the tensor product of super vector spaces V, W is

$$(9.11) \quad \begin{aligned} \sigma_{V,W}: V \otimes W &\longrightarrow W \otimes V \\ v \otimes w &\longmapsto (-1)^{|v||w|} w \otimes v \end{aligned}$$

The symmetry (9.11) is the *Koszul sign rule*; it satisfies $\sigma_{W,V} \circ \sigma_{V,W} = \text{id}_{V \otimes W}$.

(3) A *superalgebra* is a super vector space A equipped with a multiplication

$$(9.12) \quad A \otimes A \longrightarrow A$$

which is associative and unital. The unit $1 \in A$ is necessarily even. The superalgebra A is *commutative* if

$$(9.13) \quad ab = (-1)^{|a||b|}ba$$

for all homogeneous elements a, b of A .

The standard super vector space of dimension $n^0|n^1$ is denoted $\mathbb{F}^{n^0|n^1}$. The linear maps (9.10), (9.11), and (9.12) are assumed to be morphisms of super vector spaces, that is, they are parity-preserving. Note that an ordinary vector space is an even super vector space; similarly for ordinary algebras. I leave the reader to define modules over a superalgebra as well as super Lie algebras. If $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$ is a super Lie algebra, then the Lie bracket restricts to a *symmetric* bilinear pairing

$$(9.14) \quad \mathfrak{g}^1 \times \mathfrak{g}^1 \longrightarrow \mathfrak{g}^0.$$

In fact, a super Lie algebra gives rise to

- a Lie algebra $\mathfrak{g}^0$,
- a $\mathfrak{g}^0$ -module $\mathfrak{g}^1$,
- a map $\text{Sym}^2 \mathfrak{g}^1 \rightarrow \mathfrak{g}^0$ of $\mathfrak{g}^0$ -modules,

and these data satisfy

$$(9.15) \quad [x, [x, x]] = 0, \quad x \in \mathfrak{g}^1.$$

Conversely, one can prove that this data subject to (9.15) determines a super Lie algebra. We refer to [DM] for much more on superalgebra, as well as for an exposition of the theory of supermanifolds.

Remark 9.16. Definition 9.7(1) defines a linear category of super vector spaces. Definition 9.7(2) is the main data required to endow it with a symmetric monoidal structure. Its most salient feature is the Koszul sign rule (9.11). Then we can rephrase Definition 9.7(3) as follows: a superalgebra is an algebra object in the category of super vector spaces. See [DM] for this point of view.

Example 9.17.

- (1) Let U be a vector space. Then the exterior algebra $A = \bigwedge^\bullet U$ has a $\mathbb{Z}$ -grading, so too a quotient $\mathbb{Z}/2\mathbb{Z}$ -grading. The resulting superalgebra is commutative.
- (2) Let $V = V^0 \oplus V^1$ be a super vector space. Then the algebra $\text{End } V$ of *all* endomorphisms of the underlying ungraded vector space is a superalgebra:

$$(9.18) \quad \text{End } V = \text{End}^0 V \oplus \text{End}^1 V$$

where the even elements preserve parity and the odd elements reverse parity.

- (3) If $x_1, \dots, x_N$ are even indeterminates, the polynomial algebra $\mathbb{R}[x_1, \dots, x_N]$ is commutative as an ungraded algebra, but is not commutative as a superalgebra (with $\mathbb{Z}/2\mathbb{Z}$ -grading the parity of the total degree).
- (4) If $\eta_1, \dots, \eta_N$ are odd indeterminates, then $\mathbb{R}[\eta_1, \dots, \eta_N]$ is the free commutative real superalgebra they generate. It is canonically isomorphic to the exterior algebra on the free real vector space generated by $\{\eta_1, \dots, \eta_N\}$.

Remark 9.19. For even indeterminates $x_1, \dots, x_N$, the free commutative algebra $\mathbb{R}[x_1, \dots, x_N]$ is the algebra of polynomial functions on an affine space. In the odd case, we can also view $\mathbb{R}[\eta_1, \dots, \eta_N]$ as functions on a space, but now an *odd* affine space; see [F5, Lecture 1] for a heuristic introduction to odd affine spaces and more generally to supermanifolds. The text [DM] is a much more thorough and rigorous treatment.

Let $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ be a super vector space. Then

$$(9.20) \quad \text{Sym}^2 \mathcal{H} = \mathcal{H} \otimes \mathcal{H} / S,$$

where $S \subset \mathcal{H} \otimes \mathcal{H}$ is the subspace generated by

$$(9.21) \quad \psi_1 \otimes \psi_2 - (-1)^{|\psi_1||\psi_2|} \psi_2 \otimes \psi_1$$

over all homogeneous $\psi_1, \psi_2 \in \mathcal{H}$. The reader is encouraged to prove the following.

Lemma 9.22. $\text{Sym}^2 \mathcal{H} \cong [\text{Sym}^2 \mathcal{H}^0 \oplus \wedge^2 \mathcal{H}] \oplus [\mathcal{H}^0 \otimes \mathcal{H}^1]$.

If $\mathcal{H}$ is the 1-particle state space of a QM system, then $\text{Sym}^2 \mathcal{H}$ is the 2-particle state space: bosonic states are products of two bosonic states or of two fermionic states, and fermionic states are the product of one bosonic state and one fermionic state. This shows how the symmetric algebra of a super vector space captures the statistics of identical particles.

Remark 9.23. Here $\mathcal{H}$ is an algebraic super vector with no topology. If it has a topology, say a Hilbert space structure, then there are completions of the algebraic symmetric tensor powers which are also Hilbert spaces. The algebraic symmetric algebra $\text{Sym}^\bullet \mathcal{H}$ is called the *Fock space*.

9.3 The Koszul sign rule and antilinearity

Specialize to the ground field $\mathbb{C}$.

Definition 9.24.

- (1) Let A be an algebra over $\mathbb{C}$. A **-operation* on A is a real linear map

$$(9.25) \quad \begin{aligned} A &\longrightarrow A \\ a &\longmapsto a^* \end{aligned}$$

such that for all $\lambda \in \mathbb{C}$, $a, a_1, a_2 \in A$ we have

$$(9.26) \quad a^{**} = a$$

$$(9.27) \quad (\lambda a)^* = \bar{\lambda} a^*$$

$$(9.28) \quad (a_1 a_2)^* = a_2^* a_1^*$$

In brief, $*$ is an antilinear antiinvolution of A . The pair $(A, *)$ is a $*$ -algebra.

(2) Let A be a superalgebra over $\mathbb{C}$. Then a $*$ -structure is as in (1) with (9.28) replaced by

$$(9.29) \quad (a_1 a_2)^* = (-1)^{|a_1||a_2|} a_2^* a_1^*$$

where a_1, a_2 are homogeneous.

The transition from (1) to (2) is dictated by the Koszul sign rule. Notice that if A is a *commutative* $*$ -superalgebra, then

$$(9.30) \quad (a_1 a_2)^* = a_1^* a_2^*$$

for all $a_1, a_2 \in A$.

We turn now to hermitian inner products.

Definition 9.31. Let $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ be a complex super vector space. A *hermitian inner product* is a bilinear pairing

$$(9.32) \quad \langle -, - \rangle : \overline{\mathcal{H}} \times \mathcal{H} \longrightarrow \mathbb{C}$$

such that for all homogeneous $\psi_1, \psi_2 \in \mathcal{H}$ we have

$$(9.33) \quad \langle \psi_1, \psi_2 \rangle = (-1)^{|\psi_1||\psi_2|} \overline{\langle \psi_2, \psi_1 \rangle},$$

and for all nonzero homogeneous $\psi \in \mathcal{H}$ we have

$$(9.34) \quad \begin{cases} \langle \psi, \psi \rangle > 0, & \psi \in \mathcal{H}^0; \\ -i\langle \psi, \psi \rangle > 0, & \psi \in \mathcal{H}^1. \end{cases}$$

Note that (9.33) implies that for ψ odd the inner product $\langle \psi, \psi \rangle$ is purely imaginary. Observe too that $\mathcal{H}^0 \perp \mathcal{H}^1$ since (9.32) is even. The restriction of $\langle -, - \rangle$ to $\mathcal{H}^0$ is an ordinary hermitian inner product, as is the restriction of $-i\langle -, - \rangle$ to $\mathcal{H}^1$. Hence there are notions of completeness and so of a super Hilbert space.

The adjoint of a (bounded) homogeneous endomorphism $T : \mathcal{H} \rightarrow \mathcal{H}$ is characterized by

$$(9.35) \quad \langle T^* \psi_1, \psi_2 \rangle = (-1)^{|T||\psi_1|} \langle \psi_1, T \psi_2 \rangle$$

for all homogeneous ψ_1 and all $\psi_2 \in \mathcal{H}$. The map $T \mapsto T^*$ is a $*$ -structure, so this makes $\text{End } \mathcal{H}$ into a $*$ -superalgebra.

Remark 9.36. If T is odd, then T has no nonzero homogeneous eigenvectors.

Remark 9.37. There is a way around the awkward sign in (9.35) for the adjoint. That leads to a more standard convention; see [DM, §4.4], [De2, §4].

9.4 Clifford algebras

In the remainder of the lecture we introduce a particular noncommutative superalgebra: the Clifford algebra. The orthogonal group O_n is a subset of an algebra: the algebra $M_n\mathbb{R}$ of $n \times n$ matrices. The *Clifford algebra* plays a similar role for a double cover group of the orthogonal group, called the *pin* group. Its identity component, the double cover of the special orthogonal group, is the *spin group*.

We begin with a heuristic motivation for Clifford algebras. Orthogonal transformations are products of reflections. For a unit norm vector $\xi \in \mathbb{R}^n$ define

$$(9.38) \quad \rho_\xi(\eta) = \eta - 2\langle \eta, \xi \rangle \xi,$$

where $\langle -, - \rangle$ is the standard inner product.

Theorem 9.39 (Cartan–Dieudonné). *Any $g \in O_n$ is the composition of $\leq n$ reflections.*

Proof. The statement is trivial for $n = 1$. Proceed by induction: if $g \in O_n$ fixes a unit norm vector ζ then it fixes the orthogonal complement $(\mathbb{R} \cdot \zeta)^\perp$, and we are reduced to the theorem for O_{n-1} . If there are no fixed unit norm vectors, then for any unit norm vector ζ set $\xi = \frac{g(\zeta) - \zeta}{|g(\zeta) - \zeta|}$. The composition $\rho_\xi \circ g$ fixes ζ and again we reduce to the $(n-1)$ -dimensional orthogonal complement. $\square$

Now generate an algebra from the unit norm vectors, with relations inspired by those of reflections. Note immediately that the vectors $\pm\xi$ both correspond to the same reflection $\rho_\xi = \rho_{-\xi}$. Therefore, we expect from the beginning that the Clifford algebra “double counts” orthogonal transformations. Now since the square of a reflection is the identity, impose the relation

$$(9.40) \quad \xi^2 = \pm 1, \quad |\xi| = 1.$$

The sign ambiguity is that described above, and we choose a sign independent of ξ . It follows that

$$(9.41) \quad \xi^2 = \pm |\xi|^2$$

for any $\xi \in \mathbb{R}^n$. Now if $\langle \xi_1, \xi_2 \rangle = 0$, then $(\xi_1 + \xi_2)/\sqrt{2}$ has unit norm and from

$$(9.42) \quad \pm 1 = \left(\frac{\xi_1 + \xi_2}{\sqrt{2}} \right)^2 = \frac{\xi_1^2 + \xi_2^2 + (\xi_1\xi_2 + \xi_2\xi_1)}{2} = \frac{\pm 2 + (\xi_1\xi_2 + \xi_2\xi_1)}{2}$$

we deduce

$$(9.43) \quad \xi_1\xi_2 + \xi_2\xi_1 = 0, \quad \langle \xi_1, \xi_2 \rangle = 0.$$

Equations (9.40) and (9.43) are the defining relations for the Clifford algebra. Check that the reflection (9.38) is given by

$$(9.44) \quad \rho_\xi(\eta) = -\xi\eta\xi^{-1}$$

in the Clifford algebra. By composition using Theorem 9.39 we obtain the action of any orthogonal transformation on $\eta \in \mathbb{R}^n$.

Definition 9.45. For $n \in \mathbb{Z}$ define the *real Clifford algebra* Cl_n as the unital associative real algebra generated by $e_1, \dots, e_{|n|}$ subject to the relations

$$(9.46) \quad \begin{aligned} e_i^2 &= \pm 1, & i &= 1, \dots, |n| \\ e_i e_j + e_j e_i &= 0, & i &\neq j. \end{aligned}$$

The complex Clifford algebra $\text{Cl}_n^{\mathbb{C}}$ is the complex algebra with the same generators and same relations. These Clifford algebras are $\mathbb{Z}/2\mathbb{Z}$ -graded by declaring that $e_1, \dots, e_{|n|}$ are odd.

Note $\text{Cl}_0 = \mathbb{R}$ and $\text{Cl}_0^{\mathbb{C}} = \mathbb{C}$.

Example 9.47. There is an isomorphism of superalgebras $\text{Cl}_{-n}^{\mathbb{C}} \cong \text{Cl}_n^{\mathbb{C}}$ obtained by multiplying each generator e_i by $\sqrt{-1}$.

Example 9.48. Cl_{-1} can be embedded in the matrix super algebra $M_{1|1}\mathbb{R} = \text{End}(\mathbb{R}^{1|1})$ —see [Example 9.17\(2\)](#)—by setting

$$(9.49) \quad e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The same equation embeds $\text{Cl}_{-1}^{\mathbb{C}}$ in $M_{1|1}\mathbb{C}$.

Example 9.50. Construct an isomorphism of superalgebras $\text{Cl}_{-2}^{\mathbb{C}} \xrightarrow{\cong} \text{End}(\mathbb{C}^{1|1})$ by

$$(9.51) \quad e_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where $i = \sqrt{-1}$. There is *no* such isomorphism over the reals. The product

$$(9.52) \quad e_1 e_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

is $-i$ times a *grading operator* on $\mathbb{C}^2$.

Example 9.53. The real Clifford algebras Cl_1 and Cl_{-1} are not isomorphic. In particular, the doubled orthogonal group $\{\pm 1, \pm e_1\}$ is different in the two cases: in Cl_1 it is isomorphic to the Klein group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, whereas in Cl_{-1} it is cyclic of order four.

We may in the future discuss basic *Morita theory*, in which we will see that super matrix algebras are in some sense trivial. That is the spirit of the following algebraic form of *Bott periodicity*.

Theorem 9.54. *There are isomorphisms of superalgebras*

$$(9.55) \quad \begin{aligned} \text{Cl}_{-2}^{\mathbb{C}} &\xrightarrow{\cong} \text{End}(\mathbb{S}), & \dim \mathbb{S} &= 1|1, \\ \text{Cl}_{-8} &\xrightarrow{\cong} \text{End}(\mathbb{S}_{\mathbb{R}}), & \dim \mathbb{S}_{\mathbb{R}} &= 8|8. \end{aligned}$$

Proof. The complex case is Example 9.50. For the real case, let Cl_{-2} act on $\mathbb{W} = \mathbb{C}^{1|1}$ via the formulas in (9.51). This action commutes (in the graded sense) with the *odd* real structure

$$(9.56) \quad J(z^0, z^1) = (\overline{z^1}, \overline{z^0}).$$

That is, $J: \mathbb{W} \rightarrow \mathbb{W}$ is antilinear, odd, and squares to $-\text{id}_{\mathbb{W}}$. Set $\mathbb{S} = \mathbb{W}^{\otimes 4}$. It carries an action of $\text{Cl}_{-2}^{\otimes 4} \cong \text{Cl}_{-8}$ which commutes with $J^{\otimes 4}$. The latter is antilinear, even, and squares to $\text{id}_{\mathbb{S}}$, so is a real structure. $\square$

Observe that $\mathbb{W}^{\otimes 2}$ carries a *quaternionic* structure $J^{\otimes 2}$: the Koszul sign rule (9.11) implies that $J^{\otimes 2}$ squares to *minus* the identity. (Check that sign!)

Next, we indicate the definitions of the pin and spin groups, as they sit in the Clifford algebras. For $n > 0$ let $S(\mathbb{R}^n) \subset \mathbb{R}^n$ denote the sphere of unit norm vectors. Since $\mathbb{R}^n$ embeds in $\text{Cl}_{\pm n}$, so too does $S(\mathbb{R}^n)$. We assert without proof that the group it generates is a Lie group $\text{Pin}_{\pm n} \subset \text{Cl}_{\pm n}$. It follows from Theorem 9.39 that there is a surjection $\text{Pin}_{\pm n} \rightarrow O_n$ defined by composing the reflections (9.44). The inverse image $\text{Spin}_{\pm n}$ of the special orthogonal group SO_n consists of products of an even number of elements in $S(\mathbb{R}^n)$. There is an isomorphism $\text{Spin}_n \cong \text{Spin}_{-n}$, but as we saw in Example 9.53 this is not true in general for Pin.

The Clifford algebra in Definition 9.45 arises from the following question, posed by Dirac: Find a square root of the Laplace operator. We work on flat Euclidean space $\mathbb{E}^n$, which is the affine space $\mathbb{A}^n$ endowed with the translation-invariant metric constructed from the standard inner product on the underlying vector space $\mathbb{R}^n$ of translations. Let $x^1, \dots, x^n$ be the standard affine coordinates on $\mathbb{E}^n$. The Laplace operator is

$$(9.57) \quad \Delta = - \sum_{i=1}^n \frac{\partial^2}{(\partial x^i)^2}.$$

A first-order operator

$$(9.58) \quad D = \gamma^i \frac{\partial}{\partial x^i}$$

satisfies $D^2 = \Delta$ if and only if γ^i satisfy the Clifford relation

$$(9.59) \quad \gamma^i \gamma^j + \gamma^j \gamma^i = -2\delta^{ij}, \quad 1 \leq i, j \leq n,$$

as in (9.46). If we let (9.57), (9.58) act on the space $C^\infty(\mathbb{E}^n; \mathbb{S})$ of functions with values in a vector space $\mathbb{S}$, then we conclude that $\mathbb{S}$ is a Cl_{-n} -module.

²²Use (9.64) below.

9.5 Abstract Clifford algebras

For more details see [ABS, Part I], [De1, §2].

A quadratic form on a vector space V is a function $Q: V \rightarrow k$ such that

$$(9.60) \quad B(\xi_1, \xi_2) = Q(\xi_1 + \xi_2) - Q(\xi_1) - Q(\xi_2), \quad \xi_1, \xi_2 \in V,$$

is bilinear and $Q(n\xi) = n^2Q(\xi)$.

Definition 9.61. The *Clifford algebra* $Cl(V, Q) = Cl(V)$ of a quadratic vector space is an algebra equipped with a linear map $i: V \rightarrow Cl(V, Q)$ which satisfies the following universal property: If $\varphi: V \rightarrow A$ is a linear map to an algebra A such that

$$(9.62) \quad \varphi(\xi)^2 = Q(\xi) \cdot 1_A, \quad \xi \in V,$$

then there exists a unique algebra homomorphism $\tilde{\varphi}: Cl(V, Q) \rightarrow A$ such that $\varphi = \tilde{\varphi} \circ i$.

We leave the reader to prove that i is injective and that $Cl(V, Q)$ is unique up to unique isomorphism. Furthermore, there is a surjection

$$(9.63) \quad \otimes V \longrightarrow Cl(V, Q)$$

from the tensor algebra, as follows from its universal property. This gives an explicit construction of $Cl(V, Q)$ as the quotient of $\otimes V$ by the 2-sided ideal generated by $\xi^2 - Q(\xi) \cdot 1_{\otimes V}$, $\xi \in V$. The tensor algebra is $\mathbb{Z}$ -graded, and since the ideal sits in even degree the quotient Clifford algebra is $\mathbb{Z}/2\mathbb{Z}$ -graded. The increasing filtration $\otimes^0 V \subset \otimes^{\leq 1} V \subset \otimes^{\leq 2} V \subset \dots$ induces an increasing filtration on $Cl(V, Q)$ whose associated graded is isomorphic to the ($\mathbb{Z}$ -graded) exterior algebra $\bigwedge^\bullet V$. There is a canonical isomorphism

$$(9.64) \quad Cl(V' \oplus V'', Q' \oplus Q'') \cong Cl(V', Q') \otimes Cl(V'', Q''),$$

deduced from the universal property. The standard Clifford algebras in Definition 9.45 have the form $Cl(V, Q)$ for $V = \mathbb{R}^n, \mathbb{C}^n$ and Q the positive or negative definite standard quadratic form on V .

The Clifford algebras are *central simple* as $\mathbb{Z}/2\mathbb{Z}$ -graded algebras. I will leave the simplicity (there are no nontrivial 2-sided homogeneous ideals) as an exercise and here prove the centrality.

Proposition 9.65. $Cl(V, Q)$ has center k .

Proof. Suppose $x = x^0 + x^1$ is a central element. Fix an orthonormal basis $e_1, \dots, e_n$ of V . Then for every $i = 1, \dots, n$ we have

$$(9.66) \quad \begin{aligned} x^0 e_i &= e_i x^0 \\ x^1 e_i &= -e_i x^1 \end{aligned}$$

There is a unique decomposition $x^0 = a^0 + e_i b^1$ where a^0, b^1 belong to the Clifford algebra generated by the basis elements excluding e_i . Then

$$(9.67) \quad \begin{aligned} x^0 e_i &= a^0 e_i + e_i b^1 e_i = e_i a^0 - (e_i)^2 b^1 \\ e_i x^0 &= e_i a^0 + (e_i)^2 b^1. \end{aligned}$$

Since x^0 is central we have $x^0 e_i = e_i x^0$, and so (9.67) implies that $b^1 = 0$. Since this holds for every i , we conclude that x^0 is a scalar. Similarly, write $x^1 = a^1 + e_i b^0$ so that

$$(9.68) \quad \begin{aligned} x^1 e_i &= a^1 e_i + e_i b^0 e_i = -e_i a^1 + (e_i)^2 b^0 \\ -e_i x^1 &= -e_i a^1 - (e_i)^2 b^0 \end{aligned}$$

from which $x^1 = 0$. □

For a vector space L and $\theta \in L^*$ let ϵ_θ denote exterior multiplication by θ , which is an endomorphism of the exterior algebra $\bigwedge^\bullet L^*$. For $\ell \in L$ the adjoint of exterior multiplication by ℓ is contraction ι_ℓ , an endomorphism of $\bigwedge^\bullet L^*$ of degree -1 .

Proposition 9.69. *Suppose $V = L \oplus L^*$ with the split quadratic form $Q(\ell + \theta) = \theta(\ell)$, $\ell \in L$, $\theta \in L^*$. Set $\mathbb{S} = \bigwedge^\bullet L^*$ with its $\mathbb{Z}/2\mathbb{Z}$ -grading by the parity of the degree. Then the map $V \rightarrow \text{End } \mathbb{S}$*

$$(9.70) \quad \begin{aligned} \ell &\longmapsto \iota_\ell \\ \theta &\longmapsto \epsilon_\theta \end{aligned}$$

extends to an isomorphism $\text{Cl}(V) \xrightarrow{\cong} \text{End } \mathbb{S}$ of the Clifford algebra with a super matrix algebra.

Proof. Using (9.64) we reduce to the case $\dim L = 1$ which can be checked by hand; it is essentially Example 9.50. □

Sitting inside the Clifford algebra $\text{Cl}(V, Q)$ is the pin group $\text{Pin}(V, Q)$ generated by $S(V)$ and its even subgroup $\text{Spin}(V, Q) = \text{Pin}(V, Q) \cap \text{Cl}(V, Q)^0$. When V is real and Q is definite these are compact Lie groups. In that case we can average a metric over a real or complex Clifford module $\mathbb{S} = \mathbb{S}^0 \oplus \mathbb{S}^1$ so that $\text{Pin}(V, Q)$ acts orthogonally (unitarily in the complex case). It follows that $e \in S(V)$ is self- or skew-adjoint, according as Q is positive or negative definite. (Here we use a sign-adjust notion of adjoint, alluded to in Remark 9.37.)

Lecture 10: Bosons and fermions

We revisit Axiom System 1.1, which is the axiom system for states and observables. Now we incorporate two types of states: bosonic states and fermionic states as well as mixtures. The vector space of observables is now $\mathbb{Z}/2\mathbb{Z}$ -graded, and the homogeneous elements are called bosonic observables and fermionic observables. Measurement can be carried out only for bosonic observables.

Then we turn to QM systems, where the basic data is linear ([Definition 5.1](#)) or, better, projective ([Definition 8.27](#)). Here we incorporate fermionic states and observables by postulating that the (projective) Hilbert space be $\mathbb{Z}/2\mathbb{Z}$ -graded.

The bulk of the lecture gives examples of QM systems with fermions. The first is a fermionic analog of the harmonic oscillator. The harmonic oscillator—bosonic or fermionic—has a canonical projective incarnation as an irreducible representation of a Heisenberg group. We develop this idea in the bosonic case, including the intertwining projective representation of the symplectic group. We conclude by reinterpreting the QM system in §7.1 as a system with fermions, and we exhibit its supersymmetry at the quantum level, at least in special cases.

10.1 General mechanical systems with fermions

For the generalization of [Axiom System 1.1](#) to incorporate fermions we only indicate the modifications in the data, as listed there.

- (1) There are two real convex spaces $\mathcal{S}^0, \mathcal{S}^1$ whose elements are called *bosonic states* and *fermionic states*, respectively. The labels ‘0’ and ‘1’ lie in $\mathbb{Z}/2\mathbb{Z}$. A general state is a convex combination of a bosonic and fermionic state, i.e., it is a point in the *join*²³

$$(10.1) \quad \mathcal{S} = \mathcal{S}^0 * \mathcal{S}^1.$$

A pure state in $\mathcal{S}$ is either a pure state in $\mathcal{S}^0$ or a pure state in $\mathcal{S}^1$: it is either bosonic or fermionic.

- (2) The vector space $\mathcal{O}$ is $\mathbb{Z}/2\mathbb{Z}$ -graded, the real structure is even, there is a super Lie algebra structure on $\mathcal{O}^\infty$, and the map [\(1.2\)](#) applies only to even elements of $\mathcal{O}_\mathbb{R}$:

$$(10.2) \quad \begin{aligned} \text{Borel}(\mathbb{R}; \mathbb{R}) \times \mathcal{O}_\mathbb{R}^0 &\longrightarrow \mathcal{O}_\mathbb{R}^0 \\ f, A &\longmapsto f(A) \end{aligned}$$

Elements of $\mathcal{O}_\mathbb{R}^0$ are (*even*) *observables* and elements of $\mathcal{O}_\mathbb{R}^1$ are (*odd*) *observables*.

- (3) Only bosonic observables can be measured:²⁴

$$(10.3) \quad \begin{aligned} \mathcal{O}_\mathbb{R}^0 \times \mathcal{S} &\longrightarrow \text{Prob}(\mathbb{R}) \\ A, \sigma &\longmapsto \sigma_A \end{aligned}$$

- (4) Only bosonic observables—those in $(\mathcal{O}_\mathbb{R}^\infty)^0$ —generate one-parameter groups, and the action on states preserves the decomposition [\(10.1\)](#).

Remark 10.4. One should relax (1) as follows: the labeling set of the spaces $\mathcal{S}^0, \mathcal{S}^1$ can be *any* 2-element set. In other words, rather than labeling by the cyclic group $\mathbb{Z}/2\mathbb{Z}$, which has a definite identity element and a definite non-identity element, we allow the labeling to be a $\mathbb{Z}/2\mathbb{Z}$ -torsor. This allows the two sets of states to be exchanged by symmetries, and also allows such an exchange in a family of systems.²⁵ Symmetries that exchange the two spaces of states are called *supersymmetries*.

²³The join of topological spaces X, Y is the quotient of the Cartesian product $[0, 1] \times X \times Y$ by the subspaces $\{0\} \times \{x\} \times Y$ and $\{1\} \times X \times \{y\}$ for each $x \in X, y \in Y$; see [\[H, p. 9\]](#).

²⁴For QM systems with fermions, [Remark 9.36](#) indicates that measurement of fermionic observables is absurd.

²⁵If a family is parametrized by a space or stack S , then the labels form the total space of a double cover of S .

For a motion ([Definition 1.11](#)), the constituent one-parameter groups of automorphisms of $\mathcal{S}$ and of $\mathcal{O}$ are required to preserve the $\mathbb{Z}/2\mathbb{Z}$ -gradings.

10.2 Abstract QM systems with fermions

In [Lecture 1](#) we deduced the data (1)–(4) above from a separable complex Hilbert space. In [Lecture 8](#) we explored the projective nature of quantum mechanics and told how to specify a projective space rather than a Hilbert space. The analog of the latter discussion that incorporates fermionic states is interesting, and we will make comments below, but we begin with linear QM data ([Definition 5.1](#)).

Let $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ be a super Hilbert space.²⁶ Recall from the comments following ([9.34](#)) that each ungraded vector space $\mathcal{H}^0, \mathcal{H}^1$ inherits a Hilbert space structure. Define a mechanical system with fermions as follows:

- (1) The space $\mathcal{S}^0$ of bosonic states is the convex space of self-adjoint endomorphisms S of $\mathcal{H}^0$ that are nonnegative, trace class, and trace $S = 1$. Fermionic states are self-adjoint endomorphisms of $\mathcal{H}^1$ that satisfy the same conditions. Pure states are the disjoint union of two projective spaces:

$$(10.5) \quad \mathcal{PS} = \mathbb{P}\mathcal{H}^0 \amalg \mathbb{P}\mathcal{H}^1.$$

- (2) Set $\mathcal{O} = \text{End } \mathcal{H} = \text{End}^0 \mathcal{H} \oplus \text{End}^1 \mathcal{H}$. (Recall [Example 9.17\(2\)](#).) The real structure is the adjoint $A \mapsto A^*$, and the super Lie algebra structure is ([1.26](#)) modified by the Koszul sign rule ([9.11](#)):

$$(10.6) \quad [A_1, A_2]_{\hbar} = \frac{-i}{\hbar} (A_1 \circ A_2 - (-1)^{|A_1||A_2|} A_2 \circ A_1),$$

An *even* self-adjoint operator has spectral measures—self-adjoint projection-valued “probability” measures on $\mathbb{R}$ given by the spectral theorem—but now two of them: one for the operator on $\mathcal{H}^0$ and one for the operator on $\mathcal{H}^1$. So there are two spectral decompositions, one bosonic and the other fermionic, so to speak. As mentioned in footnote ²⁴, it does not make sense to measure fermionic observables.

- (3) Measurement is as in ([1.27](#)) and ([1.28](#)), though one must account for the bosonic and fermionic spectra and apply the one appropriate to the state, which is either bosonic or fermionic.
- (4) The 1-parameter unitary group U_t in [Definition 5.1](#) is parity-preserving on $\mathcal{H}$, and it determines 1-parameter groups ([1.30](#)) of states and observables that preserve parity.

10.3 The fermionic harmonic oscillator

This is an analog of the abstract simple harmonic oscillator discussed in §[6.2](#). Let A be *odd*, and posit that $[A, A^*] = 1$. More formally, let $\mathcal{A}$ be the complex $*$ -superalgebra generated by the odd

²⁶As in [Remark 10.4](#), in place of a $\mathbb{Z}/2\mathbb{Z}$ -graded Hilbert space we can grade by a $\mathbb{Z}/2\mathbb{Z}$ -torsor.

element A subject to the relation $[A, A^*] = 1$. Note the Koszul sign rule for the commutator:

$$(10.7) \quad [A, A^*] = AA^* + A^*A.$$

The Koszul sign rule also implies $A^2 = 0$, and so $\mathcal{A}$ is 4-dimensional, with basis $1, A, A^*, A^*A$. Construct a module M over $\mathcal{A}$ of dimension $1|1$ as follows. Assume $\Omega \in M$ satisfies $A\Omega = 0$. Then

$$(10.8) \quad M = \mathbb{C}\langle\Omega\rangle \oplus \mathbb{C}\langle A^*\Omega\rangle$$

and the action of $\mathcal{A}$ is determined by the commutator $[A, A^*] = 1$. There is a choice of $\mathbb{Z}/2\mathbb{Z}$ -grading of M . If Ω is assumed even, then $A^*\Omega$ is odd; if Ω is assumed odd, then $A^*\Omega$ is even. More naturally, we do not make a choice and grade M by a $\mathbb{Z}/2\mathbb{Z}$ -torsor. For the fermionic harmonic oscillator with Hamiltonian

$$(10.9) \quad H = A^*A + 1/2$$

the lines in (10.8) are eigenlines with eigenvalues $1/2, 3/2$. In terms of the axioms for a mechanical system, the spaces $\mathcal{S}^0, \mathcal{S}^1$ of bosonic and fermionic states each consist of a single point. The $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $\mathcal{O}$ is the underlying vector space of the superalgebra $\mathcal{A}$.

There is a generalization parallel to our description (§6.3) of the polarized bosonic harmonic oscillator. Let W be a finite dimensional real vector space.²⁷ Set

$$(10.10) \quad F(W) = \bigwedge^\bullet W_{\mathbb{C}}^*.$$

This exterior algebra, which is finite dimensional and $\mathbb{Z}$ -graded, is the *fermionic Fock space* of W . (Note that we only use the quotient $\mathbb{Z}/2\mathbb{Z}$ -grading of the $\mathbb{Z}$ -grading.) For $w \in W$ and $\ell \in W^*$ define linear operators A_w and M_ℓ of degrees -1 and $+1$, respectively:

$$(10.11) \quad \begin{aligned} A_w(\theta) &= \theta(w), & \theta &\in \bigwedge^1 W_{\mathbb{C}}^* \\ M_\ell(\omega) &= \ell \wedge \omega, & \omega &\in F(W), \end{aligned}$$

and extend A_w to be a derivation: $A_w(\omega_1 \wedge \omega_2) = A_w(\omega_1) \wedge \omega_2 + \omega_1 \wedge A_w(\omega_2)$ for all $\omega_1, \omega_2 \in F(W)$. Then

$$(10.12) \quad [A_w, M_\ell] = \ell(w)$$

is scalar multiplication by $\ell(w)$.

Remark 10.13.

²⁷As above, there is an important variation for infinite dimensional topological vector spaces with a polarization; see [PS, §12].

- (1) The *odd*²⁸ real vector space of operators A_w, M_ℓ is the quotient in a central extension

$$(10.14) \quad 0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{h} \longrightarrow \Pi(W \oplus W^*) \longrightarrow 0$$

of real super Lie algebras. The commutator (10.12) induces an orthogonal structure on $W \oplus W^*$ which is *split*: the signature²⁹ of the nondegenerate symmetric bilinear form is (N, N) if $\dim W = N$. There is a generalization of this model in which we replace $W \oplus W^*$ by a finite dimensional real orthogonal vector space, often taken to be positive definite.

- (2) There are operators A_w, M_ℓ for w, ℓ in the complexifications $W_{\mathbb{C}}, W_{\mathbb{C}}^*$, and they generate a finite dimensional complex algebra: the Clifford algebra.

One can define a Hamiltonian as in the bosonic case; see (6.23) and the text that follows, and also note the special case (10.9) above.

10.4 Schur's lemma

As a preliminary, we review a basic result about irreducible representations.

Let G be a (Lie) group. We work over $\mathbb{C}$ though any algebraically closed field will do. Recall that a representation on a vector space W is *irreducible* if W has no proper G -invariant subspaces. (If W has a topology, then we ask for no *closed* G -invariant subspaces.) Also, recall that a *morphism* of G -representations $\pi': G \rightarrow \text{Aut } W'$ and $\pi: G \rightarrow \text{Aut } W$ is a linear map $T: W' \rightarrow W$ such that $T \circ \pi'(g) = \pi(g) \circ T$ for all $g \in G$.

Theorem 10.15. *Suppose $\pi: G \rightarrow \text{Aut } W$ is an irreducible finite dimensional linear representation of G on a complex vector space W . Then if $T: W \rightarrow W$ is an endomorphism of π , there exists $\lambda \in \mathbb{C}$ such that $T = \lambda \text{id}_W$.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of T . The λ -eigenspace $\ker(\lambda \text{id}_W - T)$ is a nonzero G -invariant subspace of W , hence equals W by the assumption that π is irreducible. $\square$

Corollary 10.16. *Let G be a (Lie) group. Then to each isomorphism class ρ of irreducible linear representation of G is canonically attached a projective space $\mathbb{P}_\rho$ with a G -action.*

In other words, the associated projective representation is canonical (whereas the linear representation is only canonical up to tensoring by a line).

Next is Schur's lemma for unitary representations, which may be infinite dimensional.

Theorem 10.17. *Suppose $\pi: G \rightarrow U(\mathcal{H})$ is an irreducible unitary representation of G on a complex Hilbert space $\mathcal{H}$. Then if $T: \mathcal{H} \rightarrow \mathcal{H}$ is a unitary automorphism of π , there exists $\lambda \in \mathbb{T}$ such that $T = \lambda \text{id}_W$.*

²⁸Let Π denote the ground field $\mathbb{F}$, viewed as a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space which is odd. Then for any super vector space V , let $\Pi V = \Pi \otimes V$. We call ΠV the *parity-reversal* of V .

²⁹Here I use 'signature' for the pair (p, n) of dimensions of maximal positive and negative definite subspaces. Usually 'signature' is the integer $p - n$, which is zero here.

Proof. There is a spectral theorem for unitary operators, which produces a self-adjoint projection-valued measure on $\mathbb{T} \subset \mathbb{C}$. If that spectral measure of T is not concentrated at a point, then there is a self-adjoint projection whose kernel is a closed proper G -invariant subspace. (Alternative proof: Apply the spectral theorem for self-adjoint operators to $T + T^*$ and $i(T - T^*)$.) $\square$

Corollary 10.18. *Let $\pi': G \rightarrow U(\mathcal{H}')$ and $\pi: G \rightarrow U(\mathcal{H})$ be isomorphic unitary irreducible representations. Then*

- (1) *the collection of unitary morphisms $\mathcal{H}' \rightarrow \mathcal{H}$ of G -representations is a $\mathbb{T}$ -torsor; and*
- (2) *There is a unique isomorphism $\mathbb{P}\mathcal{H}' \rightarrow \mathbb{P}\mathcal{H}$ which preserves the Fubini–Study structures and commutes with the G -actions.*

See (8.19) for the definition of Fubini–Study geometry in finite dimensions. Also, there is an analog of Corollary 10.16 for unitary representations.

Corollary 10.19. *Let $\pi: G \rightarrow U(\mathcal{H})$ be an irreducible unitary representation of G on a complex Hilbert space $\mathcal{H}$, and assume given an automorphism $\alpha: G \rightarrow G$ of G such that the composition $G \xrightarrow{\alpha} G \xrightarrow{\pi} U(\mathcal{H})$ is isomorphic to π as a G -representation. Then there is a $\mathbb{T}$ -torsor of unitary operators on $\mathcal{H}$ which implement α as an intertwiner. As a consequence, there is a unique projective unitary automorphism of $\mathbb{P}\mathcal{H}$ which implements α as an intertwiner.*

Recall the projective space $\mathbb{P}_\rho$ in Corollary 10.16.

Corollary 10.20. *Let G be a (Lie) group, and let ρ be an isomorphism class of irreducible unitary representations of G . Suppose H is a group of automorphisms of G such that each $h \in H$ preserves³⁰ ρ . Then there exists a canonical extension of the G -action on $\mathbb{P}_\rho$ to an action of the semidirect product $H \ltimes G$.*

10.5 The bosonic harmonic oscillator revisited

We build on the material in §6.4.

Let A be a finite dimensional real symplectic affine space. Thus A is affine over a finite dimensional real vector space V which is equipped with a nondegenerate skew-symmetric pairing $\omega: V \times V \rightarrow \mathbb{R}$. The pairing is then a translation-invariant symplectic form on A . The symmetry groups of A and V fit into a group extension

$$(10.21) \quad 1 \longrightarrow V \longrightarrow \mathrm{Sp}(A) \longrightarrow \mathrm{Sp}(V) \longrightarrow 1$$

in which $\mathrm{Sp}(V)$ is the group of linear automorphisms of V that preserve ω , and $\mathrm{Sp}(A)$ is the group of affine automorphisms of A that preserve ω . The kernel V consists of translations in A . Define

$$(10.22) \quad \widehat{\mathrm{Sp}}(V) = \{T: V \rightarrow V \text{ real linear} : T^*\omega = \pm\omega\},$$

³⁰This means that if $\pi: G \rightarrow \mathrm{Aut} W$ is a linear representation in the isomorphism class ρ , then the composition $G \xrightarrow{h} G \xrightarrow{\pi} \mathrm{Aut} W$ is also in the isomorphism class ρ

and let $\widehat{\text{Sp}}(A)$ be the corresponding 2-component Lie group of affine symmetries. These fit into group extensions

$$\begin{aligned}
 (10.23) \quad & 1 \longrightarrow \text{Sp}(V) \longrightarrow \widehat{\text{Sp}}(V) \longrightarrow \mu_2 \longrightarrow 1 \\
 & 1 \longrightarrow \text{Sp}(A) \longrightarrow \widehat{\text{Sp}}(A) \longrightarrow \mu_2 \longrightarrow 1 \\
 & 1 \longrightarrow V \longrightarrow \widehat{\text{Sp}}(A) \longrightarrow \widehat{\text{Sp}}(V) \longrightarrow 1
 \end{aligned}$$

Remark 10.24. A particular case is the model geometry in which $V = \mathbb{R}^{2m}$ acts on $A = \mathbb{A}^{2m}$ and ω is a standard symplectic form on $\mathbb{R}^{2m}$. It is this model geometry that is used to make the canonical constructions.

Recall the Heisenberg Lie algebra $\mathfrak{h}(A)$, defined in (6.27) as the Lie algebra of affine functions $A \rightarrow \mathbb{R}$ with its Poisson bracket. The Heisenberg group $H(A)$ is the corresponding Lie group; it fits into the group extension

$$(10.25) \quad 1 \longrightarrow \mathbb{T} \longrightarrow H(A) \longrightarrow \exp V \longrightarrow 1$$

The Stone-von Neumann [Theorem 6.29](#) asserts that there is a canonical isomorphism class of irreducible unitary representations of $H(A)$ on which the center $\mathbb{T} \subset H(A)$ acts by scalar multiplication. Note that since the scalars $\mathbb{T} \subset H(A)$ act by homotheties on the unitary representation of [Theorem 6.29](#), they act trivially on the projectivization of any unitary representation in the isomorphism class. Now our discussion of Schur's lemma in §10.4—in particular, the infinite dimensional unitary analog of [Corollary 10.16](#)—immediately implies the following.

Corollary 10.26. *There is a canonical Fubini–Study projective space $\mathbb{P}_A$ attached to the symplectic vector space A , and it carries a projective unitary action of $\exp V$.*

The connected Lie group $\text{Sp}(A)$ acts by conjugation on the normal translation subgroup V , the action drops to the quotient group $\text{Sp}(V)$, and by the Stone-von Neumann theorem this action preserves the isomorphism class of the Heisenberg representation. Therefore, by [Corollary 10.19](#) there exists a canonical projective unitary action of $\text{Sp}(A)$ on $\mathbb{P}_A$, and furthermore it extends the action of V . In a linearization the symplectic group $\text{Sp}(V)$ acts through a central extension, the metaplectic group.³¹ Altogether ([Corollary 10.20](#)) we obtain a projective action of $\text{Sp}(V) \ltimes V$ on $\mathbb{P}_A$.

By our discussion of Schur's lemma we can also implement the larger group $\widehat{\text{Sp}}(A)$ as projective transformations of $\mathbb{P}_A$, but the off-component acts by projective *anti*unitaries. This follows since elements in the off-component reverse the sign of the symplectic form ω , hence map the Heisenberg Lie algebra $\mathfrak{h}(A)$ to the Lie algebra obtained from the negative symplectic form. On the group level (10.25) this action inverts the kernel $\mathbb{T}$, which acts by scalar multiplication in the Heisenberg representation: this inversion is the antilinearity. The projective space $\mathbb{P}_A$ has a projective quantum geometric structure, as in [Definition 8.27](#), but now in infinite dimensions.

This discussion is summarized as follows. Recall the model group PQ of projective unitary and antiunitary transformations in (8.48).

³¹The metaplectic group has kernel μ_2 whereas this central extension has kernel $\mathbb{T}$, so is called the “metaplectic-*c*” group by analogy with the spin-*c* group.

Theorem 10.27. *There is a canonical projective representation*

$$(10.28) \quad \widehat{\mathrm{Sp}}(\mathbb{A}^{2m}) \longrightarrow \mathrm{PQ}.$$

Remark 10.29.

- (1) The projective representation (10.28) does not lift to a linear representation into Q . The obstruction is *not* topological, though it can be measured in a suitable generalization of standard topological cohomology theories.
- (2) As explained above, the representation (10.28) associates a family of projective spaces to a fiber bundle of affine spaces via the associated principal $\widehat{\mathrm{Sp}}(\mathbb{A}^{2m})$ -bundle. For example, any symplectic manifold gives rise to such a family via its affine tangent bundle.
- (3) There is an infinite dimensional variant of (10.28); see [Sh, Se1, PS].

One can construct spaces of states and observables from the projective space, as explained in [Lecture 8](#). However, there is no canonical harmonic oscillator Hamiltonian that is invariant under $\widehat{\mathrm{Sp}}(\mathbb{A}^{2m})$. Rather, there is a class of Hamiltonians invariant under $\mathrm{Sp}(\mathbb{A}^{2m})$: elements in the (closure of the) second filtration of the dense algebraic subspace of the Heisenberg representation.

10.6 The fermionic harmonic oscillator revisited

There is a parallel discussion for the fermionic harmonic oscillator, but with some major differences: the vector space V is odd in the sense of superalgebra, the Heisenberg Lie algebra encodes canonical *anticommutation* relations, and the quantum system is $\mathbb{Z}/2\mathbb{Z}$ -graded and finite dimensional. Our discussion is abbreviated and only highlights these differences.

Let V be an odd finite dimensional real vector space, so $V = \Pi U$ for an even vector space U . Then a skew-symmetric bilinear form on V satisfies

$$(10.30) \quad \omega(v_1, v_2) = -(-1)^{|v_1||v_2|} \omega(v_2, v_1) = \omega(v_2, v_1),$$

so it comes from a symmetric bilinear form on U . We assume that this form is positive definite, i.e., is an inner product on U . For the odd vector space V , the symplectic group $\mathrm{Sp}(V)$ in (10.21) is the orthogonal group $\mathrm{O}(U)$, and for an odd affine space³² A over V the affine symplectic group is the Euclidean group of an associated even affine space. Observe that these Lie groups have two components: elements in the identity component preserve orientation and elements in the off-component reverse orientation. Now the analogs of $\widehat{\mathrm{Sp}}(V)$ and $\widehat{\mathrm{Sp}}(A)$ have 4 components: they are simply direct products of μ_2 with the unhatted groups. The Heisenberg group $H(V)$ is a super Lie group. For the analog of [Theorem 6.29](#) we need to assume that $\dim V$ is even.³³ Then the representation is on a $\mathbb{Z}/2\mathbb{Z}$ -graded finite dimensional complex Hilbert space $\mathbb{S} = \mathbb{S}^0 \oplus \mathbb{S}^1$, and [Theorem 9.54](#) states that the complex Clifford algebra on an even dimensional vector space is a super matrix algebra. The odd symplectic group $\mathrm{Sp}(V)$ (so the orthogonal group $\mathrm{O}(U)$) is

³²We have not defined this notion, so the reader may want to contemplate the definition.

³³If $\dim V$ is odd, then there is a representation on a module over the complex Clifford algebra $\mathrm{Cl}_1^{\mathbb{C}}$; see [FHT3] for a detailed discussion.

implemented projectively; it acts via the pin- c covering group. The analog of [Corollary 10.26](#) is a pair of projective spaces $\mathbb{P}_V^0, \mathbb{P}_V^1$ with an action of $\widehat{\mathrm{Sp}}(A)$. The numbering ‘0,1’ is *not* canonical: in a family whose base is not simply connected the two projective spaces can exchange as one traverses a closed loop.

Remark 10.31.

- (1) As opposed to [Remark 10.29](#)(1) in the even case, here the central extension of the orthogonal group is detected in topology: the class of the pin- c central extension is the Bockstein of the second Stiefel–Whitney class.
- (2) There is a family of odd harmonic oscillators (without a particular Hamiltonian) associated to a fiber bundle of Euclidean spaces, so in particular to a real vector bundle with an inner product. The obstruction to linearizing the family is the Bockstein of the second Stiefel–Whitney class of the vector bundle: a linearization is induced from a spin- c structure on the vector bundle. The family parametrized by a Riemannian manifold (using its tangent bundle) occurs in the simplest supersymmetric system: the superparticle.
- (3) There is an infinite dimensional variant [[PS](#), §12].

10.7 Addendum to [Section 7.1](#)

We revisit the QM system analyzed in [§7.1](#), now with the modification

$$(10.32) \quad \mathcal{H} = L^2(\mathbb{E}^1; \mathbb{C}^{1|1}) = \left\{ \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} : \psi^i \in L^2(\mathbb{E}^1; \mathbb{C}) \right\} = \mathcal{H}^0 \oplus \mathcal{H}^1.$$

In other words, we take the state space to be $\mathbb{Z}/2\mathbb{Z}$ -graded with the indicated grading. Recall the operator Q_h in [\(7.4\)](#). The adjoint is Q_h^* , but the formulas must be modified slightly due to the signs and factors of $\sqrt{-1}$ that occur in adjoints of operators on $\mathbb{Z}/2\mathbb{Z}$ -graded Hilbert spaces; see [\(9.35\)](#). Set $R_h = Q_h + Q_h^*$. Then the Hamiltonian is

$$(10.33) \quad H_h = \frac{1}{2} R_h^2.$$

Assume that R_h has no kernel, so that R_h is an isomorphism. For example, this occurs if h is a cubic polynomial ([§7.1.4](#)). Then R_h permutes the pure states (compare [\(7.5\)](#)):

$$(10.34) \quad \mathbb{P}\mathcal{H}^0 \xrightleftharpoons[R_h]{R_h} \mathbb{P}\mathcal{H}^1$$

A general bosonic or fermionic state S lies in $\mathrm{End} \mathcal{H}^0$ or in $\mathcal{H}^1$, and an observable A lies in $\mathrm{End} \mathcal{H}$. The action of R_h on pure states extends to an action

$$(10.35) \quad \begin{aligned} S &\longmapsto R_h S R_h^{-1} \\ A &\longmapsto R_h^{-1} A R_h \end{aligned}$$

on all homogeneous states and all observables. This symmetry commutes with the Hamiltonian [\(10.33\)](#), so it also commutes with the unitary flows the Hamiltonian generates. This is an example of a *supersymmetry*.

Lecture 11: Wick rotation, algebras of observables, and correlation functions

This lecture, the last on quantum mechanics *per se*, treats a few miscellaneous topics.

We begin with a brief discussion of affine time, emphasizing as always (§7.4) the symmetries. We always assume a *time-orientation*, which we sometimes call an *arrow of time*. But we also consider the larger symmetry group that reverses the arrow of time. We prove an easy, but basic, **Lemma 11.3** for symmetries of QM systems and its corollary: If the Hamiltonian H is unbounded above and bounded below, then a time-preserving symmetry acts unitarily and a time-orientation reversing symmetry acts antiunitarily. (If H is similar to $-H$, then this need not hold.)

Next, we give the definition of a *correlation function* in quantum mechanics. These are basic quantities that encode physical information.

Wick rotation applies to QM systems of positive energy (**Definition 5.7**). It expresses time evolution and correlation functions as boundary values of holomorphic expressions. In particular, the unitary time evolution is the boundary value of a holomorphic 1-parameter semigroup of contractive operators. Wick rotation [Wk] is the restriction to a real 1-parameter semigroup, an imaginary time evolution. The original context for Wick rotation is relativistic quantum field theory. In that context it leads to modern geometric axiom systems, as we shall see.

Our axiomatization of mechanical systems in general (**Axiom System 1.1**), and QM systems in particular, treats states and observables on an equal footing. With some additional hypotheses one can put observables in the primary position and treat states as a derived concept. This has a long history in quantum mechanics and quantum field theory, dating back to papers of Jordan–von Neumann–Wigner [JvNW] and Irving Segal [Seg]. Observables are assumed to form an algebra—in fact, a C^* -algebra (or perhaps another form of topological $*$ -algebra)—and states are special linear functionals on the algebra. We give a brief introduction to this viewpoint.

You may want to review the material on group extensions (§8.7.4).

11.1 Time, time-orientation, and symmetries

In mechanics, time is an oriented 1-dimensional Euclidean space M . Thus M is an affine space over a 1-dimensional oriented real inner product space U . The vector line U is the group of time translations. Note that U has a distinguished basis: a positively oriented vector of unit norm. The inner product on U can be regarded as a “clock”: it is used to measure elapsed time. The orientation of U allows to distinguish in which direction the clock is running. The symmetry group $O^\uparrow(M)$ of this structure sits in a group extension, and it is a subgroup of the group $O(M)$ of all isometries of M , time-orientation preserving or not:

$$(11.1) \quad \begin{array}{ccccccc} 1 & \longrightarrow & U & \longrightarrow & O^\uparrow(M) & \longrightarrow & O^\uparrow(U) \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & U & \longrightarrow & O(M) & \longrightarrow & O(U) \longrightarrow 1 \end{array}$$

Note that $O^\uparrow(U)$ consists only of the identity element, hence $O^\uparrow(M)$ consists only of (time) translations. The larger group $O(M)$ also contains (time) reflections.

Remark 11.2. The model geometry $G \curvearrowright F$ for time is the affine line $F = \mathbb{A}^1$ with symmetry group $G = \text{Euc}_1^\uparrow \cong \mathbb{R}$ translations of the affine line. Observe that the inner product on a real line is determined up to multiplication by a positive real number, and the symmetry groups (11.1) are independent of the inner product. Hence we need not fix an inner product on U . Indeed, an inner product fixes a clock to measure in seconds, or minutes, or hours, or light-years, depending on the scale of the inner product. It is sometimes convenient not to fix an inner product.

Let $(\mathcal{H}, U_t)$ be linear data for a QM system, and suppose H is the Hamiltonian: $U_t = e^{-itH/\hbar}$. (In terms of our previous discussion, $t \in U \cong \mathbb{R}$ is an interval of time.) Recall our discussion of quantum symmetries, and in particular Wigner's [Theorem 8.34](#). It states that the group of quantum symmetries sits in a group extension (8.48): each quantum symmetry g lifts to a real linear symmetry $\tilde{g}$ which is either unitary or antiunitary. Write $\phi(g) = \pm 1$ to track the two possibilities. Now we also consider how the symmetry acts on time, and we allow a quantum symmetry $g: \mathbb{P}\mathcal{H} \rightarrow \mathbb{P}\mathcal{H}$ to act by preserving or reversing time. Let $t(g) = \pm 1$ track this dichotomy.

Lemma 11.3.

(1)

$$(11.4) \quad \tilde{g}^{-1}H\tilde{g} = \phi(g)t(g)H.$$

(2) If H is bounded below and unbounded above, then $\phi(g) = t(g)$.

Proof. Symmetries act on time evolution by conjugation, and they may preserve or reverse the direction of that evolution. Hence for all $t \in \mathbb{R}$ we have

$$(11.5) \quad \tilde{g}^{-1}e^{-itH/\hbar}\tilde{g} = e^{t(g)(-itH)/\hbar},$$

from which we deduce

$$(11.6) \quad \tilde{g}^{-1}(iH)\tilde{g} = t(g)(iH),$$

and then (11.4). For (2) observe that under the assumptions given, H is not conjugate to $-H$. $\square$

Remark 11.7. In many quantum systems, for example most quantum field theories, the Hamiltonian satisfies the conditions in (2) and so time-preserving symmetries are unitary and time-orientation reversing symmetries are antiunitary. There are exceptions, notably in finite dimensional systems. For a more elaborate discussion of symmetry in QM systems, see [\[FM1\]](#).

11.2 Correlation functions

Suppose given a mechanical system as in [Axiom System 1.1](#). Let M be a time line. It is natural in this context to assume that the observables form a complex vector bundle $\mathcal{O} \rightarrow M$, and the smooth observables form a dense subbundle $\mathcal{O}^\infty \rightarrow M$. Time evolution lifts time translation $U \curvearrowright M$ to $U \curvearrowright \mathcal{O}$. This is the time evolution of observables: the Heisenberg picture in the case of QM systems.

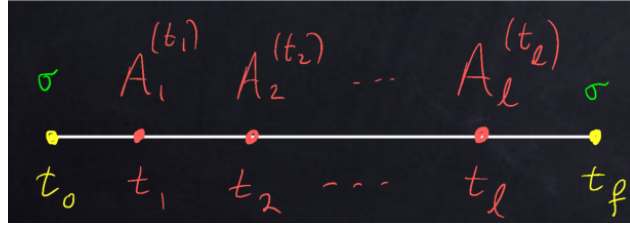


FIGURE 24. A correlation function

Suppose $(\mathcal{H}, U_t)$ is the linear data of a QM system. Fix $t_0 < t_1 < \cdots < t_\ell < t_f$ a finite set of ordered points in M . Let $A_i^{(t_i)} \in \mathcal{O}_{t_i}$, $i \in \{1, \dots, \ell\}$ be observables. Fix a state $\sigma \in \mathcal{S}$. We illustrate this data in [Figure 24](#). For simplicity assume that σ is a pure state—a line $L \subset \mathcal{H}$ —and choose a unit norm vector $\psi \in L$. Then the *correlation function* is the complex number

$$(11.8) \quad \langle A_\ell^{(t_\ell)} \cdots A_1^{(t_1)} \rangle_{[t_0, t_f]}^\sigma = \langle U_{t_0-t_f} \psi, A_\ell^{(t_\ell)} \circ \cdots \circ A_1^{(t_1)} \psi \rangle$$

Note we have translated the (final) state σ back to the initial time t_0 . (We can have different initial and final states.) Let us rewrite (11.8) in the Schrödinger picture. Assume that each observable is the time translate of an observable at initial time:

$$(11.9) \quad A_i^{(t_i)} = U_{t_0-t_i} A_i U_{t_i-t_0}, \quad i \in \{1, \dots, \ell\},$$

for some observable $A_i \in \mathcal{O}_{t_0}$. Then the correlation function is

$$(11.10) \quad \langle A_\ell^{(t_\ell)} \cdots A_1^{(t_1)} \rangle_{[t_0, t_f]}^\sigma = \langle \psi, U_{t_f-t_\ell} A_\ell \cdots A_2 U_{t_2-t_1} A_1 U_{t_1-t_0} \psi \rangle$$

The last expression reads: begin in the pure state $\mathbb{C}\langle\psi\rangle$, evolve for time $t_1 - t_0$, apply observable A_1 , evolve for time $t_2 - t_1$, apply observable A_2 , $\dots$, apply observable A_ℓ , evolve for time $t_f - t_\ell$. Often one takes σ to be a vacuum state, in which case (11.10) is called a *vacuum correlation function*.

Remark 11.11. The correlation function (11.10) only involves differences of time, which lie in U . That reflects the time translation symmetry.

11.3 Wick rotation

Let $\mathcal{H}$ be a finite dimensional hermitian vector space, and suppose $H \in \text{End } \mathcal{H}$ is self-adjoint. The real 1-parameter group³⁴ $t \mapsto e^{-itH}$, $t \in \mathbb{R}$ of unitary operators has an extension to a complex 1-parameter group of endomorphisms: simply take the parameter t to be a complex number. For each $t \in \mathbb{C}$ the operator e^{-itH} is of course bounded, since $\mathcal{H}$ is finite dimensional, though the family of operators may not be uniformly bounded. But if $\mathcal{H}$ is infinite dimensional, then the operator e^{-itH} may be unbounded for fixed t . Note that for $\lambda \in \mathbb{R}$,

$$(11.12) \quad |e^{-it\lambda}| = e^{\text{Im}(t)\lambda}.$$

If $\lambda \geq 0$ and $\text{Im}(t) < 0$, then $|e^{-it\lambda}| \leq 1$. Hence, if H is a nonnegative operator, then for $\text{Im}(t) < 0$ we have $\|e^{-itH}\| \leq 1$ in operator norm.

³⁴In the application to quantum mechanics we have set $\hbar = 1$.

Definition 11.13. Let $\mathcal{H}$ be a Hilbert space. A linear operator $C: \mathcal{H} \rightarrow \mathcal{H}$ is *contractive* if its operator norm satisfies $\|C\| \leq 1$.

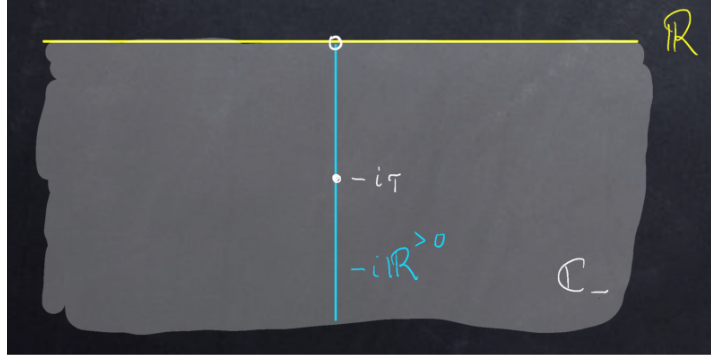


FIGURE 25. Wick rotation from real time to imaginary time

Let $(\mathcal{H}, U_t)$ be the linear data of a QM system that has *positive energy* (Definition 5.7): the Hamiltonian is nonnegative. Apply the preceding to the time evolution, as illustrated in Figure 25. Real time evolution in quantum mechanics is a 1-parameter real group of unitary operators. Under the positive energy hypothesis, this unitary evolution is the boundary value of a *holomorphic* 1-parameter semigroup³⁵ of contractive operators parametrized by

$$(11.14) \quad \mathbb{C}_- = \{t \in \mathbb{C} : \text{Im}(t) < 0\}.$$

Now restrict to the real semigroup $-i\mathbb{R}^{>0} \subset \mathbb{C}_-$. Thus for $\tau > 0$ set $t = -i\tau$ to obtain

$$(11.15) \quad \tau \mapsto C_\tau = e^{-\tau H}, \quad \tau \in \mathbb{R}^{>0}.$$

This real semigroup of contractive operators is usually better behaved analytically than the group of unitary operators. The analytic difference is more stark when Wick rotating correlation functions, especially in quantum field theory.

Example 11.16. For the quantum mechanical particle on a closed Riemannian manifold M , as in Lecture 5, the Hilbert space is $\mathcal{H} = L^2(M; \mathbb{C})$ and the Hamiltonian $H = \Delta$ is the Laplace operator. The Wick-rotated “evolution” is via the heat operator $e^{-\tau H}$, $\tau > 0$. This operator is not only contractive, but it is smoothing: the image of an L^2 function is a smooth function. In fact, it extends to distributions, which then also map to smooth functions. Of course, the unitary evolution operator does not have that property.

There is also a Wick rotation of correlation functions. For example, the correlation function (11.10) with $\ell = 1$ (for convenience) rotates to a function of imaginary time differences³⁶

$$(11.17) \quad \begin{aligned} t_1 - t_0 &= -i\tau_1 \\ t_2 - t_1 &= -i\tau_2 \\ t_f - t_2 &= -i\tau_f \end{aligned}$$

³⁵A *semigroup* is a set with an associative composition law. If it has a unit, then it is a monoid.

³⁶The times t_0, t_1, t_2, t_f live on an affine time line M ; evolution is a function of time differences in a vector line U , and these are what appear in (11.10), as noted in Remark 11.11. Their imaginary counterparts appear in the Wick-rotated correlation function (11.18).

using the contractive evolution (11.15):

$$(11.18) \quad \langle \psi, C_{\tau_f} A_2 C_{\tau_2} A_1 C_{\tau_1} \psi \rangle$$

11.4 Algebras of observables

We already listed a few of the original references [JvNW, Seg] at the beginning of the lecture; see also [Str, §1.3] and [Ha, §I.1] for further discussions. For example, there you will find elaborations of an argument justifying restriction to *bounded* operators; the argument is based on the fact that real-world measurements are bounded.

In the general **Axiom System 1.1** for mechanics, the vector space $\mathcal{O}$, whose real points are observables, is not assumed to be an algebra. In the main examples—classical or statistical mechanics on a Riemannian manifold, quantum mechanics in which we take $\mathcal{O}$ to consist of *bounded* operators—it *does* have the structure of an associative unital algebra. Also, the convex space of states is separate data, not derived from $\mathcal{O}$. Observe that a state $\sigma \in \mathcal{S}$ determines a real linear functional on observables:

$$(11.19) \quad \begin{array}{ccc} \mathcal{O}_{\mathbb{R}} & \longrightarrow & \text{Prob}(\mathbb{R}) \xrightarrow{E} \mathbb{R} \\ A & \longmapsto & \sigma_A \longmapsto \langle \sigma_A \rangle \end{array}$$

where E maps a probability distribution to its expectation value. The linearity follows from (1.4). Recall also that Borel functions of observables are observables, so moments of σ_A are computed by applying (11.19) to powers of A . A probability measure is not determined by its moments in general, though it is if the moments satisfy a bound on their growth. In any case we can take this as heuristic motivation for the idea that states can be defined as linear functionals of observables.

Based on such considerations, and many more as explained in the references, one often postulates that in quantum mechanics the complex vector space $\mathcal{O}$ (of complex observables) is a *C*-algebra*. We pause to give basic definitions.

Recall that a *Banach space* is a normed linear space which is complete in the metric topology induced by the norm.

Definition 11.20. Let $\mathcal{O}$ be a complex algebra

- (1) A *Banach algebra* structure on $\mathcal{O}$ is a Banach norm compatible with the algebra structure:

$$(11.21) \quad \begin{aligned} \|A_1 \circ A_2\| &\leq \|A_1\| \|A_2\|, & \text{for all } A_1, A_2 \in \mathcal{O}, \\ \|1\| &= 1 \end{aligned}$$

- (2) If $\mathcal{O}$ is a ***-algebra, then the Banach norm induces the structure of a *Banach *-algebra* if

$$(11.22) \quad \|A^*\| = \|A\| \quad \text{for all } A \in \mathcal{O}.$$

- (3) A Banach ***-algebra is a *C*-algebra* if, in addition,

$$(11.23) \quad \|A^* A\| = \|A\|^2 \quad \text{for all } A \in \mathcal{O}.$$

Example 11.24.

- (1) Let X be a compact Hausdorff topological space, and let $C(X; \mathbb{C})$ denote the vector space of continuous complex-valued functions on X . It is an *commutative* algebra by pointwise addition and multiplication. The sup norm ($\|f\| = \max_{x \in X} |f(x)|$) is a Banach space norm. Complex conjugation provides a $*$ -structure. Every commutative C^* -algebra is isomorphic to $C(X; \mathbb{C})$ for some compact Hausdorff space X .
- (2) Let $\mathcal{H}$ be a complex Hilbert space, finite dimensional or separable infinite dimensional. Then the $*$ -algebra $\text{End } \mathcal{H}$ of bounded linear operators on $\mathcal{H}$, equipped with the operator norm ($\|T\| = \sup_{\xi \in \mathcal{H}, \|\xi\|=1} \|T\xi\|$), is a C^* -algebra. It is not commutative as long as $\dim \mathcal{H} > 1$.

Basic operator theory of bounded operators on Hilbert space generalize to C^* -algebras. The following is just a taste.

Definition 11.25. Let $\mathcal{O}$ be a C^* -algebra. Fix $A \in \mathcal{O}$.

- (1) The *spectrum* of A is

$$(11.26) \quad \text{spec } A = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible in } \mathcal{O}\}.$$

- (2) A is *self-adjoint* if $A^* = A$.
- (3) A is *nonnegative* if A is self-adjoint and $\text{spec } A \subset \mathbb{R}$.

The spectrum is a compact subset of $\mathbb{C}$; the spectrum of a self-adjoint element is a subset of $\mathbb{R} \subset \mathbb{C}$. An element A is nonnegative iff it has the form $A = B^*B$ for some $B \in \mathcal{O}$.

The GNS (Gelfand–Naimark–Segal) construction implies that any C^* -algebra is isomorphic to a closed $*$ -subalgebra of $\text{End } \mathcal{H}$ for some Hilbert space $\mathcal{H}$, though this presentation is not canonical. There are many references on C^* -algebras; for example, Chapter 1 of the lecture notes [Pu] summarizes the basics.

With this background we can define states in terms of observables, assuming observables form a C^* -algebra.

Definition 11.27. Let $\mathcal{O}$ be a C^* -algebra. A *state* on $\mathcal{O}$ is a continuous linear map $\omega: \mathcal{O} \rightarrow \mathbb{C}$ such that (i) $\omega(A) \geq 0$ for all nonnegative $A \in \mathcal{O}$, and (ii) $\omega(1) = 1$.

States form a convex space; as before, pure states are the extreme points.

Remark 11.28. Algebra-like structures have been a recent focus in Wick-rotated field theory: factorization algebras [CG1, CG2] and, in a topological context, E_n -algebras.

Lecture 12: Affine spacetimes

So far in our discussion of mechanics we have treated time and space separately, to the extent that we've treated space at all. In this lecture we combine time and space into a single geometric

structure: spacetime. The spacetime structures we introduce are affine: structures on affine space. (As with other affine geometric structures, we can transport them to smooth manifolds.)

The classical version is a *Galilean structure*, whose signal feature is simultaneity. This is the natural home for Newton’s equations, which we touch upon only briefly: it is non-relativistic, hence non-realistic, hence non-relevant; its interest for us is mostly historical and pedagogical. (But see [Ar] for applications to Newtonian mechanics.) One of the most important developments of early 20th century physics is Einstein’s introduction of *Minkowski spacetime*, based on a Lorentz metric. This requires the specification of a universal physical constant: the speed of light c . (A Galilean spacetime has separate metrics for time and space, whereas a Lorentz metric mixes them; the speed of light is needed to convert from lengths to time intervals.) In the second part of this lecture we develop the linear algebra underpinning: the geometry of a real vector space equipped with a Lorentz metric. Some statements, such as the Cauchy–Schwarz inequality for timelike vectors, are different than in the positive definite case.

Our spacetimes are equipped with a time-orientation. This then orients worldlines that represent motions: there is a *forward* direction. In the dual spacetime the time-orientation induces a notion of *positive energy*, which we saw in [Lecture 11](#) is what enables Wick rotation. Also, typically only time-orientation preserving symmetries of spacetime act as linear symmetries of a quantum mechanical system; time-orientation preserving symmetries of spacetime typically act antilinearly (see [Lemma 11.3](#)).

12.1 Euclidean space

Fix the dimension of space to be $d \in \mathbb{Z}^{\geq 0}$.

Let W be a d -dimensional inner product space, and suppose E is an affine space over W . Then E is called a *Euclidean space*. Its symmetry group $O(E)$ is the classical group of Euclidean isometries, and it fits into a group extension

$$(12.1) \quad 0 \longrightarrow W \longrightarrow O(E) \longrightarrow O(W) \longrightarrow 1$$

in which $O(W)$ is the group of linear orthogonal transformations of the vector space W . The group $O(E)$ has two components, distinguished by whether a transformation preserves or reverses the orientation of space. Some mechanical systems have the full group $O(E)$ of isometries as symmetries; others only have the subgroup $SO(E)$ of orientation-preserving isometries.

Example 12.2.

- (1) A “free” particle moving in E (with zero potential energy function) as in [\(1.16\)](#) has the full group $O(E)$ of isometries.
- (2) Recall the particle on a ring ([Example 3.56](#)). The second term in the Lagrangian density [\(3.57\)](#) is not invariant under spatial reflection $x \mapsto -x$.

12.2 Galilean spacetime

We use the notation $n = d + 1$ for spacetime dimension.

Definition 12.3. Let V be a real vector space of dimension n . A *Galilean structure* Γ on V is: (i) a codimension one subspace $W \subset V$ equipped with an inner product; and (ii) an orientation on V/W . A *Galilean spacetime* M is an affine space over a vector space equipped with a Galilean structure.

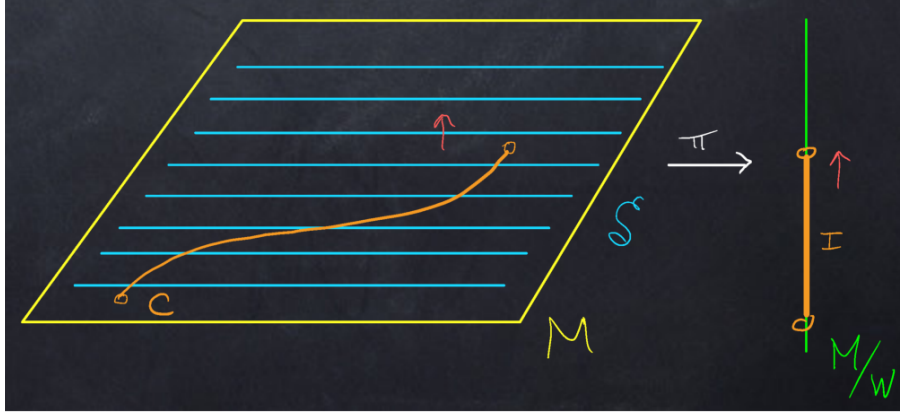


FIGURE 26. Galilean spacetime M and the simultaneity foliation $\mathcal{S}$

A Galilean spacetime M is depicted in Figure 26. The orbits of translation by $W \subset V$ on M give a codimension 1 foliation $\mathcal{S}$ by cooriented affine subspaces: the *simultaneity foliation*. Spacetime points on the same leaf represent simultaneous “events” in spacetime. The coorientation, which is induced from the orientation of V/W , gives a notion of forward time. The inner product on W induces a Euclidean structure on each leaf, which then is an affine model of space as in the previous subsection. The quotient affine space M/W is an affine model of time, as in §11.1. Note that we do not fix an inner product—a “clock”—on V/W ; see Remark 11.2. Observe that in a Galilean spacetime M there is no distinguished complement to $W \subset V$, which would give a distinguished time direction in M . Let

$$(12.4) \quad \pi: M \longrightarrow M/W$$

be the quotient map.

Definition 12.5. Let M be a Galilean spacetime. A *worldline* in M is a connected 1-dimensional submanifold $C \subset M$ that is transverse to the simultaneity foliation $\mathcal{S}$. The worldline has *constant velocity* if $C \subset M$ is an open interval in an affine line.

As depicted in Figure 26, the quotient (12.4) maps C diffeomorphically onto a submanifold $I \subset M/W$, an open interval. The inverse $\gamma: I \rightarrow C \hookrightarrow M$ is a *canonical* parametrization of C .

The tangent line U at some point of a worldline is a line in V that is transverse to W . Lines transverse to W form an affine space over $\text{Hom}(V/W, W)$. Therefore, two affine worldlines have a relative velocity in $\text{Hom}(V/W, W)$. (Observe that vectors in V/W have units of time and vectors in W have units of length, so an element of $\text{Hom}(V/W, W)$ has units of velocity. If we fix an inner product—a clock—on V/W , then there is a distinguished nonzero vector in V/W which is oriented and of unit norm, in which case we can identify the velocity as a vector in W .) However, there is

no absolute velocity of a single worldline. Let $\gamma: I \rightarrow M$ be the canonical parametrization of a worldline. Then $V/W \xrightarrow{d\gamma} V \xrightarrow{d\pi} V/W$ has constant value the identity map, hence $(V/W)^{\otimes 2} \xrightarrow{d^2\gamma} V \xrightarrow{d\pi} V/W$ is zero. So $d^2\gamma$ factors to a map $(V/W)^{\otimes 2} \rightarrow W$, which is the acceleration. In summary, acceleration is defined for a Galilean worldline, whereas velocity is not. (Zero acceleration, or constant velocity, is defined, as in [Definition 12.5](#).) As a consequence, Newtonian mechanics can be formulated in a Galilean spacetime. For example, Newton's second law $F = ma$ uses the acceleration.

Let $\mathcal{G}^\uparrow(M)$ denote the subgroup of affine symmetries of M that preserve the Galilean structure Γ . There is a larger group $\mathcal{G}(M)$ of symmetries that preserve the simultaneity foliation and inner products but not necessarily the time-orientation. These groups fit into extensions

$$(12.6) \quad \begin{array}{ccccccc} 1 & \longrightarrow & V & \longrightarrow & \mathcal{G}^\uparrow(M) & \longrightarrow & \mathcal{G}^\uparrow(V) \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & V & \longrightarrow & \mathcal{G}(M) & \longrightarrow & \mathcal{G}(V) \longrightarrow 1 \end{array}$$

If $n \geq 2$ then $\mathcal{G}^\uparrow(M)$ has two components, distinguished by whether a transformation preserves or reverses orientation of space. The group of components of $\mathcal{G}(M)$ is the Klein four-group, isomorphic to the product of two cyclic groups of order two, with the additional distinction that tracks time-orientation preservation or reversal. These symmetries include temporal and spatial translations as well as Euclidean transformations of space. There is also a distinguished subgroup of affine transformations $f: M \rightarrow M$ whose differential $df \in \text{Aut}(V)$ induces the identity map on the subspace $W \subset V$ and the identity map on the quotient V/W . These are *shearing* transformations: $df \in \text{Hom}(V/W, W)$ is a velocity that measures the distance the boost boosts in unit time. [Figure 27](#) illustrates the effect of a *Galilean boost* on a constant velocity worldline.

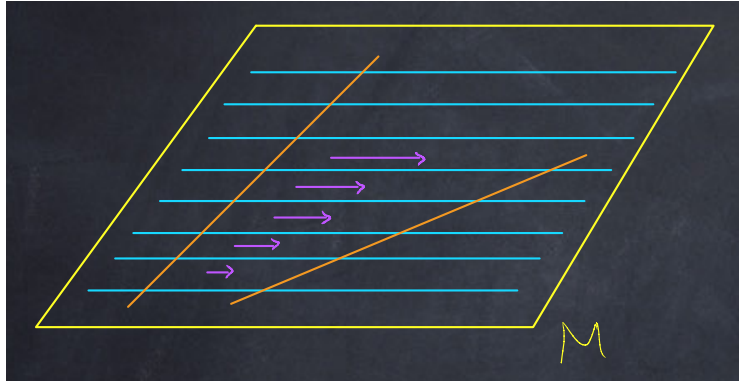


FIGURE 27. A Galilean boost

Remark 12.7. The *Galilean group* is the spin double cover of the identity component of $\mathcal{G}^\uparrow(M)$. It has a nontrivial central extension related to mass in nonrelativistic physics [\[GS\]](#), [\[DF2, §1.4\]](#). See also [\[FM1\]](#) for more about Galilean geometry and symmetry in nonrelativistic quantum mechanics.

12.3 Lorentz geometry

As preparation for our discussion of Minkowski spacetime in the next lecture we discuss the linear geometry of a Lorentz metric.

Definition 12.8. Let V be a real n -dimensional vector space. A *Lorentz structure* on V is a nondegenerate symmetric bilinear form

$$(12.9) \quad \langle -, - \rangle : V \times V \longrightarrow \mathbb{R}$$

of signature $(1, n-1)$.

The standard model is $V = \mathbb{R}^{1, n-1}$ with standard basis $e_0, e_1, \dots, e_{n-1}$ such that

$$(12.10) \quad \langle e_i, e_j \rangle = \begin{cases} 0, & i \neq j; \\ +1, & i = j = 0; \\ -1, & i = j \neq 0. \end{cases}$$

Definition 12.11. Let V be a Lorentzian vector space with metric $\langle -, - \rangle$.

- (1) For $\xi \in V$ set $|\xi| = \sqrt{|\langle \xi, \xi \rangle|}$.
- (2) Set

$$(12.12) \quad \begin{aligned} V^{>0} &= \{\tau \in V : \langle \tau, \tau \rangle > 0\}, \\ N(V) &= \{\eta \in V : \langle \eta, \eta \rangle = 0\}, \\ V^{<0} &= \{\sigma \in V : \langle \sigma, \sigma \rangle < 0\}. \end{aligned}$$

Then $V = V^{>0} \amalg N(V) \amalg V^{<0}$.

Vectors in $N(V)$ are called *null* or *lightlike*, those in $V^{>0}$ *timelike*, and those in $V^{<0}$ *spacelike*. Each of these subsets is a *cone* (i.e., is closed under scalar multiplication by $\mathbb{R}^{>0}$). In [Figure 28](#) the timelike vectors lie inside the null cone and the spacelike vectors lie outside of it.

Remark 12.13. The terminology suggests that $\langle \tau, \tau \rangle$ has units of time squared (T^2) for a timelike vector τ , whereas $\langle \sigma, \sigma \rangle$ has units of length squared (L^2) for a spacelike vector σ . We choose $\langle -, - \rangle$ to have units of L^2 , but then need a universal constant to convert from length to time, as we introduce in a Minkowski spacetime.

If $U \subset V$ is a timelike line, then there is an orthogonal splitting

$$(12.14) \quad V \cong U \oplus U^\perp$$

in which $U^\perp$ is spacelike: the metric on $U^\perp$ is negative definite; see [Figure 28](#).

Proposition 12.15. Let V be a Lorentzian vector space of dimension n .

- (1) $V^{>0}$ has two components.

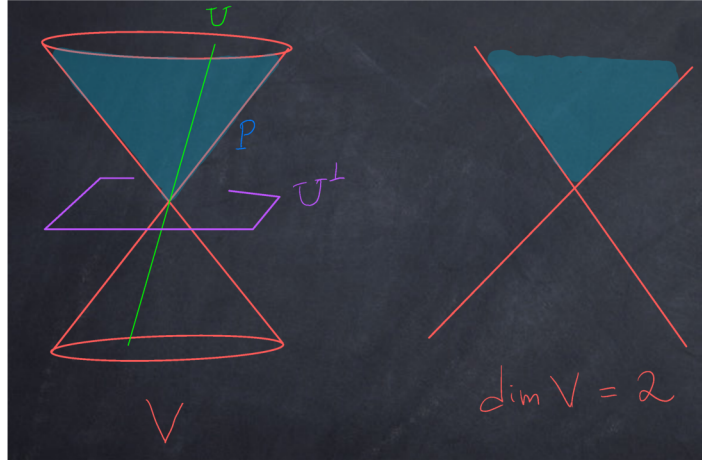


FIGURE 28. Lorentzian vector spaces

- (2) $V^{<0}$ is empty if $n = 1$, has two components if $n = 2$, and is connected if $n \geq 3$.
 (3) $N(V) \setminus \{0\}$ is empty if $n = 1$, has 4 components if $n = 2$, and has two components if $n \geq 3$.

Proof. Fix a timelike vector τ_0 , and as in (12.14) write vectors in V as a multiple of τ_0 plus an orthogonal spacelike vector. Then define deformation retractions

$$(12.16) \quad f_t(a\tau_0 + \sigma) = \begin{cases} ((1-t)a + t \operatorname{sign}(a))\tau_0 + (1-t)\sigma, & \text{case (1);} \\ (1-t)a\tau_0 + \frac{(1-t)|\sigma| + t}{|\sigma|}\sigma, & \text{case (2), } n \geq 2; \\ \frac{a\tau_0 + \sigma}{(1-t) + t|a|}, & \text{case (3), } n \geq 2 \end{cases}$$

onto S^0 , S^{n-2} , and $S^0 \times S^{n-2}$ respectively. Here $a \in \mathbb{R}$. □

Definition 12.17. Let V be a Lorentzian vector space. A choice of component $P \subset V^{>0}$ is a *time-orientation*. Vectors in P are *forward timelike*. The closure $\overline{P}$ consists of *forward vectors*, which are timelike or lightlike.

Proposition 12.18. Let V be a Lorentzian vector space of dimension $n \geq 2$. Then the orthogonal group $O(V)$ has 4 components, distinguished by the action of an orthogonal transformation on the two-element sets $\pi_0 V^{>0}$ and $\pi_0 \mathcal{B}(V)$, where $\mathcal{B}(V)$ is the space of bases of V .

In other words, we track whether an orthogonal transformation preserves time-orientation and overall orientation. (In place of the latter we often track space-orientation).

Proof. For $V = \mathbb{R}^{1,1}$ with metric $\langle e_0, e_0 \rangle = 1$, $\langle e_1, e_1 \rangle = -1$, $\langle e_0, e_1 \rangle = 0$, an orthogonal transformation has the form

$$(12.19) \quad \begin{pmatrix} \epsilon_1 \cosh \varphi & \epsilon_2 \sinh \varphi \\ \epsilon_1 \sinh \varphi & \epsilon_2 \cosh \varphi \end{pmatrix},$$

where $\varphi \in \mathbb{R}$ and $\epsilon_i = \pm 1$. For $n > 2$ we proceed by induction. Let $\dim V > 2$. The subgroup $\mathrm{SO}^\uparrow(V)$ of transformations that act trivially on both two element sets acts transitively on

the connected space $\{\sigma \in V : |\sigma|^2 = -1\}$; the stabilizer subgroup at σ is isomorphic to $\text{SO}^\uparrow(\mathbb{R}\langle\sigma\rangle^\perp)$, which is connected by induction. (Note that $\mathbb{R}\langle\sigma\rangle^\perp$ is Lorentzian.) $\square$

The next proposition is analogous to the usual Cauchy–Schwarz and triangle inequalities in a Euclidean vector space, but with an important difference: *the inequalities are reversed*.

Proposition 12.20. *Let τ, τ' be forward vectors in a time-oriented Lorentzian vector space. Then*

(i) $\langle\tau, \tau'\rangle \geq |\tau| |\tau'|$; and

(ii) $\tau + \tau'$ is a forward vector, and $|\tau + \tau'| \geq |\tau| + |\tau'|$.

Equality holds if and only if τ, τ' are proportional.

Proof. Choose τ_0 forward timelike with $|\tau_0| = 1$. Write $\tau = a\tau_0 + \zeta$ and $\tau' = a'\tau_0 + \zeta'$, where $\langle\tau_0, \zeta\rangle = \langle\tau_0, \zeta'\rangle = 0$. Then $a, a' > 0$, $a \geq |\zeta|$, and $a' \geq |\zeta'|$. Thus by the ordinary Cauchy–Schwarz inequality $aa' \geq |\zeta| |\zeta'| \geq -\langle\zeta, \zeta'\rangle$, from which $\langle\tau, \tau'\rangle = aa' + \langle\zeta, \zeta'\rangle \geq 0$ with equality if and only if τ, τ' are lightlike and proportional. Now the affine function $t \mapsto \langle\tau + t\tau', \tau_0\rangle$ is not identically zero, so for some t_0 we have $\langle\tau + t_0\tau', \tau_0\rangle = 0$. Then either $\tau + t_0\tau' = 0$ or $\tau + t_0\tau'$ is spacelike. In the former case we are done, so assume the latter. Then the quadratic function $t \mapsto \langle\tau + t\tau', \tau + t\tau'\rangle$ has a strictly negative minimum, so its discriminant $\langle\tau, \tau'\rangle^2 - \langle\tau, \tau\rangle \langle\tau', \tau'\rangle$ is strictly positive. This proves (i). The deduction of (ii) from (i) is as in the Euclidean case: expand $\langle\tau + \tau', \tau + \tau'\rangle$. $\square$

Definition 12.21. Suppose τ, τ' are forward timelike vectors in a time-oriented Lorentzian vector space. The *hyperbolic angle* φ between τ and τ' is defined by

$$(12.22) \quad \cosh \varphi = \frac{\langle\tau, \tau'\rangle}{|\tau| |\tau'|}.$$

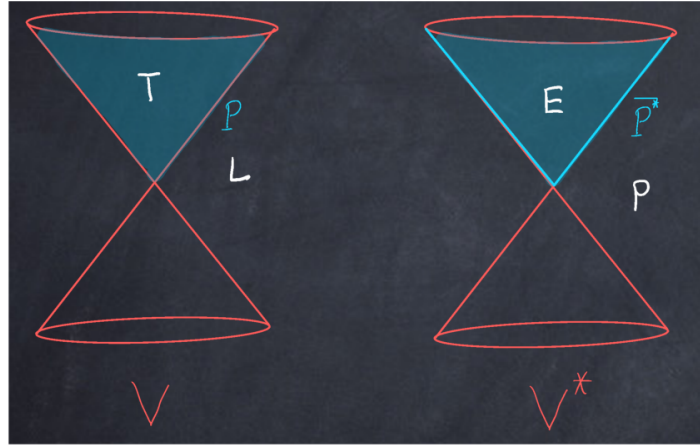


FIGURE 29. A Lorentzian vector space V and its dual V^* , with forward timelike ($P \in V$) and positive energy ($\overline{P}^* \setminus \{0\} \subset V^*$) vectors colored. Regions are labeled by units: time (T), length (L), energy (E), and momentum (p)

A nondegenerate symmetric bilinear form on a vector space V induces a canonical isomorphism

$$(12.23) \quad \phi: V \xrightarrow{\cong} V^*$$

and, by transport, a nondegenerate symmetric bilinear form on V^* . Therefore, the dual to a Lorentzian vector space V is a Lorentzian vector space V^* . The dual to time (T) has units of T^{-1} , which is *frequency*, and the dual to length (L) has units of L^{-1} , which is *wave number*. In a *quantum* theory we have Planck's constant $\hbar$ available; see (1.25). Multiplying by $\hbar$ we see that the dual to time has units ML^2/T^2 of *energy*, and the dual to length has units ML/T of *momentum*. Thus a point of V^* is sometimes called an *energy-momentum* (vector). A time-orientation $P \subset V^{>0}$ induces the same structure $P^* \subset (V^*)^{>0}$. Nonzero vectors in $\overline{P^*}$ are said to have *positive energy*. See Figure 29 for an illustration.

12.4 Spin groups

The spin group is a double cover of the identity component of an orthogonal group. For a finite dimensional real vector space equipped with a definite bilinear form, the orthogonal group is compact and consists of two components. The identity component for the standard model is the special orthogonal group SO_n , and for $n \geq 3$ the fundamental group is cyclic of order 2. In that case the spin group $Spin_n$ is a simply connected compact Lie group. The fundamental group of SO_2 is infinite cyclic, and $Spin_2$ is a connected double covering group. The group SO_1 consists only of the identity element, so $Spin_1 = \mathbb{Z}_2 = \{\pm 1\}$ is the cyclic group of order 2. In Lorentz signature $(1, n-1)$ the identity component $SO_{1,n-1}^\uparrow$ of the orthogonal group is noncompact if $n \geq 2$; its deformation retracts onto its maximal compact subgroup is SO_{n-1} . In low dimensions there are exceptional isomorphisms among Lie groups that allow us to identify the compact and Lorentz spin groups in terms of other Lie groups:

n	$Spin_n$	$Spin_{1,n-1}$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	$\mathbb{T}$	$\mathbb{R}^{>0} \times \mathbb{Z}_2$
3	SU_2	$SL_2 \mathbb{R}$
4	$SU_2 \times SU_2$	$SL_2 \mathbb{C}$
5	Sp_2	$Sp_{1,1}$
6	SU_4	$SL_2 \mathbb{H}$

12.5 The speed of light

A Galilean structure (Definition 12.3) equipped with a “clock” has two inner products: one for time (V/W) and one for space (W). They have separate units of time and length. The cornerstone of special relativity is the merging of these two inner products into a single entity: a Lorentz metric. It gives measurements in units of length, say, and therefore we need a “mechanism” to convert to units of time. That mechanism is a universal constant of nature: the speed of light c . The constancy of the speed of light was Einstein's starting point for his special theory of relativity. The constant c has units of speed, i.e., of length over time: $[c] = \frac{L}{T}$. In a quantum theory there is another universal constant, $\hbar$, and $[\hbar] = \frac{ML^2}{T}$. In a relativistic quantum theory we have both universal constants, and use them to convert

$$(12.25) \quad L \stackrel{c}{\sim} T \stackrel{\hbar/c^2}{\sim} M^{-1}$$

Bonus Lecture: Symmetry and families

Two powerful techniques that permeate mathematics are the use of symmetry and of parametrized families. Indeed, as we will see the former is a special case of the latter via the *equivariance* $\rightarrow$ *families* principle. Both are powerful problem solving techniques, and more fundamentally they capture important aspects of structure. Not surprisingly, this is true for quantum systems as well.

We begin with a few examples of symmetry and families in quantum mechanics. Then, as background and perspective, we meander lightly over their occurrence in several mathematical situations. Some of our examples resonate with ideas in quantum theory, such as phase transitions. We explain in an elementary example how to convert a situation with symmetry into a parametrized family.

We conclude the lecture with mathematical background on fiber bundles, groupoids, and stacks.

B.1 Examples in quantum mechanics

Example B.1 (symmetry in a QM system). Recall the simple harmonic oscillator, which was introduced in [Lecture 5](#). The classical theory describes a particle on the Euclidean line $\mathbb{E}^1$ with potential function $V(x) = x^2/2$. The classical Hamiltonian for a motion $\gamma: \mathbb{R} \rightarrow \mathbb{E}^1$ is

$$(B.2) \quad H = \frac{1}{2}(\dot{\gamma}^2 + \gamma^2),$$

and the classical phase space is a plane. The motion via concentric circles is depicted in [Figure 12](#). Evidently, the rotation group O_2 acts as symmetries of this system. Notice that the identity component SO_2 acts by time-preserving transformations, whereas reflections—which comprise the off-component of O_2 —act by time-orientation reversing transformations.

For the quantum harmonic oscillator, we take the state space $\mathcal{H} = L^2(\mathbb{E}^1; \mathbb{C})$ and Hamiltonian

$$(B.3) \quad H = \frac{1}{2} \left(-\frac{d^2}{dx^2} + x^2 \right),$$

though based on the analysis in [Lecture 5](#) we can give an abstract description. As we will show, the group O_2 acts on this quantum theory, but we must understand quantum theory as *projective* as explained in [Lecture 8](#). Later in this lecture we address the question: Is there a linearization of this projective action, i.e., a lift to a linear action on $\mathcal{H}$?

The next examples are of families in quantum mechanics. First, we indicate how we encounter a family in a fixed quantum mechanical (QM) system. Then we give examples of families of quantum systems. The distinction between these two paradigms is sometimes illusory; see the remark at the end of [Example B.6](#) and also [Example B.17](#).

Example B.4 (families in a fixed system). Suppose $(\mathcal{H}, H)$ defines a QM system. Recall the definition of correlation functions from the end of [Lecture 5](#). For example, fix a unit norm vector $\psi \in \mathcal{H}$, which determines a pure state, and fix times $t_0 < t_1 < t_2 < t_f$ and observables A_1, A_2 . Then the 2-point correlation function

$$(B.5) \quad f(t_1, t_2) = \langle \psi, U_{t_f-t_2} A_2 U_{t_2-t_1} A_1 U_{t_1-t_0} \psi \rangle$$

as a function of t_1, t_2 is an example of a family of computations in a single theory.

Example B.6 (families of QM systems). A system may have parameters—in quantum field theory they are often called *coupling constants*—which we allow to vary, and thus embed the system in a family. For example, the simple harmonic oscillator with mass m and spring constant k has Hamiltonian

$$(B.7) \quad H_{m,k} = \frac{1}{2}(m\dot{\gamma}^2 + k\gamma^2).$$

This defines a family of systems with parameter space $(\mathbb{R}^{>0})^{\times 2}$.

In quantum field theory it is oft fruitful to turn a coupling “constant” into a scalar field. In this way one can consider a single system with these scalar fields, or a family of system parametrized by the values of these scalar fields. (A classical analog appears in [Example B.17](#) below.)

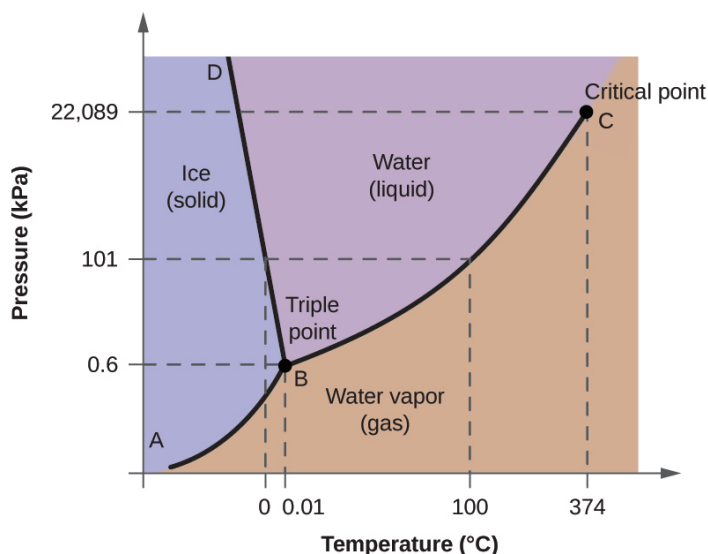


FIGURE 30. Phases of H_2O

Example B.8 (another family of quantum systems). The chemical compound H_2O can be described as a quantum system with parameters of temperature and pressure, as illustrated in [Figure 30](#). We return to this picture later in the lecture.

B.2 Examples of families in mathematics

We give simple, but illustrative, examples from different parts of mathematics.

Example B.9 (algebraic geometry). Even for a simple arithmetic equation, such as

$$(B.10) \quad x^4 + y^4 - z^4 = 0, \quad x, y, z \in \mathbb{Z},$$

it is fruitful to fiber over $\text{Spec}(\mathbb{Z})$, i.e., to consider the equation over finite fields $\mathbb{F}_p$, p prime. In complex geometry, or in algebraic geometry more generally, one encounters nice families of varieties: *flat* families. Consider a deformation of a singularity, as in the equation

$$(B.11) \quad y^2 - x^2 = \epsilon.$$

Here $x, y \in \mathbb{C}$, and for each fixed $\epsilon \in \mathbb{C}$ the equation (B.11) describes a curve in the complex affine x, y -plane. The curve is a nonsingular conic if $\epsilon \neq 0$, and it is a union of two lines if $\epsilon = 0$. Alternatively, (B.11) defines a surface in complex affine 3-space with coordinates x, y, ϵ . The surface projects to the complex affine ϵ -line, and the fibers of the projection are the plane curves just introduced; see Figure 31.

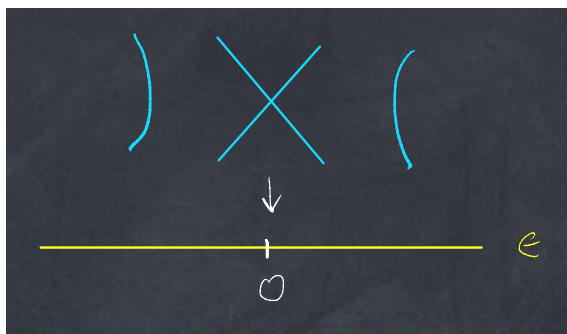


FIGURE 31. A family of conics

Example B.12 (algebra). Define

$$(B.13) \quad A = \mathbb{C}[x, \epsilon]/(x^2 - \epsilon).$$

If ϵ is a fixed complex number, then (B.13) is a family of algebras A_ϵ parametrized by $\epsilon \in \mathbb{C}$. For $\epsilon \neq 0$ the algebra A_ϵ is isomorphic to $\mathbb{C} \times \mathbb{C}$; for $\epsilon = 0$ it is the nonsemisimple algebra $\mathbb{C}[x]/(x^2)$. If we regard ϵ as an indeterminate, then (B.13) defines a single algebra which is also a module over the polynomial algebra $\mathbb{C}[\epsilon]$. Pass to a geometric picture by defining spaces on which these algebras are the algebras of complex-valued functions: the planar conic $\{x^2 - \epsilon = 0\}$ in the affine (ϵ, x) -plane projecting to the ϵ -line, as depicted in Figure 32. The fiber over $\epsilon \neq 0$ is a set of two distinct points, whereas the fiber over $\epsilon = 0$ is a double point.

Example B.14 (topology). There are many notions of a family of topological spaces. The nicest is a *fiber bundle*, in which the homeomorphism type of fibers is locally constant on the base. We emphasize that a fiber bundle is a *property* or *condition* on a map $\pi: X \rightarrow S$; it is not extra structure or data. (See Definition B.25 below.) There are weaker properties of a continuous map which also tell that its fibers vary “nicely” in some weaker, often homotopical, sense: fibration, Serre fibration, and quasifibration.

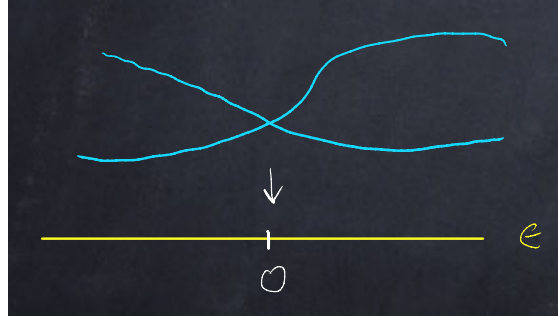


FIGURE 32. Two points degenerating to a double point

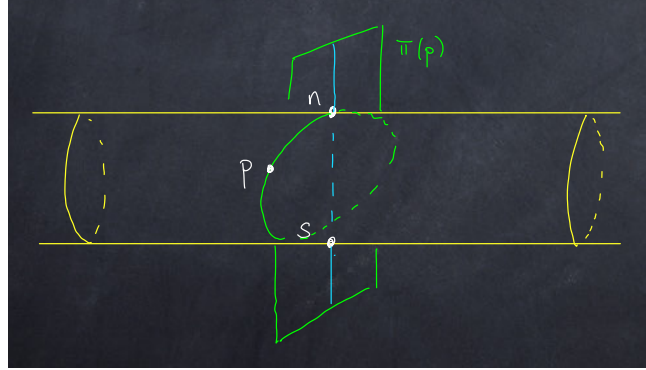


FIGURE 33. A submersion which is not a fiber bundle

Example B.15 (differential topology). The nicest notion is that of a smooth fiber bundle. The fibers of a submersion are submanifolds, but the diffeomorphism type can jump if the fibers are not all compact. (A proper submersion is a fiber bundle.) Here is an example.

Work in the affine space $\mathbb{A}^3$ with coordinates (x, y, z) . Define

$$(B.16) \quad X = \{(x, y, z) \in \mathbb{A}^3 : y^2 + z^2 = 1\} \setminus \{(0, 0, +1), (0, 0, -1)\}.$$

This is a cylinder with two points n, s deleted. Let P denote the space of affine planes in $\mathbb{A}^3$ which contain the z -axis; then P is diffeomorphic to $\mathbb{RP}^1$, the space of lines through the origin in the x, y -plane. Define $\pi: X \rightarrow P$ to be the map which takes $p \in X$ to the plane containing the distinct non-collinear points n, s, p , as depicted in Figure 33. Then π is a surjective submersion. However, π is not a fiber bundle. The typical fiber of π is an ellipse minus the points n, s , whereas the fiber over the x, z -plane $\Pi_{x,z}$ is the union of two affine lines minus n, s . The former has two components while the latter has four components. The special fiber over $\Pi_{x,z}$ is not diffeomorphic to the other fibers, and therefore π cannot be locally trivial at $\Pi_{x,z}$. Variation: compactify the generic fiber by blowing up the points n, s into circles to obtain a new submersion whose generic fiber is a single circle and whose special fiber is two lines.

Example B.17 (differential equations). First, *ordinary* differential equations (ODEs). A vector field ξ on a smooth manifold M determines the first-order ODE

$$(B.18) \quad \dot{\gamma}(t) = \xi(\gamma(t))$$

for a motion $\gamma: \mathbb{R} \rightarrow M$. (Assume M is compact or ξ is suitably bounded so that integral curves exist for infinite time.) An initial condition is $\gamma(0) = m$, $m \in M$, so we obtain a family of initial value problems parametrized by M . Consider now a family of vector fields $\xi(s)$, $s \in S$, for S a smooth manifold. This defines a family of ODEs parametrized by S . Consider the vector field $\eta(s, m) = (0, \xi(s)(m))$ on the Cartesian product manifold $S \times M$. The ODE at a particular s is solved by integral curves of η on $\{s\} \times M$. More generally, if $\sigma: \mathbb{R} \rightarrow S$ is a “scalar field” with values in S , there is an ODE

$$(B.19) \quad \dot{\gamma}(t) = \xi(\sigma(t))(\gamma(t))$$

which we solve as follows: pull back η via $\sigma \times \text{id}_M: \mathbb{R} \times M \rightarrow S \times M$ to a vector field η_σ on $\mathbb{R} \times M$, and construct integral curves of $\partial/\partial t + \eta_\sigma$. This is a classical dynamical systems variant of a maneuver in quantum field theory.

The *continuity method* (or method of continuity) for *partial* differential equations [GT] is another instance of families. Consider a PDE $\mathcal{E}$ that one wants to prove has solutions. The technique is to embed it into a family $\mathcal{E}_t$, $t \in [0, 1]$, in which $\mathcal{E}_1 = \mathcal{E}$ and existence for $\mathcal{E}_0$ is straightforward. Then

$$(B.20) \quad \{t \in [0, 1] : \mathcal{E}_t \text{ has a solution}\}$$

is proved to be open by an implicit function theorem argument. The art comes in proving that (B.20) is also closed, which usually requires delicate *a priori* estimates.

B.3 Moduli spaces

Families often arise from considering (equivalence classes of) mathematical objects of a specified type. Here are a few examples.

Example B.21. Fix a vector space V . Consider the family of lines $L \subset V$. The parameter space for lines is the projective space $\mathbb{P}V$. Furthermore, there is a universal line bundle $\mathcal{L} \rightarrow \mathbb{P}V$ whose fiber at $L \rightarrow \mathbb{P}V$ is canonically identified with the line $L \subset V$. From that description, it is clear that $L \rightarrow \mathbb{P}V$ comes as a subbundle $L \subset \underline{V}$ of the constant vector bundle $\underline{V} \rightarrow \mathbb{P}V$ with fiber V .

One can generalize to the moduli space of subspaces of V of any fixed dimension. The parameter space is the *Grassmannian*, and again there is a universal vector bundle over the Grassmannian, and it is embedded in the constant vector bundle with fiber V .

This example illustrates that in many moduli problems one fixes *discrete parameters*, here the dimension of a subspace.

Important moduli spaces in geometry include the moduli space of algebraic curves of a fixed genus, the moduli space of algebraic surfaces with fixed topological invariants, etc.

The following example from differential geometry is a geometric cousin of moduli spaces of QM systems.

Example B.22 (1-dimensional Riemannian manifolds). Consider the heuristic “moduli space of 1-dimensional spaces with metric data”. One interpretation is the moduli space of smooth 1-dimensional Riemannian manifolds. To begin, fix the diffeomorphism type to be a circle. Then the

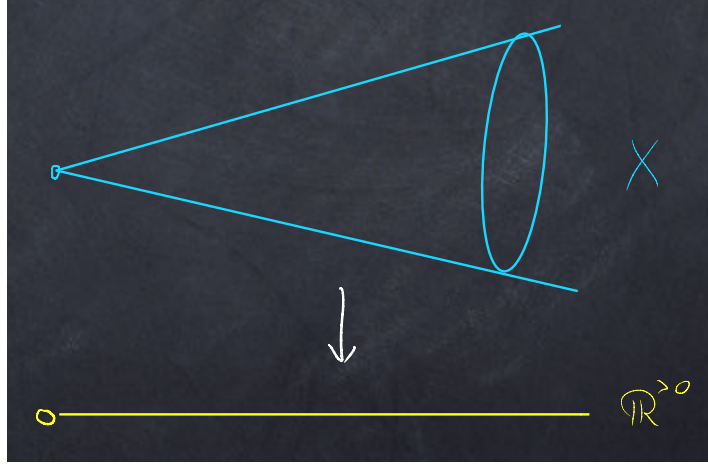


FIGURE 34. The universal fiber bundle of Riemannian circles

Riemannian structure has a single invariant: the total length. (This follows using parametrization by arclength.) So the moduli space is $\mathbb{R}^{>0}$ and the universal Riemannian manifold X is a cone minus the cone point. The resulting fiber bundle³⁷ is depicted in Figure 34. There is a similar picture for the diffeomorphism type $(S^1)^{\amalg 2}$ of two disjoint circles: the moduli space is $(\mathbb{R}^{>0})^{\times 2}$.

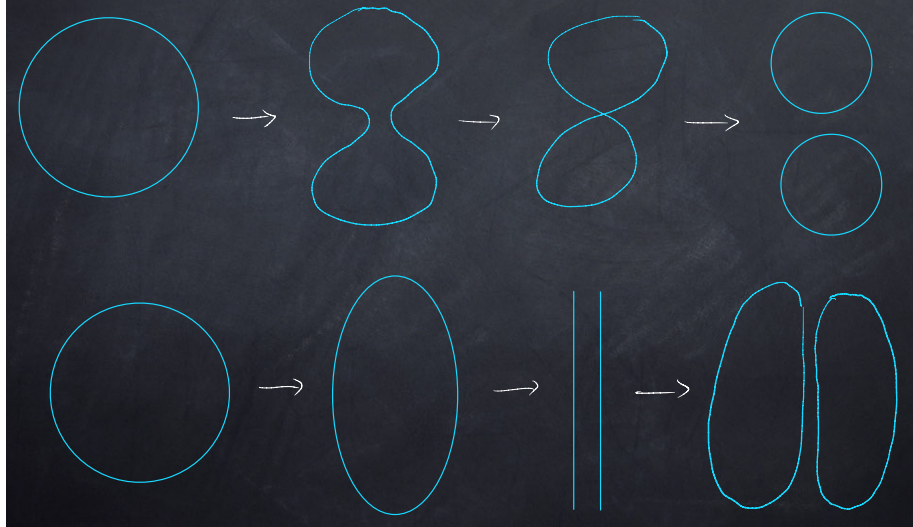


FIGURE 35. Two processes that turn 1 circle into 2 circles

Are there more general “moduli spaces of 1-dimensional spaces with metric data” in which a single circle can be connected to two circles by a smooth path? Figure 35 illustrates two ways in which one circle can morph into two circles.³⁸ The first mechanism passes through a compact

³⁷This should be a *Riemannian* fiber bundle, but what is a Riemannian structure on a fiber bundle? It must generalize a Riemannian structure on a single Riemannian manifold—a Riemannian manifold over a point. Over a general base S , a Riemannian structure on a fiber bundle $\pi: X \rightarrow S$ should certainly include an inner product on the relative tangent bundle $T(X/S) = \ker \pi_* \rightarrow X$. But that is not sufficient to produce a parametrized Levi-Civita covariant derivative. For that one also needs a *horizontal distribution* on the fiber bundle π : a subbundle of $TX \rightarrow X$ transverse to the relative tangent bundle. A horizontal distribution—compatible with other structures present—is usually required on a fiber bundle equipped with differential geometric data.

³⁸Moduli of 1-dimensional spaces embedded into a Euclidean plane might be a route to tighten these heuristics.

singular space, the figure 8. The second scenario is essentially that in [Example B.15](#); it contains only smooth spaces, but passes through a noncompact manifold. The first scenario is analogous to a first-order phase transition in quantum mechanics. The second is analogous to a higher-order phase transition. Note in the second that the spectrum of the Laplace operator is pure point off of the noncompact fiber of the family. At the noncompact fiber the spectral gap disappears, and it is continuous spectrum which has violated the gap. One might imagine that in the first scenario the gap is violated by an additional eigenvalue which comes down to the minimum.

The general lesson is that a moduli space $\mathcal{M}$ may contain a *singular locus* $\Delta \subset \mathcal{M}$, and often the moduli space of interest is $\mathcal{M} \setminus \Delta$. In [Example B.22](#) we do not pin down a moduli space $\mathcal{M}$ which includes noncompact 1-manifolds, but one can imagine such a space to be path connected via processes analogous to the second one in [Figure 35](#). (That one may get a connected moduli space when allowing certain “surgery-like” processes is familiar in other contexts. For example, Reid [[Re](#)] conjectured such a connectedness statement for the moduli of Calabi–Yau 3-folds.) The moduli space $\mathcal{M} \setminus \Delta$ of closed smooth Riemannian 1-manifold has a nontrivial set of components in bijection with $\mathbb{Z}^{\geq 0}$. In fact, the isomorphism $\pi_0(\mathcal{M} \setminus \Delta) \xrightarrow{\cong} \mathbb{Z}^{\geq 0}$ takes a manifold M to the cardinality of $\pi_0 M$. This is typical: information about the homotopy type of the moduli space $\mathcal{M}$ (or $\mathcal{M} \setminus \Delta$) can often be expressed in terms of topological invariants of the objects it parametrizes.

Another general lesson. There are two general types of “questions” we might ask and express in terms of a moduli space. *Geometric* questions lead to functions which are not locally constant. In [Example B.22](#) the total length of a closed Riemannian 1-manifold is such a function. *Topological* questions are those that only depends on the underlying homotopy type of the moduli space. The first such query is the set of *deformation classes* of parametrized objects, which is the set of path components of the moduli space. That remembers coarse structure, in [Example B.22](#) the number of circles (rather than their total length).

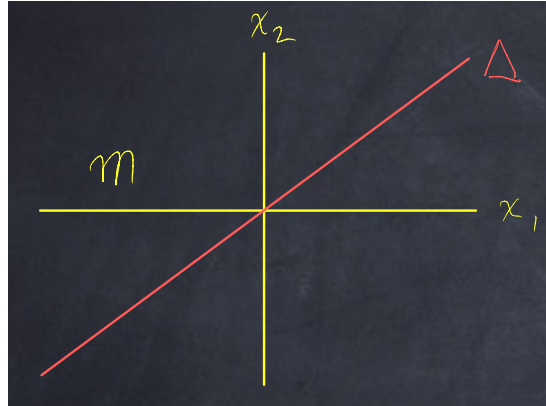


FIGURE 36. The moduli space of ordered pairs of real numbers

Example B.23. A simple illustration of these general features is the moduli space $\mathcal{M} \cong \mathbb{R}^2$ of ordered pairs (x_1, x_2) of real numbers. One might consider the diagonal $\Delta \subset \mathcal{M}$ of pairs (x, x) , $x \in \mathbb{R}$, to be singular, as in [Figure 36](#). This is the “gapless” subspace on which the distance between the points on the real line vanishes. Thus $\mathcal{M} \setminus \Delta$ parametrizes gapped pairs. A geometric

function is $x_1 - x_2: \mathcal{M} \setminus \Delta \rightarrow \mathbb{R}^{\neq 0}$. The composition with $\mathbb{R}^{\neq 0} \rightarrow \pi_0 \mathbb{R}^{\neq 0}$ factors to an isomorphism $\pi_0(\mathcal{M} \setminus \Delta) \rightarrow \pi_0 \mathbb{R}^{\neq 0}$ which only remembers the ordering of the points.

B.4 Symmetry as families

The cyclic group C_2 of order 2 acts on $\mathcal{M}$ by exchanging the order of the points. The action is reflection about the diagonal $\Delta \subset \mathcal{M}$, as depicted in [Figure 37](#). The fixed point set is Δ , and the action is free on $\mathcal{M} \setminus \Delta$. The quotient *space* is contractible. However, the quotient space loses information about the stabilizer subgroups at points where the action fails to be free. For example, consider the path in $\mathcal{M}$ in [Figure 37](#); it is a loop in the quotient. One can imagine the path as moving in a double cover of the quotient, and as it crosses the fixed point set Δ it should jump sheets. In other words, the lift of the loop in the quotient to the double cover is not a closed loop. This indicates that the quotient should have nonzero fundamental group, so in particular cannot be contractible.

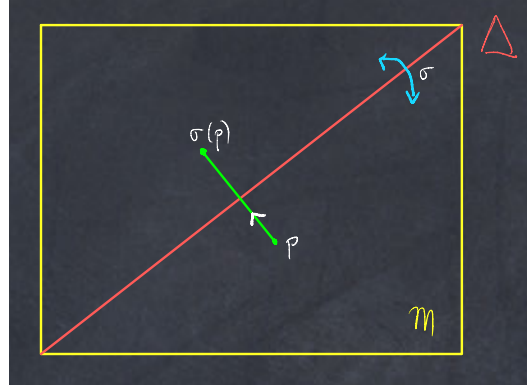


FIGURE 37. The C_2 -action on $\mathcal{M}$; a loop in the quotient

There is a quotient construction which captures this information: the *action groupoid* or, better, the *quotient stack*. See [§B.6.2](#) for an introduction to groupoids and stacks. The intuitive idea of the action groupoid is that if X is a space equipped with the action of a group G , then instead of identifying points x and $g \cdot x$ for $x \in X$, $g \in G$, we adjoin a morphism from x to $g \cdot x$. The morphism is labeled by the group element, and composition of morphisms follows the group law in G . We use the notation ' $X//G$ ' for the action groupoid. Notice that in the example at hand, there are nonidentity automorphisms of points of Δ . A *stack* is the invariant object underlying a groupoid, much as a smooth manifold is the invariant object underlying an open cover and its gluings. (In fact, the latter is an example of how a groupoid determines a stack.) One can define groupoids and stacks for discrete sets, topological spaces, and smooth manifolds (and beyond). We often implicitly use the smooth version. Associated to a groupoid or stack is a topological space, its *geometric realization*, and so too is associated a homotopy type.

The quotient groupoid $*//G$ of a group G acting on a point $*$ is a rigid version of the *classifying space* of a group. Its geometric realization has the usual homotopy type of the classifying space BG . If $G \curvearrowright X$ is a space with a left group action, then there is a fiber bundle

$$(B.24) \quad X//G \longrightarrow *//G$$

of stacks, and the fiber over the basepoint $* \hookrightarrow */G$ is the space X . This is the strong form of a group action: an object³⁹ fibered over the classifying space $*/G$. This is the *equivariance* $\longrightarrow$ *families* principle that sees objects with symmetry as special cases of parametrized families.

B.5 A moduli space in quantum mechanics

After this long digression we return to physics, specifically to the family of QM systems pictured in [Figure 30](#). This is the phase diagram of H_2O . Now we understand it as a moduli space $\mathcal{M}$ of quantum systems. One can ask geometric questions, which often lead to functions on $\mathcal{M}$, such as correlation functions as functions of temperature and pressure. But one can also ask topological questions, such as: What is the set of *phases*? Here the phase of a quantum system is defined to be a path component, and so the set of phases is π_0 of the moduli space. This moduli space is manifestly contractible, so there seem to be no interesting phases. However, there is a singularity locus Δ in this problem, namely the locus of points in $\mathcal{M}$ at which the quantum system is gapless. (Recall the basic dichotomy of [Definition 5.7\(2\)](#).) In [Figure 30](#) these points are on the dark black curve, which extends to the origin and off to infinite pressure past the point D in the figure. The other branch terminates in a “critical point” C. These gapless points are where the system undergoes a *phase transition*. Here the transition is first-order except at the point C. The first-order and higher-order phase transitions are analogous to the two geometric transitions in [Figure 35](#). The set $\pi_0(\mathcal{M} \setminus \Delta)$ of phases has cardinality 2. Note, however, if we restrict to standard room pressure (which is 101 kPa in the diagram), then there are the 3 standard phases of our everyday experience: solid, liquid, gas.

There are similar families in other parts of quantum theory: quantum field theory and string theory, for example. One usually fixes discrete parameters, such as dimension and symmetry type. The set of phases is a macroscopic quantity, as topological quantities (independent of scale) tend to be.

B.6 Mathematical background

B.6.1 Fiber bundles. The following, which is due to Steenrod [[Ste](#)] can be given in the topological or smooth context. For definiteness we work in the smooth category.

Definition B.25. A smooth map $\pi: X \rightarrow S$ of smooth manifolds is a *fiber bundle* if for all $s \in S$ there exists an open neighborhood $U \subset S$ of s and a diffeomorphism which fits into the commutative diagram

$$(B.26) \quad \begin{array}{ccc} U \times \pi^{-1}(s) & \xrightarrow{\approx} & \pi^{-1}(U) \\ & \searrow \text{pr}_1 & \swarrow \pi \\ & U & \end{array}$$

We might also require that π be surjective. Notice that the only data in [Definition B.25](#) is the smooth map π . The definition is a condition on a smooth map, as are the definitions of an immersion, embedding, submersion, etc.

³⁹The picture is quite general: the object may be a (higher) category in place of a topological space.

B.6.2 Groupoids and stacks. I have gone on too long in these notes, so I will simply give references. You will find a leisurely exposition of categories, classifying spaces, groupoids, etc. in the course notes [F4]. I also recommend [FHT1, §§A.1–A.2], which emphasizes the topological case. For smooth groupoids, called *Lie groupoids*, and the associated stacks see [BX]. I also recommend the expository article [We1].

Lecture 13: The geometry of Minkowski spacetime

In the first part of the lecture we consider the geometry of special relativity, whose home is *Minkowski spacetime*. A Minkowski spacetime is a Lorentz version of Euclidean space, and it is also equipped with a time-orientation and a speed of light $c \in \mathbb{R}^{>0}$. We study worldlines of particles, for which we define acceleration as the second fundamental form. We also define their energy-momentum, which lives in the dual space to translations. The limit as $c \rightarrow \infty$ of a Minkowski spacetime is a Galilean spacetime, as we briefly show. (See [DF2, §1.4] for a more generous exposition.) We also introduce the *Poincaré group*, which is a double cover of the identity component of isometries of Minkowski spacetime.

13.1 Minkowski spacetime

Relativistic spacetime is an affine space over a Lorentzian vector space. We include as well the speed of light and a time-orientation.

Definition 13.1. Let V be a Lorentzian vector space. A *Minkowski spacetime* (M, P, c) over V is an affine space M over V , a time-orientation $P \subset V^{>0}$, and positive real number $c > 0$.



FIGURE 38. A Minkowski spacetime M over a Lorentzian vector space V

See Figure 38 for a depiction. The standard example $\mathbb{M}^n$ is standard affine space $\mathbb{A}^n$ with standard affine coordinates $t, x^1, \dots, x^{n-1}$ and the translationally invariant metric

$$(13.2) \quad c^2 dt^2 - (dx^1)^2 - (dx^2)^2 - \dots - (dx^{n-1})^2.$$

Write $x^0 = ct$. Forward timelike vectors $\tau \in \mathbb{R}^{1,n-1}$ are required to have positive $\partial/\partial t$ component. We typically abbreviate (M, P, c) as M .

Definition 13.3. Let M be a Minkowski spacetime over a Lorentzian vector space V . Suppose $p, q \in M$. Write $q = p + \xi$ for $\xi \in V$.

- (i) Then we say that p, q are *timelike/lightlike/spacelike separated* according as ξ is timelike/lightlike/spacelike.
- (ii) If p, q are spacelike separated, define the *distance* between p and q as

$$(13.4) \quad \text{dist}(p, q) = \sqrt{-\langle \xi, \xi \rangle}.$$

- (iii) If p, q are timelike separated, define the *proper time* between p and q as

$$(13.5) \quad \text{ptime}(p, q) = \frac{1}{c} \sqrt{\langle \xi, \xi \rangle}.$$

The proper time between two points (spacetime events) on the path is the elapsed time an observer traveling on this path would measure on his clock. Consider a triangle p, q, r such that any pair of vertices are timelike separated. Then the reverse triangle inequality in Proposition 12.20 implied that

$$(13.6) \quad \text{ptime}(p, q) + \text{ptime}(q, r) < \text{ptime}(p, r)$$

unless q lies on the affine line connecting p and r . This is known as the *twin paradox*: an evil twin traveling in a spaceship from p to q to r ages less than the hapless chap traveling the straight and narrow from p to r .

The following is the Minkowski spacetime analog of Definition 12.5.

Definition 13.7. Let M be a Minkowski spacetime over a Lorentzian vector space V . A *worldline* is a connected 1-dimensional submanifold $C \subset M$ such that $T_p C \subset V$ is timelike for all $p \in C$. The worldline has *constant velocity* if $C \subset M$ is an open interval in an affine line.

A constant velocity motion is sometimes called an *inertial path* or *inertial motion*.

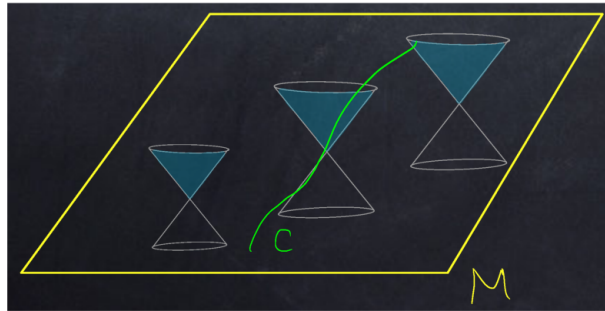


FIGURE 39. A worldline C in a Minkowski spacetime M

Lemma 13.8. *Up to time translation, a worldline has a unique parameterization $s \mapsto p(s)$ such that $|\dot{p}(s)| = c$ for all s and $\dot{p}(s)$ is forward timelike.*

The proof is essentially the same as that of unit speed parametrizations of motions on Euclidean space. The parameter s is called the *proper time* along C ; it is determined up to shifts by a constant.

For $C \subset M$ a worldline, let $U_p = T_p C \subset V$ denote the tangent line at $p \in C$. Then U_p is a timelike line in V ; it is independent of p for inertial paths. Let $\tau_p \in U_p$ denote the forward vector of proper time $|\tau_p| = c$. Note that $\tau_p = \dot{p}(s)$ for any distinguished parametrization, where $p(s) = p$. This is not a velocity. (Recall from the discussion following [Definition 12.5](#) that worldlines in Galilean spacetimes do not have a velocity either.) The second derivative of a distinguished parametrization satisfies $\ddot{p}(s) \in U_{p(s)}^\perp$. The subspace $U_p^\perp \subset V$ is spacelike, and $\ddot{p}(s)$ is the *acceleration* of the worldline at $p = p(s)$. Note that it vanishes identically iff the motion is inertial.

The following is a preliminary to an intrinsic definition of acceleration. Let M be an affine space over a vector space V , and suppose $C \subset M$ is a submanifold of dimension k . Let $\text{Gr}_k(V)$ be the *Grassmannian* of k -dimensional subspaces of V . Define the *Gauss map*

$$(13.9) \quad \begin{aligned} \Gamma: C &\longrightarrow \text{Gr}_k(V) \\ p &\longmapsto T_p C \end{aligned}$$

The Gauss map is constant iff C lies in an affine subspace of dimension k . If $U \in \text{Gr}_k(V)$, then there is a canonical isomorphism $T_U \text{Gr}_k(V) \cong \text{Hom}(U, V/U)$. Use it to write the differential of the Gauss map at $p \in C$ as a linear map

$$(13.10) \quad d\Gamma_p: T_p C \longrightarrow \text{Hom}(T_p C, V/T_p C).$$

Equivalently, this is a bilinear map

$$(13.11) \quad \Pi_p: T_p C \times T_p C \longrightarrow V/T_p C.$$

Lemma 13.12. *The bilinear form Π_p is symmetric.*

We leave the proof to the homework. The symmetric bilinear form Π_p is the *second fundamental form* of $C \subset M$ at p . It is typically defined when M is a Euclidean space, in which case we identify the quotient $V/T_p C$ with the normal space $T_p C^\perp \subset V$. In this form it generalizes to submanifolds of Riemannian manifolds.

Return now to a worldline $C \subset M$ in a Minkowski spacetime. The second fundamental form at $p \in C$ is a linear map

$$(13.13) \quad \alpha_p: U_p^{\otimes 2} \rightarrow U_p^\perp,$$

where the Lorentz metric identifies the quotient V/U_p with the spacelike hyperplane $U_p^\perp \subset V$. Then α_p is the (intrinsic) *acceleration*. Observe that it has units L/T^2 since U_p has units of time and $U_p^\perp$ has units of length. We leave the reader to reconcile this definition with the one above using a distinguished parametrization.

If we attach a mass m to a worldline, then we can define its *energy-momentum*. Recall [\(12.23\)](#) that the Lorentz metric determines an isomorphism $\phi: V \xrightarrow{\cong} V^*$.

Definition 13.14. Let M be a Minkowski spacetime over a Lorentzian vector space V , and let $C \subset M$ be the worldline of a particle with (rest) mass m . Then the *energy-momentum* of (C, m) is

$$(13.15) \quad \begin{aligned} \mu: C &\longrightarrow V^* \\ p &\longmapsto m\phi(\tau_p) \end{aligned}$$

where $\tau_p \in T_p C \subset V$ is the forward timelike vector with $|\tau_p| = c$.

Note that ϕ is an isometry, hence the norm square of $\mu(p)$ is the constant $m^2 c^2$. At $p \in C$ we have the instantaneous time-space splitting $V = U_p \oplus U_p^\perp$, and so a decomposition $\mu(p) = E\tau_p + \vec{p}$ into energy plus momentum. The norm square is then

$$(13.16) \quad E^2/c^2 - |\vec{p}|^2 = m^2 c^2.$$

This is the famous equation of Einstein.

Remark 13.17. The map ϕ is structural; it has no units. The vector τ_p has units of velocity (L/T), since we normalized it via $|\tau_p| = c$. Hence $\mu(p)$ in (13.15) has units of momentum (ML/T).

The energy-momentum has constant norm, interpreted as the mass, and it has positive energy so lies in the cone $P^* \subset V^*$ of positive norm square vectors that evaluate positively on vectors in $P \subset V$.

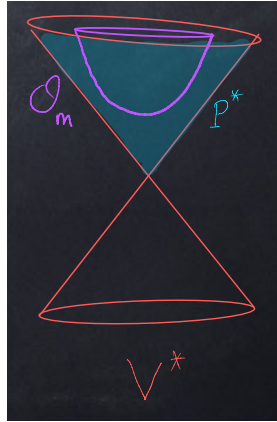


FIGURE 40. The mass shell (mass hyperboloid) $\mathcal{O}_m$

Definition 13.18. The *mass shell* or *mass hyperboloid* is the subset of positive energy vectors of fixed norm,⁴⁰

$$(13.19) \quad \mathcal{O}_m = \{\theta \in P^* : |\theta| = mc\}.$$

The mass shell is depicted in Figure 40. The image of a timelike line in V under $\phi: V \rightarrow V^*$ intersects $\mathcal{O}_m$ in a single point, and this intersection defines a diffeomorphism of the manifold of timelike lines in V with $\mathcal{O}_m$.

⁴⁰Recall that the norm on V^* has units of momentum.

13.2 The nonrelativistic limit

Consider the family of standard Minkowski spacetimes $(\mathbb{M}^n, c)$ parametrized by $c \in \mathbb{R}^{>0}$. The nonrelativistic limit is $c \rightarrow \infty$. The metric (13.2) blows up, but the inverse metric

$$(13.20) \quad \frac{1}{c^2} \left(\frac{\partial}{\partial t} \right)^2 - \left(\frac{\partial}{\partial x^1} \right)^2 - \left(\frac{\partial}{\partial x^2} \right)^2 - \cdots - \left(\frac{\partial}{\partial x^{n-1}} \right)^2$$

has a limit which is a degenerate symmetric bilinear form on V^* , where $V = \mathbb{R}^{1, n-1}$. Its kernel is a line $K \subset V^*$ whose annihilator is a codimension one subspace $W \subset V$. The time orientation P induces an orientation of V/W . Furthermore, W inherits a definite inner product from the $c \rightarrow \infty$ limit of (13.20). In other words, we obtain a Galilean structure (Definition 12.3). Therefore, Minkowski spacetime is a deformation of Galilean spacetime. The latter should be regarded as singular, whereas Minkowski spacetime is generic (or stable). We refer to [DF2, §1.4] for more details.

13.3 The Poincaré group

Let M be a Minkowski spacetime over a Lorentzian vector space V . The group $O(V)$ of linear isometries of V has a subgroup $O^\uparrow(V)$ of isometries that preserve the forward timelike cone $P \subset V$. If the dimension n is at least 2, then $O(V)$ has 4 components and $O^\uparrow(V)$ has two components. The identity component of $O(V)$ (and of $O^\uparrow(V)$) is $SO^\uparrow(V)$. These groups are depicted in Figure 41. There are corresponding affine isometry groups $SO^\uparrow(M) \subset O^\uparrow(M) \subset O(M)$. The double cover of $SO^\uparrow(V)$ is the spin group $\text{Spin}(V)$; see §12.4 for a very brief introduction.

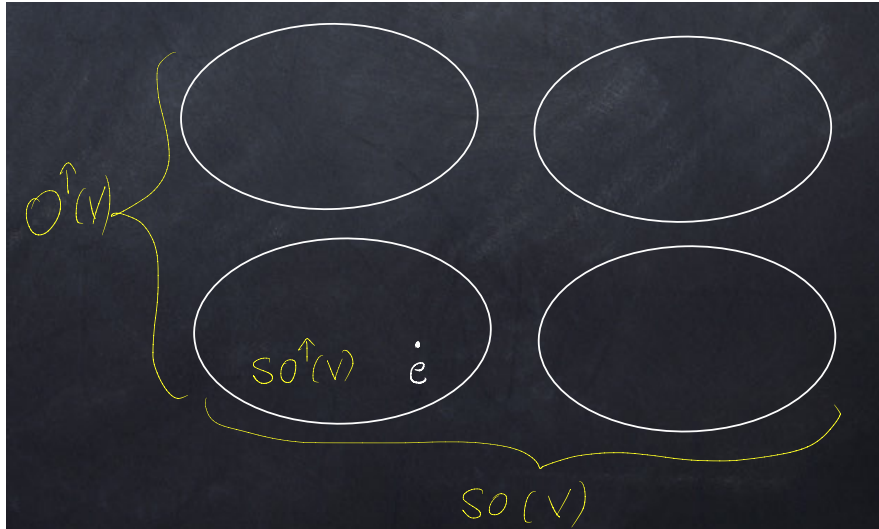


FIGURE 41. The four components of $O(V)$ (assuming $n \geq 2$)

Definition 13.21. Let M be a Minkowski spacetime over a Lorentzian vector space V . The *Poincaré group* $\mathcal{P}(M)$ is a double cover $\mathcal{P}(M) \rightarrow SO^\uparrow(M)$ that projects to the spin double cover $\text{Spin}(V) \rightarrow SO^\uparrow(V)$.

These isometry groups of Minkowski spacetime fit into the following diagram of group extensions:

$$(13.22) \quad \begin{array}{ccccccccc} 1 & \longrightarrow & V & \longrightarrow & \mathcal{P}(M) & \longrightarrow & \text{Spin}(V) & \longrightarrow & 1 \\ & & \parallel & & \downarrow 2:1 & & \downarrow 2:1 & & \\ 1 & \longrightarrow & V & \longrightarrow & \text{SO}^\uparrow(M) & \longrightarrow & \text{SO}^\uparrow(V) & \longrightarrow & 1 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & V & \longrightarrow & \text{O}^\uparrow(M) & \longrightarrow & \text{O}^\uparrow(V) & \longrightarrow & 1 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & V & \longrightarrow & \text{O}(M) & \longrightarrow & \text{O}(V) & \longrightarrow & 1 \end{array}$$

Each of the groups $\text{SO}^\uparrow(M) \subset \text{O}^\uparrow(M) \subset \text{O}(M)$ acts *effectively* on M : every non-identity group element moves some point of M . The Poincaré group $\mathcal{P}(M)$ has a special non-identity element ϵ that covers the identity element of $\text{SO}^\uparrow(M)$; it projects to a similar element of $\text{Spin}(V)$. Then ϵ acts as the identity map on M : it does not move any points of spacetime. Such a symmetry is called *internal*. We consider more general groups of symmetries, including more elaborate internal symmetries, in upcoming lectures.

Remark 13.23. In Galilean geometry there is a distinguished subgroup of *boost* symmetries; see [Figure 27](#) and the nearby text. There is no such distinguished subgroup in Minkowski geometry unless we fix a time-space splitting [\(12.14\)](#).

Lecture 14: Classical relativistic particle; relativistic systems; relativistic QM

In the first part of this lecture we consider a classical particle in Minkowski spacetime. The classical motions have worldlines that are timelike affine lines. The collection of such lines is a fiber bundle of affine spaces, and it carries a symplectic form for which the fibers are Lagrangian. We indicate a Lagrangian density from which this symplectic manifold can be derived, and we also show how its nonrelativistic limit is the standard Lagrangian of Newtonian mechanics.

The classical relativistic particle is an example of a *relativistic system*. This is a modification of [Definition 1.11](#) that replaces time evolution of states and observables by an action of the Poincaré group on states and observables. (Later we consider more general symmetry groups.) Then we specialize to relativistic quantum mechanics, for which the data is a (projective) Hilbert space and a unitary representation of the Poincaré group. We recall the spectral theorem for the restriction of the representation to the subgroup of translations. The spectrum of the representation is the support of the spectral measure, which is a subset of energy-momentum vectors. In subsequent lectures we explore relativistic QM systems in more depth and give examples. (For instance, the next lecture begins with the relativistic quantum particle.)

We conclude the lecture with mathematical background on smooth densities.

14.1 The classical relativistic particle

Let M be a Minkowski spacetime over a vector space V with Lorentzian inner product $\langle -, - \rangle$. This data includes a convex cone $P \subset V$ of forward timelike vectors and a constant $c \in \mathbb{R}^{>0}$, the speed of light. Recall (Definition 13.7) that a particle trajectory in this geometry is a *worldline*: a smooth connected 1-dimensional submanifold $C \subset M$ whose tangent lines are timelike.

Definition 14.1. A *classical trajectory* is an affine worldline $C \subset M$. Denote the space of all classical trajectories as $\mathcal{M}$.

Thus $\mathcal{M}$ is the manifold of all timelike affine lines in M . It is the *phase space* of the classical relativistic particle. We will see that it has a nice geometric structure. A classical trajectory is depicted in Figure 42.

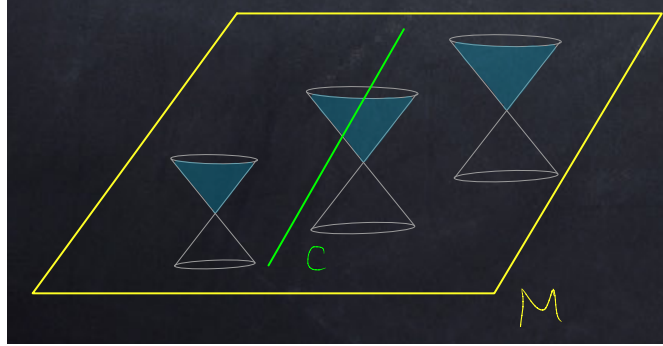


FIGURE 42. A classical trajectory

Fix a mass m , which is a positive real number, and suppose $C \subset M$ is a worldline. We interpret C as the record of a spacetime motion of a particle of mass m . Introduce the (Lagrangian) density

$$(14.2) \quad -mc \, d\mu_C$$

on C , where $d\mu_C$ is the smooth density of the induced Riemannian metric (§14.4.1). The *action* on a compact connected subset $J \subset C$ is $-mc \, \text{Length}(J)$. Therefore, the worldlines that extremize the action are locally length-minimizing, which in affine space implies that they are affine lines, i.e., classical trajectories as in Definition 14.1. It is convenient to rewrite (14.2) in terms of an arbitrary parametrization $\gamma: I \rightarrow M$ of C :

$$(14.3) \quad -mc \, d\mu_C = -mc \, \langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle^{1/2} ds, \quad s \in I.$$

Introduce standard coordinates $t, x^1, \dots, x^{n-1}$ on M so that the metric is

$$(14.4) \quad c^2 dt^2 - (dx^1)^2 - \dots - (dx^{n-1})^2.$$

Parametrize C by t , so via functions $t, x^1(t), \dots, x^{n-1}(t)$. Write

$$(14.5) \quad v = [(\dot{x}^1)^2 + \dots + (\dot{x}^{n-1})^2]^{1/2}.$$

Then write (14.3) as

$$(14.6) \quad \begin{aligned} -mc d\mu_C &= -mc^2 \sqrt{1 - v^2/c^2} dt \\ &= \left[-mc^2 + \frac{1}{2}mv^2 + O\left(\frac{v^4}{c^2}\right) \right] dt. \end{aligned}$$

The first two terms in brackets are the standard action for a Newtonian/Galilean particle of mass m and potential energy mc^2 (recall (13.16)).

Let $C \subset M$ be a classical trajectory, i.e., a timelike affine line. Its tangent line $U \subset V$ is a timelike line. Write $\tau_U \in U$ for the forward vector with $|\tau_U| = c$. The energy-momentum of C is the constant $\theta_U = m\phi(\tau_U)$ given by (13.15); note that $|\theta_U| = mc$. Recall the mass shell defined introduced in Definition 13.18. The map $U \mapsto \theta_U$ from timelike affine lines in V to $\mathcal{O}_m$ is a diffeomorphism, and so there is a surjective map

$$(14.7) \quad \pi: \mathcal{M} \longrightarrow \mathcal{O}_m$$

that takes a classical trajectory to its energy-momentum. In fact, π is a fiber bundle of affine spaces. Namely, the fiber over θ_U is the space of affine lines parallel to U , which is an affine space over V/U .

Proposition 14.8. *$\mathcal{M}$ carries a symplectic form with respect to which the fibers of π are Lagrangian.*

The key observation is that π is an affine bundle over the cotangent bundle of $\mathcal{O}_m$. Then we transport the natural symplectic form on the total space of the cotangent bundle to the total space of π via a natural family of sections of π , after proving that the result is independent of the section.

Proof. First, recall that if X is a smooth manifold, then the total space T^*X of its cotangent bundle $\varpi: T^*X \rightarrow X$ carries a canonical 1-form γ ; see Example 2.29(2). Namely, at a cotangent vector $\lambda \in T^*X$ we have

$$(14.9) \quad \gamma_\lambda(\dot{\lambda}) = \lambda(\varpi_* \dot{\lambda}), \quad \dot{\lambda} \in T_\lambda^*(T^*X).$$

The differential $d\gamma$ is a nondegenerate closed 2-form on T^*X , i.e., it is a symplectic form. (We say that T^*X is an *exact symplectic manifold* since the symplectic form is exact.) Note that the fibers of the cotangent bundle $\varpi: T^*X \rightarrow X$ are Lagrangian.

Next, suppose that $\pi: E \rightarrow X$ is an affine bundle over $\varpi: T^*X \rightarrow X$, i.e., for each $x \in X$ the fiber E_x is an affine space over T_x^*X . A section s of π induces an identification $\varphi_s: E \xrightarrow{\cong} T^*X$, and the difference of two sections s', s is a section of ϖ , i.e., a 1-form α . An easy check shows that

$$(14.10) \quad \varphi_{s'}^* \gamma - \varphi_s^* \gamma = \alpha.$$

(Of course, one can carry out these constructions locally on X .) Now if given a set $\mathcal{S}$ of sections of π such that the 1-form difference between any $s', s \in \mathcal{S}$ is *closed*, then it follows from (14.10) that the transport $\varphi_s^*(d\gamma)$ of the symplectic form to E is independent of $s \in \mathcal{S}$.

Apply this to the affine bundle (14.7), whose total space is the space of classical trajectories of a particle of mass m in a Minkowski spacetime M . First, $\mathcal{O}_m$ is defined by the equation

$$(14.11) \quad \langle \theta, \theta \rangle = m^2 c^2, \quad \theta \in V^*,$$

so differentiating we find a canonical identification $T_{\theta_U} \mathcal{O}_m \cong (U^*)^\perp \cong (U^\perp)^*$. Hence we can and do identify $T_{\theta_U}^* \mathcal{O}_m \cong U^\perp \cong V/U$. The fiber of (14.7) at $\theta_U \in \mathcal{O}_m$ is affine over V/U ; see the text following (14.7). Hence we are in the position envisioned in the previous paragraph. Furthermore, $\mathcal{S} = M$ parametrizes a natural space of sections of π : given a timelike line $U \subset V$ and a point $p \in M$, there is a unique timelike affine line parallel to U that passes through p . If $p', p \in M$, then write $p' = p + \xi$ for $\xi \in V$. Now ξ is a constant 1-form on V^* , so it restricts to a *closed* 1-form on $\mathcal{O}_m \subset V^*$, and this is the difference of the sections labeled by p', p . This completes the construction of a symplectic form on $\mathcal{M}$. The fact that the fibers of π are Lagrangian follows from the corresponding fact for the cotangent bundle to $\mathcal{O}_m$. $\square$

Observe that the isometry group $O(M)$ of Minkowski spacetime acts on the manifold $\mathcal{M}$ of timelike affine lines. We leave as an exercise that the subgroup $O^\uparrow(M)$ acts preserving the symplectic form and its complement reverses the sign of the symplectic form.

The symplectic manifold $\mathcal{M}$ almost defines a classical mechanical system, according to the discussion in [Lecture 1](#). What is missing is a 1-parameter group of symplectic diffeomorphisms of $\mathcal{M}$. What is naturally there instead is the action of $O^\uparrow(M)$ by symplectic diffeomorphisms. If we fix a timelike affine line $U_0 \subset V$, and so obtain a splitting $V = U_0 \oplus U_0^\perp$ as in (12.14), then we can restrict the action to $U \subset O^\uparrow(M)$ to obtain time evolution. But in relativistic systems we shun time-space splittings, and so we are led a relativistic modification of a mechanical system.

14.2 Relativistic systems

The following is a modification of [Definition 1.11](#). Recall [Definition 13.1](#) of a Minkowski spacetime and [Definition 13.21](#) of the Poincaré group.

Definition 14.12. Let M be a Minkowski spacetime. A *relativistic system* on M consists of DOSO (data of states and observables) as in [Axiom System 1.1](#) together with a representation of the Poincaré group $\mathcal{P}(M)$ by automorphisms of states and observables.

Remark 14.13.

- (1) As mentioned several times, we will define a more general notion of symmetry group of a relativistic system, and there is a corresponding variation of [Definition 14.12](#) in which that symmetry group acts on states and observables.
- (2) If $\dim M = 1$, then [Definition 14.12](#) reduces to [Definition 1.11](#).
- (3) As opposed to a mechanical system, a relativistic system can admit a *local* theory of observables. We will develop this for relativistic quantum mechanical systems in a future lecture.

[Definition 3.1](#) of a Hamiltonian system also generalizes to relativistic systems.

Definition 14.14. Let M be a Minkowski spacetime.

- (1) The data of a *classical relativistic system* on M is a symplectic manifold (N, ω) and an action of the Poincaré group $\mathcal{P}(M)$ by symplectic automorphisms of N .
- (2) A system is *free* if (N, ω) is an affine symplectic space and $\mathcal{P}(M)$ acts by affine symplectic automorphisms.

The classical relativistic particle is a classical relativistic system, but it is not free according to **Definition 14.14**. We will see free classical systems of *fields*.

Remark 14.15. Just as there may be a Lagrangian formulation of a Hamiltonian mechanical system (§8.2), so too there may be a Lagrangian formulation of a classical relativistic system.

Definition 5.1 of a linear QM system also has a relativistic generalization.

Definition 14.16. Let M be a Minkowski spacetime. The linear data of a *relativistic quantum mechanical (QM) system* is a pair $(\mathcal{H}, U)$ consisting of a complex separable Hilbert space $\mathcal{H}$ and a strongly continuous unitary representation $U: \mathcal{P}(M) \rightarrow \text{U}(\mathcal{H})$ of the Poincaré group.

We explore **Definition 14.16** in the next section.

Remark 14.17.

- (1) The discussion in **Lecture 8** is centered around the notion that quantum theory is projective—recall the slogan (8.1)—hence we should better say that a relativistic QM system is a *projective* unitary representation of $\mathcal{P}(M)$. However, any central extension by $\mathbb{T}$ of $\mathcal{P}(M)$ splits, and except in low dimensions there are no nontrivial characters—the splitting of the central extension is unique—so there is no loss in using linear representations. For other symmetry groups, there may be nontrivial central extensions. In any case conceptually one should replace the linear space $\mathcal{H}$ by a projective space. We have chosen to ease the exposition by using linear spaces, but the reader should keep this important point in mind.
- (2) We should also recall the discussion in **Lecture 10**, which allows bosonic and fermionic states and so replaces $\mathcal{H}$ by a $\mathbb{Z}/2\mathbb{Z}$ -graded Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. The unitary representation U is by even operators, so it preserves each homogeneous subspace.

14.3 Relativistic quantum mechanics

Recall the discussion of spectral decompositions in **Lecture 4**.

Suppose given a relativistic QM system on a Minkowski spacetime M , as in **Definition 14.16**. One might construct the spectrum of a representation of $\mathcal{P}(M)$ by decomposing it into irreducible representations. Such a decomposition may have discrete and continuous parts. Instead, it is advantageous to restrict to the *abelian* translation subgroup $V \subset \mathcal{P}(M)$ and decompose according to unitary characters of V , which are identified with V^* via the isomorphism:

$$(14.18) \quad \begin{aligned} V^* &\longrightarrow \text{Hom}(V, \mathbb{T}) \\ \theta &\longmapsto (\xi \mapsto e^{-i\theta(\xi)/\hbar}) \end{aligned}$$

(Observe $[\xi] = L$ and $[\theta] = ML/T$, so the exponent is dimensionless.)

Theorem 14.19. *Let $U: V \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the vector group V . Then there exists a spectral measure π_U on V^* such that*

$$(14.20) \quad U(\xi) = \int_{V^*} e^{-i\theta(\xi)/\hbar} d\pi_U(\theta)$$

for all $\xi \in \mathcal{H}$.

Let $P \subset V$ denote the cone of forward timelike vectors, part of the data of the Minkowski spacetime M . Its dual $P^* \subset V^*$ is the cone of energy-momentum vectors of positive energy. The support of a measure is the complement of the union of open subsets with measure zero; see [Definition 1.55](#).

Definition 14.21. Let $U: \mathcal{P}(M) \rightarrow \mathcal{U}(\mathcal{H})$ be a unitary representation of the Poincaré group $\mathcal{P}(M)$.

- (1) The *spectrum* $\text{spec}(U) \subset V^*$ of U is the support of the spectral measure π_U .
- (2) We say that U has *positive energy* if $\text{spec}(U) \subset \overline{P^*}$.
- (3) A *relativistic particle* is an irreducible unitary positive energy representation of $\mathcal{P}(M)$.

The spectrum of U is a closed subset of V^* , and it is invariant under the action of $\text{SO}^\uparrow(V)$. We study irreducible representations of $\mathcal{P}(M)$ in a future lecture.

The decomposition of $\overline{P^*}$ into $\text{SO}^\uparrow(V)$ -orbits is roughly given by the mass; see [Figure 43](#). If $n = \dim M \geq 3$, then that decomposition is precisely

$$(14.22) \quad \overline{P^*} = \{0\} \sqcup N(V^*)^+ \sqcup \bigsqcup_{m>0} \mathcal{O}_m,$$

where $N(V^*)^+ = (N(V^*) \cap \overline{P^*}) \setminus \{0\}$ is the space of positive energy null vectors. (For $\dim M = 2$ the positive energy null vectors $N(V^*)^+$ decompose into two orbits, each of which is a ray, and for $\dim M = 1$ there are no nonzero null vectors: $N(V^*)^+ = \emptyset$.)

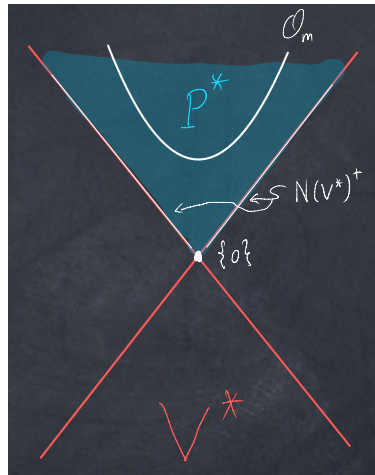


FIGURE 43. The decomposition of $\overline{P^*}$ into $\text{SO}^\uparrow(V)$ -orbits

There is an important dichotomy of relativistic quantum mechanical systems: gapped vs. gapless.

Definition 14.23. Let $U: \mathcal{P}(M) \rightarrow U(\mathcal{H})$ be a unitary representation of the Poincaré group. Then U is *gapped* if the minimal mass subset is isolated in $\text{spec}(U)$ and the restriction of U to the corresponding closed subspace of $\mathcal{H}$ is a finite sum of irreducible representations. If U is not gapped, then we say that U is *gapless*.

If $U: V \rightarrow U(\mathcal{H})$ is a unitary representation of the translation group, then elements of $\text{Sym}^k(V)$ also act by unitary operators for any $k \in \mathbb{Z}^{\geq 1}$: the symmetric product $\xi_1 * \cdots * \xi_k$ acts by the composition $U(x_1) \circ \cdots \circ U(x_k)$. The inverse Lorentz metric on V^* is an element of $\text{Sym}^2(V)$; the corresponding unitary operator M is the mass square operator (after multiplication by $\hbar^2/c^2$). It commutes with any representation U of the Poincaré group. If U is a positive energy representation of Poincaré and $0 \in \text{spec}(U)$, then $0 \in \text{spec}(M)$, and U is gapped iff M has a spectral gap in the sense of [Definition 5.7\(2\)](#).

Finally, we remark that some positive energy unitary representations of $\mathcal{P}(M)$ extend to a positive energy unitary representation of a double cover of $O^\uparrow(M)$. (Recall the various groups in [\(13.22\)](#).) There are two such double covers, known as pin groups, and they embed in Clifford algebras. See [\[FH1, Appendix A\]](#) for more detail.

14.4 Mathematical background

14.4.1 Smooth densities. Let V be a finite dimensional real vector space of dimension N . Canonically associated to V are several real lines. For example, the *determinant line* $\text{Det } V = \bigwedge^N V$ is the highest exterior power of V . A nonzero vector in $\text{Det } V$ represents an oriented parallelepiped in V . A nonzero vector in $\text{Det } V^*$ is a notion of oriented volume on V . There is also a “line” of *densities*, though it is *not* a vector space; it is diffeomorphic to the real line and carries an analogous trichotomy of positive/zero/negative numbers.

Definition 14.24.

- (1) Let V be a finite dimensional real vector space of dimension N , and let $\mathcal{B}(V)$ denote its right $\text{GL}_N \mathbb{R}$ -torsor of bases. The *line of densities* on V is

$$(14.25) \quad \begin{aligned} |\text{Det } V^*| &= \{ \mu: \mathcal{B}(V) \longrightarrow \mathbb{R} : \mu(b \cdot g) = |\det g| \mu(b) \text{ for all } b \in \mathcal{B}, g \in \text{GL}_N \mathbb{R} \} \\ &= \mathcal{B}(V) \times_{\text{GL}_N \mathbb{R}} \mathbb{R}, \end{aligned}$$

where in the second line $\text{GL}_N \mathbb{R}$ acts on $\mathbb{R}$ by $g \mapsto |\det g|$. There is a partition

$$(14.26) \quad |\text{Det } V^*| = |\text{Det } V^*|^+ \amalg \{0\} \amalg |\text{Det } V^*|^-$$

according to the sign of μ . Elements of $|\text{Det } V^*|^+$ are *positive densities*.

- (2) Let M be a smooth manifold. Then a *smooth positive density* on M is a section of the fiber bundle $|\text{Det } T^*M|^+ \rightarrow M$.

A positive density on M is also called a *smooth measure*. A partition of unity argument proves that smooth measures exist; the fibers of $|\text{Det } T^*M|^+ \rightarrow M$ are convex.

If G is a Lie group that acts smoothly and transitively on M , then a G -invariant measure exists iff for any $x \in M$ the action of the stabilizer subgroup $G_x \subset G$ on $|\text{Det } T^*M|^+$ is trivial. (In general, the action is by a character $G_x \rightarrow \mathbb{R}^{>0}$. For example, if G_x is compact, then the action is necessarily trivial. This argument proves the existence of Haar measure on compact Lie groups.)

Now suppose that V is an inner product space. Since the representation $g \mapsto |\det g|$ restricts to the trivial representation on $O_N \subset \text{GL}_n \mathbb{R}$, the line $|\text{Det } V^*|^+$ is canonically diffeomorphic to $\mathbb{R}^{>0}$. There is a canonical element: $1 \in \mathbb{R}^{>0}$. This density encodes the notion of volume on V induced from the lengths and angles computed via the inner product. Applied pointwise on a Riemannian manifold M , we learn that M carries a canonical smooth measure, the *Riemannian measure* μ_M .

Lecture 15: Relativistic quantum mechanics

We already gave basic definitions in §14.3. Namely, the data of a relativistic QM system is a unitary representation of the Poincaré group. Its restriction to the vector space of translations can be “diagonalized” according to Theorem 14.19. The spectral measure that expresses this diagonalization has a support, which leads to the notion of a *positive energy* representation (Definition 14.21). We always consider relativistic QM systems that satisfy the positive energy condition.

We begin in this lecture with an example: the quantum relativistic particle of mass m . We treated the classical relativistic particle in §14.1, where we construct a symplectic manifold $\mathcal{M}$ equipped with an action of time-orientation preserving isometries of Minkowski spacetime. Here we construct a quantum system which is meant to be its quantization. This is an *irreducible* representation of the isometry group, so is a relativistic particle (Definition 14.21). In subsequent examples we construct examples of relativistic QM systems from *fields* (rather than particles).

We then turn to a few other general topics in relativistic quantum mechanics. The first are constraints when one has the underlying quantum mechanical system of a relativistic quantum field theory. It is only in subsequent lectures that we define the latter, but here it is convenient to state the *spin-statistics theorem* and the misnamed *CPT⁴¹ theorem*. We conclude with a general discussion of affine symmetry types and of a relativistic symmetry type.

15.1 The quantum relativistic particle

We do not give a systematic discussion of quantization in these lectures, so only loosely discuss the passage from the symplectic manifold $\mathcal{M}$ of classical trajectories (Definition 14.1) of a particle

⁴¹‘T’ is ‘time-reversal’ and ‘P’ is ‘parity-reversal’. We replace ‘P’ with ‘R’, which is ‘reflection’, since in any dimension reflection in a hyperplane reverses orientation, whereas parity-reversal—reflection in the origin—preserves or reverses orientation depending on the parity of the dimension. The ‘C’ is ‘conjugation’. Usually this is ‘charge conjugation’, but there is no charge to conjugate for representations of the Poincaré group. Rather, we can think it is ‘complex conjugation’ which evokes the antilinearity of time-orientation reversing transformations, but in truth that is not its intended meaning. Best to simply call it the ‘RT theorem’. Also, ‘C’, ‘R’, and ‘T’ do not refer to particular elements of the Poincaré group but rather to entire components.

of mass m in a Minkowski spacetime M to a unitary representation of the Poincaré group $\mathcal{P}(M)$. It turns out that this representation is irreducible.

Recall that $\mathcal{M}$ is the total space of an affine bundle over the mass shell $\mathcal{O}_m$. An important ingredient in the quantization of a symplectic manifold is a *polarization*, which in this case can be taken to be a foliation by (real) Lagrangian submanifolds, namely the fibers of $\pi: \mathcal{M} \rightarrow \mathcal{O}_m$. Then, roughly, one builds a Hilbert space of functions which are constant along the leaves of the foliation, which in this case amounts to a Hilbert space of functions on $\mathcal{O}_m$. As we show in the next lecture, there is a ray of $\mathrm{SO}^\uparrow(V)$ -invariant measures on $\mathcal{O}_m$, and we let the quantization $\mathcal{H} = L^2(\mathcal{O}_m; \mathbb{C})$ be the Hilbert space of complex-valued L^2 functions on $\mathcal{O}_m$. (There are only bosonic states in this system: $\mathcal{H}$ is even.) Note that each $\xi \in V$ determines a smooth function

$$(15.1) \quad \theta \longmapsto e^{-i\theta(\xi)/\hbar}$$

on $\mathcal{O}_m$, and $\xi \in V$ acts as multiplication by this function. To see the action of the entire Poincaré group, it is convenient to make a different model. Observe that $\mathcal{P}(M)$ acts on the Cartesian product $\mathcal{O}_m \times M$, and so too acts linearly on functions on $\mathcal{O}_m \times M$. These actions factor through the quotient $\mathcal{P}(M) \rightarrow \mathrm{SO}^\uparrow(M)$. Restrict to functions f that are L^2 in the $\mathcal{O}_m$ -direction and that satisfy

$$(15.2) \quad f(\theta_U, p + \xi) = e^{i\theta_U(\xi)/\hbar} f(\theta_U, p), \quad (\theta_U, p) \in \mathcal{O}_m \times M.$$

We can interpret these functions as L^2 sections of a hermitian line bundle over $\mathcal{O}_m$. The isometry group $\mathrm{SO}^\uparrow(M)$ manifestly acts on this Hilbert space. This is the desired representation.

The spectrum of this representation is the mass shell $\mathcal{O}_m \subset V^*$. We prove in the next lecture that $\mathrm{SO}^\uparrow(V)$ acts transitively on $\mathcal{O}_m$. It follows that this representation is irreducible, so defines a relativistic particle.

15.2 Two theorems in quantum field theory

Relativistic quantum field theories form a special class of relativistic quantum mechanical theories. For these, there are two basic theorems which restrict the representations of the Poincaré group that occur.

Recall from (10.1) that states form a disjoint union of two convex spaces: bosonic states and fermionic states. A state is either bosonic or fermionic, a feature which is termed its *statistics*.

An irreducible representation of the spin group $\mathrm{Spin}(V)$ has a spin, which is a half-integer. We define it in the next lecture. For now we just need the value modulo the integers. Recall that the central element $\epsilon \in \mathrm{Spin}(V)$, which generates the kernel of the double cover $\mathrm{Spin}(V) \rightarrow \mathrm{SO}^\uparrow(V)$, has order 2.

Definition 15.3. Let $\epsilon \in \mathrm{Spin}(V)$ be the central element. An irreducible representation of $\mathrm{Spin}(V)$ has *integer spin* if ϵ acts as $+1$, and it has *half-integer spin* if ϵ acts as -1 .

A representation of integer spin factors to a representation of $\mathrm{SO}^\uparrow(V)$.

Theorem 15.4 (spin-statistics). *Suppose a positive energy unitary representation $U: \mathcal{P}(M) \rightarrow U(\mathcal{H}^0 \oplus \mathcal{H}^1)$ is the underlying QM systems a relativistic quantum field theory. Then $U(\epsilon)$ is the grading operator*

$$(15.5) \quad \text{id}_{\mathcal{H}^0} \oplus -\text{id}_{\mathcal{H}^1}$$

on $\mathcal{H}^0 \oplus \mathcal{H}^1$.

In other words, $U(\epsilon) = +1$ on $\mathcal{H}^0$ and $U(\epsilon) = -1$ on $\mathcal{H}^1$. In the physics literature, the notation ‘ $(-1)^F$ ’ is often used for the grading operator (15.5).

The second theorem tells about symmetries that reverse time-orientation. For simplicity we consider quantum field theories that only have bosonic states, so by Theorem 15.4 are representations of the quotient $\text{SO}^\uparrow(M)$ of the Poincaré group. Note that $\text{SO}^\uparrow(M)$ is the identity component of the 2-component Lie group $\text{SO}(M)$ whose off-component consists of isometries of M that reverse time-orientation (T) and reverse space-orientation (R). The following theorem states that time-orientation reversing symmetries are always implemented in a quantum field theory, albeit by antilinear operators—see Lemma 11.3.

Theorem 15.6 (CRT theorem). *Suppose a positive energy unitary representation $U: \text{SO}^\uparrow(M) \rightarrow U(\mathcal{H})$ is the underlying QM systems a relativistic quantum field theory. Then U extends to a homomorphism $\text{SO}(M) \rightarrow Q(\mathcal{H})$ in which elements of $\text{SO}(M) \setminus \text{SO}^\uparrow(M)$ act by an antiunitary transformation. If the quantum field theory has a positive energy unitary representation of the Lie group $\text{O}^\uparrow(M)$, then there is an extension to $\text{O}(M) \rightarrow Q(\mathcal{H})$ such that time-orientation reversing transformations act antiunitarily.*

Recall the Lie group Q in (8.48) which is the union of unitary and antiunitary transformations. There is a version of Theorem 15.6 for representations of the Poincaré group as well; the extensions are representations of Lie groups whose quotients by translations embed in Clifford algebras; see [FH1, Appendix A]. That reference discusses the extension of the theorem to arbitrary relativistic symmetry types.

15.3 Relativistic symmetry types

We continue the discussion of §7.4.

Let M be a space on which we are doing some geometry. Imagine, for example, that M is a topological space and our geometry consists of a finite cover $\pi: \widetilde{M} \rightarrow M$. Then symmetries of our geometry induce symmetries of M . In other words, there is a homomorphism from the symmetry group $\mathcal{G}$ of the geometric structure to the group $\text{Aut}(M)$ of symmetries of M . In our motivating example of a finite cover, this is a homomorphism $\text{Aut}(\pi) \rightarrow \text{Homeo}(M)$ from the group of homeomorphisms of $\widetilde{M}$ that preserve the fibers of π to the group of homeomorphisms of the base M . In general, one has an exact sequence of group homomorphisms

$$(15.7) \quad 1 \longrightarrow K \longrightarrow \mathcal{G} \longrightarrow \text{Aut}(M)$$

in which the kernel K consists of symmetries that fix M pointwise. These are *internal symmetries* that act on internal geometric structures without moving the points of M .

Specialize now to the case of an affine space M over a vector space V . Affine symmetries lie in a group extension

$$(15.8) \quad 1 \longrightarrow V \longrightarrow \text{Aut}(M) \longrightarrow \text{Aut}(V) \longrightarrow 1$$

in which the kernel is the normal subgroup of translations and the quotient is the group of linear symmetries of the vector space of translations. Affine geometry is the geometry of global parallelism, and therefore any geometric structure on M should be invariant under translation. So a Lie group $\mathcal{G}$ of symmetries of such a structure contains V as a normal subgroup, and so fits into a diagram

$$(15.9) \quad \begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ & & K & \xlongequal{\quad} & K & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & V & \longrightarrow & \mathcal{G} & \longrightarrow & G \longrightarrow 1 \\ & & \parallel & & \downarrow \tilde{\lambda} & & \downarrow \lambda \\ 1 & \longrightarrow & V & \longrightarrow & \text{Aut}(M) & \longrightarrow & \text{Aut}(V) \longrightarrow 1 \end{array}$$

in which the long rows are group extensions and the columns are exact: the group K of internal symmetries is the kernel of the homomorphisms $\lambda, \tilde{\lambda}$. The groups $\mathcal{G}$ of affine symmetries and G of linear symmetries determine each other, as do the group homomorphisms $\tilde{\lambda}$ and λ .

Example 15.10. If V carries an inner product, then the inclusion $\lambda: \text{O}(V) \rightarrow \text{Aut}(V)$ encodes the induced Euclidean structure on M . A spin structure is encoded in a homomorphism $\lambda: \text{Spin}(V) \rightarrow \text{Aut}(V)$ which is a nontrivial⁴² double cover of its image $\text{SO}(V)$, though in this case it is less clear how to describe the geometric structure preserved by the group $\text{Spin}(V)$. As another example, we might have the projection $\lambda: \text{Aut}(V) \times K \rightarrow \text{Aut}(V)$ for some Lie group K . This is the symmetry of a geometry $M \times F$ in which F is some manifold with K acting as a group of symmetries.

As in §7.4 we use a model space to define a general type of geometry.

Definition 15.11. Let n be a positive integer. An *affine symmetry type* of dimension n consists of a Lie group G_n and a group homomorphism $\lambda_n: G_n \rightarrow \text{GL}_n \mathbb{R}$.

If this data exists coherently for all positive integers n , then we say that the affine symmetry type is *stable*. A Euclidean structure is an example:

$$(15.12) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & \text{O}_n & \longrightarrow & \text{O}_{n+1} & \longrightarrow & \cdots \\ & & \downarrow \lambda_n & & \downarrow \lambda_{n+1} & & \\ \cdots & \longrightarrow & \text{GL}_n \mathbb{R} & \longrightarrow & \text{GL}_{n+1} \mathbb{R} & \longrightarrow & \cdots \end{array}$$

⁴²at least if $\dim V \geq 2$.

Use [Definition 7.45](#) to define (families of) geometries of type λ_n .

A symmetry type for relativistic quantum mechanics is a special case, but we insist that the symmetry group maps onto the identity component of the isometry group of Minkowski spacetime. This ensures that the symmetry group describes “relativistic invariance”.

Definition 15.13. Let n be a positive integer. A *relativistic symmetry type* of spacetime dimension n is a Lie group G_n equipped with a Lie group homomorphism $\lambda_n: G_n \rightarrow O_{1,n-1}^\uparrow$ whose image contains the identity component $SO_{1,n-1}^\uparrow \subset O_{1,n-1}^\uparrow$.

The kernel $K \subset G_n$ of λ_n is the model group of *internal symmetries*. Let $\mathcal{I}_{1,n-1}^\uparrow = O^\uparrow(\mathbb{M}^n)$ denote the group of isometries of model Minkowski spacetime $\mathbb{M}^n$ that preserve time-orientation. Then analogous to [\(15.9\)](#) is the commutative diagram

$$(15.14) \quad \begin{array}{ccccccc} & & & 1 & & 1 & \\ & & & \downarrow & & \downarrow & \\ & & & K & \xlongequal{\quad} & K & \\ & & & \downarrow & & \downarrow & \\ 1 & \longrightarrow & \mathbb{R}^{1,n-1} & \longrightarrow & \mathcal{G}_n & \longrightarrow & G_n \longrightarrow 1 \\ & & \parallel & & \downarrow \tilde{\lambda}_n & & \downarrow \lambda_n \\ 1 & \longrightarrow & \mathbb{R}^{1,n-1} & \longrightarrow & \mathcal{I}_{1,n-1}^\uparrow & \longrightarrow & O_{1,n-1}^\uparrow \longrightarrow 1 \end{array}$$

in which the long rows are group extensions and the columns are exact.

Remark 15.15. Relativistic symmetry types partition into two classes according to the image of λ_n , which is either $SO_{1,n-1}^\uparrow$ or $O_{1,n-1}^\uparrow$. There are various appellations one can ascribe to these classes. Perhaps the most common is telling whether there are ($O_{1,n-1}^\uparrow$) or are not ($SO_{1,n-1}^\uparrow$) time-orientation reversing symmetries in the theory. This is counterintuitive at first, and one can tell instead whether or not there are symmetries that reverse the orientation of space. For relativistic *field* theories, [Theorem 15.6](#) provides the time-orientation reversal symmetries if orientation-reversing symmetries of space are present.

Example 15.16. The Poincaré group $\mathcal{G}_n = \mathcal{P}_n$ provides an example of a relativistic symmetry type. Namely, we take

$$(15.17) \quad G_n = \text{Spin}_{1,n-1} \longrightarrow SO_{1,n-1}^\uparrow \subset O_{1,n-1}^\uparrow.$$

The internal symmetry group is the center $\mathbb{Z}_2 \subset \text{Spin}_{1,n-1}$ of the spin group, generated by the element ϵ in [Definition 15.3](#). Another important example is

$$(15.18) \quad G_n = \text{Spin}_{1,n-1}^c \longrightarrow SO_{1,n-1}^\uparrow \subset O_{1,n-1}^\uparrow$$

in which the internal symmetry group is the circle group $\mathbb{T}$. Theories with this symmetry type have an integer-valued charge operator, the representative of a nonzero element in the Lie algebra of $\mathbb{T}$.

Analogous to [Definition 14.16](#) is the following.

Definition 15.19. Let $\lambda_n: G_n \rightarrow O_{1,n-1}^\uparrow$ be a relativistic symmetry type, and suppose M is a Minkowski spacetime with λ_n -structure. The linear data of a *relativistic quantum mechanical (QM) system of type λ_n* on M is a pair $(\mathcal{H}, U)$ consisting of a complex separable Hilbert space $\mathcal{H}$ and a strongly continuous unitary representation $U: \mathcal{G}(M) \rightarrow U(\mathcal{H})$.

As in [Definition 7.45](#) the Minkowski spacetime is determined by a right $\mathcal{G}_n$ -torsor, and that is also used to define the Lie group $\mathcal{G}(M)$ of symmetries.

Remark 15.20. It is oft said that the Poincaré group is a *subgroup* of the group $\mathcal{G}$ of symmetries of a relativistic QM system on a Minkowski spacetime M . This discussion emphasizes that, rather, the quotient group $SO^\uparrow(M)$ receives a map from $\mathcal{G}$ or from an index 2 subgroup.

There are analogs of the spin-statistics [Theorem 15.4](#) and CRT [Theorem 15.6](#) for general symmetry groups. The first appears in [\[FH1, §2\]](#) as part of a structure theory for relativistic symmetry types. The second is [\[FH1, Theorem A.23\]](#).

Lecture 16: The Lorentz group action on energy-momentum vectors

Recall ([Definition 14.21](#)) that a relativistic particle is an irreducible unitary positive energy representation of the Poincaré group. The Poincaré group ([13.22](#)) and its cousins ([15.14](#)) sit in extensions with abelian kernel. We use this to analyze representations in two stages. First, the restriction of a representation to the kernel decomposes into characters, or rather the spectral theorem ([Theorem 14.19](#)) gives a spectral measure on the space of characters. Second, we look at the action of the quotient group on the characters and its implementation on the representation. As preparation for this analysis in the next lecture, in this lecture we study a finite group analog that eliminates the infinite dimensional linear analysis. One point is that the action of the quotient is via a *central extension*, not of the group but rather of a *groupoid* associated to the extension. Therefore, we begin with a preliminary discussion on groupoids,^{[43](#)} useful in its own right.

In the second part of the lecture we deduce the orbits and stabilizers of the group $SO_{1,n-1}^\uparrow$ acting on positive energy vectors $\overline{P_{1,n-1}^*} \subset (\mathbb{R}^{1,n-1})^*$. (These are roughly the quotient group and characters in the group extension in which Poincaré sits.) The mass shells $\mathcal{O}_m$ for $m \in \mathbb{R}^{>0}$ are orbits, and the stabilizer subgroup is isomorphic to SO_{n-1} . The situation is slightly more complicated for null vectors, which split into a few orbits, depending on the dimension. The stabilizer groups are noncompact, and for a nonzero null vector the stabilizer is a form of a Euclidean group. This deduction fits into conformal geometry, and we digress to expose the beautiful quadric that conformally compactifies a real vector space with a conformal structure (of any signature).

16.1 Groupoids, central extensions, and vector bundles

We begin with the definition of a groupoid.

⁴³Groupoids appeared briefly in [§B.4](#) in the bonus lecture.

Definition 16.1. A *groupoid* $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ is the data of a pair of sets (objects and morphisms), a map $\mathcal{G}_0 \rightarrow \mathcal{G}_1$ (identity morphisms), two maps $\mathcal{G}_1 \rightarrow \mathcal{G}_0$ (source and target), a map $\mathcal{G}_1 \rightarrow \mathcal{G}_1$ (inverse), and a map $\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \rightarrow \mathcal{G}_1$ (composition). There are several several conditions: associativity, identity morphisms, inverse morphisms, etc.

The fiber product $\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1$ is the set of composable pairs of morphisms. A groupoid is a category in which every morphism is invertible. If $\mathcal{G}_0 = *$ is a point, then $\mathcal{G}$ is a group. One can define groupoids in categories other than the category of sets. For example, see [FHT1, Appendix A] for topological groupoids. The data of a groupoid is depicted thus:

$$(16.2) \quad \begin{array}{c} \mathcal{G}_0 \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{t} \end{array} \mathcal{G}_1 \begin{array}{c} \xleftarrow{m} \\ \xrightarrow{i} \end{array} \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \end{array}$$

Here s, t are *source* and *target* maps, i is inversion, and m is the composition law (multiplication).

Example 16.3. A group G is a groupoid with $\mathcal{G}^0 = *$ a point.

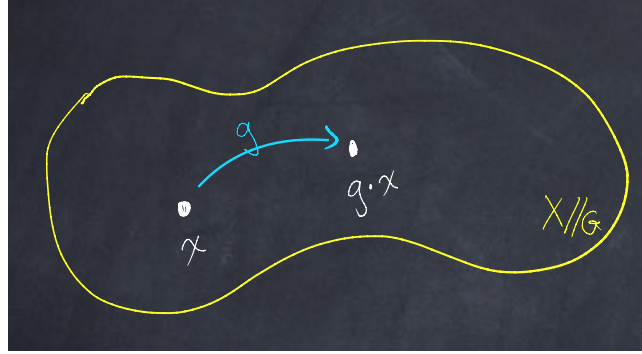


FIGURE 44. The action groupoid

Example 16.4. Suppose X is a set equipped with the left action of a group G . The *action groupoid* of $G \curvearrowright X$ is a groupoid with $\mathcal{G}^0 = X$ and $\mathcal{G}^1 = G \times X$: the source and target maps are $s(g, x) = x$, $t(g, x) = g \cdot x$; see Example 11.16.

As a preliminary, let A be an abelian group. Then multiplication $A \times A \rightarrow A$ is a group homomorphism. Hence if L_1, L_2 are right A -torsors, then so too is

$$(16.5) \quad L_1 \otimes L_2 := (L_1 \times L_2) \times_{(A \times A)} A.$$

For $A = \mathbb{C}^\times$ this duplicates the usual tensor product of the associated complex lines.

In the next definition we can use either $\mathbb{T}$ -torsors or hermitian lines.

Definition 16.6. Let $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ be a groupoid. A *central extension* $\tau = (L, \theta)$ of $\mathcal{G}$ (by $\mathbb{T}$) is a principal $\mathbb{T}$ -bundle $L \rightarrow \mathcal{G}_1$ and an isomorphism θ of principal $\mathbb{T}$ -bundles over $\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1$ that maps

$$(16.7) \quad \theta_{\gamma_2, \gamma_1} : L_{\gamma_2} \otimes L_{\gamma_1} \xrightarrow{\cong} L_{\gamma_2 \circ \gamma_1}$$

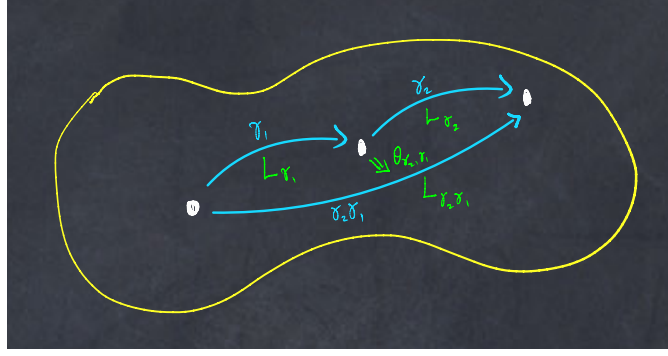


FIGURE 45. A central extension of a groupoid

For each triple $\cdot \xrightarrow{\gamma_1} \cdot \xrightarrow{\gamma_2} \cdot \xrightarrow{\gamma_3} \cdot$ of composable arrows, impose the condition that the diagram

$$(16.8) \quad \begin{array}{ccc} L_{\gamma_3 \circ \gamma_2 \circ \gamma_1} & \xrightarrow{\theta_{\gamma_2, \gamma_1}} & L_{\gamma_3} \otimes L_{\gamma_2 \circ \gamma_1} \\ \theta_{\gamma_3, \gamma_2} \downarrow & & \downarrow \theta_{\gamma_3, \gamma_2 \circ \gamma_1} \\ L_{\gamma_3 \circ \gamma_2} \otimes L_{\gamma_1} & \xrightarrow{\theta_{\gamma_3 \circ \gamma_2, \gamma_1}} & L_{\gamma_3} \otimes L_{\gamma_2} \otimes L_{\gamma_1} \end{array}$$

commute.

See Figure 45 for an illustration. Definition 16.6 reduces to a central extension of the group $\mathcal{G}^1$ in case $\mathcal{G}^0 = *$.

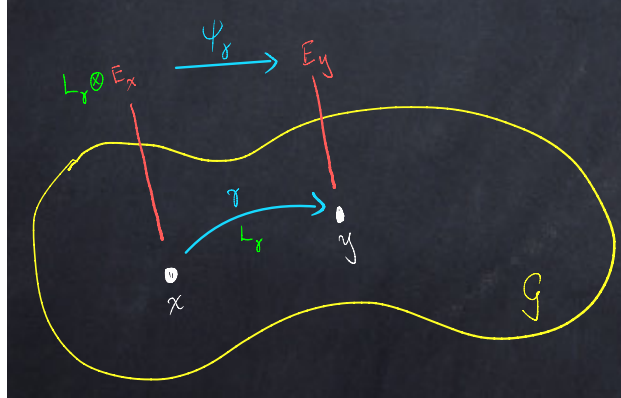


FIGURE 46. A (twisted) vector bundle over a groupoid

Definition 16.9. Let $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ be a groupoid.

(1) A *vector bundle* $E \rightarrow \mathcal{G}$ is a vector bundle $E \rightarrow \mathcal{G}_1$ together with isomorphisms

$$(16.10) \quad \psi_\gamma: E_x \xrightarrow{\cong} E_y, \quad (x \xrightarrow{\gamma} y) \in \mathcal{G}_1,$$

that satisfy $\psi_{\gamma_2 \circ \gamma_1} = \psi_{\gamma_2} \circ \psi_{\gamma_1}$ for all pairs of composable arrows $x \xrightarrow{\gamma_1} y \xrightarrow{\gamma_2} z$. A vector bundle over $\mathcal{G}$ is also called a *representation* of $\mathcal{G}$.

- (2) Let $\tau = (L, \theta)$ be a central extension of $\mathcal{G}$. Then a τ -twisted vector bundle over $\mathcal{G}$ is a vector bundle $E \rightarrow \mathcal{G}_1$ together with isomorphisms

$$(16.11) \quad \psi_\gamma: L_\gamma \otimes E_x \xrightarrow{\cong} E_y, \quad (x \xrightarrow{\gamma} y) \in \mathcal{G}_1,$$

that are required to satisfy the condition that the diagram

$$(16.12) \quad \begin{array}{ccc} L_{\gamma_2} \otimes L_{\gamma_1} \otimes E_x & \xrightarrow{\psi_{\gamma_1}} & L_{\gamma_2} \otimes E_y \\ \theta_{\gamma_2, \gamma_1} \downarrow & & \downarrow \psi_{\gamma_2} \\ L_{\gamma_2 \circ \gamma_1} \otimes E_x & \xrightarrow{\psi_{\gamma_2 \circ \gamma_1}} & E_z \end{array}$$

commute for each pair of composable arrows $x \xrightarrow{\gamma_1} y \xrightarrow{\gamma_2} z$. A τ -twisted vector bundle is also called a τ -twisted representation of $\mathcal{G}$.

If $\mathcal{G} = X//G$ is the action groupoid of $G \curvearrowright X$, then a vector bundle over $\mathcal{G}$ is a G -equivariant vector bundle over X .

Next, we prove an equivalence between equivariant vector bundles and representations of a stabilizer subgroup. Let G_0 be a compact Lie group, and denote by $\text{Irr}_f(G_0)$ the set of isomorphism classes of finite dimensional irreducible representations of G_0 . We require compactness so that $\text{Irr}_f(G_0)$ is a finite or countable discrete set.

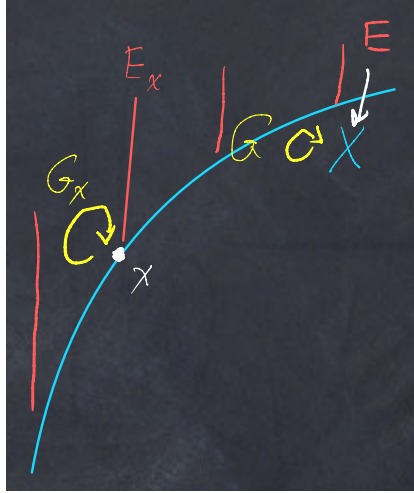


FIGURE 47. Equivariant vector bundles $\leftrightarrow$ representations of stabilizer

Proposition 16.13. *Let X be a smooth manifold, and suppose a Lie group G acts smoothly and transitively on X with compact stabilizers. Let $G_x \subset G$ be the stabilizer subgroup at $x \in X$. Then evaluation at x is a 1:1 correspondence*

$$(16.14) \quad \begin{aligned} \text{Vect}_{\text{irr}}(X//G) &\longrightarrow \text{Irr}_f(G_x) \\ E &\longmapsto E_x \end{aligned}$$

between isomorphism classes of irreducible finite rank vector bundles over $X//G$ and isomorphism classes of irreducible finite dimensional representations of G_x .

Definition 11.27 illustrates the correspondence (16.14). As corollaries, $\text{Vect}_{\text{irr}}(X//G)$ is discrete and $\text{Irr}_f(G_x)$ is canonically independent of x , which in any case is easy to verify directly.

Proof. The map

$$(16.15) \quad \begin{aligned} G &\longrightarrow X \\ g &\longmapsto g \cdot x \end{aligned}$$

is a principal G_x -bundle. The inverse map to (16.14) applies the mixing construction: an irreducible representation $G_x \subset \mathbb{E}_x$ maps to the G -equivariant vector bundle $G \times_{G_x} \mathbb{E}_x \rightarrow X$. $\square$

16.2 Finite group extensions

Consider a group extension

$$(16.16) \quad 1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1$$

of finite groups in which G' is abelian. Our task is to classify irreducible unitary⁴⁴ representations of G . A complex irreducible representation of G' is 1-dimensional, so is given by a unitary character $\chi: G' \rightarrow \mathbb{T}$. Let X be the abelian group of such characters, the *Pontrjagin dual group*⁴⁵ to G' . Use the extension (16.16) to construct a homomorphism

$$(16.18) \quad \alpha: G'' \rightarrow \text{Aut}(G').$$

Namely, if $g'' \in G''$ and $g \in G$ is a lift of g'' , then conjugation by g is an automorphism of the normal subgroup G' , and since G' is abelian this automorphism of G' is independent of the lift. Now use (16.18) to construct a *right* action of G'' on X : if $\chi \in X$ and $g'' \in G''$, then define $\chi \cdot g''$ as the composition

$$(16.19) \quad \chi \cdot g'': G' \xrightarrow{\alpha(g'')} G' \xrightarrow{\chi} \mathbb{T}$$

Example 16.20. The symmetric group on 3 letters is the smallest nonabelian group. It fits into the group extension

$$(16.21) \quad 1 \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow \text{Sym}_3 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

⁴⁴Every representation (real or complex) of a finite group is unitarizable: average an inner product over the group. We treat only complex representations; one can also investigate real and quaternionic representations.

⁴⁵The relation between G' and X is symmetric: the nondegenerate pairing

$$(16.17) \quad \begin{aligned} X \times G' &\longrightarrow \mathbb{T} \\ \chi, g' &\longmapsto \chi(g') \end{aligned}$$

identifies each group as the Pontrjagin dual of the other.

The Pontrjagin dual X to $\mathbb{Z}/3\mathbb{Z}$ is the cyclic group $\mu_3 = \{1, \omega, \omega^2\}$ of cube roots of unity. (For definiteness, take $\omega = e^{2\pi i/3}$.) Observe that $\mathbb{Z}/3\mathbb{Z}$ and μ_3 are isomorphic, but not canonically so.⁴⁶ The action of the non-identity element of $\mathbb{Z}/2\mathbb{Z}$ on X fixes 1 and exchanges $\omega \leftrightarrow \omega^2$. The quotient groupoid $X // (\mathbb{Z}/2\mathbb{Z})$ is depicted in [Figure 48](#).

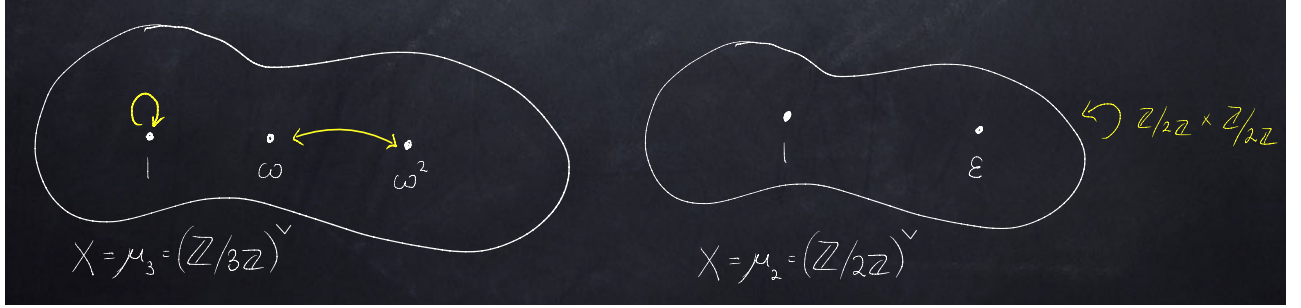


FIGURE 48. The groupoids induced from (16.21) and

Example 16.22. The 8-element quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ sits in a group extension

$$(16.23) \quad 1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow Q \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

The groupoid of the action of the quotient on the Pontrjagin dual to the kernel is also indicated in [Figure 48](#).

Now suppose $\rho: G \rightarrow \text{U}(V)$ is a unitary representation of the finite group G in the extension (16.16). “Sheafify” the extension over X as follows. For each $\chi \in X$ define the isotypical subspace

$$(16.24) \quad V_\chi = \text{Hom}_G(\mathbb{C}_\chi, V) \otimes \mathbb{C}_\chi \subset V,$$

where $\mathbb{C}_\chi$ is the line $\mathbb{C}$ with the linear G -action defined by the character χ . The vector spaces V_χ are the fibers of a vector bundle $E_V \rightarrow X$, and the space $\bigoplus_\chi V_\chi$ of sections of $E_V \rightarrow X$ is canonically isomorphic to V . The support of $E_V \rightarrow X$ (the subset of X for which the fiber is nonzero) is a union of G'' -orbits, and if ρ is an irreducible representation then the support is a single G'' -orbit. However, the action of G'' on X does not necessarily lift to the total space E_V .

Example 16.25. The symmetric group Sym_3 has 3 irreducible representations $V^{(1)}, V^{(\epsilon)}, V^{(\sigma)}$. For the trivial representation $V^{(1)}$ the vector bundle $E_{V^{(1)}} \rightarrow X$ has support $\{1\} \subset X$ and the fiber is 1-dimensional. Similarly for the sign representation $V^{(\epsilon)}$, which is pulled back from the nontrivial character of the quotient $\mathbb{Z}/2\mathbb{Z}$ in (16.21). The action of the stabilizer μ_2 at $1 \in \chi$ is trivial for the trivial representation and is nontrivial for $V^{(\epsilon)}$. The representation $V^{(\sigma)}$ is 2-dimensional, the support of $E_{V^{(\sigma)}}$ is $\{\omega, \omega^2\} \subset X$, and the fibers of $E_{V^{(\sigma)}} \rightarrow X$ are 1-dimensional at each of ω, ω^2 . However, there is no natural action of the quotient $\mathbb{Z}/2\mathbb{Z}$ in (16.21) on $E_{V^{(\sigma)}} \rightarrow X$.

⁴⁶It is not always true that an abelian group and its Pontrjagin dual are isomorphic: the Pontrjagin dual to $\mathbb{Z}$ is $\mathbb{T}$. But for a finite group there are such (noncanonical) isomorphisms.

Example 16.26. The quaternion group Q has 5 irreducible representations. Four of them are characters, pulled back from characters of the quotient group—the Klein 4-group—in (16.23). The remaining irreducible complex representation is 2-dimensional: it is the complex representation that underlies the action of Q on the quaternions $\mathbb{H}$ by left multiplication. For that representation the bundle $E \rightarrow X$ has support $\{\epsilon\} \subset X$ and the fiber is 2-dimensional. But notice that the stabilizer group $G'' = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ does *not* act linearly on the fiber—it only acts through a central extension

$$(16.27) \quad 1 \longrightarrow \mathbb{T} \longrightarrow \widetilde{G''} \longrightarrow G'' \longrightarrow 1,$$

namely the central extension induced from the central extension (16.23) by extension from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{T}$.

The example of the quaternion group typifies the general case: the group extension (16.16) determines a central extension of the groupoid $X//G''$, and it is this central extension that lifts to act linearly on $E_V \rightarrow X$. We defer to [FM1, §9] for the construction and its generalizations to extensions with nonabelian kernel and to extensions for which the kernel or quotient is a compact Lie group or lattice. What is more immediate to construct is that, in particular, for each $\chi \in X$ the stabilizer $G''_\chi \subset G''$ acts on the fiber $(E_V)_\chi = V_\chi$ through a centrally extended group $\widetilde{G''}_\chi$. That extension is defined via the diagram

$$(16.28) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \widetilde{G''}_\chi & \longrightarrow & G''_\chi \longrightarrow 1 \\ & & \uparrow \chi & & \uparrow & & \parallel \\ 1 & \longrightarrow & G' & \longrightarrow & G_\chi & \longrightarrow & G''_\chi \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & G'' \longrightarrow 1 \end{array}$$

The second row is the restriction of the third row over the stabilizer group, and the first row is a “pushout” using the character χ . That is, the image of G' in $G_\chi \times \mathbb{T}$ is normal, and $\widetilde{G''}_\chi$ is defined to be the quotient group. A splitting of the group extension (16.16) induces a splitting of the central extension of G''_χ .

Remark 16.29. In our application to the Poincaré group, the extension

$$(16.30) \quad 1 \longrightarrow V \longrightarrow \mathcal{P}(M) \longrightarrow \text{Spin}(V) \longrightarrow 1$$

splits—but not canonically—and so the action of the stabilizers also splits noncanonically.

This discussion results in the following.

Proposition 16.31. *There is a bijection between (i) isomorphism classes of irreducible unitary representations ρ of G , and (ii) pairs $(\mathcal{O}, \lambda)$ of a G'' -orbit on the set X of characters of G' and an isomorphism class of projective representations of the stabilizer group G''_χ for some $\chi \in \mathcal{O}$. The projectivity of the representation of the stabilizer is computed in (16.28).*

Remark 16.32. One can pass from representations to the representation ring of virtual representations. Then the bijection becomes an isomorphism of the representation ring of G with a twisted equivariant K -theory group of X , as explained in [FM1, §9].

16.3 Conformal groups and conformal compactification

This is a digression, another preliminary to the central material in this lecture.

Let V be a real vector space with a conformal structure, i.e., with a nondegenerate symmetric bilinear form up to multiplication by a positive real number. Suppose the form has signature $(p+1, q+1)$ for $p, q \in \mathbb{Z}^{\geq 0}$. In particular, the form is indefinite. Let $Q(V) \subset \mathbb{P}V$ be the projectivization of the null cone $N(V) \subset V$. Recall that any $N \in \mathbb{P}V$ there is a natural identification $T_N \mathbb{P}V \cong \text{Hom}(N, V/N)$. Differentiate the equation that cuts out $N(V) \subset V$ to deduce that if $N \subset V$ is a *null* line, then

$$(16.33) \quad T_N Q(V) \cong \text{Hom}(N, N^\perp/N).$$

The conformal structure on V induces a conformal structure on $N^\perp/N$: the form restricts on $N^\perp$ to have kernel N , and the induced form on the quotient induces a form on $N^\perp/N$ up to a positive constant; its signature is (p, q) . Evaluation at a nonzero vector in N produces an isomorphism $\text{Hom}(N, N^\perp/N) \cong N^\perp/N$, and the transport of the conformal structure on $N^\perp/N$ to $\text{Hom}(N, N^\perp/N)$ is independent of the choice of nonzero vector in N . Altogether, then, $Q(V)$ has a natural conformal structure of signature (p, q) . The projective orthogonal group $\text{PO}(V) = \text{O}(V)/\mu_2$ acts by conformal symmetries on the quadric $Q(V)$. Observe that $Q(V) \subset \mathbb{P}V$ is a closed submanifold of a compact manifold, hence is a compact manifold. In fact, $Q(V) \approx S^p \times S^q / \mu_2$ where the involution is the simultaneous antipodal map, as we leave the reader to verify. Then $\text{PO}(V) \cong \text{O}_{p+1, q+1} / \mu_2$.

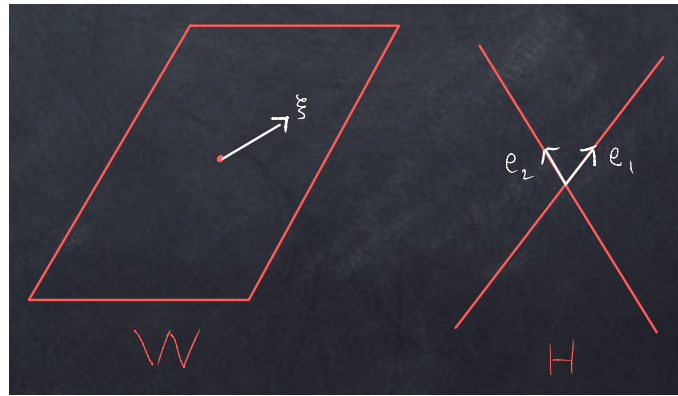


FIGURE 49. $V = W \oplus H$

Let H be a hyperbolic plane—a 2-dimensional real vector space with a bilinear form of signature $(1, 1)$. Let W be a finite dimensional real vector space equipped with a nondegenerate symmetric bilinear form, and set $V = W \oplus H$; see Figure 49. The quadric $Q(H)$ consists of two points. Fix a basis $\{e_1, e_2\}$ of H of null vectors with $\langle e_1, e_2 \rangle = 1$. The following proposition exhibits $Q(V)$ as a *conformal compactification* of W .

Proposition 16.34. *There is a partition*

$$(16.35) \quad Q(V) = W \amalg N(W) \amalg Q(W),$$

where

$$(16.36) \quad \begin{array}{lll} W \hookrightarrow Q(V) & N(W) \hookrightarrow Q(V) & Q(W) \hookrightarrow Q(V) \\ \xi \mapsto [\xi; 1, -\|\xi\|^2] & \xi \mapsto [\xi; 0, 1] & \xi \mapsto [\xi; 0, 0] \end{array}$$

In the conformal compactification $Q(V)$, the codimension one submanifold $N(W)$ and the codimension two submanifold $Q(W)$ live at infinity. If W has a positive definite conformal structure, then V is Lorentzian. In this case $N(W) = \{0\}$ is a single point, $Q(W) = \emptyset$ is empty, and the conformal compactification of W is the sphere.

Remark 16.37. Let $N = \mathbb{R} \cdot e_1$ be the span of $e_1 \in H$. Then there is a natural isomorphism $N^\perp/N \cong W$, and so we can identify $Q(V)$ as the conformal compactification of $N^\perp/N$.

16.4 Orbits and stabilizers of the $\mathrm{SO}^\uparrow(V)$ -action on $\overline{P^*} \subset V^*$

Let V be a Lorentz vector space equipped with a time-orientation $P \subset V$; the dual space V^* has a cone $P^* \subset V^*$ of positive energy-momentum vectors. Since $\mathrm{SO}^\uparrow(V)$ acts on V^* by isometries, the norm square is preserved. Recall the definitions of the mass shell (13.19) for $m \in \mathbb{R}^{>0}$ and of the positive null cone (16.38):

$$(16.38) \quad \begin{aligned} \mathcal{O}_m &= \{\theta \in P^* : |\theta| = mc\}, \\ N(V^*)^+ &= \{\theta \in \overline{P^*} \setminus \{0\} : \langle \theta, \theta \rangle = 0\}. \end{aligned}$$

The following was stated loosely in (14.22). As usual, that $n = \dim M$.

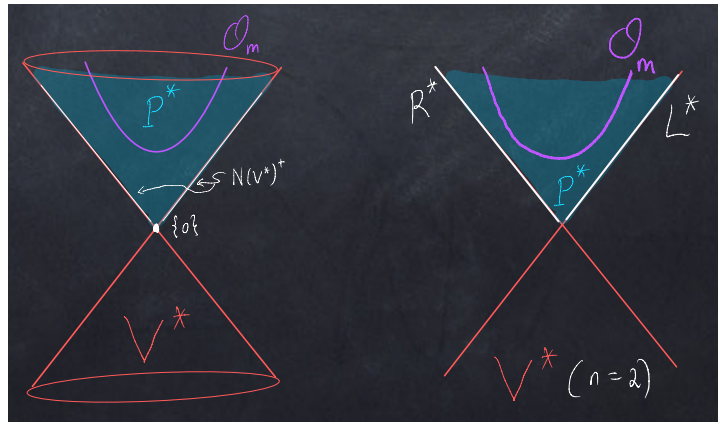


FIGURE 50. The orbits of $\mathrm{Spin}(V)$ on $\overline{P^*}$

Proposition 16.39.

- (1) $\text{SO}^\uparrow(V)$ acts transitively on $\mathcal{O}_m$.
- (2) $\text{SO}^\uparrow(V)$ acts transitively on $N(V^*)^+$ if $n \geq 3$. For $n = 2$ there are two orbits

$$(16.40) \quad N(V^*)^+ = L^* \sqcup R^*,$$

each of which is a ray. For $n = 1$, $N(V^*)^+ = \emptyset$.

This leads to the enumeration of $\text{SO}^\uparrow$ -orbits in $\overline{P}^*$, depicted in Figure 50:

$$(16.41) \quad \overline{P}^* = \begin{cases} \{0\} \sqcup N(V^*)^+ \sqcup \bigsqcup_{m>0} \mathcal{O}_m, & n \geq 3; \\ \{0\} \sqcup L^* \sqcup R^* \sqcup \bigsqcup_{m>0} \mathcal{O}_m, & n = 2; \\ \{0\} \sqcup \bigsqcup_{m>0} \mathcal{O}_m, & n = 1. \end{cases}$$

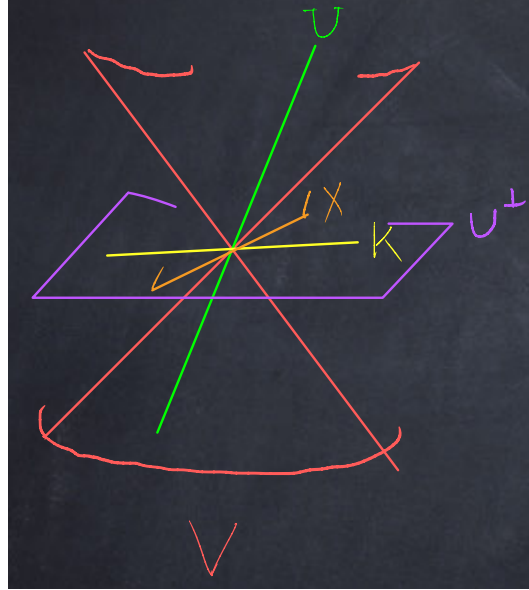


FIGURE 51. The decompositions used in the proof of Proposition 16.39

Proof. For (1) choose a splitting $V = U \oplus U^\perp$ with U a timelike line, and then there is an induced dual splitting $V^* = U^* \oplus (U^\perp)^*$. Let $\tau_U \in U$ be the forward timelike vector with $|\tau_U| = c$, and let $\theta_U \in U^*$ be the dual positive energy vector: $|\theta_U| = c$. For $\theta = \frac{E}{c}\theta_U + \theta^\perp \in \mathcal{O}_m$ compute

$$(16.42) \quad m^2 c^2 = |\theta|^2 = \frac{E^2}{c^2} - |\theta^\perp|^2.$$

For fixed E , the subgroup $\text{SO}(U^\perp) \subset \text{SO}^\uparrow(V)$ acts transitively on the sphere in $U^\perp$ of radius $|\theta^\perp|$. This allows us to reduce to the 2-dimensional case. Namely, fix a line $K \subset U^\perp$ and assume, by

the aforementioned transitivity, that $\theta^\perp \in K$. Fix $\ell_{K^*} \in K$ such that $|\ell_{K^*}| = 1$. An element of $\text{SO}^\uparrow(U \oplus K)$ acts on $U^* \oplus K^*$ by

$$(16.43) \quad \begin{pmatrix} \cosh \varphi & \sinh \varphi \\ \sinh \varphi & \cosh \varphi \end{pmatrix}, \quad \varphi \in \mathbb{R},$$

in the basis $\{\theta_U, \ell_{K^*}\}$. These matrices act transitively on $\mathcal{O}_m \cap (U^* \oplus K^*)$:

$$(16.44) \quad \begin{pmatrix} mc \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} mc \cosh \varphi \\ mc \sinh \varphi \end{pmatrix}$$

For (2) use the orthogonal decompositions (see [Figure 51](#))

$$(16.45) \quad \begin{aligned} V &= U \oplus K \oplus X \\ V^* &= U^* \oplus K^* \oplus X^* \end{aligned}$$

and observe that any vector in $N(V^*)^+$ transforms to a vector with vanishing X^* -component under a suitable element of $\text{SO}(K \oplus X)$. In the chosen basis, an element of $N(V^*)^+ \cap (U^* \oplus K^*)$ has the form $\begin{pmatrix} E/c \\ p \end{pmatrix}$ where $E/c = \pm p$. The matrices (16.43) act transitively on the two rays of $N(V^*)^+$, and if $n \geq 3$ we can use an element of $\text{SO}(K \oplus X)$ to transform from one ray to the other. $\square$

Next we turn to the stabilizer groups, continuing with the notation in the proof of [Proposition 16.39](#).

Proposition 16.46. *The stabilizer groups of the $\text{SO}^\uparrow(V)$ -action on $\overline{P^*}$ are as follows.*

- (1) *At the point $m\theta_U \in \mathcal{O}_m$, the stabilizer group is $\text{SO}(U^\perp)$.*
- (2) *If $\theta \in N(V^*)^+$ spans a null line N^* , for $N \subset V$ a null line, then the stabilizer group is $\text{Euc}^0(N^\perp/N)$, where Euc^0 is the component of the identity of the Euclidean group.*

Proof. The proof of (1) is essentially contained in the proof of [Proposition 16.39](#)(1): a time-orientation preserving isometry that fixes a forward vector in U also fixes U , hence is an isometry of $U^\perp$.

For (2), the projectivization $Q(V^*) = \mathbb{P}N(V^*) \subset \mathbb{P}V^*$ is diffeomorphic to the $(n-2)$ -sphere, the conformal compactification of $N^\perp/N$; see [Remark 16.37](#). $\text{SO}^\uparrow(V)$ acts as the group of orientation-preserving conformal transformations, as in [§16.3](#). The stabilizer of a point on the sphere—a lightlike line—is the identity component of the Euclidean group together with dilations. The stabilizer group of a point on the lightlike ray does not include the dilations. $\square$

Finally, the stabilizer group at $\{0\} \subset V^*$ of the action of $\text{SO}^\uparrow(V)$ is the entire group $\text{SO}^\uparrow(V)$.

Lecture 17: Irreducible representations of the Poincaré group

In relativistic quantum mechanics a *relativistic particle* is an irreducible positive energy unitary representation of the Poincaré group ([Definition 14.21](#)(3)). In this lecture we give the classification

of these representations, which is usually attributed to Wigner [Wig]. It fits into a general theory developed for finite groups by Frobenius and generalized to locally compact groups in a long series of papers by Mackey; see [Ma2, Ma3, Ma4] for a sample. In brief, this is part of the theory of *induced representations*. We construct the representations and sketch an argument for why these are all irreducible positive energy unitary representations.

Let M be a Minkowski spacetime over a Lorentzian vector space V . Recall that the Poincaré group fits into the group extension

$$(17.1) \quad 1 \longrightarrow V \longrightarrow \mathcal{P}(M) \longrightarrow \text{Spin}(V) \longrightarrow 1$$

in which the kernel V is abelian, and the extension is split by a choice of $p \in M$. (Hence $\mathcal{P}(M)$ is a semidirect product group, though not canonically.) We begin with a definition of the *little groups*. Then we discuss the classification in question. We make the hypothesis that the stabilizer subgroup acts via a *finite dimensional* representation, which leads to a brief discussion of irreducible representations of the spin group. We review the constraints from quantum field theory already introduced in [Lecture 15](#)—the spin-statistics theorem and the CRT theorem—and we introduce another constraint due to Weinberg–Witten [WW]. We indicate how the theory generalizes to other symmetry groups in relativistic quantum mechanics.

17.1 Little groups and invariant measures

We resume the usual relativistic setup and notation.

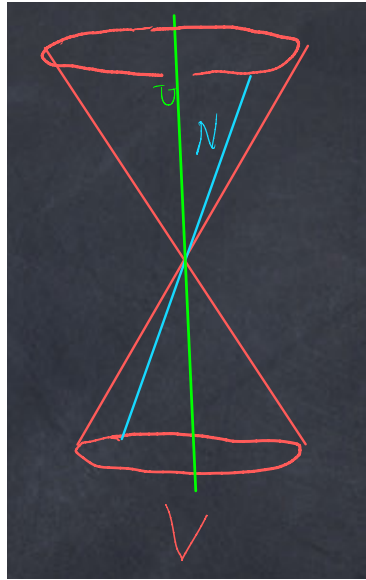


FIGURE 52. A timelike line U and a null line N

Recall from §16.4 that there are three basic types of orbits of the action of $\text{SO}^\uparrow(V) \curvearrowright \overline{P^*}$: (1) the mass shell $\mathcal{O}_m$ of positive mass $m \in \mathbb{R}^{>0}$, (2) an orbit in $N(V^*)^+$ of positive energy null energy momentum vectors⁴⁷, and (3) the singleton orbit $\{0\}$. The stabilizer groups of $\text{SO}^\uparrow(V) \curvearrowright \overline{P^*}$ are

⁴⁷If $n \geq 3$ then $\text{SO}^\uparrow(V)$ acts transitively on $N(V^*)^+$, for $n = 2$ there are two orbits, and $N(V^*)^+ = \emptyset$ for $n = 1$.

computed in [Proposition 16.46](#). For representations of the Poincaré group $\mathcal{P}(M)$ we are interested in the action $\text{Spin}(V) \curvearrowright \overline{P}^*$, and for this spin action the stabilizers are double covers of the previous. Hence in case (1), at $m\theta_U \in \mathcal{O}_m$ the stabilizer group is

$$(17.2) \quad \text{Spin}(U^\perp) \cong \text{Spin}_{n-1},$$

a compact spin group. In case (2) suppose $N \subset N(V)$ is a null line, $N^* \subset N(V^*)$ is the dual null line, and $\theta \in N^* \cap N(V^*)^+$. Then the stabilizer group at θ fits into the group extension

$$(17.3) \quad 1 \longrightarrow N^\perp/N \longrightarrow \widetilde{\text{Euc}}^0(N^\perp/N) \longrightarrow \text{Spin}(N^\perp/N) \longrightarrow 1;$$

it is the double cover of the identity component of the Euclidean group of the $(n-2)$ -dimensional inner product space $N^\perp/N$. In case (3) the stabilizer group is the noncompact Lorentz spin group $\text{Spin}(V)$.

We are interested in finite dimensional unitary representations of the stabilizer groups. In case (2) we have the following.

Lemma 17.4. *A finite dimensional unitary representation of $\widetilde{\text{Euc}}^0(N^\perp/N)$ factors through $\text{Spin}(N^\perp/N)$.*

Proof. Any unitary representation restricts to a representation of the translation subgroup $N^\perp/N$, and its spectrum is a closed subset of the Pontrjagin dual group $(N^\perp/N)^*$ of characters. That is, the spectrum is a union of orbits of $\text{SO}(N^\perp/N) \curvearrowright N^\perp/N$, i.e., is a union of concentric spheres. If the representation is finite dimensional, the spectrum must be $\{0\}$. $\square$

In our study of irreducible representations of the Poincaré group $\mathcal{P}(M)$ it is the action of the spin group $\text{Spin}(V)$ on V^* which enters; see [\(17.1\)](#). The stabilizers are the double covers of the groups in [Proposition 16.46](#). Only finite dimensional representations of the stabilizers are relevant, as we will explain, and this motivates the following.

Definition 17.5. The *little group* at $\theta \in \overline{P}^* \setminus \{0\}$ is $\text{Spin}(U^\perp)$ for $\theta = m\theta_U$ and $\text{Spin}(N^\perp/N)$ for $\theta \in N^* \cap \overline{P}^*$ a nonzero null vector.

These groups are isomorphic to Spin_{n-1} and Spin_{n-2} , respectively, but not canonically.

Remark 17.6. The identification of a stabilizer group with the standard group Spin_{n-1} or Spin_{n-2} is well-defined up to the outer automorphism induced by an element in the off-component of the corresponding orthogonal group, i.e., up to conjugation by a reflection. That automorphism can act nontrivially on $\text{Irr}_f(\text{Spin}_m)$. For example, if m is even it exchanges the self-dual and anti-self-dual irreducible components of the middle exterior power of the vector representations; it also exchanges the two half-spin representations. Also, if $m = 2$ it maps the representation of spin j to the representation of spin $-j$ (see below). In theories which require a spatial orientation—for example, quantum field theories which are not time-orientation reversal invariant—there is no ambiguity and we can label vector bundles by isomorphism classes of representations of Spin_m .

The stabilizer group of $\text{Spin}(V) \curvearrowright \overline{P}^*$ at $\{0\} \subset \overline{P}^* \subset V^*$ is the entire group $\text{Spin}(V)$.

Proposition 17.7. *If $n \geq 3$, then the only unitary finite dimensional irreducible unitary representation of $\text{Spin}(V)$ is the one-dimensional representation on which every element of $\text{Spin}(V)$ acts as the identity.*

The proof is left as a homework problem.

Now we turn to invariant smooth measures on the orbits.

Proposition 17.8. *Let $\mathcal{O} \subset \overline{P^*}$ be a nonzero orbit of $\text{Spin}(V)$. Then there is a ray of $\text{Spin}(V)$ -invariant smooth measures on $\mathcal{O}$.*

Proof. By the argument in the penultimate paragraph of §14.4.1, if $\mathcal{O} = \mathcal{O}_m$ is a mass shell for some $m > 0$, then the invariant smooth measure exists since the stabilizer group (17.2) is compact. For a positive energy null vector, the stabilizer group is the double cover of the identity component of the Euclidean group in (17.3), and it acts on the ray of positive densities on the tangent space to the orbit, which is $N^\perp$ in the notation of (17.3). Recall the discussion in §16.3: the quadric $Q(V^*)$ of null lines is an $(n-2)$ -sphere, and if $N^* \in Q(V^*)$ is identified as the point at infinity, then the subspace $N^\perp/N$ in (17.3) acts on the tangent space at infinity as infinitesimal translations at zero, which act as zero. So the action factors to the quotient spin group, which is compact and so acts trivially on densities on the tangent space. $\square$

17.2 Irreducible representations of Poincaré

We first construct irreducible unitary representations. Then we state a theorem that these are all of them. The discussion in §16.2 motivates restricting a unitary representation of $\mathcal{P}(M)$ to the vector subgroup V , and then Theorem 14.19 determines a spectral measure on V^* . If the representation is irreducible and has positive energy, then the support of the measure is a single $\text{Spin}(V)$ -orbit in $\overline{P^*}$. This is the starting point of the construction. (Recall the analog Proposition 16.31 for finite groups.)

Fix a $\text{Spin}(V)$ -orbit $\mathcal{O} \subset \overline{P^*}$. By Proposition 16.13 a finite rank $\text{Spin}(V)$ -equivariant hermitian vector bundle $E \rightarrow \mathcal{O}$ is equivalent to a finite dimensional unitary representation of the stabilizer subgroup at any point of $\mathcal{O}$. If $\mathcal{O} = \{0\}$ and $n \geq 3$, then by Proposition 17.7 the only finite dimensional irreducible unitary representation is the one-dimensional trivial representation. The corresponding representation of $\mathcal{P}(M)$ is also the trivial one-dimensional representation. If $\mathcal{O} \neq \{0\}$, then any finite dimensional unitary representation of the stabilizer factors through the little group, which is a spin group. Fix an invariant smooth measure on $\mathcal{O}$, which exists and is unique up to a positive scalar by Proposition 17.8. Then $\text{Spin}(V)$ acts unitarily on the Hilbert space

$$(17.9) \quad \mathcal{H} = L^2(\mathcal{O}; E)$$

of L^2 sections of $E \rightarrow \mathcal{O}$.

Recall the construction. On the space $C^\infty(\mathcal{O}; E)$ of smooth sections, define the hermitian form

$$(17.10) \quad \langle \psi_1, \psi_2 \rangle = \int_{\mathcal{O}} \langle \psi_1(\theta), \psi_2(\theta) \rangle d\mu_{\mathcal{O}}(\theta), \quad \psi_1, \psi_2 \in C^\infty(\mathcal{O}; E),$$

where the integrand is the pointwise hermitian inner product and $\mu_{\mathcal{O}}$ is the invariant measure on the orbit. Then (17.9) is the Hilbert space completion of the hermitian vector space $C^\infty(\mathcal{O}; E)$. If $\psi: \mathcal{O} \rightarrow E$ is a smooth section of $E \rightarrow \mathcal{O}$, and if $g \in \text{Spin}(V)$, then

$$(17.11) \quad (g\psi)(\theta) = g \cdot \psi(g^{-1} \cdot \theta), \quad \theta \in \mathcal{O}.$$

The action extends to the Hilbert space completion.

Next, a vector $\xi \in V$ determines a smooth function

$$(17.12) \quad \begin{aligned} V^* &\longrightarrow \mathbb{C} \\ \theta &\longmapsto e^{-i\theta(\xi)/\hbar} \end{aligned}$$

whose value at $\theta \in V^*$ is the unitary character θ on the vector ξ . The restriction of (17.12) to $\mathcal{O} \subset V^*$ is also smooth, and smooth complex-valued functions on $\mathcal{O}$ act as multiplication operators on L^2 sections of vector bundles.

The actions (17.11) of $\text{Spin}(V)$ and (17.12) of V do not commute. Rather, an easy computation shows that they combine to an action of the semidirect product group

$$(17.13) \quad \text{Spin}(V) \ltimes V$$

Remark 17.14. The Poincaré group $\mathcal{P}(M)$ is not canonically this semidirect product; it is naturally the extension (17.1). The discussion in §16.2 illustrates how to construct representations of group extensions with abelian kernels. Namely, the extension determines a central extension of the action groupoid $V^*//\text{Spin}(V)$, and it is vector bundles over this central extension that are needed. Irreducible positive energy bundles have support on a positive energy orbit, and they are determined by an irreducible extension of the central extension of stabilizer subgroup; see (16.28). In this case that is a central extension of a spin group,⁴⁸ which necessarily splits. Or, more simply, we observe that the Poincaré group is isomorphic to the semidirect product (17.13) upon fixing a point $p_0 \in M$. So the classification of representations of $\mathcal{P}(M)$ is equivalent to the classification of representations of the semidirect product (17.13).

Let $\mathcal{U}_{\mathcal{P}(M)}^+$ denote the set of isomorphism classes of irreducible positive energy unitary representations of the Poincaré group $\mathcal{P}(M)$. The main theorem, which we do not prove here, asserts that the foregoing construction accounts for all desired representations.

Remark 17.15. We assume that the representations are induced from *finite dimensional* representations of the stabilizer group, i.e., are presented as sections of a *finite rank* vector bundle over an orbit of $\text{Spin}(V)$ on V^* . This finite dimensionality is equivalent to the assertion that there are a finite number of states at a fixed energy-momentum.

Theorem 17.16. *There is a surjective map*

$$(17.17) \quad \mathcal{U}_{\mathcal{P}(M)}^+ \longrightarrow \mathbb{R}^{\geq 0}.$$

The fiber at $m > 0$ is isomorphic to $\text{Vect}_{\text{irr}}(\mathcal{O}_m//\text{Spin}(V))$. If $n \geq 3$, then the fiber over $m = 0$ is the union $\text{Vect}_{\text{irr}}(N(V^)^+//\text{Spin}(V)) \cup \{*\}$. For $n = 2$ the fiber over $m = 0$ has 5 points.*

⁴⁸or for nonzero null vectors of a double cover of a Euclidean group

Here ‘ $*$ ’ denotes the trivial representation. For $n = 2$ there are three massless orbits, and the nonzero ones each admit two equivariant line bundles: the stabilizer subgroup μ_2 acts trivially or by the sign representation.

I would like to see a proof of [Theorem 17.16](#) along the lines of the outline in the first paragraph of this section. Wigner’s paper is [\[Wig\]](#), and there are several papers and books of Mackey that discuss induced representations for semidirect products and group extensions; a sample is [\[Ma2, Ma3, Ma4, Ma5, Ma6\]](#). There is also a discussion in [\[Sg, §3.9\]](#) in a volume that is of broader interest.

Remark 17.18. As explained in [Lecture 10](#) a quantum mechanical system has bosonic and fermionic states, so all representations should be on $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces. This amounts to a pair of representations of the Poincaré group, and so a pair of equivariant vector bundles over an orbit.

17.3 Representations of the spin group

So the classification comes down to a classification of representations of the compact spin group Spin_m for $m = n - 1$ (massive representations) and $m = n - 2$ (massless representations). We only scratch the surface.

First, since $\text{Spin}_2 \cong \mathbb{T}$ is abelian its irreducible complex representations are 1-dimensional. The character group of $\mathbb{T}$ is infinite cyclic, and for Spin_2 we identify that infinite cyclic group with $\frac{1}{2}\mathbb{Z}$ and so label an irreducible representation by its *spin* $j \in \frac{1}{2}\mathbb{Z}$: the representation labeled by j is $\lambda \mapsto \lambda^{2j}$. The spin 0 representation is the trivial one and the spin 1/2 representation is the representation on which $\text{Spin}_2 \cong \mathbb{T}$ acts by scalar multiplication.⁴⁹ For $m \geq 2$ we define the spin of an irreducible representation as follows.

Definition 17.19. Let $\text{Spin}_m \rightarrow U(W)$ be an irreducible unitary representation. The *spin* of the representation is the largest spin $|j|$ that occurs in the decomposition into irreducibles under restriction to $\text{Spin}_2 \subset \text{Spin}_m$.

The decomposition is independent of the choice of $\text{Spin}_2 \subset \text{Spin}_m$. The spin is a measure of the complexity of the representation. The spin modulo integers agrees with [Definition 15.3](#): an irreducible representation of integer spin factors through $\text{Spin}_m \rightarrow \text{SO}_m$ whereas in a representation with half-integer spin the central element $\epsilon \in \text{Spin}_m$ acts as -1 .

Remark 17.20. For *massless* little group representations, the ‘spin’ is usually called ‘helicity’.

There is a unique spin 0 irreducible representation of Spin_m : the trivial representation.

A spin 1/2 representation is called a *spin representation*; it extends to a module over the even Clifford algebra. For m odd there is a unique irreducible spin representation $\mathbb{S}$ of dimension $2^{(m-1)/2}$; for m even there are two “half-spin” representations $\mathbb{S}^\pm$, each of dimension $2^{m/2-1}$. In low dimensions we can use the special isomorphisms [\(12.24\)](#) to describe/construct the spin representations. Thus $\text{Spin}_1 \cong \mu_2$ and the spin representation is the sign representation on $\mathbb{S} = \mathbb{C}$. As above, $\text{Spin}_2 \cong \mathbb{T}$ and the two half-spin representations are the spin $\pm 1/2$ representations on $\mathbb{C}$; they are

⁴⁹That is not a very helpful description without an explicit isomorphism $\text{Spin}_2 \xrightarrow{\cong} \mathbb{T}$. Better: Spin_2 embeds in the real Clifford algebra Cliff_{-2} generated by e_1, e_2 with $e_i^2 = -1$ as elements $\cos(\theta/2) + \sin(\theta/2)e_1e_2$, $0 \leq \theta \leq 4\pi$, and the spin j representation sends the element labeled θ to multiplication by $e^{\sqrt{-1}j\theta/2}$. See [\[ABS\]](#) for the basics of Clifford algebras and spin groups.

complex conjugates. Then $\text{Spin}_3 \cong \text{SU}_2$, and the spin representation $\mathbb{S} = \mathbb{C}^2$ is the defining representation of SU_2 . Finally, $\text{Spin}_4 \cong \text{SU}_2 \times \text{SU}_2$, and the two half-spin representations $\mathbb{S}^\pm \cong \mathbb{C}^2$ factor through projection onto a factor. Each of these representations is self-conjugate (quaternionic).

The basic spin 1 representation is the vector representation $\text{Spin}_m \rightarrow \text{SO}_m \curvearrowright \mathbb{C}^m$. The exterior power representations

$$(17.21) \quad \bigwedge^\ell \mathbb{C}^m, \quad \ell = 1, 2, \dots, \left\lfloor \frac{m}{2} \right\rfloor$$

also have spin 1. If m is even, then the middle exterior power representation splits into a sum

$$(17.22) \quad \bigwedge^{m/2} \mathbb{C}^m = \bigwedge_+^{m/2} \mathbb{C}^m \oplus \bigwedge_-^{m/2} \mathbb{C}^m$$

of irreducibles: the *self-dual* and *anti-self-dual* representations.

The symmetric power $\text{Sym}^s \mathbb{C}^m$, $s \in \mathbb{Z}^{\geq 0}$ has spin s .

Remark 17.23. For the Poincaré group, then, a particle—an irreducible positive energy unitary representation—has a mass $m \in \mathbb{R}^{>0}$ and a spin $j \in \frac{1}{2}\mathbb{Z}^{\geq 0}$. These are the *quantum numbers* of the particle. However, these quantum numbers do not determine the particle, as the examples above illustrate.

17.4 Constraints from relativistic quantum field theory

The two basic theorems of relativistic quantum field theory—[Theorem 15.4](#) and [Theorem 15.6](#)—constrain the particle representations that occur in relativistic quantum field theory.

First, as in [Remark 17.18](#) a quantum field theory has a $\mathbb{Z}/2\mathbb{Z}$ -graded unitary representation $\rho: \mathcal{P}(M) \rightarrow \text{U}(\mathcal{H}^0 \oplus \mathcal{H}^1)$, which amounts to a pair of unitary representations. The spin-statistics theorem implies the following.

Corollary 17.24. *In a quantum field theory the little group (Spin_m) representation has integer spin for bosonic states and half-integer spin for fermionic states.*

The CRT theorem states that in a quantum field theory the representation of the Poincaré group extends to a representation of a 2-component Lie group that double covers the middle group in

$$(17.25) \quad 1 \longrightarrow V \longrightarrow \text{SO}(M) \longrightarrow \text{SO}(V) \longrightarrow 1$$

and that the off-component acts antiunitarily. Thus the $\text{SO}^\uparrow(V)$ -action on V^* extends to an action of $\text{SO}(V)$, but since the off-component is to act antilinearly, it conjugates the character and so the usual action of the off-component on V^* is multiplied by -1 . If n is even, then $-1 \in \text{SO}(V)$, and so the stabilizer group of the $\text{SO}(V)$ -action at a point $\theta \in \overline{P}^*$ is the direct product of the $\text{SO}^\uparrow(V)$ stabilizer group and the cyclic group μ_2 . Furthermore, the generator of μ_2 acts antilinearly on the fiber of an equivariant vector bundle. In other words, that generator is a real structure that commutes with the little group representation. For the Poincaré group we obtain a central extension, and the new generator may act as a quaternionic structure. This proves the following.

Corollary 17.26. *If n is even, then in a quantum field theory the little group representation is self-conjugate.*

This is an important constraint. For example, the self-dual and anti-self-dual representations (17.22) of SO_2 are *not* self-conjugate (they are each other's conjugate), so they cannot occur, whereas they are self-conjugate in the case of SO_4 .⁵⁰

There is another basic constraint due to Weinberg–Witten [WW]. It states, roughly, that massless particles in a quantum field theory have helicity ≤ 1 (and in a theory of quantum gravity they have helicity ≤ 2).

17.5 More examples

We illustrate the classification by exhibiting continuous *families* of representations. Recall that the representations of a compact Lie group do not deform: the space of isomorphism classes of irreducible representations is discrete. Not so for a noncompact Lie group, such as the Poincaré group. We also illustrate Corollary 17.26 in this discussion. The basic idea is to consider a continuous family of massive representations with $m > 0$ and to consider the limit $m \rightarrow 0$. This produces a massless (mass zero) representation. We are led to construct (different) massless representations that do not deform to massive representations. We only describe a few illustrative cases.

Example 17.27. Set $n = 4$ so that the massive little group is isomorphic to $\mathrm{Spin}_3 \cong \mathrm{SU}_2$ and the massless little group is isomorphic to $\mathrm{Spin}_2 \cong \mathbb{T}$. Every representation of SU_2 is self-conjugate—either real or quaternionic—and so there is no constraint on massive representations from Corollary 17.26. However, *irreducible* massless representations are parametrized by $j \in \frac{1}{2}\mathbb{Z}$, and if $j \neq 0$ the representation is not self-conjugate: the conjugate of the helicity j representation is the helicity $-j$ representation. On the other hand, for any $j \in \frac{1}{2}\mathbb{Z}^{>0}$ the 2-dimensional representation $\underline{j} \oplus \underline{-j}$ is self-conjugate: it admits either a real or a quaternionic structure.

There is a unique spin 0 particle for any mass $m \geq 0$ and these form a continuous family of irreducible representations.

The vector representation of the massive little group is 3-dimensional, so gives rise to a rank 3 equivariant vector bundle over each mass shell $\mathcal{O}_m$, $m > 0$, and these form a continuous family. The limit $m \rightarrow 0$ is again rank 3, now supported over the positive energy null vectors $N(V^*)^+$. The corresponding representation of the massless little group is reducible: it is the sum of the vector representation of Spin_2 and the trivial 1-dimensional representation.

We claim that the 2-dimensional vector representation of Spin_2 , which is real,⁵¹ gives rise to a massless representation that does not deform to a massive representation. For if it did, the central element $\epsilon \in \mathrm{Spin}_3$ acts trivially, and so the representation drops to SO_3 . However, there is no 2-dimensional real representation of SO_3 .

This means that the massless vector particle is rigid: it does not deform. (Note in general that the space of isomorphism classes of irreducible representations of Poincaré with fixed mass is discrete.) We will see that this representation occurs in 4-dimensional *Maxwell theory*, the quantum theory of electromagnetism. We conclude: that theory does not have a massive deformation.

⁵⁰The latter occur as massless particles in some 6-dimensional quantum field theories. There are similar massless particles in 10-dimensional quantum field theory approximations to superstring theories.

⁵¹It is $\underline{1} \oplus \underline{-1}$ in the notation above.

Example 17.28. Now set $n = 3$. The massless vector particle is induced from the trivial 1-dimensional representation of the little group $\text{Spin}_1 \cong \mu_2$. This representation *does* deform to the massive vector particle induced from either the spin 1 or the spin -1 representation of the little group $\text{Spin}_2 \cong \mathbb{T}$. (These representations are not self-conjugate, but the restriction in [Corollary 17.26](#) from CRT only applies in even spacetime dimensions.) We will see (in a homework problem) that this massive deformation comes about from a Chern–Simons term that is added to the Maxwell theory.

17.6 General relativistic symmetry types

Recall from [Definition 15.13](#) that the general symmetry type in n -dimensional relativistic quantum mechanics is specified by a Lie group G_n and a homomorphism $\lambda: G_n \rightarrow O_{1,n-1}^\uparrow$ whose image contains the identity component $SO_{1,n-1}^\uparrow \subset O_{1,n-1}^\uparrow$. The homomorphism λ fits into the diagram

$$(17.29) \quad \begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ & & K & \xlongequal{\quad} & K & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{R}^{1,n-1} & \longrightarrow & \mathcal{G}_n & \longrightarrow & G_n \longrightarrow 1 \\ & & \parallel & & \downarrow \tilde{\lambda}_n & & \downarrow \lambda_n \\ 1 & \longrightarrow & \mathbb{R}^{1,n-1} & \longrightarrow & \mathcal{I}_{1,n-1}^\uparrow & \longrightarrow & O_{1,n-1}^\uparrow \longrightarrow 1 \end{array}$$

in which the rows are group extensions and K is the kernel of λ . The group K is the group of internal symmetries. If K is a compact Lie group, as often occurs, then the (new) little group representations at fixed positive energy-momentum are still discrete, but there are additional discrete invariants—quantum numbers—that label the representations. The induced irreducible positive energy unitary representations of $\mathcal{G}$ are the particles for this symmetry type, and they carry the additional quantum numbers.

Example 17.30. Consider $G_n = \text{Spin}_{1,n-1} \times_{\mu_2} \mathbb{T} = \text{Spin}_{1,n-1}^c$, and so $\mathcal{G}_n = \mathcal{P}_n \times_{\mu_2} \mathbb{T}$. Note that $\text{Spin}_{1,n-1}^c$ is noncompact; it is the Lorentz spin- c group. On the other hand, the little group is the *compact* Lie group Spin_m^c , where as usual $m = n - 1$ in the massive case and $m = n - 2$ in the massless case. Irreducible representations of $\text{Spin}_m^c = \text{Spin}_m \times_{\mu_2} \mathbb{T}$ are the tensor product of an irreducible representation of Spin_m and an irreducible representation of $\mathbb{T}$. The latter is indexed by an integer, which in many contexts is an *electric charge*. Regardless, that integer is a new quantum number for this symmetry type. That the central element $\epsilon \in \text{Spin}_m$ and $-1 \in \mathbb{T}$ act equally in the representation enforces a constraint, once we apply [Corollary 17.24](#): bosonic states have even electric charge and fermionic states have odd electric charge.

Lecture 18: Locality and fields

A *relativistic quantum mechanical system* (Definition 14.16) is a unitary positive energy representation of the Poincaré group, or of some other symmetry group of a system on a Minkowski spacetime. *Quantum field theories* are an important class of examples; they have more data than the representation. In particular, they have *fields*. In this lecture we introduce a very general concept of a field, and then we give an example of a free classical field theory, the theory of a real massive scalar field. In the next lecture we quantize this theory and we also give examples of other free field theories.

The signal property of a field is *locality*. Local fields also occur in differential geometry (vector fields, tensor fields, spinor fields), and so we begin with a general discussion of local objects in geometry: sheaves. Sheaves on a fixed space are more familiar than sheaves on a class of spaces—here manifolds of a fixed dimension—and so we introduce that more general concept. Fields in both geometry and physics often have internal symmetries—gauge symmetries—and so form groupoids or higher groupoids. The associated sheaf theory has a homotopical flavor, which we only hint at here and leave for references (and references in references).

Following the general discussion we specialize to fields on Minkowski spacetime, which have an additional property: Poincaré invariance. (As in many situations, what sounds like a condition is in fact data.) The classical wave equations that cut out classical fields are also local, and so classical solutions form a subsheaf of all fields. The space of global classical solutions on Minkowski spacetime has a symplectic structure. We illustrate all of this for the massive scalar field.

18.1 Fields as sheaves

Here are some “local quantities” on a smooth manifold:

- (1) real-valued functions
- (2) double covers
- (3) orientations
- (4) Riemannian metrics
- (5) connections on principal G -bundles for a fixed Lie group G
- (6) spin structures

How do we encode locality in a mathematical structure? One common feature is a GLOBAL $\rightarrow$ LOCAL operation: given any of these objects on a manifold, we can restrict to an open submanifold (open subset). What is more or less the same, we can pull back under a local diffeomorphism. Some of these—(1), (2), (5)—admit a pullback operation for *any* smooth map of smooth manifolds, but that is not true of the others. A second common feature is a LOCAL $\rightarrow$ GLOBAL operation. Given an open cover of a manifold, an object of one of these types on each open set, and isomorphisms on intersections of open sets, then we can glue the local objects uniquely into a global object.

Example 18.1. Consider the map $M \mapsto H^1(M; \mathbb{Z}/2\mathbb{Z})$ which assigns to a smooth manifold M the abelian group of *isomorphism classes* of double covers over M . This has a GLOBAL $\rightarrow$ LOCAL structure: isomorphism classes of double covers restrict to isomorphism classes of double covers. But it does not satisfy a LOCAL $\rightarrow$ GLOBAL condition. For example, cover the circle S^1 by two overlapping open intervals $\mathcal{U}_1, \mathcal{U}_2$, as in Figure 53. Then $H^1(-; \mathbb{Z}/2\mathbb{Z}) = 0$ on $\mathcal{U}_1, \mathcal{U}_2$, and $\mathcal{U}_1 \cap \mathcal{U}_2$,

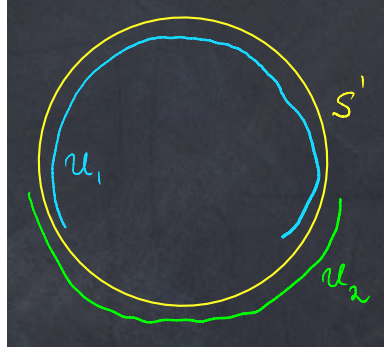


FIGURE 53. A covering of the circle by two open sets

but globally we have $H^1(S^1; \mathbb{Z}/2\mathbb{Z}) \neq 0$. To construct the nontrivial double cover of S^1 from local data we must remember how that local data is glued, not just that there exists a gluing. In other words, isomorphisms of double covers must be part of the data, not merely a condition. Yet again: double covers are *local* objects only if we treat the collection of double covers as a groupoid, not as a set. This lesson—that isomorphisms and automorphisms (gauge transformations) must be tracked and not merely noted—is crucial for building *local* quantum field theories out of connections (gauge fields) and their cousins of higher categorical depth.

Fix one of the “local quantities” in the list above, and denote by $\mathcal{F}(M)$ the collection of all such objects on a smooth manifold M . What mathematical structure does $\mathcal{F}(M)$ have? In cases (1), (3), and (4) it is a set. But in cases (2), (5), and (6) the objects in $\mathcal{F}(M)$ have internal symmetries: a double cover has deck transformations, and a connection may have automorphisms.⁵² In these cases $\mathcal{F}(M)$ is a groupoid. There are other examples of fields that have additional layers of structure—automorphisms of automorphisms of $\dots$, etc.—and in those cases the collection of fields is a higher groupoid. Technically, it is easier to use simplicial sets than higher groupoids; see the discussion in [FH2, §4] and the references therein.

The following very general definition is used to encode the GLOBAL $\rightarrow$ LOCAL property.

Definition 18.2. Let C be a category, let **Set** be the category of sets. A *presheaf of sets* on C is a functor $\mathcal{F}: C^{\text{op}} \rightarrow \mathbf{Set}$.

The *opposite category* C^{op} has the same objects and morphisms as C , but the source and target of a morphism are reversed, as is the order of composition. There is also a category **Gpd** of groupoids, a category **Set** $_{\Delta}$ of simplicial sets,⁵³ and corresponding *presheaves of groupoids* and *simplicial presheaves*

$$(18.3) \quad \begin{aligned} \mathcal{F}: C^{\text{op}} &\longrightarrow \mathbf{Gpd} \\ \mathcal{F}: C^{\text{op}} &\longrightarrow \mathbf{Set}_{\Delta} \end{aligned}$$

⁵²A double cover is the special case of (5) in which G is cyclic of order 2. If M is connected, then the automorphism group of a G -connection on M is noncanonically isomorphic to the centralizer in G of the holonomy group with respect to some basepoint. A spin structure is a “twisted version” of a double cover; it too has automorphisms.

⁵³Let Δ be the category whose objects are nonempty totally ordered finite sets and whose morphisms are order-preserving maps. A *simplicial set* is a functor $\Delta^{\text{op}} \rightarrow \mathbf{Set}$, which explains the notation **Set** $_{\Delta}$.

Example 18.4. Let X be a smooth manifold. Define the category $\mathbf{Open}(X)$ whose objects are open subsets of X and whose morphisms are inclusions of open sets. A *presheaf on X* is a presheaf on the category $\mathbf{Open}(X)$. For example, there are presheaves of sets $\mathcal{F}_1, \mathcal{F}_2$ which assign to an open set $\mathcal{U} \subset X$

$$(18.5) \quad \begin{aligned} \mathcal{F}_1(\mathcal{U}) &= \{\text{orientations of } \mathcal{U}\} \\ \mathcal{F}_2(\mathcal{U}) &= \{\text{Riemannian metrics on } \mathcal{U}\} \end{aligned}$$

Example 18.6. Let $C = \mathbf{Man}$ be the category whose objects are smooth manifolds and whose morphisms are smooth maps. The following are presheaves of sets or groupoids on $\mathbf{Man}$:

$$(18.7) \quad \begin{aligned} \mathcal{F}_1(M) &= \Omega_M^\bullet \\ \mathcal{F}_2(M) &= \text{Map}(M, X) \\ \mathcal{F}_3(M) &= \{\text{groupoid of } G\text{-connections on } M\} \\ \mathcal{F}_4(M) &= H^1(M; \mathbb{Z}/2\mathbb{Z}) \\ \mathcal{F}_5(M) &= \{\text{groupoid of double covers of } M\} \end{aligned}$$

Here X is a fixed smooth manifold and G is a fixed Lie group. The following assignments are *not* presheaves on $\mathbf{Man}$ since there is no pullback under arbitrary smooth maps:

$$(18.8) \quad \begin{aligned} M &\longmapsto \{\text{Riemannian metrics on } M\} \\ M &\longmapsto \{\text{orientations of } M\} \\ M &\longmapsto \{\text{groupoid of spin structures on } M\} \end{aligned}$$

Example 18.9. For $n \in \mathbb{Z}^{\geq 0}$ let $C = \mathbf{Man}_n$ be the category whose objects are smooth manifolds of dimension n and whose morphisms are local diffeomorphisms. There is an “inclusion” functor $\mathbf{Man}_n \rightarrow \mathbf{Man}$, so any presheaf on $\mathbf{Man}$, such as those in (18.7), pulls back to a presheaf on $\mathbf{Man}_n$. In addition, the assignments (18.8) are presheaves of sets or groupoids on $\mathbf{Man}_n$.

Remark 18.10.

- (1) In [Example 18.6](#) and [Example 18.9](#) we can think of a presheaf as analogous to a distribution, but on manifolds rather than on functions, in which case M may be regarded as a “test manifold”.
- (2) Fields in relativistic field theory fall under [Example 18.4](#) (X is Minkowski spacetime), and fields in a Wick-rotated n -dimensional field theory form a presheaf on $\mathbf{Man}_n$, as in [Example 18.9](#).

Now we turn to the LOCAL $\rightarrow$ GLOBAL condition: the *sheaf* condition. In the general setting of presheaf on an arbitrary category C additional structure on C is required: a *Grothendieck topology*. For the examples $C = \mathbf{Open}(X)$, $C = \mathbf{Man}$, $C = \mathbf{Man}_n$ the definition is more familiar.

Definition 18.11. Let $\mathcal{F}: \mathbf{Man}^{\text{op}} \rightarrow \mathbf{Set}$ be a presheaf. Then $\mathcal{F}$ is a *sheaf* if for every manifold M and every open cover $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ of M

$$(18.12) \quad \mathcal{F}(M) \longrightarrow \prod_{\alpha_0} \mathcal{F}(\mathcal{U}_{\alpha_0}) \rightrightarrows \prod_{\alpha_0, \alpha_1} \mathcal{F}(\mathcal{U}_{\alpha_0} \cap \mathcal{U}_{\alpha_1})$$

is an equalizer diagram.

This is the usual gluing property of a sheaf: given $x_{\alpha_0} \in \mathcal{F}(\mathcal{U}_{\alpha_0})$ such that the restrictions of x_{α_0} and x_{α_1} to $\mathcal{U}_{\alpha_0} \cap \mathcal{U}_{\alpha_1}$ agree, there is a unique $x \in \mathcal{F}(M)$ which restricts on $\mathcal{U}_{\alpha_0}$ to x_{α_0} . We often say $\mathcal{F}(U)$ is the set of sections of the sheaf $\mathcal{F}$ on the open set U . There is a similar definition for sheaves of sets on $\mathbf{Open}(X)$ or on $\mathbf{Man}_n$.

For a sheaf of groupoids or a sheaf of simplicial sets the analog of Definition 18.11 uses a bit of homotopy theory: see [FH2, §5].

Example 18.13. The presheaves in (18.5), (18.7), and (18.8) are all sheaves except for $\mathcal{F}_4$ in (18.7), as already emphasized in Example 18.1.

18.2 Fields on Minkowski spacetime

Recall the background data of a relativistic field theory: a Lorentzian vector space V , an affine space M over V , a time-orientation $P \subset V$, a speed of light $c \in \mathbb{R}^{>0}$, and a relativistic symmetry type (Definition 15.13): a Lie group G and a homomorphism $\lambda: G \rightarrow \mathbf{O}^\uparrow(V)$ whose image contains $\mathbf{SO}^\uparrow(V)$. There results a Lie group $\mathcal{G}$ and a homomorphism $\tilde{\lambda}: \mathcal{G} \rightarrow \mathbf{O}^\uparrow(M)$ that fit into the diagram (15.14).

A relativistic field theory also needs fields. Recall that sheaves push forward under maps.⁵⁴

Definition 18.14. A (collection of) *relativistic field(s)* on M is a simplicial sheaf $\mathcal{F}: \mathbf{Open}(M)^{\text{op}} \rightarrow \mathbf{Set}_\Delta$ which is $\mathcal{G}$ -invariant in the sense that it is equipped with isomorphisms

$$(18.15) \quad \mathcal{F} \xrightarrow{\cong} \tilde{\lambda}(g)_* \mathcal{F}, \quad g \in \mathcal{G},$$

that satisfy appropriate transitivity for $g_1, g_2 \in \mathcal{G}$.

For sheaves of sets the transitivity is a simple equality; for sheaves of groupoids or simplicial sets it involves more data to express higher coherence.

Example 18.16. Here are typical examples of fields. The symmetry types are either $\mathbf{O}^\uparrow(V) \xrightarrow{\text{id}} \mathbf{O}^\uparrow(V)$ or $\mathbf{Spin}(V) \rightarrow \mathbf{O}^\uparrow(V)$. Variations and many more examples exist.

- (1) Fix a smooth manifold X and let $\mathcal{F}(\mathcal{U}) = \text{Map}(\mathcal{U}, X)$. In general, this is called a *scalar field* or *σ -model field*. For $X = \mathbb{R}$ it is a *real scalar field*, for $X = \mathbb{C}$ it is a *complex scalar field*, for $X = \mathfrak{g}$ a Lie algebra it is a *Lie algebra-valued scalar field*, etc.

⁵⁴A map $X \rightarrow Y$ induces a pullback $\mathbf{Open}(Y) \rightarrow \mathbf{Open}(X)$ on open sets, and a presheaf $\mathbf{Open}(X)^{\text{op}} \rightarrow \mathbf{Set}_\Delta$ pushes forward to $\mathbf{Open}(Y)^{\text{op}} \rightarrow \mathbf{Set}_\Delta$ by composition with $\mathbf{Open}(Y)^{\text{op}} \rightarrow \mathbf{Open}(X)^{\text{op}}$.

- (2) Now suppose X is a smooth manifold equipped with an action of G . Define the sheaf $\mathcal{F}(\mathcal{U}) = \text{Map}(\mathcal{U}, X)$ as in (1), but now $g \in G$ acts on $\mathcal{F}(U)$ by

$$(18.17) \quad \begin{aligned} \mathcal{F}(U) &\xrightarrow{\cong} \mathcal{F}(\tilde{\lambda}(g)U) \\ \phi &\longmapsto \left(p \mapsto \pi(g) \cdot \phi(\tilde{\lambda}(g)^{-1} \cdot p) \right) \end{aligned}$$

where $\pi(g) \in G$ is the image of g under the homomorphism $\pi: \mathcal{G} \rightarrow G$.

- (3) In classical Maxwell theory we have the classical electromagnetic field $\mathcal{F}(\mathcal{U}) = \Omega_{\text{closed}}^2(\mathcal{U})$. This description does not include gauge invariance, so in many ways it is advantageous to take instead the groupoid-valued field $\mathcal{F}(\mathcal{U}) = \{\mathbb{R}^{>0}\text{-connections on } \mathcal{U}\}$.
- (4) For a Lie group G we have the groupoid-valued sheaf $\mathcal{F}(\mathcal{U}) = \{G\text{-connections on } \mathcal{U}\}$ of *gauge fields* with gauge group G .
- (5) This is an important special case of (2). Let $\mathbb{S}$ be a real *spinor representation* of $\text{Spin}(V)$, i.e., a real module over a real Clifford algebra built from the Lorentzian vector space V . Set $\mathcal{F}(\mathcal{U}) = \text{Map}(\mathcal{U}, \mathbb{S})$. Then $\mathcal{F}$ is called a *spinor field*. It should be treated as an odd (fermionic) field and so has a quite different nature from bosonic fields.⁵⁵ The symmetry type here is $\text{Spin}_{1,n-1} \rightarrow \text{O}_{1,n-1}^\uparrow$. Note that the $\text{Spin}(M)$ action is used to push forward $\mathcal{F}$, as in (18.15), and in addition the isomorphism (18.15) uses the $\text{Spin}(V)$ -action on $\mathbb{S}$.

There are both *classical* and *quantum* relativistic field theories. A sequence of data/steps in the specification of and passage from a classical field theory to a quantum field theory is:

- (A) A sheaf $\mathcal{F}$ of relativistic fields
- (B) A subsheaf $\mathcal{M} \subset \mathcal{F}$ of “classical solutions”, often cut out by a wave equation
- (C) A symplectic structure on $\mathcal{M}(M)$
- (D) A positive energy polarization of the symplectic manifold $\mathcal{M}(M)$
- (E) Quantize!

In §18.3 we stop short of (D) and (E), which we take up in the next lecture.

Remark 18.18.

- (1) The data (B) and (C) is often derived from an action principle: a Lagrangian. It gives more than a symplectic structure (see §3.4). Namely, one constructs a complex line bundle $\mathcal{L} \rightarrow \mathcal{M}(M)$ with covariant derivative whose curvature is the symplectic form (up to a constant). In the purely symplectic theory this is called a “prequantization line bundle”. This geometric prequantization is discussed in many sources, for example [Ko, SiWo].
- (2) The positive energy condition on the polarization (D) is crucial; positive energy typically characterizes the polarization.
- (3) When $\mathcal{M}(M)$ is an affine or linear symplectic space, then there is a good mathematical theory of quantization. Affine spaces of classical solutions are a signal of a “free” theory; see Definition 14.14(2). The theory of geometric quantization [BW] also illuminates quantization in many other situations, but the old adage still applies:

Quantization is an art, not a functor

⁵⁵We do not elaborate on fermionic fields here; see [F5, §1] for a heuristic account.

18.3 The massive scalar field: classical theory

Fix the relativistic symmetry type

$$(18.19) \quad \lambda: \mathrm{O}^\uparrow(V) \times \mu_2 \longrightarrow \mathrm{O}^\uparrow(V)$$

which is projection onto the first factor. The internal symmetry group $K = \mu_2$ is cyclic of order 2. The corresponding affine symmetry group is

$$(18.20) \quad \mathcal{G} = \mathrm{O}^\uparrow(M) \times \mu_2.$$

The field $\mathcal{F}$ is a real scalar field, defined as $\mathcal{F}(\mathcal{U}) = C^\infty(\mathcal{U}; \mathbb{R})$. (We choose C^∞ functions for classical fields.) The internal symmetry acts on a scalar field ϕ by $\phi \mapsto -\phi$. For the real scalar field, $\mathcal{F}$ is a sheaf of real vector spaces.

The inverse metric—the Lorentz inner product on the dual space V^* —is an $\mathrm{O}^\uparrow(V)$ -invariant vector in $\mathrm{Sym}^2 V$. View V as the space of translation-invariant vector fields on M , and so the inverse metric defines an $\mathrm{O}^\uparrow(M)$ -invariant second order differential operator on M : the *wave operator* $\square$. For standard Minkowski spacetime with coordinates $x^0 = ct, x^1, \dots, x^{n-1}$ and metric

$$(18.21) \quad c^2 dt^2 - (dx^1)^2 - \dots - (dx^{n-1})^2,$$

we have

$$(18.22) \quad \square = \frac{1}{c^2} \partial_t^2 - \partial_1^2 - \dots - \partial_{n-1}^2.$$

The key ingredient in the massive scalar field theory is the *Klein–Gordon equation* for mass $m > 0$:

$$(18.23) \quad \left(\square + \frac{c^2}{\hbar^2} m^2 \right) \phi = 0.$$

The equation is local, so defines a subsheaf $\mathcal{M} \subset \mathcal{F}$: for an open $\mathcal{U} \subset M$, the subset $\mathcal{M}(\mathcal{U}) \subset \mathcal{F}(\mathcal{U})$ consists of C^∞ functions $\phi: \mathcal{U} \rightarrow \mathbb{R}$ that satisfy (18.23). Equation 18.23 is invariant under $\phi \mapsto -\phi$, and so is compatible with the symmetry (18.19). Also, (18.23) is a linear equation, hence $\mathcal{M}$ is a sheaf of real vector spaces.

The last ingredient of the classical theory is a symplectic structure on the real vector space $\mathcal{M}(M)$ of global classical solutions. As mentioned in Remark 18.18(1), the symplectic structure can be derived from a Lagrangian; see [DF2, §3.2]. It turns out to be the 2-form

$$(18.24) \quad \omega(\dot{\phi}_1, \dot{\phi}_2) = \int_N * d\dot{\phi}_1 \cdot \dot{\phi}_2 - * d\dot{\phi}_2 \cdot \dot{\phi}_1,$$

where $N \subset M$ is a spacelike affine hyperplane and $*$ is derived from the Lorentz inner product: it maps q -forms to orientation-twisted $(n - q)$ -forms, hence the integrand in (18.23) restricts to a density on N . Also, the “variations” $\dot{\phi}_1, \dot{\phi}_2$ are assumed to have compact spatial support.

Remark 18.25. This compact support condition reflects an important feature of field theory that we have not attended to. There is not a single classical field theory on M , but rather a family of classical field theories parametrized by the asymptotic values of the field at infinity in M , and furthermore the values at infinity are constrained to have finite energy. (Here again, energy is derived from the Lagrangian; see [DF2, §2.10].) Since the fields satisfy a wave equation (18.24), it suffices to specify the values at *spatial* infinity. These families of theories often encode symmetries.

It is convenient to view the global space of classical solutions $\mathcal{M}(M)$ via Fourier transform. For that we choose a basepoint $p_0 \in M$ and so identify

$$(18.26) \quad \begin{aligned} V &\longrightarrow M \\ \xi &\longmapsto p_0 + \xi \end{aligned}$$

Finite energy imposes a decay condition at spatial infinity, which makes the Fourier transform well-behaved. First, there is an $O(V)$ -invariant measure $d\xi$ on V from the Lorentz metric: it assigns volume 1 to a parallelepiped spanned by an orthonormal basis. There is a dual measure $d\theta$ on V^* . Then the Fourier transform of $\phi = \phi(\xi)$ is a function $\hat{\phi} = \hat{\phi}(\theta)$ on V^* .⁵⁶

$$(18.27) \quad \hat{\phi}(\theta) = \frac{1}{(2\pi)^{n/2}} \int_V d\xi e^{+i\theta(\xi)/\hbar} \phi(\xi)$$

$$(18.28) \quad \phi(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{V^*} d\theta e^{-i\theta(\xi)/\hbar} \hat{\phi}(\theta)$$

The Fourier transform of the second-order ordinary differential operator in (18.23) is multiplication by a quadratic polynomial; the Fourier transformed Klein–Gordon equation is

$$(18.29) \quad \left(-\frac{|\theta|^2}{\hbar^2} + \frac{m^2 c^2}{\hbar^2} \right) \hat{\phi}(\theta) = 0.$$

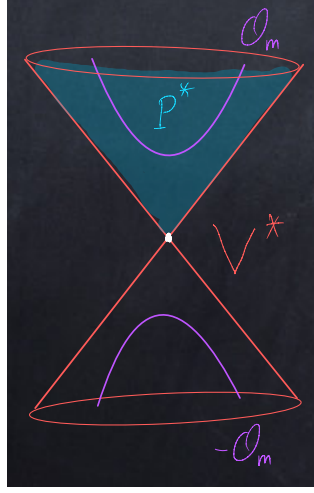
Also, the Fourier transform of a real-valued function ϕ satisfies the reality constraint

$$(18.30) \quad \hat{\phi}(-\theta) = \overline{\hat{\phi}(\theta)}.$$

If $\hat{\phi}$ satisfies (18.29), then its support lies in $\mathcal{O}_m \cup -\mathcal{O}_m$, where the mass shell $\mathcal{O}_m \subset P^*$ is defined in (13.19); see Figure 54. In this way we identify $\mathcal{M}(M)$ as the real points in the complex vector space of suitable functions $\hat{\phi}: \mathcal{O}_m \cup -\mathcal{O}_m \rightarrow \mathbb{C}$ with real structure $\hat{\phi} \mapsto \iota^* \hat{\phi}$, where $\iota: V^* \rightarrow V^*$ is reflection in the origin: multiplication by -1 .

The generator of the internal symmetry group μ_2 acts on $\mathcal{M}(M)$ by $\phi \mapsto -\phi$ and so too $\hat{\phi} \mapsto -\hat{\phi}$. Also, under Fourier transform the multiplication in the symplectic form (18.24) becomes convolution, and the set of $\hat{\phi}$ with support on $\mathcal{O}_m$ is a complex Lagrangian subspace, as is the set of $\hat{\phi}$ with support on $-\mathcal{O}_m$.

⁵⁶The signs are chosen so that (18.28) expresses ϕ as a linear combination of characters of V : recall that $\theta \in V^*$ labels the character $\xi \mapsto e^{-i\theta(\xi)/\hbar}$.

FIGURE 54. The support of $\hat{\phi}$ for a solution to the massive wave equation

Lecture 19: Quantization of free fields

We discuss the quantization of free field theories. This is the passage from classical waves to quantum particles.

We begin with a general discussion of the quantization of a symplectic vector space, a topic we have already treated in the finite dimensional setting. The main point is the introduction of a polarization. Most relevant for free fields are the Kähler polarizations. There is also a quantization theory for odd symplectic vector spaces, but here we restrict to the even case.

In §19.2 we resume our discussion of the massive scalar field, now taking up its quantization. We briefly comment on the massless scalar field as well. The electromagnetic field is another example of a free field theory; the classical version is based on Maxwell equations. The new point here is gauge symmetry. Nonabelian gauge theories are mentioned at the end of the lecture as an example of a non-free quantum theory.

19.1 Quantization of a symplectic vector space

We recount the algebraic construction of a Heisenberg representation, which we apply in the next section of the massive scalar field. The finite dimensional version of this construction appears in Lecture 6: see (6.17).

Let W be a real symplectic vector space with symplectic form ω . If W is infinite dimensional, then we assume that it is equipped with a suitable topology and that ω is continuous. Define the *Weyl algebra* (see Remark 6.21(2)) on the dual space W^* :

$$(19.1) \quad \mathcal{W}(W^*) = \otimes^\bullet(W^*)/I,$$

where I is the 2-sided ideal in the tensor algebra $\otimes^\bullet(W^*)$ generated by

$$(19.2) \quad \lambda_1 \otimes \lambda_2 - \lambda_2 \otimes \lambda_1 - \omega(\lambda_1, \lambda_2) \cdot 1, \quad \lambda_1, \lambda_2 \in W^*.$$

Here we use the induced symplectic form on the dual space. Replace ω by $t\omega$ for $t \in \mathbb{R}^{\neq 0}$ to obtain a family of algebras $\mathcal{W}_t(W^*)$. There is a limiting algebra as $t \rightarrow 0$: the symmetric algebra $\text{Sym}^\bullet(W^*)$. This is the algebra of polynomial, or algebraic, functions on W . Hence the Weyl algebra is a noncommutative deformation of the commutative algebra of functions on W . Recall from (1.18) that functions on W are the observables of the classical system derived from W . The Weyl algebra plays the role of algebraic observables in a *deformation quantization* of this classical system. A complete quantization includes states as well as observables. For a C^* -algebra of observables we defined states in Definition 11.27. Here we have a nontopological algebra $\mathcal{W}(W^*)$ of observables, and the first step is to construct an *irreducible* module over this algebra. Then we introduce a hermitian inner product and complete the module to a Hilbert space $\mathcal{H}$ and construct states from $\mathcal{H}$ as usual (§1.3).⁵⁷

The key ingredient is a polarization.

Definition 19.3. Let (W, ω) be a real symplectic vector space. An *isotropic subspace* is a subspace $L \subset W$ on which the symplectic form vanishes. If L is maximal among isotropic subspaces, then it is a *Lagrangian subspace*. A *polarization* of W is a choice of Lagrangian subspace. A *complex polarization* is a (complex) Lagrangian subspace $L \subset W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$. A complex polarization is real if $\bar{L} = L$. A complex polarization is of *Kähler type* if $L \cap \bar{L} = 0$, in which case $W_{\mathbb{C}} = L \oplus \bar{L}$.

A complex polarization of Kähler type induces a complex structure on W compatible with the symplectic structure, i.e., a symplectic automorphism I of W such that $I^2 = -\text{id}_W$. Conversely, such a complex structure induces a Lagrangian decomposition of the complexification of W according to the eigenvalues of $I \otimes \mathbb{C}$:

$$(19.4) \quad W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C} = W_+ \oplus W_-, \quad W_- = \overline{W_+}.$$

(In the notation of Definition 19.3, $W_+ = L$ and $W_- = \bar{L}$.) There results a hermitian form

$$(19.5) \quad \langle \bar{w}_1, w_2 \rangle = 2i\omega(\bar{w}_1, w_2), \quad w_1, w_2 \in W_+,$$

on W_+ , and we say the complex structure is *positive* if this form is positive definite. Assume so.

Next, construct a *complex* module $\mathcal{D}$ over $\mathcal{W}(W^*)$. Namely, define

$$(19.6) \quad \mathcal{D} = \text{Sym}^\bullet(W_+^*), \quad \Omega = 1 \in \text{Sym}^0(W_+^*).$$

The vector Ω is called the *vacuum*. The module structure is the algebra homomorphism

$$(19.7) \quad \begin{aligned} \mathcal{W}(W^*) \otimes \mathbb{C} &\longrightarrow \text{End } \mathcal{D} \\ \lambda_+ &\longmapsto M_{\lambda_+} \\ \lambda_- &\longrightarrow \iota_{\lambda_-} \end{aligned}$$

⁵⁷Some elements of $\mathcal{W}(W^*)$ act on $\mathcal{H}$ as *unbounded* operators. The problem is cleanest in the finite dimensional odd case. Then $\mathcal{W}(W^*)$ is replaced by a Clifford algebra, and the problem is to write its complexification as endomorphisms of a super vector space. If the Clifford algebra has an odd number of generators, its commutant in the algebra is isomorphic to $\mathbb{C}\ell_1^{\mathbb{C}}$ and the resulting $\mathbb{Z}/2\mathbb{Z}$ -graded Hilbert space is viewed as a module over this commutant.

where $\lambda_+ \in W_+^*$, $\lambda_- \in W_-^*$, the operator M_{λ_+} is multiplication by $\lambda_+ \in \text{Sym}^1(W_+^*)$, and ι_{λ_-} is the contraction operator of degree -1 defined by

$$(19.8) \quad \iota_{\lambda_-}(\lambda_+^{(1)} \cdots \lambda_+^{(k)}) = \sum_{i=1}^k \omega(\lambda_-, \lambda_+^{(i)}) \lambda_+^{(1)} \cdots \cancel{\lambda_+^{(i)}} \cdots \lambda_+^{(k)}.$$

We call M_{λ_+} a *creation operator* and ι_{λ_-} an *annihilation operator*. The annihilation operators annihilate the vacuum ($W_-^* \cdot \Omega = 0$) and the creation operators generate the vector space $\mathcal{D}$ ($W_+^* \cdot \Omega = \mathcal{D}$).

Use (19.5) to induce a hermitian form on $\mathcal{D}$, and then complete to obtain a Hilbert space $\mathcal{H}$. It is this Hilbert space which is the quantization, but the dense algebraic subspace is useful for computations. We can also complete $W_+^* = \text{Sym}^1(W_+^*)$ to a Hilbert space. In the context of quantum field theory this is the *1-particle state space*. We can complete its symmetric powers to obtain *multi-particle state spaces*. The direct sum of these multi-particle state spaces is dense in $\mathcal{H}$; it is called the *Fock space*. It is an intermediate space between $\mathcal{D}$ and $\mathcal{H}$.

Remark 19.9. There are analogous constructions for an *odd* symplectic vector space that is used to construct free fermionic theories. In the finite dimensional case it appears at the end of [Lecture 10](#). See [\[KoSt\]](#), [\[Kaz, §3\]](#) for the infinite dimensional case.

19.2 The quantum massive scalar field

Resume our relativistic setup, with its Minkowski spacetime M , vector space V of translations, forward timelike cone $P \subset V$, and speed of light c . Recall the classical massive scalar field, as discussed in §18.3. Solutions to the classical linear wave equation (18.23) form a real symplectic vector space, denoted $\mathcal{M}(M)$ in our exposition, which is identified by Fourier transform as a real subspace of a vector space of complex-valued functions on the union $\mathcal{O}_m \cup -\mathcal{O}_m$ of a mass shell and its negative; see [Figure 54](#).

Now apply §19.1 to the symplectic vector space $\mathcal{M}(M)$. The key point is that we choose a polarization (19.4) in which W_+ consists of *positive energy* fields. First, observe that the complexification of $\mathcal{M}(M)$ consists of *complex-valued* functions on $\mathcal{O}_m \cup -\mathcal{O}_m$ without any reality condition (18.30). Choose W_+ to be the subspace of functions with support on the positive energy mass shell $\mathcal{O}_m$. This is where the crucial positive energy condition enters. Then set $\mathcal{H}_1$ to be the Hilbert space completion of W_+^* . By [Theorem 17.16](#) it carries an irreducible representation of $\text{SO}^\uparrow(M)$, the representation of the massive scalar *particle* of mass m . This is precisely the representation obtained in [Lecture 14](#) by quantizing the particle of mass m ; see (15.2) and the surrounding text. As mentioned above, $\mathcal{H}_1$ is the *1-particle state space*.

The *Fock space* is the symmetric algebra $\text{Sym}^\bullet \mathcal{H}_1$. The Hilbert space completion of the subspace $\text{Sym}^k \mathcal{H}_1$ is the *k-particle state space*. The Hilbert space completion

$$(19.10) \quad \mathcal{H} = \overline{\text{Sym}^\bullet \mathcal{H}_1}$$

of the symmetric algebra is the Hilbert space of the relativistic quantum mechanical system that represents the massive scalar field. It carries a representation of the symmetry group (18.20). The

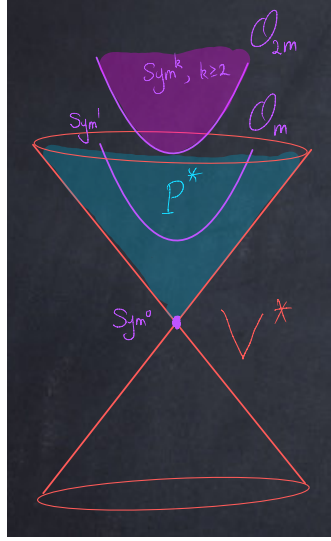


FIGURE 55. The spectrum of the massive scalar field

generator of the internal symmetry group μ_2 acts as $+1$ on $\text{Sym}^{\text{even}} \mathcal{H}_1$ and as -1 on $\text{Sym}^{\text{odd}} \mathcal{H}_1$. In other words, it gives a $\mathbb{Z}/2\mathbb{Z}$ -grading by parity of particle number. Observe that the integer particle number is defined on the Fock space by the grading of the symmetric algebra, but it does not extend to the Hilbert space completion (19.10), whereas the mod 2 particle number does extend. The spectrum of the theory is depicted in Figure 55. There is a unique vacuum state $\text{Sym}^0 \mathcal{H}_1 \subset \mathcal{H}$, which sits discretely in the spectrum. There is a spectral gap. Next in the spectrum, ordered by mass, is the irreducible scalar representation of mass m , which sits discretely in the spectrum of representations of $O^\uparrow(M)$. Above that is continuous spectrum: the convex hull of the mass shell $\mathcal{O}_{2m}$. Each representation $\overline{\text{Sym}^k \mathcal{H}_1}$, $k \in \mathbb{Z}^{\geq 0}$ is a closed subspace of $\mathcal{H}$.

Remark 19.11. The quantization of a *field*—in this case the massive scalar field—leads to *particle* representations of the symmetry group. This “duality” between fields and particles is a basic feature of quantum field theory. In more complicated quantum field theories the 1-particle subspace may not be reducible, but it is a sum of a *finite* number of irreducible representations. Those irreducible representations comprise the *particle spectrum* of the theory.

19.3 The massless scalar field

This is ostensibly similar to the massive case—simply set $m = 0$ in (18.23)—but the result is quite different. Assume the dimension $n \geq 3$; there are special features in $n = 1$ and $n = 2$. We simply report on the positive energy unitary representation of $O^\uparrow(M) \times \mu_2$.

The 1-particle space $\mathcal{H}_1$ is an irreducible representation of $O^\uparrow(M)$, but now a massless representation: it sits in the fiber over 0 in (17.17). This is the massless scalar particle: the representation of the little group is trivial. The Hilbert space $\mathcal{H}$ is the completion of the Fock space, as in (19.10). The k -particle state space $\overline{\text{Sym}^k \mathcal{H}_1} \subset \mathcal{H}$ is a closed subspace. There is a unique vacuum $\text{Sym}^0 \mathcal{H}_1$, but the system is *gapless*—there is no spectral gap. The spectrum of the 1-particle space is $N(V^*)^+$, which contains $\{0\}$ in its closure. The spectrum of the entire theory $\mathcal{H}$ is the closed cone $\overline{P^*}$ of positive energy vectors.

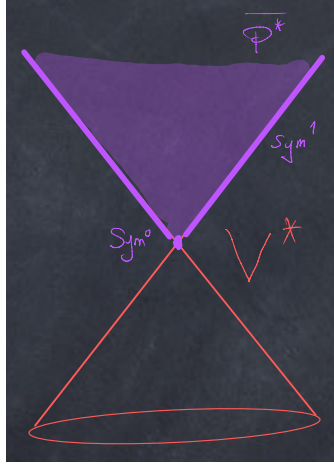


FIGURE 56. The spectrum of the massless scalar field

19.4 Maxwell theory: classical electromagnetism

We work in any dimension n , though of course the electromagnetism of our world is $n = 4$, and we initially go beyond the setting of Minkowski spacetime. Let (N, g_N) be a Riemannian manifold. Define the split spacetime

$$(19.12) \quad M = \mathbb{M}^1 \times N$$

with Lorentz metric $(dx^0)^2 - g_N$, where $x^0 = ct$ is the standard coordinate on the time line $\mathbb{M}^1$. The classical *electric* and *magnetic* fields are differential forms on N that depend on time:

$$(19.13) \quad \begin{aligned} E: \mathbb{M}^1 &\longrightarrow \Omega_N^1 \\ B: \mathbb{M}^1 &\longrightarrow \Omega_N^2 \end{aligned}$$

The electromagnetic field is the 2-form on M that combines the electric and magnetic fields:

$$(19.14) \quad F = B - dt \wedge E.$$

Let $*_N$ be the Hodge star operator for the positive definite metric on N . Then if $*$ is the Hodge $*$ operator on $\mathbb{M}^1 \times N$ relative to the Lorentz metric, we compute

$$(19.15) \quad *F = \frac{1}{c} *_N E + c dt \wedge *_N B.$$

Maxwell's equations in empty space are:

$$(19.16) \quad \begin{aligned} dB &= 0 & dE &= -\frac{\partial B}{\partial t} \\ d*_N E &= 0 & c^2 d*_N B &= *_N \frac{\partial E}{\partial t} \end{aligned}$$

They combine to the simple equations

$$(19.17) \quad \begin{aligned} dF &= 0 \\ d * F &= 0 \end{aligned}$$

for the 2-form F on spacetime M .

In fact, we demand a stronger form of the first equation in (19.17), which states that F is closed: we demand that F be exact. In other words, we postulate a 1-form $A \in \Omega_M^1$ such that

$$(19.18) \quad F = F_A = dA.$$

The 1-form A is determined up to a closed 1-form, so lives in the quotient

$$(19.19) \quad A \in \Omega_M^1 / \Omega_{M,\text{closed}}^1.$$

We take (19.19) to be the fundamental field in Maxwell theory, and use the decomposition (19.14) to define the electric field $E = E_A$ and magnetic field $B = B_A$. The second equation in (19.17) is a wave equation for A :

$$(19.20) \quad d * dA = 0.$$

Remark 19.21. The first Maxwell equation in (19.17) follows from (19.18). The second is the Euler–Lagrange equation for the Lagrangian density

$$(19.22) \quad L = -\frac{1}{2} \langle F_A, F_A \rangle \mu_M = \frac{1}{2} \left(\frac{\|E_A\|^2}{c^2} - \|B_A\|^2 \right) \mu_M,$$

where the norms of the forms E_A, B_A are computed in the positive definite metric on N . A multiplicative constant is usually inserted in L , depending on the units used; we omit it.⁵⁸

Now resume the relativistic setup on a Minkowski spacetime M , but without the splitting (19.12). Since M is contractible, closed 1-forms are exact. Hence the space of solutions to (19.20) in the quotient (19.19) is the first cohomology group of the complex

$$(19.23) \quad \Omega_M^0 \xrightarrow{d} \Omega_M^1 \xrightarrow{*d*d} \Omega_M^1$$

As in §18.3 rewrite the complex after Fourier transform: for $\theta \in V^*$, $f \in \Omega_V^0$, and $A \in \Omega_V^1$ we have

$$(19.24) \quad \begin{aligned} (df)^\wedge(\theta) &= \sqrt{-1} \hat{f}(\theta) \theta \\ (*d * d\alpha)^\wedge(\theta) &= |\theta|^2 \hat{A}(\theta) - \langle \hat{A}(\theta), \theta \rangle \theta. \end{aligned}$$

⁵⁸In the “mks” system of units, the constant is written $\epsilon_0 c^2 = 10^7/4\pi$.

Then one can show that the first cohomology of (19.23) is isomorphic to the space of real functions $\{\hat{A}: N(V^*) \rightarrow V^* \otimes \mathbb{C}\}$ on the lightcone that satisfy $\langle \hat{A}(\theta), \theta \rangle = 0$ modulo the space of functions $\{\hat{A}(\theta) = \hat{f}(\theta)\theta\}$, where $\hat{f}$ ranges over the complex functions on the lightcone. The reality condition is $\hat{A}(-\theta) = \hat{A}(\theta)$. Polarize by positive energy. The resulting complex Lagrangian W_+^* is the space of sections of the complex vector bundle $E \rightarrow N(V^*)^+$ whose fiber at $\theta \in N(V^*)^+$ is $N^\perp/N \otimes \mathbb{C}$, where $N^* = \mathbb{R} \cdot \theta$. It follows from Proposition 16.46(2) that the 1-particle space—the Hilbert space of L^2 sections of $E \rightarrow N(V^*)^+$ —is the irreducible vector representation of spin 1. This is the “vector particle” that corresponds to the Maxwell field.

Remark 19.25. If $n = 3$, then the little group is $\text{Spin}_1 \cong \mu_2$, and the representation is the trivial representation. So in this dimension the vector particle representation reduces to the scalar particle representation. This is a manifestation of *electromagnetic duality*.

We can rewrite Maxwell theory as a gauge theory with gauge group $\mathbb{R}^{>0}$, the connected multiplicative group of real numbers. (One can also impose Dirac’s quantization of charge, which is encoded by choosing the *compact* gauge group $\mathbb{T}$, but we do not do so here.) For $\mathcal{U} \subset \mathbf{Open}(M)$ put $\mathcal{F}(\mathcal{U}) = \{\text{groupoid of } \mathbb{R}^{>0}\text{-connections on } \mathcal{U}\}$. Note that any principal bundle over the contractible space M admits a section, and that section pulls back a connection to a real 1-form $A \in \Omega_M^1$. A change of section is given by a function $g: M \rightarrow \mathbb{R}^{>0}$, and it induces the gauge transformation $A \mapsto A + dg/g$. Any such function is uniquely $g = e^f$ for some $f \in \Omega_M^0$, and then $A \mapsto A + df$.

The subsheaf $\mathcal{M} \subset \mathcal{F}$ of classical solutions is cut out by the equation

$$(19.26) \quad d * F_A = 0,$$

where $*$ is the Lorentz signature star operator and $F_A \in \Omega_M^2$ is the curvature of the connection A .

19.5 A word about nonabelian gauge theories

Let G be a compact Lie group and let $\langle -, - \rangle$ be a bi-invariant inner product on the Lie algebra $\mathfrak{g}$. In a pure G -gauge theory the fields are $\mathcal{F}(\mathcal{U}) = \{\text{groupoid of } G\text{-connections on } \mathcal{U}\}$. If A is a connection with curvature F_A , then the Bianchi identity and Yang–Mills equations of the pure gauge theory—the equation that cuts out classical solutions—are

$$(19.27) \quad \begin{aligned} d_A F_A &= 0 \\ d_A * F_A &= 0 \end{aligned}$$

This classical theory is conjectured to have a quantum counterpart.

Conjecture 19.28 (Clay Problem). *There is a quantum field theory that corresponds to classical Yang–Mills theory, and it is gapped.*

The task is to construct a *quantum field theory*, not just a relativistic quantum mechanical system, and to prove that it is gapped. We refer to [JW] for background and a precise formulation of the conjecture.

Remark 19.29. The classical theory is not free. Namely, the curvature

$$(19.30) \quad F_A = F_{A_0} + d_{A_0} \alpha + \frac{1}{2} [\alpha \wedge \alpha]$$

of a perturbation $A = A_0 + \alpha$ of a reference connection has quadratic terms, and so the Yang–Mills equation has cubic terms—it is not a linear equation. This nonlinearity, or non-freeness, means that the free theory is not necessarily a good guide, even qualitatively, to the quantum theory. In this case the quantum theory is expected to acquire a mass gap.

Lecture 20: Local observables

A symplectic manifold N determines the data of a classical system; a Hilbert space $\mathcal{H}$, or a projective Hilbert space, determines the data of a quantum system. That is, from N or $\mathcal{H}$ we define a theory of states and observables. For theories on spacetime we should think of these as *global*: N or $\mathcal{H}$ is constructed from global data. We can also contemplate a theory of observables which is *local* on spacetime. Such local theories of observables are the subject of this lecture. In the previous lecture we defined fields as local objects in spacetime, so it is not a surprise that in field theory local observables can be constructed from local fields, an idea we develop in the next lecture. Locality of states is more subtle in quantum field theory; in discrete quantum mechanical systems it is more evident, as we see in (20.6). However, in general locality of states is more subtle; see [Remark 20.8\(2\)](#). The locality of states in Wick-rotated field theory is a subject for next semester.

The starting point in this lecture is classical field theory on Minkowski spacetime, where observables on an open subset form a *commutative* algebra. We then move on to a discrete quantum model—the toric code is used as a concrete illustration—where there are *noncommutative* algebras of observables that are local in space. Basic observables in this theory may be regarded as *point operators*⁵⁹ in the sense that their support is adjacent to a single vertex or face. These observables do not separate vacua, however; there are *line observables*⁶⁰ that do so.

Finally, we present a version of an axiom system for local observables in quantum field theory due to Haag–Kastler [[HK](#), [Ha](#)]. We note that this approach—based as it is on *bounded operators*—follows in the tradition of Irving Segal [[S](#)]. We only give highlights and defer to the literature for details. In the next lectures we construct local observables in a field theory. The signal feature of local observables in Minkowski spacetime is *causality*: observables with spacelike separated support commute. There is an analogy between commutators in algebras and *causal complements* in spacetime.

We work mostly in a Minkowski spacetime M over a Lorentzian vector space V , equipped with the usual accouterments: a cone P of forward timelike vectors, a speed of light c , and a relativistic symmetry type $\lambda: G \rightarrow \mathrm{O}^\uparrow(V)$. Recall that λ induces a homomorphism $\tilde{\lambda}: \mathcal{G} \rightarrow \mathrm{O}^\uparrow(M)$.

⁵⁹The words ‘observable’, ‘operator’, and also ‘defect’ are often used interchangeably in this context. In this lecture we stick to ‘observable’.

⁶⁰A geometer’s note. Affine subspaces in affine geometry: point, line, plane. Submanifolds in differential geometry: point, curve, surface. Standard physics terminology for the support of observables: point, line, surface.

20.1 Local observables in classical field theory

Suppose that $\mathcal{F}$ is a collection of relativistic fields (Definition 18.14), which is a sheaf on M . In a classical field theory, a local equation—for example, a wave equation—cuts out a subsheaf $\mathcal{M} \subset \mathcal{F}$, and there is a symplectic structure on $N = \mathcal{M}(M)$. The subsheaf $\mathcal{M}$ is assumed $\mathcal{G}$ -invariant (in the sense of (18.15)). From the symplectic manifold N one constructs global states and global observables, as usual.

For $\mathcal{U} \subset M$ an open set, let⁶¹

$$(20.1) \quad \mathcal{A}^{\text{cl}}(\mathcal{U}) = C^\infty(\mathcal{M}(\mathcal{U}), \mathbb{C}).$$

Complex conjugation provides a real structure on $\mathcal{A}^{\text{cl}}(\mathcal{U})$, and the real points are the classical observables on $\mathcal{U}$. If $\mathcal{U}_1 \subset \mathcal{U}_2$, then there is a restriction map $\mathcal{M}(\mathcal{U}_2) \rightarrow \mathcal{M}(\mathcal{U}_1)$, and so an inclusion map $\mathcal{A}^{\text{cl}}(\mathcal{U}_1) \rightarrow \mathcal{A}^{\text{cl}}(\mathcal{U}_2)$: an observable on $\mathcal{U}_1$ is also an observable on the larger set $\mathcal{U}_2$. This structure is dual to a presheaf; recall Definition 18.2 and Definition 18.11.

Definition 20.2. Let C be a category, let **Set** be the category of sets.

- (1) A *precosheaf of sets* on C is a functor $\mathcal{A}: C \rightarrow \mathbf{Set}$.
- (2) Let M be a topological space, and let **Open**(M) be the category of open subsets and inclusions. A precosheaf $\mathcal{A}: \mathbf{Open}(M) \rightarrow \mathbf{Vect}$ of vector spaces is a *cosheaf* if for every open set $U \subset M$ and every open cover $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ of U the diagram

$$(20.3) \quad \mathcal{A}(U) \longleftarrow \bigoplus_{\alpha_0} \mathcal{A}(\mathcal{U}_{\alpha_0}) \longleftarrow \bigoplus_{\alpha_0, \alpha_1} \mathcal{A}(\mathcal{U}_{\alpha_0} \cap \mathcal{U}_{\alpha_1})$$

is a coequalizer diagram.

The cosheaf condition (2) also applies to precosheaves of abelian groups.

Example 20.4. There is a cosheaf $\mathcal{U} \mapsto C_c^0(\mathcal{U}; \mathbb{C})$ that assigns to each open subset $\mathcal{U} \subset M$ the space of *compactly supported* continuous functions on $\mathcal{U}$. Also, singular cochains on $\mathcal{U}$ form a cosheaf. If M is a smooth manifold, then distributions also form a cosheaf.

The precosheaf $\mathcal{A}^{\text{cl}}$ satisfies the gluing condition (20.3): a coherent finite set of observables glues to a single observable. Furthermore, the cosheaf $\mathcal{A}^{\text{cl}}$ is $\mathcal{G}$ -invariant: if $g \in \mathcal{G}$, then there are coherent isomorphisms $\mathcal{A}^{\text{cl}}(\mathcal{U}) \xrightarrow{\cong} \mathcal{A}^{\text{cl}}(\lambda(g)(\mathcal{U}))$, $\mathcal{U} \in \mathbf{Open}(M)$. Finally, $\mathcal{A}^{\text{cl}}$ is a cosheaf of *commutative* algebras. For convenience, here is an enumeration of these properties:

- (A) $\mathcal{A}^{\text{cl}}(\mathcal{U})$ is a commutative algebra
- (B) $\mathcal{A}^{\text{cl}}$ is a cosheaf
- (C) $\mathcal{A}^{\text{cl}}$ is $\mathcal{G}$ -invariant

We turn now to an example of local observables in a discrete quantum system.

⁶¹As in (1.18) we could take $\mathcal{A}^{\text{cl}}$ to consist of Borel functions and define a dense subset $(\mathcal{A}^{\text{cl}})^\infty$ of C^∞ functions.

20.2 Local observables in the toric code

Kitaev's toric code [Ki] was introduced in Section 5.5. It is formulated on a closed surface Y equipped with an embedded finite graph Λ , as in Figure 11. There is a finite set $\mathcal{D}(\Lambda, \Lambda^0)$ of isomorphism classes of double covers of Λ equipped with trivializations over the vertices, and the finite dimensional Hilbert space $\mathcal{H}$ of toric code is the space (5.21) of complex functions on $\mathcal{D}(\Lambda, \Lambda^0)$. The Hilbert space of this theory has a local description. Recast $\mathcal{D}(\Lambda, \Lambda^0)$ as the set of μ_2 -valued functions on the edges of Λ , as in (5.20), and set

$$(20.5) \quad \mathcal{H}_e = \text{Map}(\mu_2, \mathbb{C}), \quad e \in \text{Edge}(\Lambda).$$

Then there is a natural isomorphism

$$(20.6) \quad \mathcal{H} \cong \bigotimes_e \mathcal{H}_e.$$

This expresses the global state space $\mathcal{H}$ as a *finite* tensor product of local state spaces $\mathcal{H}_e$. As for the algebra $\text{End } \mathcal{H}$, whose real points (self-adjoint operators) are observables, we have

$$(20.7) \quad \text{End } \mathcal{H} \cong \bigotimes_e \text{End } \mathcal{H}_e.$$

Remark 20.8.

- (1) There is an infinite version of the toric code on a noncompact surface, for example on an affine plane. This beautiful theory is developed in [Naa]. Our remark here is that an infinite analog of (20.7) is well-defined—one takes products of operators in $\text{End } \mathcal{H}_e$ such that for all but finitely many e the operator equals id . Then one can complete to a C^* -algebra and construct a good theory of observables. But there is no infinite analog of (20.6) since there is no distinguished vector in $\mathcal{H}_e$. Therefore, one does not construct states directly from a Hilbert space, as we do in the finite model, but rather one constructs states as linear functionals on the algebra of complex observables, as in Definition 11.27.
- (2) The tensor product decomposition (20.6) expresses a locality of states in space. There is an analog in quantum field theory, but one does not have empty space between the sites of the local states and so the decomposition is more subtle. Locality of states is cleanest in *topological* field theories, as we shall see next semester, but even there the state space is not a simple tensor product as in (20.6).

Definition 20.9. Let $\mathcal{U} \subset Y$ be an open set in Y . An operator in (20.7) has *support* in $\mathcal{U}$ if it lives in the subalgebra

$$(20.10) \quad \mathcal{A}(\mathcal{U}) = \bigotimes_{e \in \mathcal{U}} \text{End } \mathcal{H}_e.$$

The embedding $\mathcal{A}(\mathcal{U}) \subset \text{End } \mathcal{H}$ is achieved by tensoring with $\text{id} \in \text{End } \mathcal{H}_{e'}$ for all $e' \notin \mathcal{U}$. The algebra $\mathcal{A}(\mathcal{U})$ is the complex algebra of observables on the open set $\mathcal{U}$.

The assignment $\mathcal{U} \mapsto \mathcal{A}(\mathcal{U})$ has the following properties:

- (A) $\mathcal{A}(\mathcal{U})$ is an algebra
- (B) $\mathcal{A}$ is a cosheaf
- (C) If $\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$, then $\mathcal{A}(\mathcal{U}_1 \cup \mathcal{U}_2) = \mathcal{A}(\mathcal{U}_1) \otimes \mathcal{A}(\mathcal{U}_2)$; in particular, $\mathcal{A}(\mathcal{U}_1)$ and $\mathcal{A}(\mathcal{U}_2)$ commute

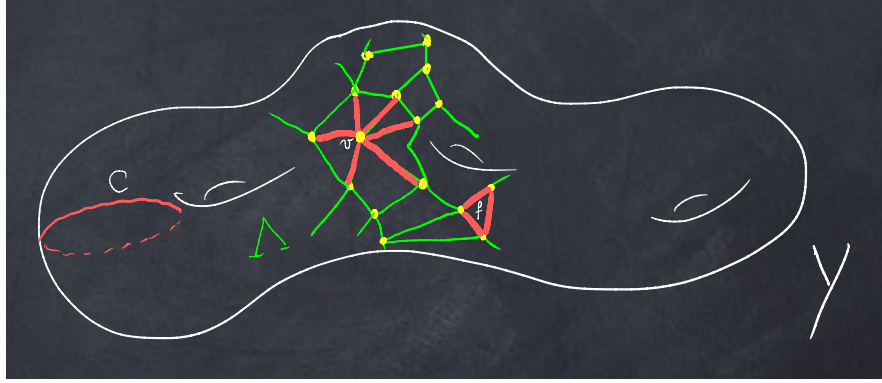


FIGURE 57. The support of point observables H_v, H_f and line observables H_C

The toric code has fundamental observables H_v, H_f associated to a vertex v and a face f ; they are defined in (5.24). Their supports are depicted in Figure 57. From a distance—if the lattice $\Lambda \subset Y$ is “thick”—these supports appear pointlike: the support consists of edges adjacent to a single vertex or a single face. The vacuum subspace $\mathcal{H}_0 \subset \mathcal{H}$, which is the kernel of the toric code Hamiltonian (5.25), is the simultaneous kernel of the operators H_v, H_f over all vertices and faces. It is isomorphic to the vector space of complex-valued functions on $\pi_0 \mathcal{D}(Y)$, the set of isomorphism classes of double covers of the surface Y . Typically it has dimension > 1 —the vacuum is not unique—and clearly the *point observables* or *point observables* H_v, H_f do not separate vacua. Therefore, we introduce *line observables*: observables whose support is a 1-dimensional submanifold.⁶² Let $C \subset Y$ be a closed loop, and define

$$(20.11) \quad h_C: \mathcal{D}(\Lambda, \Lambda^0) \longrightarrow \{0, 1\}$$

to be the function that returns the isomorphism type of the restriction of a double cover to C . Multiplication by h_C is a self-adjoint operator H_C . These observables, taken over all $C \subset Y$, separate vacua in the toric code.

Remark 20.12. The line observables H_C are *topological* when restricted to vacua. More precisely, they are invariant under deformation of C , in fact under homologies of C . This is another indication that the low energy, or vacuum, “sector” of the toric code is a topological theory; see Remark 5.28(2).

20.3 Causal structure of spacetime

In Minkowski spacetime M the analog of property (C) above depends on the causal structure of spacetime. Introduce the notation $p \perp q$ for spacelike separated points $p, q \in M$.

⁶²In general, observables can have support on a stratified manifold.

Definition 20.13. Let $\mathcal{U} \subset M$ be a subset.

(1) The *causal complement* to $\mathcal{U}$ is

$$(20.14) \quad \mathcal{U}' = \{q \in M : p \perp q \text{ for all } p \in \mathcal{U}\}.$$

(2) The *causal completion* of $\mathcal{U}$ is $\mathcal{U}''$.

(3) $\mathcal{U}$ is *causally complete* if $\mathcal{U}'' = \mathcal{U}$.

(4) Subsets $\mathcal{U}_1, \mathcal{U}_2 \in M$ are *spacelike separated* if $\mathcal{U}_1 \subset \mathcal{U}_2'$ or equivalently $\mathcal{U}_2 \subset \mathcal{U}_1'$. In that case we write $\mathcal{U}_1 \perp \mathcal{U}_2$.

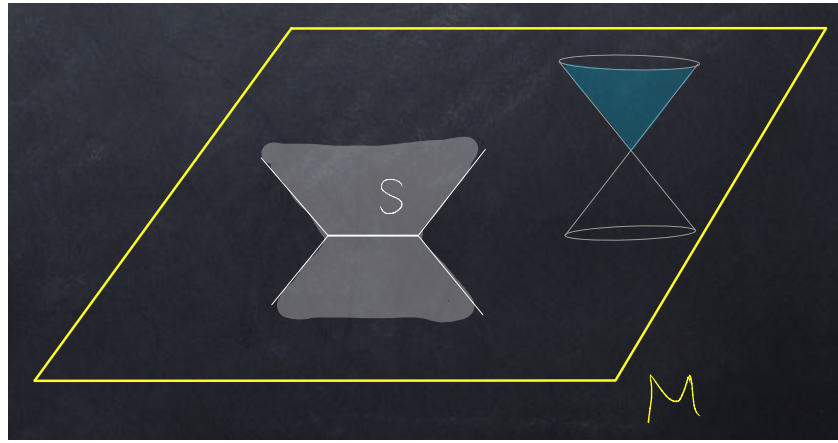


FIGURE 58. A causally complete region S in Minkowski spacetime M

It follows immediately from (20.14) that $\mathcal{U} \subset \mathcal{U}''$ and that $\mathcal{U}' = \mathcal{U}'''$, i.e., $\mathcal{U}'$ is causally complete.

There are parallel definitions for algebras. Let $\mathcal{A}$ be an algebra. Introduce the notation $a \perp b$ for commuting elements $a, b \in \mathcal{A}$.

Definition 20.15. Let $\mathcal{A}$ be an algebra and $S \subset \mathcal{A}$ a subset.

(1) The *commutant* of S is

$$(20.16) \quad S' = \{b \in \mathcal{A} : a \perp b \text{ for all } a \in S\}.$$

(2) Subsets $S_1, S_2 \in \mathcal{A}$ *commute* if $S_1 \subset S_2'$ or equivalently $S_2 \subset S_1'$. In that case we write $S_1 \perp S_2$.

There are analogs of Definition 20.13(2,3) as well, though I do not know a term for the double commutant S'' of a subset $S \subset \mathcal{A}$. Note that $S \subset S''$ and $S' = S'''$.

If $\mathcal{A} = \text{End } \mathcal{H}$ is the $*$ -algebra of bounded operators in a Hilbert space $\mathcal{H}$, then the commutant S' of a subset $S \subset \text{End } \mathcal{H}$ is a *von Neumann algebra*. The algebra S'' is the *von Neumann closure* of S . Recall (§11.4) that a C^* -algebra is closed in the norm topology. A von Neumann algebra satisfies a stronger condition: it is closed in the weak topology. For basics on von Neumann algebras, including the von Neumann double commutant theorem used here implicitly, see [Jo].

20.4 Local observables in Minkowski spacetime

The perspective here is due to Haag–Kastler; it is often called *algebraic quantum field theory*. The original reference is [HK] and there are many since. The book [Ha] provides a thorough account. The recent paper [Ded] contains a summary as well as extensive references.⁶³ Our account makes no pretensions to completeness. Also, we begin with a relativistic quantum mechanical system—a positive energy unitary representation—and take local observables to be subalgebras of global observables, whereas typically the net of local observables takes values in abstract C^* -algebras.

Resume the setup at the beginning of the lecture: a Minkowski spacetime M , a relativistic symmetry type $\lambda: G \rightarrow O^\uparrow(V)$, and the rest. In the following definition the subsets $\mathcal{U} \subset M$ are sometimes required to be causally closed; one can use axiom (E) to reduce to that case.

Definition 20.17. Suppose $\rho: \mathcal{G} \rightarrow \text{Aut } \mathcal{H}$ is a relativistic quantum mechanical system on M . A *net of local observables* for ρ is an assignment

$$(20.18) \quad \mathcal{U} \longmapsto \mathcal{A}(\mathcal{U})$$

to each bounded open subset $\mathcal{U} \subset M$ of a subalgebra $\mathcal{A}(\mathcal{U}) \subset \text{End } \mathcal{H}$ such that:

- (A) $\mathcal{A}(\mathcal{U})$ is a C^* -algebra
- (B) If $\mathcal{U}_1 \subset \mathcal{U}_2$, then $\mathcal{A}(\mathcal{U}_1) \subset \mathcal{A}(\mathcal{U}_2)$
- (C) If $\mathcal{U}_1 \perp \mathcal{U}_2$, then $\mathcal{A}(\mathcal{U}_1) \perp \mathcal{A}(\mathcal{U}_2)$
- (D) $\mathcal{A}$ is $\mathcal{G}$ -invariant: $\mathcal{A}(\lambda(g)\mathcal{U}) = \rho(g)\mathcal{A}(\mathcal{U})\rho(g)^{-1}$, $g \in \mathcal{G}$
- (E) $\mathcal{A}(\mathcal{U}'') = \mathcal{A}(\mathcal{U})$

Note that (C) is equivalent to $\mathcal{A}(\mathcal{U}') \subset \mathcal{A}(\mathcal{U})'$. The net satisfies *Haag duality* if

$$(20.19) \quad \mathcal{A}(\mathcal{U}') = \mathcal{A}(\mathcal{U})',$$

perhaps for some restricted class of $\mathcal{U}$. For those opens $\mathcal{U}$ that satisfy (20.19), the C^* -algebra $\mathcal{A}(\mathcal{U}) = \mathcal{A}(\mathcal{U}'') = \mathcal{A}(\mathcal{U})''$ is a von Neumann algebra.

Lecture 21: Fields and local observables: the Wightman picture

We began our study of relativistic quantum systems in [Lecture 14](#) with a straightforward generalization ([Definition 14.16](#)) of a quantum mechanical system, namely a positive energy unitary representation of the Poincaré group (or of a more general symmetry group). In [Lecture 20](#) we introduced ([Definition 20.17](#)) local theories of observables à Haag–Kastler. Recall that we introduced a general concept of fields in [Lecture 18](#), and sketched there how to construct a relativistic quantum mechanical system by quantizing a classical theory of fields. In this lecture we complete our survey of relativistic quantum systems by constructing local observables from fields, à la Wightman.

⁶³The paper surveys various attempts to axiomatize quantum field theory.

We begin by reprising the massive scalar field. First, in the *classical* theory we introduce commutative algebras of local observables defined as functions of fields. There are classical correlation functions of these observables. Free field quantization promotes the classical observables to quantum observables, but to get well-defined operators we smear against test functions. These *field operators* satisfy causality: they commute if the smearing functions have spacelike separated supports.

We defer to references, such as [SW, Ha, Kaz] for detailed accounts of the Wightman theory. In this lecture we sketch the crucial consequence of the positive energy hypothesis: correlation functions in Minkowski spacetime are boundary values of holomorphic functions in some complex domain. Furthermore, in that domain we can restrict to correlation functions on Euclidean space. *Wick rotation* is the process of passing from Minkowski spacetime to Euclidean space, here for correlation functions. In the next lecture Wick rotation of the representation of the Poincaré group leads us to the Segal axioms for quantum field theory. We conclude this lecture by commenting on Wick rotation of relativistic symmetry types and the reflection positivity property that is the Wick rotation of unitarity.

There is much more to say about relativistic quantum field theory, of course, both from the more mathematical point of view—results such as the Reeh–Schlieder theorem, explained in [W2], for example—and from real physics, for which there are many standard texts, e.g. [Wei2, Wei3, Wei4]. There is also a whole literature of constructions of quantum field theories in the Wightman framework; see [GJ, Ja] for an entrée. A major challenge is to construct 4-dimensional Yang–Mills theory (Conjecture 19.28); see [Ch] for a recent status update and survey of the literature.

21.1 Classical massive scalar field redux

This theory was discussed in Lecture 18. We work on a Minkowski spacetime M over a Lorentzian vector space V of dimension n . Real scalar fields form a sheaf

$$(21.1) \quad \begin{aligned} \mathcal{F}: \mathbf{Open}(M)^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ \mathcal{U} &\longmapsto C^\infty(\mathcal{U}, \mathbb{R}) \end{aligned}$$

and classical solutions comprise the subsheaf $\mathcal{M} \subset \mathcal{F}$ of real-valued functions that satisfy the Klein–Gordon equation

$$(21.2) \quad \left(\square + \frac{c^2}{\hbar^2} m^2 \right) \phi = 0.$$

The global solutions $\mathcal{M}(M)$ to the Klein–Gordon equation form a symplectic vector space, from which we construct the complex algebra

$$(21.3) \quad \mathcal{A}^{\mathrm{cl}}(M) = C^\infty(\mathcal{M}(M), \mathbb{C}).$$

This is a commutative $*$ -algebra; its real points are the *global* observables. Following (20.1) we define *local* observables on an open subset $\mathcal{U} \subset M$:

$$(21.4) \quad \mathcal{A}^{\mathrm{cl}}(\mathcal{U}) = C^\infty(\mathcal{M}(\mathcal{U}), \mathbb{C}).$$

The definitions (21.3) and (21.4) apply to any classical field theory. For the real scalar field, a typical local observable is O_p^{cl} for $p \in \mathcal{U} \subset M$:

$$(21.5) \quad O_p^{\text{cl}}(\phi) = \phi(p), \quad \phi \in \mathcal{M}(\mathcal{U}).$$

A “smeared out” variant attaches to a real-valued function $f \in C_c^\infty(\mathcal{U})$ of compact support in $\mathcal{U}$ the local observable

$$(21.6) \quad O_f^{\text{cl}}(\phi) = \int_{\mathcal{U}} f(p) \phi(p) d\mu_M(p), \quad \phi \in \mathcal{M}(\mathcal{U}),$$

where μ_M is the $O(M)$ -invariant measure constructed from the Lorentz metric. Both the “sharp” observable O_p^{cl} and the smeared out observable O_f^{cl} are well-defined in the classical theory.

Recall that a *state* in the classical theory is a probability measure ν on $\mathcal{M}(M)$. (A pure state is a point measure supported at some $\phi_0 \in \mathcal{M}(M)$.) Fix a state ν and fix distinct points $p_1, \dots, p_k \in \mathcal{U} \subset M$. Then the classical *correlation function* of the local observables $O_{p_1}^{\text{cl}}, \dots, O_{p_k}^{\text{cl}}$ is

$$(21.7) \quad \langle O_{p_1}^{\text{cl}} \cdots O_{p_k}^{\text{cl}} \rangle_\nu = \int_{\mathcal{M}(M)} O_{p_1}^{\text{cl}}(\phi) \cdots O_{p_k}^{\text{cl}}(\phi) d\nu(\phi).$$

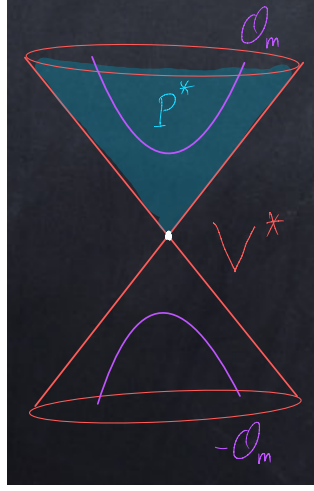
For fixed ν this is a C^∞ function of $p_1, \dots, p_k \in \mathcal{U}$. If ν is the pure state at $\phi_0 \in \mathcal{M}(M)$, then the correlation function reduces to

$$(21.8) \quad \langle O_{p_1}^{\text{cl}} \cdots O_{p_k}^{\text{cl}} \rangle_{\phi_0} = \phi_0(p_1) \cdots \phi_0(p_k).$$

Remark 21.9. Our comment that “for fixed ν this is a C^∞ function of $p_1, \dots, p_k \in \mathcal{U}$ ” indicates that we must consider correlation functions as defined in families, and we should axiomatize the smoothness in parameters. This is clear in the Wightman theory, but is not usually formulated explicitly in other axiom systems, such as the Segal axioms we come to soon.

21.2 Field operator of the quantum massive scalar field

Recall the quantization (19.10) of the massive scalar field. The procedure is to fix a basepoint $p_0 \in M$, identify $M \approx V$, and then Fourier transform to identify $\mathcal{M}(M) \otimes \mathbb{C}$ as a vector space $W_{\mathbb{C}}$ of complex-valued functions on $\mathcal{O}_m \cup -\mathcal{O}_m \subset V^*$, as in Figure 59. There is a natural polarization (19.4) by positive energy: let $W_+ \subset W_{\mathbb{C}}$ be the subspace of functions supported on $\mathcal{O}_m$. The 1-particle Hilbert space $\mathcal{H}_1$ is the Hilbert space completion of W_+^* , which is the space of continuous linear functionals on $L^2(\mathcal{O}_m; \mathbb{C})$. The state space $\mathcal{H}$ of the theory is the Hilbert space completion of $\text{Sym}^\bullet \mathcal{H}_1$. Furthermore, under quantization (19.7), continuous linear functionals on $L^2(\mathcal{O}_m; \mathbb{C})$ act on $\mathcal{H}$ as multiplication operators, whereas continuous linear functionals on $L^2(-\mathcal{O}_m; \mathbb{C})$ act on $\mathcal{H}$ as contraction operators.

FIGURE 59. Polarization of $\mathcal{M}(M)$

Can one quantize O_p^{cl} to an operator O_p on $\text{Sym}^\bullet \mathcal{H}_1$? To investigate, rewrite the functional $\phi \mapsto \phi(p)$ in (21.5) as a functional of the Fourier transform $\hat{\phi}$ using the inverse Fourier transform (18.28):

$$(21.10) \quad O_p^{\text{cl}}: \hat{\phi} \mapsto \frac{1}{(2\pi)^{n/2}} \int_{V^*} d\theta e^{-i\theta(\xi)/\hbar} \hat{\phi}(\theta),$$

where $p = p_0 + \xi$ for $\xi \in V$. The integral reduces to an integral over $\mathcal{O}_m \cup -\mathcal{O}_m$ since $\hat{\phi}$ is supported there. But there is already an issue in the definition (21.5), even before passing to the Fourier transform: O_p^{cl} is not well-defined on L^2 functions. Neither, then, is (21.10). So O_p^{cl} has no good quantization. Consider instead the smeared operator (21.6), where f has compact support. Since an L^2 function $f\phi$ with compact support is also L^1 , the integral in (21.6) is well-defined. As a functional of the Fourier transform, the classical observable O_f^{cl} is

$$(21.11) \quad O_f^{\text{cl}}: \hat{\phi} \mapsto \int_{V^*} d\theta \hat{f}(-\theta) \hat{\phi}(\theta).$$

Decompose $\int_{V^*} = \int_{\mathcal{O}_m} + \int_{-\mathcal{O}_m}$ and so decompose $O_f^{\text{cl}} = O_f^{\text{cl},+} + O_f^{\text{cl},-}$. Then $O_f^{\text{cl},+}$ is a continuous linear functional on $L^2(\mathcal{O}_m; \mathbb{C})$, i.e., $O_f^{\text{cl},+}$ lies in the 1-particle Hilbert space $\mathcal{H}_1$. The corresponding quantum observable O_f^+ acts on the Fock space $\text{Sym}^\bullet \mathcal{H}_1$ as a multiplication operator. It extends to an unbounded operator on the Hilbert space completion $\mathcal{H}$. Similarly, $O_f^{\text{cl},-}$ is a continuous linear functional on $L^2(-\mathcal{O}_m; \mathbb{C})$, i.e., $O_f^{\text{cl},-}$ lies in the complex conjugate space $\overline{\mathcal{H}_1}$. The corresponding quantum observable O_f^- acts on the Fock space $\text{Sym}^\bullet \mathcal{H}_1$ as a contraction operator, and it too extends to an unbounded operator on the Hilbert space completion $\mathcal{H}$. The sum $O_f = O_f^+ + O_f^-$ is the *quantum field operator* that corresponds to the classical observable O_f^{cl} .

Remark 21.12.

- (1) The smearing by compactly supported functions in $\mathcal{U}$ is a general feature of field operators in quantum field theory. The operators obtained are *unbounded*, so are only defined on a dense

domain. Hence they do not fit directly into the Haag–Kastler framework (Definition 20.17) of *bounded* operators. Rather, apply the spectral theorem to construct bounded operators from the unbounded field operators.

- (2) The quantum field is an operator-valued distribution: the function f is a “test function”.

21.3 Wightman functions and Wick rotation

Continue with a Minkowski spacetime M over a Lorentzian vector space V of dimension n , and assume given as usual a cone P of forward timelike vectors, a speed of light c , and a relativistic symmetry type $\lambda: G \rightarrow \mathrm{O}^\uparrow(V)$; the latter induces an affine symmetry type $\lambda: \mathcal{G} \rightarrow \mathrm{O}^\uparrow(M)$. The basic data of a Wightman quantum field theory is:

- a positive energy unitary representation⁶⁴ $U: \mathcal{G} \rightarrow \mathrm{U}(\mathcal{H})$
- $\mathcal{D} \subset \mathcal{H}$ a $\mathcal{G}$ -invariant dense subspace
- A $\mathcal{G}$ -invariant vector $\Omega \in \mathcal{D}$
- an $\mathrm{End} \mathcal{D}$ -valued distribution on M

The vector Ω is called the *vacuum vector*,⁶⁵ and the operator-valued distribution on M is the *field*. It is defined in terms of a real linear representation R of $\mathrm{Spin}(V)$ and R^* -valued test functions; see [Kaz] for details. Here, for ease of notation, take $R = \mathbb{R}$; this choice applies to the scalar field. Let $\mathcal{O}_f$ denote the observable attached to the test function f (of compact support). The Wightman “function” is a distribution on $M^{\times k}$:

$$(21.13) \quad \mathcal{W}(f_1 \times \cdots \times f_k) = \langle \Omega, \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_k} \Omega \rangle.$$

It has a Schwartz kernel $\mathcal{W}_k(p_1, \dots, p_k)$:

$$(21.14) \quad \mathcal{W}(f_1 \times \cdots \times f_k) = \int_{M^{\times k}} d\mu_M(p_1) \cdots d\mu_M(p_k) \mathcal{W}_k(p_1, \dots, p_k) f_1(p_1) \cdots f_k(p_k).$$

Invariance under V acting on $M^{\times k}$ acting diagonally implies that $\mathcal{W}_k$ descends to a distribution W_k on the quotient affine space $M^{\times k}/V$, which—after a choice—is identified with the vector space $V^{\times(k-1)}$. The Fourier transform $\widehat{W}_k$ is a distribution on $(V^*)^{\times(k-1)}$. Define the *backward tube* as the open subset

$$(21.15) \quad \mathcal{T} = V - iP \subset V_{\mathbb{C}}$$

of the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ of V .

Lemma 21.16. $\mathrm{supp} \widehat{W}_k \subset (\overline{P^*})^{\times(k-1)}$.

Corollary 21.17. W_k is the boundary value of a homomorphic function on $\mathcal{T}^{\times(k-1)}$.

⁶⁴One should take $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ to allow for fermionic states.

⁶⁵The vacuum is a state, the line $\mathbb{C}\langle\Omega\rangle$.

The support condition on $\widehat{W}_k$ is a consequence of the positivity of energy of the representation U , as we now sketch. The corollary is standard from there.

To give an idea of the proof, consider $k = 2$ for convenience of notation. Then for any $p \in M$ and $\xi \in V$ we compute (heuristically with distributions—use test functions!)

$$\begin{aligned}
 W_2(\xi) &= \langle \Omega, \phi(p)\phi(p + \xi)\Omega \rangle \\
 &= \langle \phi(p)\Omega, \phi(p + \xi)\Omega \rangle \\
 (21.18) \quad &= \langle \phi(p)\Omega, U(\xi)\phi(p)U(\xi)^{-1}\Omega \rangle \\
 &= \langle \phi(p)\Omega, U(\xi)\phi(p)\Omega \rangle \\
 &= \langle \Psi, U(\xi)\Psi \rangle,
 \end{aligned}$$

where $\Psi = \phi(p)\Omega \in \mathcal{H}$. To pass to the second line use the self-adjointness of the real operator $\phi(p)$, and to pass to the fourth line use the V -invariance of the vacuum vector Ω . The representation U , restricted to the translation subgroup $V \subset \mathcal{G}$, has a spectral representation

$$(21.19) \quad U(\xi) = \int_{V^*} e^{-i\theta(\xi)/\hbar} d\pi(\theta),$$

where π is a self-adjoint projection-valued measure on V^* with support on $\overline{P^*}$. Hence

$$(21.20) \quad W_2(\xi) = \int_{V^*} e^{-i\theta(\xi)/\hbar} \langle \Psi, d\pi(\theta)\Psi \rangle.$$

In other words, the measure $\langle \Psi, d\pi(\theta)\Psi \rangle$ is the Fourier transform of W_2 at $\theta \in V^*$, and it vanishes unless $\theta \in \overline{P^*}$ since $\text{supp } \pi \subset \overline{P^*}$.

For the corollary, observe that the character $\xi \mapsto e^{-i\theta(\xi)/\hbar}$ of V is the boundary value of a holomorphic function on $\mathcal{T} = V - iP$ if $\theta \in \overline{P^*}$. (The crucial estimate is (11.12).)

The holomorphic function W_2 extends to the larger domain

$$(21.21) \quad D_2 = \text{SO}_n(\mathbb{C})(\mathcal{T}) \cup -\text{SO}_n(\mathbb{C})(\mathcal{T}) \subset V_{\mathbb{C}}.$$

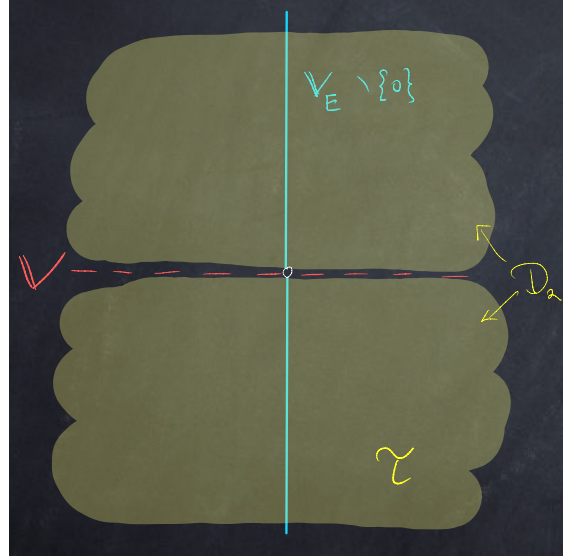
The domain D_2 in the case $n = 1$ (quantum mechanics) is illustrated in Figure 60. There are analogous domains D_k for all $k \in \mathbb{Z}^{\geq 2}$ and in all dimensions. They are defined in [SW, §3-3] [Kaz, §2.1], for example, and the following lemma and its extension to higher k is proved. Part (1) of the lemma is due to Jost.

Lemma 21.22. *If $n \geq 3$, then*

- (1) *any real spacelike $\xi \in V$ lies in D_2 , and*
- (2) *D_2 is connected.*

Note from Figure 60 that both parts of Lemma 21.22 fail for $n = 1$. There is a variant for $n = 2$ in [SW, §3-3].

Here is a brief enumeration of some important theorems in the Wightman theory. Two of the most important have already been stated in §15.2. A thorough treatment may be found in [SW, Kaz].

FIGURE 60. The domain D_2 for $n = 1$

- (1) A Wightman quantum field theory can be reconstructed from the Wightman functions $\{\mathcal{W}_k\}$.
- (2) W_k extends to a holomorphic function $W_k^{\mathbb{C}}$ on a domain $D_k \subset V_{\mathbb{C}}^{\times(k-1)}$. Both D_k and $W_k^{\mathbb{C}}$ are $G(\mathbb{C})^{\times(k-1)}$ -invariant, where $G(\mathbb{C})$ is the complexification of the Lie group G .
- (3) Write $V = U \oplus U^{\perp}$ for a timelike line $U \subset V$, and set $V_E = \sqrt{-1}U \oplus U^{\perp} \subset V_{\mathbb{C}}$. The Lorentz inner product on V complexifies to a nondegenerate complex symmetric bilinear form on $V_{\mathbb{C}}$ which then restricts to a (negative) definite inner product on V_E . The case $n = 1$ is illustrated in Figure 60. The theorem is that

$$(21.23) \quad V_E^{\times(k-1)} \setminus \text{diagonals} \subset D_k.$$

The restriction of $W_k^{\mathbb{C}}$ to $V_E^{\times(k-1)}$ minus diagonals is the *Schwinger function*, also called the *Euclidean correlation function*.

The vector space V_E contains imaginary time translation iU . There is a Euclidean space E over V_E which is the Wick rotation of the Minkowski spacetime M .⁶⁶ The Euclidean correlation function on $V_E^{\times(k-1)}$ minus diagonals lifts to $E^{\times k}$ minus diagonals.

Osterwalder–Schrader [OS] axiomatized properties of the Euclidean correlation functions; see [Kaz, §2.2]. They proved an important reconstruction theorem. The Osterwalder–Schrader theorem recovers a Wightman field theory on Minkowski spacetime from a Euclidean field theory on Euclidean space.

The passage from quantum field theory on Minkowski spacetime to quantum field theory on Euclidean space is *Wick rotation*. In Section 11.3 we discussed the Wick rotation of the positive energy unitary representation of time translation. Here we have Wick-rotated correlation functions. In doing so we pass from Minkowski spacetime to a complex domain to Euclidean space:

$$(21.24) \quad M \rightsquigarrow D_k \rightsquigarrow E$$

⁶⁶We leave the reader to use standard models to systematize this Wick rotation.

There is an additional passage $E \rightsquigarrow X$ from Euclidean space to a Riemannian manifold that we take up soon.

Remark 21.25. Fields in Wightman theory are assumed to be linear; they are functions on Minkowski spacetime with values in a real vector space R , as defined before (21.13). But there fields in σ -models and in gauge theories are not of this type. We gave a very general definition of a field in [Lecture 18](#), and it is this general notion that will be used in the axiom system for Wick-rotated field theory on compact manifolds. There are other limitations of the Wightman theory.

21.4 Relativistic symmetry types and Wick rotation

Recall ([Definition 15.13](#)) that a *relativistic symmetry type* is a homomorphism⁶⁷ $\lambda: G_n \rightarrow O_{1,n-1}^\uparrow$ whose image contains the identity component $SO_{1,n-1}^\uparrow$. The kernel K of λ is the group of internal symmetries. Recall from ([17.29](#)) that there is a corresponding affine symmetry type $\tilde{\lambda}: \mathcal{G}_n \rightarrow \mathcal{I}_{1,n-1}^\uparrow$.

Wick rotation of λ is encoded in the diagram

$$(21.26) \quad \begin{array}{ccccccc} 1 & \longrightarrow & K & \longrightarrow & G_n & \xrightarrow{\lambda} & O_{1,n-1}^\uparrow \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & K(\mathbb{C}) & \longrightarrow & G_n(\mathbb{C}) & \longrightarrow & O_n(\mathbb{C}) \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & K & \longrightarrow & G_n^E & \xrightarrow{\lambda^E} & O_n \end{array}$$

The top row is the relativistic symmetry type; the bottom row is the Wick-rotated *Euclidean symmetry type*. Wick rotation passes through the complex Lie group $G_n(\mathbb{C})$, which acts as symmetries of the holomorphic correlation functions $W_k^{\mathbb{C}}$. Also, the internal symmetry group K does not change under Wick rotation: it is the same in the top and bottom rows.

If K is a compact Lie group, then so too is G_n^E . The rigidity of Lie groups leads to a structure theory for these relativistic symmetry types and their Wick rotations [[FH1](#), §2]. A few highlights are expressed in the following.

Theorem 21.27. *Suppose $n \geq 3$.*

- (1) *The pullback of G_n^E over Spin_n splits into a direct product $\text{Spin}_n \times K$.*
- (2) *The Lie algebra $\mathfrak{g}$ of G splits as a direct sum $\mathfrak{g} = \mathfrak{so}_{1,n-1} \times \mathfrak{k}$.*
- (3) *There exists $k_0 \in K$ of order dividing 2 such that the pullback of G_n^E over SO_n is*

$$(21.28) \quad \text{Spin}_n \times K / \langle (\epsilon, k_0) \rangle.$$

⁶⁷We have often used the orthogonal group of a specific Lorentz vector space, but as here we use the model space $\mathbb{R}^{1,n-1}$ for a symmetry *type* and then we can transport using spaces associated to right torsors; recall §7.4.

The pullback, say over Spin_n , means pullback in the diagram

$$(21.29) \quad \begin{array}{ccc} ??? & \dashrightarrow & \text{Spin}_n \\ \downarrow & & \downarrow \\ G_n^E & \xrightarrow{\lambda^E} & O_n \end{array}$$

Remark 21.30. The splittings in [Theorem 21.27](#) are a version of the Coleman–Mandula theorem [\[CM\]](#), but from a different perspective.

There are generalizations of the spin-statistics theorem ([Theorem 15.4](#)) and the CRT theorem ([Theorem 15.6](#)) to general relativistic symmetry types. We assume both bosonic and fermionic states. The general spin-statistics theorem is as follows.

Theorem 21.31. *Consider a Wightman quantum field theory which includes the positive energy unitary representation $U: \mathcal{G} \rightarrow \text{U}(\mathcal{H}^0 \oplus \mathcal{H}^1)$. Let $k_0 \in K$ be the special element in [Theorem 21.27\(3\)](#). Then $U(k_0)$ is the grading operator*

$$(21.32) \quad \text{id}_{\mathcal{H}^0} \oplus -\text{id}_{\mathcal{H}^1}$$

on $\mathcal{H}^0 \oplus \mathcal{H}^1$.

The statement of the general CRT theorem is more complicated, and we defer to [\[FH1, Appendix A\]](#) for a detailed discussion and proof.

21.5 Euclidean quantum field theory

This is the field theory of Osterwalder–Schrader mentioned earlier. We work on a Euclidean space E over an inner product space V_E of dimension n . The Euclidean symmetry type is a group homomorphism $\lambda^E: G_n^E \rightarrow O_n$ such that the image of λ^E is either SO_n or O_n . We might also assume that G_n^E is a *compact* Lie group. (As in the relativistic case, it could be a super group, a homotopical group, etc.) There is a corresponding affine group $\mathcal{G}_n^E$ which acts on Euclidean space:

$$(21.33) \quad \begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ & & K & \xlongequal{\quad} & K & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & V_E & \longrightarrow & \mathcal{G}_n^E & \longrightarrow & G_n^E \longrightarrow 1 \\ & & \parallel & & \downarrow \tilde{\lambda}^E & & \downarrow \lambda^E \\ 1 & \longrightarrow & V_E & \longrightarrow & \text{Euc}_n & \longrightarrow & O_n \longrightarrow 1 \end{array}$$

There is no unitary representation of $\mathcal{G}_n^E$, though a subsemigroup of V_E acts by contractive operators. There are observables and correlation functions, the Schwinger functions mentioned above. And

there is a local theory of observables: a vector space $\mathcal{A}(\mathcal{U})$ for open sets $\mathcal{U} \subset E$. One of the key properties—a Wick-rotated version of unitarity—is *reflection positivity*. As in [Figure 61](#), let $\Pi \subset E$ be an affine hyperplane, and let $\mathcal{U} \subset E$ be an open set supported on one side of Π . Define $\bar{\mathcal{U}} \subset E$ to be the reflection of $\mathcal{U}$ in Π . If $\mathcal{O} \in \mathcal{A}(\mathcal{U})$ is an observable in $\mathcal{U}$, then there is a reflected observable $\bar{\mathcal{O}} \in \bar{\mathcal{U}}$. Reflection positivity is the statement

$$(21.34) \quad \langle \bar{\mathcal{O}} \mathcal{O} \rangle \geq 0.$$

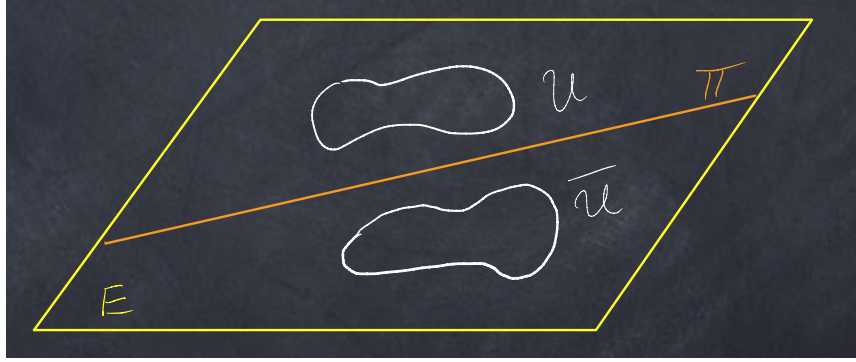


FIGURE 61. Reflection positivity

For a heuristic derivation of [\(21.34\)](#) from the path integral, see [\[Detal, Problem FP19\]](#).

Lecture 22: More Wick rotation; mathematical background

Our first encounter with Wick rotation was in quantum mechanics in [Lecture 11](#). We showed how positivity of energy leads to a 1-parameter holomorphic semigroup of contractive operators such that time evolution is a boundary limit. Then restriction to a real 1-parameter semigroup is the Wick-rotated representation [\(11.15\)](#) of time translation. In relativistic quantum field theory ([Lecture 21](#)) we Wick-rotated correlation functions rather than the unitary representation of the symmetry group. Here we begin by recounting Segal's Wick rotation of the positive energy unitary representation of translations in dimension 2, but for quantum field theory in general, not just for conformal field theory. The first move is to compactify space, which is a step toward compactifying Wick-rotated time as well, a step we take in the next lecture.

Then we turn to mathematical background for the next lecture. The first topic is the passage from affine geometries to geometries on smooth manifolds, à la Cartan. This may be regarded as a continuation of the discussion in [§7.4](#). The second topic is the notion of a symmetric monoidal structure on a category. In the next lecture we express a Wick-rotated theory as a homomorphism of such categories.

22.1 Wick rotation in 2 dimensions

Wick rotation of time evolution in quantum mechanics is summarized in [Figure 25](#). Time translations form the vector space $V = \mathbb{R}$ and are represented unitarily on a Hilbert space $\mathcal{H}$ by a representation $t \mapsto e^{-itH/\hbar}$, where H is the Hamiltonian. The positive energy assumption is that the spectrum of H is contained in ⁶⁸ $\overline{P^*} \subset V^* = \mathbb{R}^*$, where $\overline{P^*} = (\mathbb{R}^*)^{>0}$. Then there is a contractive holomorphic semigroup on $V - iP = \mathbb{C}_-$ whose boundary value on V is the unitary time evolution. Restriction to the real semigroup $-iP \cong P$ is the Wick-rotated evolution $\tau \mapsto e^{-\tau H/\hbar}$.

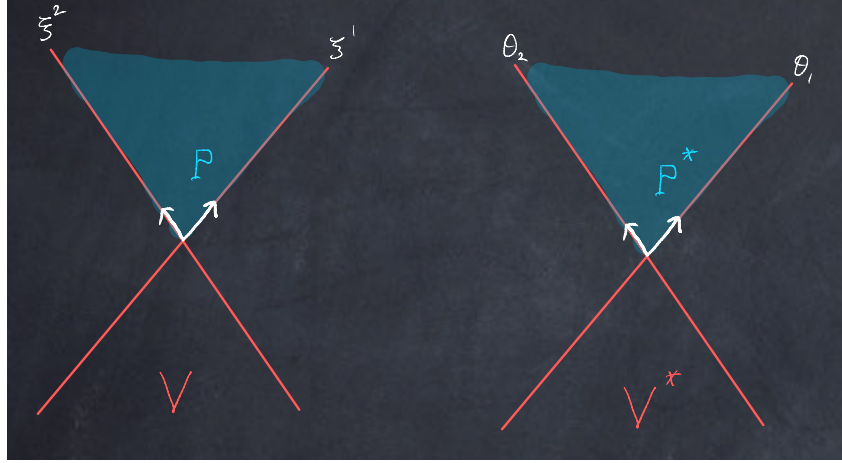


FIGURE 62. Translation group V , the forward timelike cone P , and their duals

Consider standard Minkowski spacetime $M = \mathbb{A}^2$ with coordinates t, x and Lorentz metric

$$(22.1) \quad ds^2 = c^2 dt^2 - dx^2,$$

where c is the speed of light. The translation group $V = \mathbb{R}^2$ acts, and we use lightcone coordinates ξ^1, ξ^2 relative to the lightlike basis of V indicated in [Figure 62](#). The positive timelike vectors (ξ^1, ξ^2) satisfy $\xi^1, \xi^2 > 0$. The dual space $V^* = (\mathbb{R}^2)^*$ has a dual basis, and the dual cone P^* is the set of vectors (θ_1, θ_2) with $\theta_1, \theta_2 > 0$.

Compactify space: identify $x \sim x + 2\pi$. The quotient of M by this discrete group of translations is diffeomorphic to $\mathbb{R} \times S^1$. It is a torsor over the group T which is the quotient of V defined by

$$(22.2) \quad \begin{aligned} 1 &\longrightarrow 2\pi\mathbb{Z} \longrightarrow V \longrightarrow T \longrightarrow 1 \\ 2\pi n &\longmapsto (\pi n, -\pi n) \end{aligned}$$

The Pontrjagin dual group $T^\vee$ of characters of T sits as the sub in the group extension

$$(22.3) \quad \begin{aligned} 1 &\longrightarrow T^\vee \longrightarrow V^* \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 1 \\ (\theta_1, \theta_2) &\longmapsto \frac{\theta_1 - \theta_2}{2} \pmod{\mathbb{Z}} \end{aligned}$$

⁶⁸Of course $\mathbb{R}^* = \mathbb{R}$, but we use the $*$ to emphasize that $\mathbb{R}^*$ is the space of smooth unitary characters of $\mathbb{R}$.

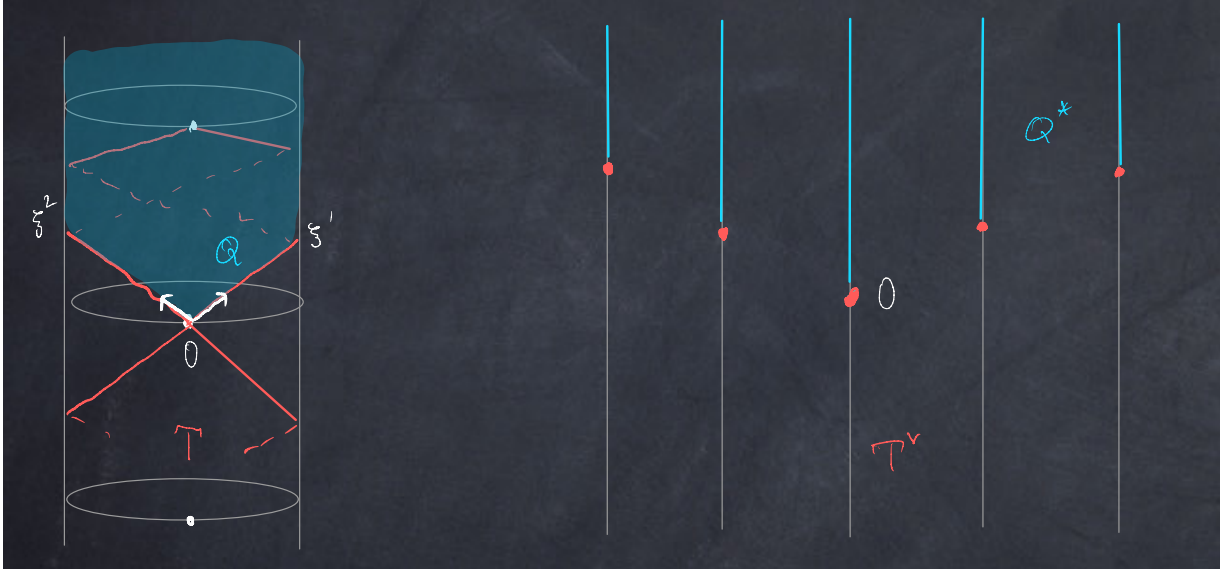
FIGURE 63. Translation group T , the forward timelike cone Q , and their duals

Figure 63 depicts T and $T^\vee$. Each has a subsemigroup of positive elements, which in these coordinates are

$$(22.4) \quad \begin{aligned} Q &= \{(\xi^1, \xi^2) : \xi^1, \xi^2 > 0\} \\ Q^* &= \{(\theta_1, \theta_2) : \theta_1, \theta_2 > 0\} \end{aligned}$$

Compactification of space leads to discretization of momentum.

The crucial point is that a character in $\overline{Q^*}$ is the boundary value of a holomorphic semicharacter of the complex semigroup $T - iQ$. The restriction to the real semigroup $-iQ \cong Q$ is the semicharacter

$$(22.5) \quad \begin{aligned} Q &\longrightarrow \mathbb{R}^{>0} \\ (\tau^1, \tau^2) &\longmapsto e^{-(\theta_1 \tau^1 + \theta_2 \tau^2)/\hbar} \end{aligned}$$

A 2-dimensional positive energy unitary representation of T has spectrum supported in $\overline{Q^*}$, so it has a Wick rotation which is a contractive real semigroup of operators parametrized by Q .

22.2 Cartan: from symmetry types to structures on manifolds

For an introduction to Cartan's theory of G -structures, see [Che, Sb]. We barely touch the surface of this beautiful theory.

Definition 22.6. An n -dimensional *symmetry type* is a pair (G_n, λ_n) in which G_n is a Lie group and $\lambda_n: G_n \rightarrow \mathrm{GL}_n \mathbb{R}$ is a homomorphism of Lie groups. The kernel $\ker(\lambda_n)$ is called the *internal symmetry group*.

We neither assume that λ_n is injective nor that λ_n is surjective. (Traditionally, Cartan's notion of G -structure requires that λ_n be injective.) A symmetry type induces the action $\tilde{\lambda}_n$ of a group of

affine symmetries of the model space $\mathbb{A}^n$ via pullback from the group Aff_n of affine symmetries:

$$(22.7) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{R}^n & \dashrightarrow & \mathcal{G}_n & \dashrightarrow & G_n \longrightarrow 1 \\ & & \parallel & & \downarrow \tilde{\lambda}_n & & \downarrow \lambda_n \\ 1 & \longrightarrow & \mathbb{R}^n & \longrightarrow & \text{Aff}_n & \longrightarrow & \text{GL}_n \mathbb{R} \longrightarrow 1 \end{array}$$

In terms of the Erlangen program (Definition 7.45), (G_n, λ_n) -geometry is the geometric type $\mathcal{G}_n \curvearrowright \mathbb{A}^n$. In other words, (G_n, λ_n) -geometry is the study of structures/properties/quantities in the model affine space $\mathbb{A}^n$ that are invariant under the $\mathcal{G}_n$ -action. We leave the reader to think through basic examples: Euclidean geometry ($G_n = O_n$), affine symplectic geometry ($G_{2m} = \text{Sp}_{2m} \mathbb{R}$), affine geometry with a volume ($G_n \subset \text{GL}_n \mathbb{R}$ the group of matrices with determinant ± 1), affine spin geometry ($G_n = \text{Spin}_n$), affine geometry with a translation-invariant foliation, etc. This formalism covers *first-order* geometric structures; there are also higher order geometric structures, such as conformal structures.

As a preliminary, we recall change of structure group in a principal bundle. Let G be a Lie group, and recall that a principal G -bundle is a fiber bundle of right G -torsors. For a single G -torsor, we described the *mixing construction* in Definition 7.41. There is an analog for principal G -bundles, but we only need a special case.

Definition 22.8. Let $\lambda: G' \rightarrow G$ be a homomorphism of Lie groups, and suppose $\pi': P' \rightarrow X$ is a principal G' -bundle. The *associated* principal G -bundle $\pi: \lambda(P') \rightarrow X$ has total space

$$(22.9) \quad P' \times_{G'} G = (P' \times G) / G',$$

where the right G' -action on $P' \times G$ is $(p', g) \cdot g' = (p' \cdot g', \lambda(g')^{-1}g)$.

Definition 22.10. Let $\lambda: G' \rightarrow G$ be a homomorphism of Lie groups, and suppose $\pi: P \rightarrow X$ is a principal G -bundle. A *lift* or *reduction* (π', θ) of π along λ is a principal G' -bundle $\pi': P' \rightarrow X$ and an isomorphism

$$(22.11) \quad \begin{array}{ccc} P & \xrightarrow{\theta} & \lambda(P') \\ & \searrow \pi & \swarrow \lambda(\pi') \\ & X & \end{array}$$

of principal G -bundles.

The following definition is due to Élie Cartan. (I do not know an exact reference, but surely this should be ascribed to him). A smooth n -dimensional manifold carries a canonical principal $\text{GL}_n \mathbb{R}$ -bundle

$$(22.12) \quad \pi: \text{GL}(X) \longrightarrow X,$$

its *frame bundle*. A point of $\text{GL}(X)$ is a point $x \in X$ and a basis $\mathbb{R}^n \xrightarrow{\cong} T_x X$ of the tangent space.

Definition 22.13. Let (G_n, λ_n) be an n -dimensional symmetry type, and suppose X is a smooth n -dimensional manifold with frame bundle $\pi: \mathrm{GL}(X) \rightarrow X$. A (G_n, λ_n) -*structure on X* is a reduction of π along $\lambda_n: G_n \rightarrow \mathrm{GL}_n \mathbb{R}$.

We often use ‘ G_n -structure’ when the homomorphism λ_n is an inclusion or is unambiguous.

Example 22.14.

- (1) An O_n -structure on a smooth n -manifold is equivalent to a Riemannian structure.
- (2) Let $G_n \subset \mathrm{GL}_n \mathbb{R}$ be the stabilizer subgroup of $\mathbb{R}^k \subset \mathbb{R}^n$:

$$(22.15) \quad G_n = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

A G_n -structure on a smooth n -manifold is a k -dimensional distribution.

- (3) Suppose $n = 2m$ is even. Then a U_m -structure on a smooth n -manifold is usually called an *almost complex structure*.
- (4) Let K be a Lie group, and let $\lambda_n: O_n \times K \rightarrow \mathrm{GL}_n \mathbb{R}$ be the composition of projection onto the first factor and the inclusion $O_n \hookrightarrow \mathrm{GL}_n \mathbb{R}$. The corresponding structure on a smooth n -manifold is a Riemannian metric together with a principal K -bundle.
- (5) The reader should spell out spin and spin^c structures. What is the internal symmetry group in each case?

There is a category of (G_n, λ_n) -manifolds, which we encourage the reader to define carefully. For example, an Sp_{4k} -manifold is a hyperkähler manifold, and a morphism, or map, of Sp_{4k} -manifolds is what is often called a ‘triholomorphic map’.

In the model affine space $\mathbb{A}^n$, the group $\mathcal{G}_n$ in (22.7) acts by global *symmetries*. On a (G_n, λ_n) -manifold the linear symmetry group G_n becomes an infinitesimal *structure*. In view of the group extension (22.7), one can ask for the curved manifold manifestation of the translation subgroup $\mathbb{R}^n$. It too becomes infinitesimal data. Infinitesimal translation on a smooth manifold is encoded in a linear connection, here required to be compatible with the (G_n, λ_n) -structure

Definition 22.16. Let (G_n, λ_n) be an n -dimensional symmetry type, and suppose X is a smooth n -dimensional (G_n, λ_n) -manifold. A *differential (G_n, λ_n) -structure on X* is a connection on the principal G_n -bundle $P \rightarrow X$ that defines the (G_n, λ_n) -structure.

A differential (G_n, λ_n) -structure induces a connection on the frame bundle, and one can further ask that this induced linear connection be *torsionfree*. There is an existence and uniqueness theory for torsionfree differential (G_n, λ_n) -structures [Joy, §2.6]. The obstructions to torsionfree connections are familiar: Frobenius tensor of a distribution, Nijenhuis tensor of an almost complex structure, differential of an almost symplectic form, The existence of a torsionfree (G_n, λ_n) -connection is an *integrability condition* on the (G_n, λ_n) -structure.

22.3 Symmetric monoidal categories

For your convenience I reproduce part of [F4, Lecture 13] here. Those notes on bordism contain much relevant material for the next lecture and especially for next semester, particularly concerning topological field theories, and there is much more background about categories as well in those notes.

A category is an enhanced version of a set; a *symmetric monoidal category* is an enhanced version of a commutative monoid. Just as a commutative monoid has data (composition law, identity element) and conditions (associativity, commutativity, identity property), so too does a symmetric monoidal category have data and conditions. Only now the conditions of a commutative monoid become data for a symmetric monoidal category, and there are new and numerous conditions. We do not spell them all out, but defer to the references.

If C', C'' are categories, then there is a *Cartesian product* category $C = C' \times C''$. The set of objects is the Cartesian product $C_0 = C'_0 \times C''_0$ and the set of morphisms is likewise the Cartesian product $C_1 = C'_1 \times C''_1$. There is a transposition functor $\tau: C' \times C'' \rightarrow C'' \times C'$.

Definition 22.17. Let C be a category. A *symmetric monoidal structure* on C consists of an object

$$(22.18) \quad 1_C \in C,$$

a functor

$$(22.19) \quad \otimes: C \times C \longrightarrow C$$

and natural isomorphisms

$$(22.20) \quad \begin{array}{ccc} & \xrightarrow{-\otimes(-\otimes-)} & \\ C \times C \times C & \uparrow \alpha & C \\ & \xleftarrow{(-\otimes-)\otimes-} & \end{array}$$

$$(22.21) \quad \begin{array}{ccc} & \xrightarrow{(-\otimes-)\circ\tau} & \\ C \times C & \uparrow \sigma & C \\ & \xleftarrow{-\otimes-} & \end{array}$$

and

$$(22.22) \quad \begin{array}{ccc} & \xrightarrow{\text{id}_C} & \\ C & \uparrow \iota & C \\ & \xleftarrow{1_C \otimes -} & \end{array}$$

The quintuple $(1_C, \otimes, \alpha, \sigma, \iota)$ is required to satisfy the axioms indicated below.

As indicated above, the functor τ in (22.20) is transposition:

$$(22.23) \quad \begin{aligned} \tau: C \times C &\longrightarrow C \times C \\ y_1, y_2 &\longmapsto y_2, y_1 \end{aligned}$$

A crucial axiom is that

$$(22.24) \quad \sigma^2 = \text{id}.$$

Thus for any $y_1, y_2 \in C$, the composition

$$(22.25) \quad y_1 \otimes y_2 \xrightarrow{\sigma} y_2 \otimes y_1 \xrightarrow{\sigma} y_1 \otimes y_2$$

is $\text{id}_{y_1 \otimes y_2}$. The other axioms express compatibility conditions among the extra data (22.18)–(22.22). For example, we require that for all $y_1, y_2 \in C$ the diagram

$$(22.26) \quad \begin{array}{ccc} & (1_C \otimes y_1) \otimes y_2 & \\ \alpha \swarrow & & \searrow \iota \\ 1_C \otimes (y_1 \otimes y_2) & \xrightarrow{\quad \iota \quad} & y_1 \otimes y_2 \end{array}$$

commutes. We can state the axioms informally as asserting the equality of any two compositions of maps built by tensoring α, σ, ι with identity maps. These compositions have domain a tensor product of objects $y_1, \dots, y_n$ and any number of identity objects 1_C —ordered and parenthesized arbitrarily—to a tensor product of the same objects, again ordered and parenthesized arbitrarily. Coherence theorems show that there is a small set of conditions which needs to be verified; then arbitrary diagrams of the sort envisioned commute. You can find precise statements and proof in [Mac, JS]

A symmetric monoidal functor is a homomorphism between symmetric monoidal categories, but as is typical for categories the fact that the identity maps to the identity and tensor products to tensor products is expressed via data, not as a condition. Then there are higher order conditions.

Definition 22.27. Let C, D be symmetric monoidal categories. A *symmetric monoidal functor* $F: C \rightarrow D$ is a functor with two additional pieces of data, namely an isomorphism

$$(22.28) \quad 1_D \longrightarrow F(1_C)$$

and a natural isomorphism

$$(22.29) \quad \begin{array}{ccc} & F(- \otimes -) & \\ \curvearrowright & \uparrow \psi & \curvearrowleft \\ C \times C & & D \end{array}$$

$F(-) \otimes F(-)$

There are many conditions on this data.

The first condition expresses compatibility with the associativity morphisms: for all $y_1, y_2, y_3 \in C$ the diagram

$$(22.30) \quad \begin{array}{ccc} (F(y_1) \otimes F(y_2)) \otimes F(y_3) & \xrightarrow{\psi} & F(y_1 \otimes y_2) \otimes F(y_3) \\ \alpha_D \downarrow & & \downarrow \psi \\ F(y_1) \otimes (F(y_2) \otimes F(y_3)) & & F((y_1 \otimes y_2) \otimes y_3) \\ \psi \downarrow & & \downarrow F(\alpha_C) \\ F(y_1) \otimes F(y_2 \otimes y_3) & \xrightarrow{\psi} & F(y_1 \otimes (y_2 \otimes y_3)) \end{array}$$

is required to commute. Next, there is compatibility with the identity data ι : for all $y \in C$ we require that

$$(22.31) \quad \begin{array}{ccc} F(1_C) \otimes F(y) & \xrightarrow{F(\psi)} & F(1_C \otimes y) \\ \textcolor{red}{(22.28)} \uparrow & & \downarrow F(\iota) \\ 1_D \otimes F(y) & \xrightarrow{\iota} & F(y) \end{array}$$

commute. The final condition expresses compatibility with the symmetry σ : for all $y_1, y_2 \in C$ the diagram

$$(22.32) \quad \begin{array}{ccc} F(y_1) \otimes F(y_2) & \xrightarrow{\sigma_D} & F(y_2) \otimes F(y_1) \\ \psi \downarrow & & \downarrow \psi \\ F(y_1 \otimes y_2) & \xrightarrow{F(\sigma_C)} & F(y_2 \otimes y_1) \end{array}$$

We leave the reader to define a symmetric monoidal natural transformation of symmetric monoidal functors.

Lecture 23: Segal axioms

In this lecture we discuss the axioms ([Axiom System 23.14](#)) for Wick-rotated field theory introduced by Graeme Segal [[Se2](#)] in the 1980s and elaborated recently in a joint paper with Maxim Kontsevich [[KS](#)]. (The 2011 Felix Klein lectures [[Se3](#)] by Segal remain also a valuable resource.) Segal's original context is 2-dimensional *conformal* field theory. By the end of the 1980s, Atiyah [[A1](#)] adopted the Segal axioms to *topological* field theory, and he also made a strong connection with bordism theory. Over the ensuing decade it became apparent that the point of view in this axiom system—Wick-rotated field theory is a linear representation of a bordism category—applies to general quantum field theories. The Segal axioms provide a pathway into quantum field theory which

has been of enormous value and influence. At the same time they codify structures that have appeared in many mathematical contexts: low dimensional topology, symplectic geometry, geometric representation theory, higher algebra, number theory, etc. We remark that at the time Segal wrote his first paper, 2-dimensional conformal field theory was very much in the collective consciousness due to the first superstring revolution. Segal’s motivations—as stated in [Se2]—came not only from theoretical physics, but also from relations to concrete problems in mathematics: representations of the monster finite group, the theory of positive energy representations of loop groups, the geometry of moduli spaces of Riemann surfaces and relations to $\text{Diff}^+(S^1)$, and elliptic cohomology. Thus from the beginning of this axiom system, physics and mathematics were intertwined.

The route we have taken in these lectures to the Segal axiom system is through quantum mechanics and relativistic quantum field theory. As a codification of quantum field theory in general, the axioms should be treated as malleable, or at the very least incomplete.⁶⁹ It is early days, still, and one glaring problem is the dearth of rigorously constructed examples. Nonetheless, this framework has led to interesting applications in quantum theory. And, apart from the physics, the axioms are a natural mathematical structure that has wide applicability, as alluded to above. We emphasize that the axioms go beyond what is strictly required by Wick rotating relativistic quantum field theories. The additional generalizations are part of their power in mathematics.

In the previous lecture (§22.1) we reviewed Wick rotation of the unitary representation of time translation in quantum mechanics. Then we made an analogous construction in 2-dimensional quantum field theory. The first move in that Wick rotation, taken in the previous lecture—is compactification of space. (This is a step toward compactifying Wick-rotated time as well. Indeed, it is an important feature of the Segal axioms that theories are only evaluated on *compact* manifolds.) The Wick-rotated representation of translations in Minkowski spacetime is a contractive representation of a certain real semigroup. The second move is to identify this semigroup with a particular semigroup of *Riemannian* cylinders. That is the starting point in this lecture. The identification of these Wick rotations of semigroups of spacetime translations as semigroups of Riemannian manifolds is a crucial step. And the generalization to *arbitrary* Riemannian manifolds is yet a further step toward the Segal axiom. The third move is to allow the topology of space to change. This leads to a variant of Milnor’s bordism category [Mi2] that includes Riemannian metrics. (Technically, we obtain a *semicategory*: identity maps are not included, at least at first glance.)

At this point we have arrived at the concise Segal axioms (Axiom System 23.14): a field theory is a linear representation of a Riemannian bordism category. This is analogous to the linear representation of a group, so fits into a long line of mathematical structures.

The abstract Segal axioms seem far from their physics origins. To make the connection, we derive local observables, correlation functions, and Hamiltonians in the Segal picture. We illustrate the singular nature of local observables in the case of quantum mechanics.

One can regard the Segal axioms a first step toward localization of *states*. (Recall that we defined local theories of *observables* in Lecture 20.) Namely, there is a state space assigned to every closed Riemannian $(n - 1)$ -manifold in an n -dimensional theory, and these $(n - 1)$ -manifolds undergo

⁶⁹This is one reason we use ‘Axiom System’ in place of ‘Definition’: the latter sounds much too definitive. Specifically, one needs to assume some good short range behavior if a quantum field theory is to have good analytic properties, much the same as one needs well-posedness in PDE: the backward heat equation satisfies the definition of a PDE, but one cannot evolve far with it.

topology change: Morse surgeries. The situation is not as local as it might be—in topological theories we will localize states completely—but regardless of the depth of localization, states are paramount in Segal’s approach. (This is in contrast to the primacy of observables in the Irving Segal, Haag–Kastler, Costello–Gwilliam line.)

There are three further topics in this lecture: background fields, topological vector spaces, and reflection positivity. The first two elaborate on the domain and codomain, respectively, of a field theory. The third is the Wick rotation of unitarity.

We introduced the notion of a *relativistic symmetry type* in [Definition 15.13](#). It controls whether a theory has time-orientation reversing symmetries, what the internal symmetries are, and how they interact with spacetime symmetries. The Wick rotation to a Euclidean symmetry type for Euclidean field theory is depicted in the diagram [\(21.26\)](#). Recall from [Theorem 21.27](#) that Wick-rotated symmetry types are highly constrained. In the previous lecture ([§22.2](#)) we reviewed the passage from a Euclidean symmetry type to structure on curved manifolds. Here we morph the result and generalize to *background fields*. This allows for local background structures not induced by symmetry, such as scalar fields with values in nonconnected manifolds.

The codomain of a quantum field theory [\(23.15\)](#) is a category of topological vector spaces and continuous linear maps. The linear maps need to be “small” to model the contractive nature of Wick-rotated time evolution in quantum mechanics; see [\(11.15\)](#). The germ surrounding objects in the bordism category ([Remark 8.39](#)) leads to two canonical systems of bordisms, and under limits we obtain a canonical pair of *nuclear* topological vector spaces attached to the object. It is these vector spaces that play a crucial role, and we give some indication of their properties.

Finally, unitarity is not part of [Axiom System 23.14](#), just as the definition of a linear representation of a Lie group (think $\mathrm{SL}_2\mathbb{R}$) does not include unitarity. Indeed, for representations of semisimple Lie groups, unitarizability is an interesting question; the subclass of unitary representations has special properties. The Wick-rotated field theory analog of unitarity is *reflection positivity*, as stated in [\(21.34\)](#). For field theory on curved manifolds—that is, as a function out of a bordism category—*reflection* is an extra structure, both on the background fields and on the symmetric monoidal functor that is the field theory. Then *positivity* is a condition on the reflection structure.

We conclude with further mathematical background: basic definitions for topological vector spaces and some material on duality in symmetric monoidal categories.

These notes have far more than we were able to include in the last lecture of the semester.

23.1 Semigroups of Riemannian manifolds

In 1-dimensional quantum mechanics, Wick rotation produces a semigroup of contractive operators parametrized by $\mathbb{R}^{>0}$; in a 2-dimensional theory with space diffeomorphic to S^1 , the relevant semigroup is Q as defined in [\(22.4\)](#). We now identify each as a semigroup of Riemannian manifolds, with composition by gluing.

Remark 23.1. I know no analog of *semigroups* of Riemannian manifolds that models *groups* of spacetime translations.

For the 1-dimensional semigroup $\mathbb{R}^{>0}$, observe that a Riemannian metric on a manifold diffeomorphic to $[0, 1]$ has a single invariant: length. Furthermore, the length adds under gluing

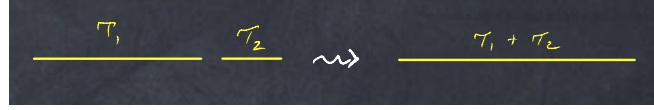


FIGURE 64. Gluing of Riemannian intervals

of Riemannian intervals—see [Figure 64](#). Therefore, the semigroup of Riemannian closed intervals under gluing—concatenation—is isomorphic to $\mathbb{R}^{>0}$. Wick-rotated time evolution in quantum mechanics is a contractive representation of this semigroup.

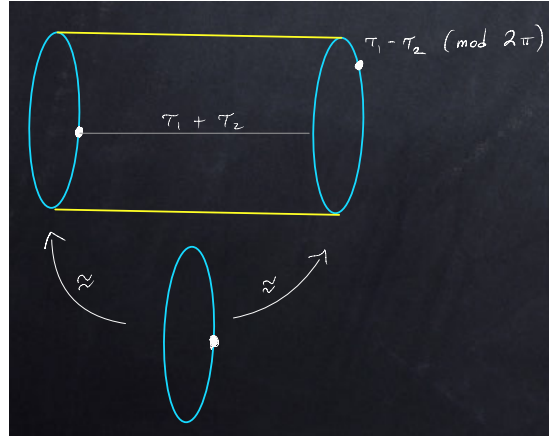


FIGURE 65. A semigroup of Riemannian cylinders

For the 2-dimensional real semigroup Q there is no pithy isomorphic semigroup of Riemannian manifolds. The underlying smooth manifold should be diffeomorphic to a cylinder $[0, 1] \times S^1$, but the space of isometry classes of Riemannian metrics on the cylinder is infinite dimensional: the Gauss curvature is an invariant and already there is an infinite dimensional space of those. We must standardize space to be the circle $S^1(2\pi)$ of length 2π , certainly for the initial and final spaces—the boundary of the cylinder—and so we demand an isometry of each boundary circle to a standard circle of length 2π . To realize Q , then, restrict to cylinders isometric to $[0, \tau] \times S^1(2\pi)$, as depicted in [Figure 65](#), together with an isometry of standard $S^1(2\pi)$ onto each boundary component, as indicated by the images of a basepoint in standard $S^1(2\pi)$.

Remark 23.2.

- (1) This realization of Q has a preferred splitting as a Cartesian product of Wick-rotated time and space. On general Riemannian manifolds, to which we generalize forthwith, there is no such splitting.
- (2) The analog of the real semigroup Q in 2-dimensional conformal field theory is infinite dimensional, and is a main ingredient in [\[Se2\]](#). (It was introduced independently by Kontsevich and Neretin [\[Ne\]](#).) Note that every conformal cylinder, or annulus, is conformally equivalent to a standard annulus $\{z \in \mathbb{C} : r \leq |z| \leq 1\}$ for some $r \in (0, 1)$ by uniformization; the semigroup in question also includes parametrizations of the two boundary circles. That 2-dimensional conformal field theory has a contractive representation of this *infinite dimensional* semigroup makes it more tractable than most other quantum field theories: there is more symmetry to deploy.

In the 2-dimensional case it is natural to consider not only representations of the real semigroup Q obtained by Wick rotation of the semigroups of spacetime translations, but rather representations of the much larger semigroup of all Riemannian cylinders. (In the 1-dimensional case we already have the semigroup of all Riemannian intervals.) Similarly, in arbitrary dimension n , for any closed $(n - 1)$ -manifold Y we consider arbitrary Riemannian metrics on the cylinder $[0, 1] \times Y$.

Remark 23.3. In relativistic quantum field theory, one usually assumes the existence of an *energy-momentum* tensor. This may be interpreted as encoding an infinite order formal deformation of the flat Lorentz metric on Minkowski spacetime. A stronger form of that assumption is the formulation of the theory on arbitrary Lorentz manifolds. We make the corresponding assumption in the Wick-rotated setting.

23.2 Topology change

The final conceptual input is topology change. In dimension 2 one motivation comes from string theory, where topology change can be viewed as the splitting and joining of strings. In any dimension we can think that this splitting and gluing of space is an expression of the locality of state spaces in the quantum theory.



FIGURE 66. Topology change of “space” in quantum mechanics

Remark 23.4. In dimension 1—quantum mechanics—letting in topology change has the consequence that the “Wick-rotated processes” depicted in Figure 66 are allowed. In the first process, two copies of the system come together and annihilate; in the second, two copies are created out of nothing. Or, we can view the illustration as a single process from two copies to two copies. As we will see ‘copies’ means the conjunction (oft called ‘stacking’) of a system with itself without interaction: the state space of the conjunction is the tensor product of the individual state spaces, and the Hamiltonian of the conjunction is the sum of the individual Hamiltonians.

In his lectures on the h-cobordism theorem [Mi2], Milnor introduced the mathematical structure that tracks topology change: *bordism categories*. First, recall the classical definition of a bordism.

Definition 23.5. Fix $d \in \mathbb{Z}^{\geq 0}$. Closed d -manifolds Y_0, Y_1 are *bordant* if there exists a compact $(d + 1)$ -manifold X with boundary such that⁷⁰ $Y_0 \sqcup Y_1 \approx \partial X$.

⁷⁰For smooth manifolds the symbol ‘ $\approx$ ’ means ‘is diffeomorphic to’.

The manifold X is called a *bordism*. The condition in [Definition 23.5](#) defines an equivalence relation on diffeomorphism classes of closed d -manifolds; the set of equivalence classes is denoted Ω_d . Furthermore, disjoint union passes through this equivalence relation, where it becomes a commutative, associative composition law on Ω_d . The class of the empty manifold $\emptyset^d$ is a unit for this composition law. Finally, every element has order 2: the manifold $X = [0, 1] \times Y$ is a null bordism of $Y^{\sqcup 2}$. In summary, Ω_d is an abelian group.

Remark 23.6.

- (1) There are variants in which the manifolds carry additional topological structure, such as an orientation. Inverses still exist, but manifolds may not be their own inverse. For example, the inverse of a closed oriented d -manifold is the oppositely oriented manifold. A standard reference on bordism theory is [\[Sto\]](#).
- (2) A homomorphism from a bordism group to an abelian group is a *bordism invariant*. For example, the signature of a manifold of dimension $4k$ is a homomorphism $\Omega_{4k}(SO) \rightarrow \mathbb{Z}$ that is a crucial invariant in differential topology.

Bordism groups are classical; Milnor's idea was to “categorify” by promoting the bordism in [Definition 23.5](#) from a condition to data. In Wick-rotated field theory, the manifolds Y play the role of space and the manifolds X play the role of a Wick-rotated spacetime. Hence introduce $n = d + 1$ for the dimension of X .

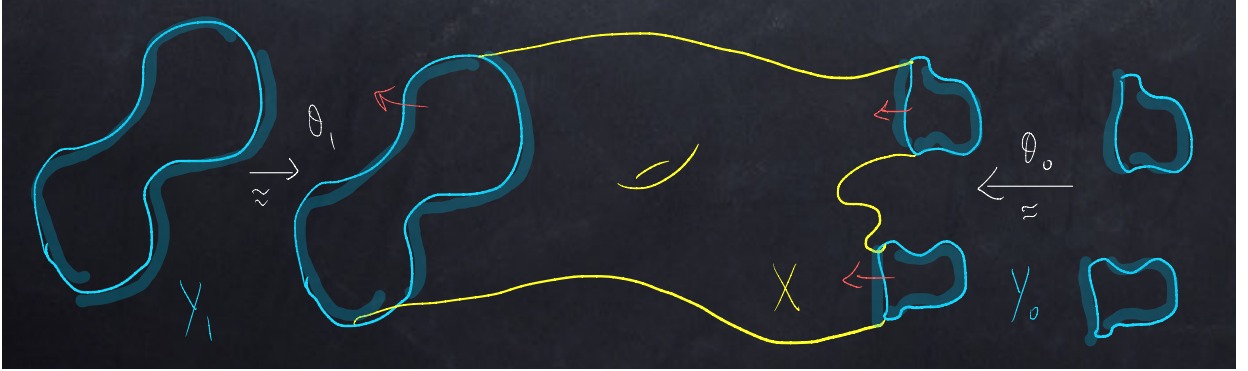


FIGURE 67. A bordism $X: Y_0 \rightarrow Y_1$

Definition 23.7. Let Y_0, Y_1 be closed $(n - 1)$ -manifolds. A *bordism* $(X, p, \theta_0, \theta_1)$ from Y_0 to Y_1 consists of a compact n -manifold X with boundary, a partition $p: \partial X \rightarrow \{0, 1\}$ of its boundary, and disjoint embeddings

$$(23.8) \quad \theta_0: [0, +1) \times Y_0 \longrightarrow X$$

$$(23.9) \quad \theta_1: (-1, 0] \times Y_1 \longrightarrow X$$

such that $\theta_i(0, Y_i) = (\partial X)_i$, $i = 0, 1$, where $(\partial X)_i = p^{-1}(i)$.

A bordism is depicted in [Figure 67](#). The function p is depicted by an arrow on each boundary component that points in (incoming=domain) or out (outgoing=codomain). One should include 2-sided collars to facilitate gluing, or even better *germs* of 2-sided collars.

Remark 23.10. In the application to Wick-rotated field theory, the arrow at a boundary component is a Wick-rotated time-orientation.

Definition 23.11 (Milnor). The *bordism category* Bord_n is the category whose objects are closed $(n - 1)$ -manifolds and whose morphism sets are equivalence classes of bordisms:

$$(23.12) \quad \text{Hom}_{\text{Bord}_n}(Y_0, Y_1) = \{(X, p, \theta_0, \theta_1)\} / \sim$$

An equivalence of bordisms is a diffeomorphism of manifolds X that commutes with p, θ_0, θ_1 . As in [Remark 23.6\(1\)](#) there are variants for manifolds equipped with tangential structures, such as spin structures, framings, etc. Observe that the identity map in $\text{Hom}_{\text{Bord}_n}(Y, Y)$ is represented by a cylinder $[0, 1] \times Y$ with identity boundary identifications. A diffeomorphism of Y is represented by the same cylinder, but with the boundary identification $\theta_1 : Y \xrightarrow{\sim} \{1\} \times Y$ equal to the inverse of the diffeomorphism. Two diffeomorphisms are equivalent if they are *pseudoisotopic*; see [\[Mi2, §1\]](#).

The composition law in the abelian bordism group—disjoint union—categorifies to a *symmetric monoidal structure* on the category Bord_n . We provide background material on symmetric monoidal structures in [Section 22.3](#). A monoidal structure on a category is an associative composition law on objects and morphisms, but now associativity on objects is data rather than a condition and there is another layer that brings in a condition. A symmetry of the monoidal product is data that expresses commutativity.



FIGURE 68. A closed $(n - 1)$ -manifold Y embedded in a germ U with coorientation

Now, motivated by our previous discussions, we augment Milnor’s bordism category Bord_n to include Riemannian metrics. An object in $\text{Bord}_n(\text{Riem})$ is a closed $(n - 1)$ -manifold Y , a germ U of an n -manifold into which Y is embedded as a submanifold, a coorientation of $Y \subset U$, and a Riemannian metric on U . See [Figure 68](#) for an illustration. One can take U to be an n -manifold diffeomorphic to $(-1, 1) \times Y$. The germ is the inverse limit of an inverse system of such Riemannian manifolds U containing Y as a cooriented codimension one submanifold. We overload ‘ Y ’ to mean the manifold Y together with the germ, coorientation, and Riemannian metric. Morphisms in $\text{Bord}_n(\text{Riem})$ are equivalence classes of bordisms, only now the bordisms are also equipped with a Riemannian metric and the diffeomorphisms θ_i are *isometries* of germs of n -manifolds containing closed cooriented codimension one submanifolds. (So now the shading in [Figure 67](#) is 2-sided, as in [Figure 68](#).)

Remark 23.13.

- (1) The bordism category with metrics is discussed in [KS] and is elaborated in [Wed, §2.1].
- (2) As remarked in [KS, §3], the endomorphism sets in $\text{Bord}_n(\text{Riem})$ do not contain identities, and so $\text{Bord}_n(\text{Riem})$ is a *semicategory* rather than a category, in line with the *semigroups* $\mathbb{R}^{>0}$ and Q that appeared earlier. One can rectify this by adjoining identity maps, and as well one should adjoin isometries of objects to the category—they too should act in a field theory. (They are Wick-rotated remnants of isometries of space in Minkowski spacetime, once one fixes a spacelike codimension one affine hyperplane.)
- (3) In the application to Wick-rotated field theory, a morphism X in $\text{Bord}_n(\text{Riem})$ is not a “spacetime”: it carries a Riemannian metric, not a Lorentzian metric. We could call it a ‘Wick-rotated spacetime’ or, more simply, say that we study field theory on X .

23.3 Axiom system for field theory

Recall that the Wick rotation of the unitary positive energy representation of the translation group of Minkowski spacetime is a contractive linear representation of a real semigroup. Based on this, we now give a provisional definition of a field theory, provisional subject to more elaboration of the domain and codomain. (There are more substantial provisos: footnote ⁶⁹, [Remark 23.16](#), [Remark 23.18](#), [Remark 23.23\(1\)](#), and [Remark 23.31](#).) Let $t\text{Vect}$ be a “suitable” category of complex topological vector spaces and continuous linear maps, which we suppose has a suitable tensor product. Crucially, the linear maps are *nuclear*, which is a class of linear maps analogous to contractive maps between Banach spaces. The following is what we have been calling the ‘Segal axioms’. The first occurrence in print that I am aware of is [KS, §3].

Axiom System 23.14. A (*Wick-rotated*) *field theory* is a homomorphism

$$(23.15) \quad F: \text{Bord}_n(\text{Riem}) \longrightarrow t\text{Vect}.$$

The symmetric monoidal structure on $\text{Bord}_n(\text{Riem})$ is disjoint union; the symmetric monoidal structure on $t\text{Vect}$ is tensor product. A homomorphism in this context is oft called a symmetric monoidal functor. It is a map that preserves all algebraic structures, but as is characteristic in category theory that “preservation” includes data as well as conditions. So F maps compositions of bordisms to compositions of linear maps, it maps disjoint unions to tensor products, and it maps⁷¹ the empty $(n-1)$ -manifold to the vector space $\mathbb{C}$. An endomorphism of $\emptyset^{n-1}$ in $\text{Bord}_n(\text{Riem})$ is a closed Riemannian n -manifold X , and $F(X)$ is a complex number, the *partition function*⁷² of X .

Remark 23.16. In [Lecture 8](#) we emphasized that quantum theory is a *projective* system, not a *linear* system, and we illustrated that principle in quantum mechanics. (Recall the slogan (8.1).) Hence we should replace [Axiom System 23.14](#) by a projective representation of the bordism category, i.e., by a homomorphism into a symmetric monoidal category of projective spaces.⁷³ We will explain

⁷¹The empty set is a manifold of any dimension and it is the unit for disjoint union. The vector space of scalars $\mathbb{C}$ is the unit for tensor product.

⁷²The nomenclature derives from statistical mechanics and reflects a relationship between Wick-rotated quantum field theory and statistical mechanics.

⁷³The precise construction does not have this exact description.

how to do that in a lecture next semester, and then we will measure the projectivity by a special type of *invertible* field theory, which in this context is called the *anomaly*. In the meantime, we persist with the linear representation (23.15).

A field theory assigns a vector space $F(Y)$ to each closed $(n-1)$ -manifold Y (with germ, coorientation, and Riemannian metric). This can be regarded as the state space of a quantum mechanical system;⁷⁴ states and maps of states have center stage in the Segal axioms. Observables are derived quantities. In the remainder of this lecture we explain how to tease out observables and correlation functions from **Axiom System 23.14**. We refer to [KS] for more exposition on these matters.

23.4 Hamiltonians

In the Hamiltonian setting of relativistic mechanics, spacetime is a Cartesian product of space and time. So too in Wick-rotated field theory we only expect time evolution in the presence of a product structure. Hence let Y be an object in $\text{Bord}_n(\text{Riem})$ —a closed $(n-1)$ -manifold—which is embedded in the Cartesian product $(-1, 1) \times Y$ equipped with a Riemannian metric that is the product of the standard metric on $(-1, 1)$ and a Riemannian metric on Y . A field theory F assigns to this data a topological vector space $F(Y)$. For $\tau \in \mathbb{R}^{>0}$ consider the bordism

$$(23.17) \quad X_\tau = [0, \tau] \times Y : Y \rightarrow Y$$

with the product Riemannian metric. Then $F(X_\tau) : F(Y) \rightarrow F(Y)$ is a 1-parameter semigroup of (nuclear) linear maps. This is the Wick rotation of time evolution. One can hope to construct a continuous linear operator $H_Y : F(Y) \rightarrow F(Y)$ such that $F(X_\tau) = e^{-\tau H_Y/\hbar}$ if one can give meaning to the exponential. For example, in a *unitary* theory—unitarity is discussed in the next lecture—there is a Hilbert space associated to $F(Y)$, and then it is reasonable to carry this out. The operator H_Y is the Hamiltonian.

Remark 23.18. In this argument we evaluate F on a family $\{X_\tau\}$ of bordisms; recall the discussion in the bonus lecture. Clearly **Axiom System 23.14** should be expanded to include smoothness in families. See [ST] for an implementation in Euclidean field theory.

23.5 Local observables

Next, we construct a Wick-rotated analog of a local net of observables (**Definition 20.17**). Let F be an n -dimensional field theory, and suppose X is a closed Riemannian n -manifold. Let $\mathcal{U} \subset X$ be an open subset whose closure $\overline{\mathcal{U}}$ is a smooth n -manifold with boundary. To each such *smooth* open subset⁷⁵ assign the topological vector space

$$(23.19) \quad \mathcal{A}(\mathcal{U}) = F(\partial \overline{\mathcal{U}}).$$

⁷⁴There is also a Hamiltonian if Y is embedded in a germ with a product metric, as we show below.

⁷⁵'Smooth open subset' is not a standard term; I use it only here and only for convenience.

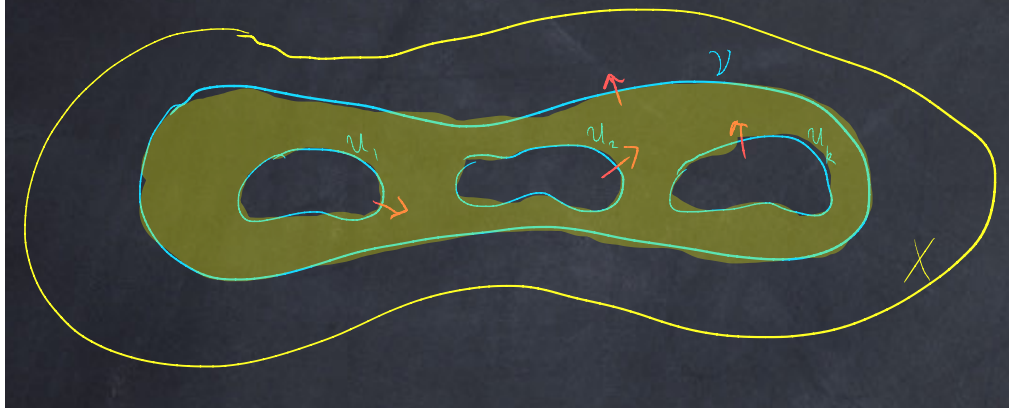


FIGURE 69. The prefactorization algebra structure on observables

If, as in [Figure 69](#), $\mathcal{U}_1, \dots, \mathcal{U}_k \subset \mathcal{V} \subset X$ is an inclusion of disjoint smooth open subsets $\mathcal{U}_i$ in an open subset $\mathcal{V}$, then

$$(23.20) \quad \mathcal{V} \setminus (\mathcal{U}_1 \sqcup \dots \sqcup \mathcal{U}_k): \partial \overline{\mathcal{U}}_1 \sqcup \dots \sqcup \partial \overline{\mathcal{U}}_k \longrightarrow \partial \overline{\mathcal{V}}$$

is a morphism in $\text{Bord}_n(\text{Riem})$, and the value of F is a continuous linear map

$$(23.21) \quad \mathcal{A}(\mathcal{U}_1) \otimes \dots \otimes \mathcal{A}(\mathcal{U}_k) \longrightarrow \mathcal{A}(\mathcal{V}).$$

This satisfies an operadic property: if for each $i \in \{1, \dots, k\}$ we have given disjoint smooth open subsets $\mathcal{U}_i^1, \dots, \mathcal{U}_i^{j_i} \subset \mathcal{U}_i$, then there is an identity between [\(23.21\)](#) and the maps

$$(23.22) \quad \mathcal{A}(\mathcal{U}_i^1) \otimes \dots \otimes \mathcal{A}(\mathcal{U}_i^{j_i}) \longrightarrow \mathcal{A}(\mathcal{U}_i)$$

that expresses associativity. An assignment $\mathcal{A}$ with these properties is called a *prefactorization algebra* (on the category $s\mathbf{Open}(X)$ of smooth open subsets of X and inclusion maps).

Remark 23.23.

- (1) A *factorization algebra* is a prefactorization algebra that satisfies a local-to-global condition analogous to a sheaf condition. I do not know if it follows from the Segal axioms as stated; presumably one needs additional conditions. Factorization algebras generalize the *chiral algebras* of Beilinson–Drinfeld [\[BD\]](#), which were inspired by 2-dimensional conformal field theory. The theory of factorization algebras is developed by Costello–Gwilliam in [\[CG1\]](#).
- (2) In relativistic field theory, a net of observables ([Definition 20.17](#)) assigns an *algebra* to each open set, and the algebras of spacelike separated sets commute. In a prefactorization algebra, by contrast, to each smooth open set is attached a topological vector space; the algebra structure is replaced by the maps [\(23.21\)](#). Furthermore, the analog of commutation holds for *every* pair of disjoint open sets: the tensor product in [\(23.21\)](#) is symmetric monoidal. Observe that in Euclidean space—and on Riemannian manifolds—two distinct points are spacelike separated: the displacement vector between distinct points in Euclidean space is spacelike since the inner product is negative definite.

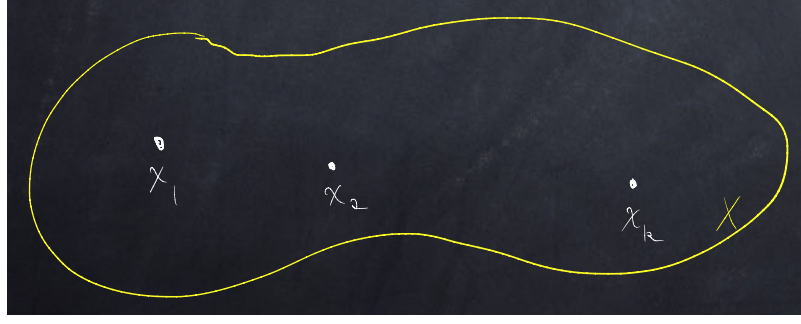


FIGURE 70. Correlation function of point observables

23.6 Correlation functions

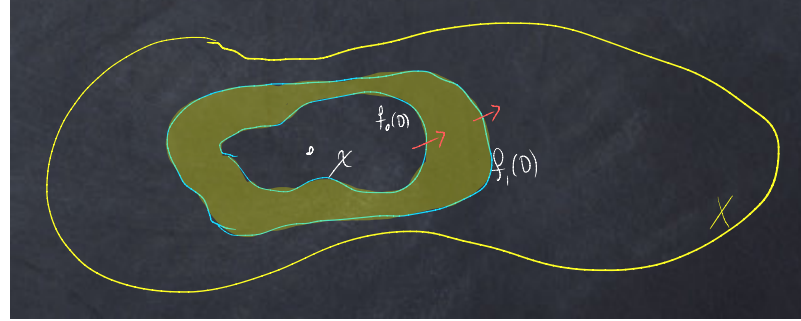
Next, we illustrate how to recover correlation functions from [Axiom System 23.14](#), following discussions in [\[KS\]](#). Given a field theory F and a closed Riemannian manifold X , we seek:

- (1) a topological vector space $\mathcal{O}_x$ of point observables for each $x \in X$, and
- (2) a continuous multilinear function

$$(23.24) \quad \mathcal{O}_{x_1} \times \cdots \times \mathcal{O}_{x_k} \longrightarrow \mathbb{C}$$

for each subset $\{x_1, \dots, x_k\} \subset X$ of distinct points.

The main idea is an inverse limit construction.


 FIGURE 71. Inverse limit construction of $\mathcal{O}_x$

Let $D = D^n$ be the standard closed n -disk, and let $\mathring{D}$ denote its interior. For $x \in X$ set

$$(23.25) \quad \mathcal{D}_x = \{f: D \hookrightarrow X : f \text{ is an embedding with } x \in f(\mathring{D})\}.$$

If $f_0, f_1 \in \mathcal{D}_x$ satisfy $f_0(D) \subset f_1(\mathring{D})$, as depicted in [Figure 71](#), then there is a bordism

$$(23.26) \quad f_1(D) \setminus f_0(\mathring{D}): f_0(\partial D) \longrightarrow f_1(\partial D).$$

These bordisms correspond to morphisms in an inverse system whose set of objects is $\mathcal{D}_x$. Define the point observables as an inverse limit in $t\mathbf{Vect}$:

$$(23.27) \quad \mathcal{O}_x = \varprojlim_{f \in \mathcal{D}_x} F(f(\partial D)).$$

This construction gives a continuous linear map

$$(23.28) \quad \mathcal{O}_x \longrightarrow F(\partial D)$$

for each $f \in \mathcal{D}_x$: an observable at x induces an observable in the smooth open set $f(\mathring{D})$.

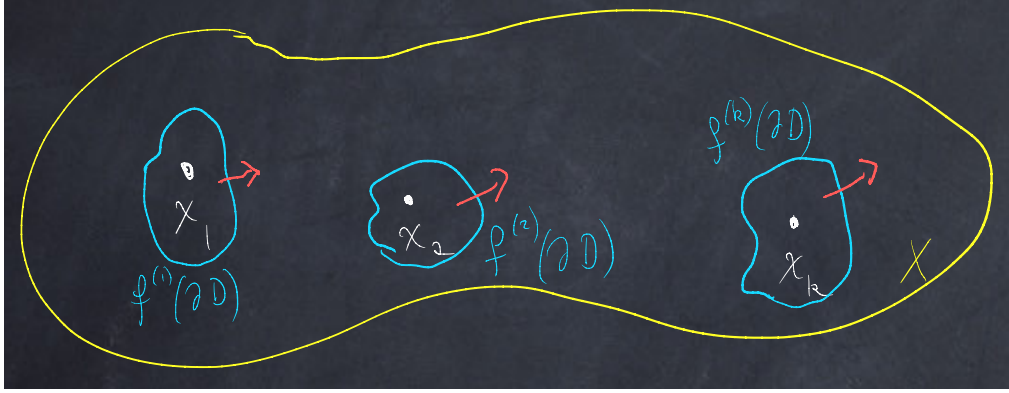


FIGURE 72. Correlation functions

To construct the correlation function (23.24), choose $f^{(i)} \in \mathcal{D}_{x_i}$ with disjoint images. Then apply F to the bordism depicted in Figure 72,

$$(23.29) \quad X \setminus \bigsqcup_i f^{(i)}(\mathring{D}) : \bigsqcup_i f^{(i)}(\partial D) \longrightarrow \emptyset^{n-1},$$

and compose with the maps (23.28).

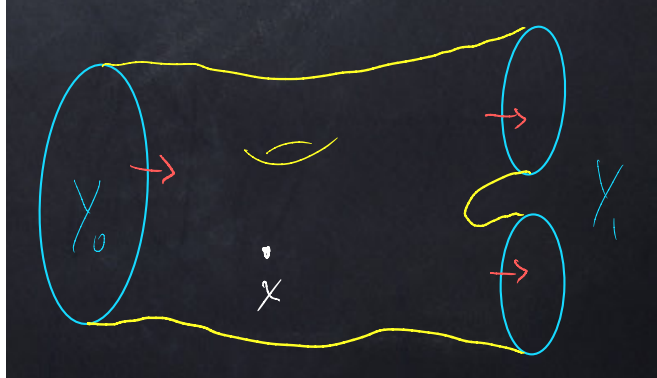


FIGURE 73. Observable as linear operator

Finally, recall from Remark 21.12(2) that in a relativistic field theory a *quantum field* is an operator-valued distribution. An analog in our Wick-rotated setting is as follows. Suppose $X : Y_0 \rightarrow Y_1$ is a morphism in $\text{Bord}_n(\text{Riem})$. Let $x \in \mathring{X}$ be an interior point, as in Figure 73. Apply F to obtain a continuous linear map

$$(23.30) \quad \mathcal{O}_x \longrightarrow \text{Hom}(F(Y_0), F(Y_1))$$

from observables at x to continuous linear maps.

Remark 23.31. As in [Remark 23.18](#) we should evaluate F on the family of bordisms parametrized by $\mathring{X}$. Smoothness of that evaluation should include a vector bundle structure on the family of vector spaces $\{\mathcal{O}_x\} \rightarrow \mathring{X}$, and furthermore that vector bundle should carry a flat covariant derivative. That given, one can hope to make sense of operator-valued distributions. We refer to [\[KS\]](#) for further discussion.

23.7 Wick-rotated quantum mechanics in the axiom system

This material is from [\[Se3, Lecture 2\]](#).

Quantum mechanics—in Wick-rotated form—is the case $n = 1$ of [Axiom System 23.14](#). An object in $\text{Bord}_1(\text{Riem})$ is a finite set of points embedded in a germ of a Riemannian 1-manifold, and equipped with a coorientation. Consider a single point. Using parametrization by arclength for Riemannian 1-manifolds, we see that the germ is isometric to the germ of $\{0\} \subset (-1, 1)$ with the standard metric and standard coorientation. A theory F maps this to a topological vector space which, assuming unitarity (as will be discussed below), we can replace with a Hilbert space $\mathcal{H}$. Then, as in [\(23.17\)](#), the bordisms $[0, \tau]: \{0\} \rightarrow \{\tau\}$, $\tau \in \mathbb{R}^{>0}$, map under F to a 1-parameter semigroup of contractive operators $e^{-\tau H/\hbar}$ on $\mathcal{H}$.

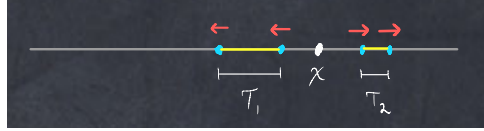


FIGURE 74. The inverse limit that computes observables

The topological vector space of observables is defined in [\(23.27\)](#). Let $x \in (0, \tau)$. Then the boundary of an embedded disk containing x in its interior is $\{x - \delta_1, x + \delta_2\}$ for some $\delta_1, \delta_2 > 0$. A bordism [\(23.26\)](#) is depicted in [Figure 74](#). Under F it evaluates to the map

$$(23.32) \quad T \mapsto e^{-\tau_2 H/\hbar} \circ T \circ e^{-\tau_1 H/\hbar}$$

on endomorphisms of $\mathcal{H}$. The inverse limit of $\text{End } \mathcal{H}$ under these maps consists of singular operators.

Example 23.33. Consider a particle on a real analytic Riemannian manifold M with state space $\mathcal{H} = L^2(M; \mathbb{C})$ and Hamiltonian the Hodge laplacian $H = \Delta$, as in [Lecture 5](#). The heat operator $e^{-\tau \Delta/\hbar}$ maps distributions (in the dual space to analytic functions) to analytic functions: it is a smoothing operator. So the inverse limit under [\(23.32\)](#) contains operators that map analytic functions to distributions. These are unbounded operators on L^2 functions, which is what one expects for observables in quantum mechanics.

23.8 Euclidean symmetry types and background fields

Recall Wick rotation of symmetry types from [§21.4](#). A *relativistic symmetry type* is a homomorphism of Lie groups $\lambda: G_n \rightarrow \text{O}_{1,n-1}^\uparrow$ whose image contains the identity component $\text{SO}_{1,n-1}^\uparrow$. (The latter ensures relativistic invariance: the symmetry group cannot be too small.) The kernel K of λ is the group of internal symmetries. The Wick rotation of (G_n, λ) appears in [\(21.26\)](#). For convenience, we formalize the definition.

Definition 23.34. An n -dimensional *Euclidean symmetry type* is a pair (G_n^E, λ) in which G_n^E is a Lie group and $\lambda: G_n^E \rightarrow O_n$ is a homomorphism of Lie groups whose image contains the identity component SO_n . The kernel K of λ is the group of *internal symmetries*.

We might require that K , and hence G_n^E , be compact. As commented earlier, we can also loosen the requirement that K be a Lie group and allow super Lie groups as well as homotopical generalizations of groups.

Notice that a (G_n^E, λ) -structure on a smooth manifold X induces a Riemannian structure on X , and so a principal O_n -bundle

$$(23.35) \quad \pi: \mathcal{B}_O(X) \longrightarrow X$$

of orthonormal frames. Then (23.35) carries a unique torsionfree connection, the *Levi-Civita connection*. We require that the connection on a differential G_n^E -manifold (Definition 22.16) induce the Levi-Civita connection on the orthonormal frame bundle (23.35).

So far, in Axiom System 23.14, we defined a field theory as a homomorphism out of a bordism category of Riemannian manifolds. Given a Euclidean symmetry type (G_n^E, λ) , we can form a bordism category $\text{Bord}_n(G_n^E)$ of *differential G_n^E -manifolds*. Objects are closed $(n-1)$ -manifolds embedded in a germ of a differential G_n^E -manifold of dimension n . Bordisms are compact n -manifolds equipped with a differential G_n^E -structure. Then we can modify (23.15) to be a homomorphism

$$(23.36) \quad F: \text{Bord}_n(G_n^E) \longrightarrow t\text{Vect}.$$

This is sufficient for many field theories.

Example 23.37. Consider the Euclidean symmetry type $G_n^E = SO_n \times K$ with $\lambda: G_n^E \rightarrow O_n$ given as projection to SO_n followed by inclusion into O_n . Then $\text{Bord}_n(G_n^E)$ is the domain of Wick-rotated n -dimensional theories with no time-orientation reversing symmetry and an internal symmetry group K which, in its relativistic incarnation, does not interact with symmetries that move points of spacetime. Objects and morphisms in the bordism category come equipped with a principal K -bundle with connection, which is called a “background gauge field” for the symmetry in the physics literature.

However, this does not cover all background fields that occur in field theories, both in their physical incarnation as Wick rotations of relativistic quantum field theories and in their mathematical applications. For example, we might have a background field that is a smooth map to the two-point set $\{A, B\}$. This amounts to having two theories—Type A and Type B—and allowing different components of a manifold to have different theories on them. (A physical example is the Type II superstring.) It is also useful to allow “ σ -model fields” into an arbitrary smooth manifold that may have more than two points.

We already introduced a general notion of ‘field’ in Lecture 18. Recall that $\mathbf{Man}_n$ is the category of smooth n -dimensional manifolds and local diffeomorphisms, and $\mathbf{Set}_\Delta$ is the category of simplicial sets and simplicial maps between them.

Definition 23.38. A (set of) n -dimensional *background fields* is a sheaf

$$(23.39) \quad \mathcal{F}: \mathbf{Man}_n \longrightarrow \mathbf{Set}_\Delta.$$

For example, the Wick-rotated quantum theory of a free spinor field has

$$(23.40) \quad \mathcal{F} = \{\text{Riemannian metrics, spin structures}\}.$$

In other words, for M a smooth n -manifold an object of the groupoid, or simplicial set, $\mathcal{F}(M)$ is a pair of a Riemannian metric on M and a spin structure on M . This is a differential G_n^E -structure on M for the Euclidean symmetry type $\text{Spin}_n \rightarrow \text{O}_n$. There are examples not coming from Euclidean symmetry types. A Wick-rotated *classical* σ -model into a fixed Riemannian manifold P might have $\mathcal{F} = P$, meaning $\mathcal{F}(M) = C^\infty(M, P)$ is the set of smooth maps $M \rightarrow P$. There are also background fields of a more topological nature. For example, the sheaf of n -framings has $\mathcal{F}(M)$ equal to the set of trivializations of the tangent bundle $TM \rightarrow M$, i.e., the set of sections of the frame bundle (22.12). It corresponds to the affine symmetry type $\{e\} \hookrightarrow \text{O}_n$, which does not fit Definition 23.34.

Remark 23.41. Replacing Euclidean symmetry types by more general affine symmetry types, and even more generally by background fields, is one way in which the axiom system goes beyond Wick-rotated relativistic field theory.

The domain of a Wick-rotated field theory is a bordism category of manifolds equipped with background fields from an arbitrary sheaf on $\mathbf{Man}_n$. This leads to the following variation of Axiom System 23.14.

Axiom System 23.42. Fix a positive integer n and a simplicial sheaf $\mathcal{F}$ on $\mathbf{Man}_n$. A (*Wick-rotated*) n -dimensional field theory over $\mathcal{F}$ is a symmetric monoidal functor

$$(23.43) \quad F: \text{Bord}_n(\mathcal{F}) \longrightarrow t\text{Vect}.$$

We sometimes use the language ‘a field theory on $\mathcal{F}$ ’.

Remark 23.44. The dimension n and the background fields $\mathcal{F}$ tell what “type” of field theory we have. It is useful to always articulate explicitly what n and $\mathcal{F}$ are for a given field theory.

23.9 Topological vector spaces

Now we turn to the codomain $t\text{Vect}$ of a Wick-rotated field theory. It is a symmetric monoidal category of topological vector spaces, and we need to specify it precisely. I will make some remarks in this direction following [KS, §3]. The PhD thesis [Wed] of Richard Wedeen has a more complete development of these ideas. Some basic definitions can be found in §23.13.1.

A topological vector space is the marriage of a linear space and a topological space. We require (Definition 23.66) that the vector space operations be continuous and that the topology be Hausdorff. The simplest class of topological vector spaces is the collection of *Hilbert spaces*. On

a smooth manifold M these include spaces of L^2 functions and Sobolev spaces. Many theorems about finite dimensional inner product spaces have generalizations to Hilbert spaces, and the study of linear operators is rich and detailed, including for example the spectral theorem for self-adjoint operators ([Theorem 1.46](#)). The next simplest class of topological vector spaces is the collection of *Banach spaces*. A Banach space is a complete normed linear spaces. Already here the geometry can be quite varied; Banach spaces provide a sufficiently rich playground for deep mathematical inquiry. The bounded continuous functions on any Hausdorff locally compact topological space form a Banach space, and on a smooth manifold so too do the C^k functions for any $k \geq 0$. Perhaps the next simplest class of topological vector spaces is the collection of *Fréchet spaces* ([Definition 23.67](#)). They are locally convex, metrizable, and complete, but in general their topology is not given by a norm. (It is given by a *countable* family of *seminorms*; see [Definition 23.68](#).) The canonical example of a Fréchet space is the space of smooth functions on a smooth manifold.

Remark 23.45. One can develop calculus on affine spaces over these topological vector spaces, and then define infinite dimensional manifolds locally modeled on them. Calculus on affine spaces over Banach spaces behaves much like ordinary calculus. In particular, the implicit function theorem holds. But, say, spaces of smooth maps between smooth manifolds are not Banach manifolds—they are Fréchet manifolds—and for these the implicit function theorem is much more delicate. Even on Banach spaces one may not have bump functions, so partitions of unity exist only on certain Banach manifolds, e.g., Hilbert manifolds.

The key property of *Wick-rotated* time evolution in quantum mechanics ([11.15](#)) is that it is by *contractive* operators on a Hilbert space. A contractive operator is “small”, and there are different kinds of smallness for an operator on Hilbert space: compact, Hilbert–Schmidt, trace class, etc. The Hilbert space structure is tied to unitarity, and in the general [Axiom System 23.42](#) we do not assume unitarity. So the topological vector spaces in the image of a Wick-rotated field theory are not assumed to be Hilbert spaces, and we need a notion of smallness for linear operators between more general topological vector spaces. This was a notion introduced by Grothendieck in his first major work [[Gr](#)]: *nuclear operators*. He also introduced the concept of a *nuclear space*, which is a special type of topological vector space, and also a good theory of tensor products of nuclear spaces. In fact, there is a symmetric monoidal category whose objects are nuclear vector spaces and whose morphisms are continuous linear maps; the monoidal structure is the tensor product. The definitions are complicated, and we do not tackle them in this version of the notes. An excellent summary of nuclear spaces is [[Co](#), Appendix 2] to which we refer the reader.

Example 23.46. Let M be a smooth manifold. Then the space $C^\infty(M)$ of smooth functions on M is nuclear, as is its dual $C^{-\infty}(M)$, which is a space of distributions. If M is real analytic, then the space $C^\omega(M)$ of real analytic functions is nuclear, as is its dual space. These spaces behave well under the tensor product of nuclear spaces. For example, if M_1, M_2 are smooth manifolds, then there is an isomorphism

$$(23.47) \quad C^\infty(M_1 \times M_2) \cong C^\infty(M_1) \otimes C^\infty(M_2).$$

Of particular importance to us are *nuclear Fréchet* spaces and their duals, *dual nuclear Fréchet* spaces. (The dual of a Fréchet space is not necessarily metrizable, hence is not necessarily Fréchet.) The basic properties we need are as follows:

- (1) The dual of a nuclear Fréchet space is dual nuclear Fréchet, and the dual of a dual nuclear Fréchet space is a nuclear Fréchet space.
- (2) The inverse limit of a countable sequence of nuclear maps of Fréchet spaces is a nuclear Fréchet space.
- (3) The direct limit (colimit) of a countable sequence of nuclear maps of Fréchet spaces is a dual nuclear Fréchet space.
- (4) A continuous map from a dual nuclear Fréchet space to a nuclear Fréchet space is nuclear.

Example 23.48. For a smooth manifold M , the space $C^\infty(M)$ is nuclear Fréchet and the dual space $C^{-\infty}(M)$ is dual nuclear Fréchet. For a real analytic manifold M , the space $C^\omega(M)$ of real analytic functions is dual nuclear Fréchet and the dual space $C^{-\omega}(M)$ is nuclear Fréchet.

23.10 The codomain of a field theory

Now we apply the foregoing in the context of Wick-rotated field theory. Again, I view this material as provisional.

The first task is to define a symmetric monoidal category $t\mathbf{Vect}$ which serves as the codomain of a Wick-rotated field theory, as in [Axiom System 23.42](#). In [\[KS\]](#) they take the codomain to be the category of Fréchet spaces and nuclear maps. That seems reasonable, as it gives latitude in quantum mechanics, say, to take C^∞ functions, L^2 functions, Sobolev spaces of functions, etc. However, there is not a good theory of tensor products of general Fréchet spaces. See [\[Wed\]](#) for much more in this direction.

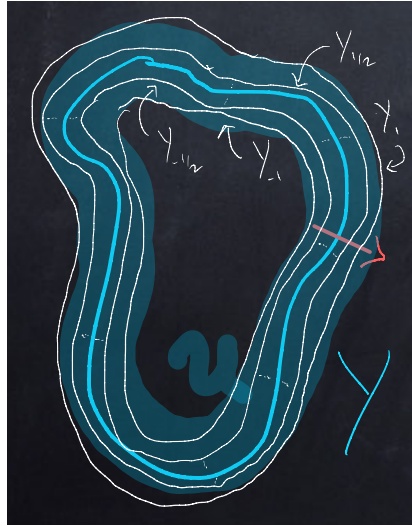


FIGURE 75. A direct system and an inverse system centered on Y

Assume given a functor (not symmetric monoidal) $F: \mathbf{Bord}_n(\mathcal{F}) \rightarrow t\mathbf{Vect}$ with values in this category. Let Y be a closed $(n-1)$ -manifold equipped with a germ U of an n -manifold, a coorientation of $Y \subset U$, and a background field $\phi \in \mathcal{F}(U)$. Represent the germ by an actual n -manifold U . Since $Y \subset U$ is cooriented and has codimension one, its normal bundle is trivializable. The tubular neighborhood theorem gives an embedding of the tubular neighborhood in U and so a foliation of

Example 23.55. For a quantum mechanical particle moving on a smooth Riemannian manifold M , as in [Example 23.33](#),⁷⁷ if $Y = \text{pt}$ then $\check{F}(Y)$ is a space of real analytic functions (a notion determined by the Hodge Laplacian) and $\hat{F}(Y)$ is its dual space. One has flexibility in choosing the Fréchet space $F(Y)$ in between.

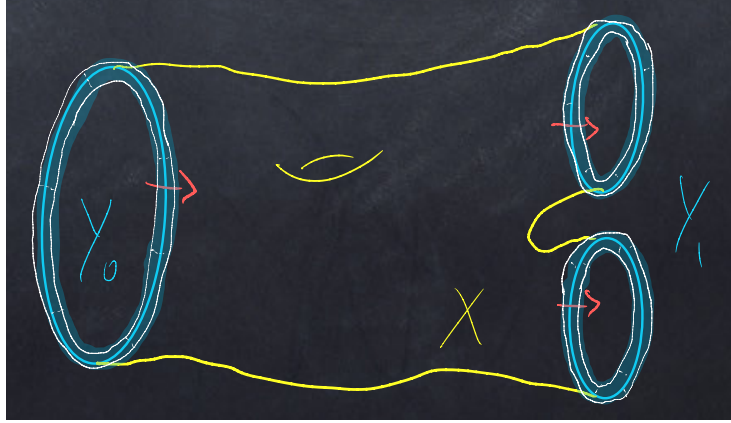


FIGURE 76. A bordism $X: Y_0 \rightarrow Y_1$

Consider now a bordism $X: Y_0 \rightarrow Y_1$, as in [Figure 76](#). Not depicted is the background field $\phi \in \mathcal{F}(X)$, which restricts on the germ of the boundaries Y_0, Y_1 to the background fields that are part of those objects. (More precisely, there is a specified isomorphism.) The functor F assigns to X a continuous linear map⁷⁸ $F(X): F(Y_0) \rightarrow F(Y_1)$. The construction of the direct and inverse limits, and the use of bordisms embedding in X between the various parallel copies of the boundary components, gives a sequence of continuous linear maps

$$(23.56) \quad \check{F}(Y_0) \longrightarrow F(Y_0) \longrightarrow \hat{F}(Y_0) \longrightarrow \check{F}(Y_1) \longrightarrow F(Y_1) \longrightarrow \hat{F}(Y_1)$$

In particular, we have a commutative diagram

$$(23.57) \quad \begin{array}{ccc} \check{F}(Y_0) & \xrightarrow{\check{F}(X)} & \check{F}(Y_1) \\ \downarrow & \nearrow \hat{F}(X) & \downarrow \\ \hat{F}(Y_0) & \xrightarrow{\hat{F}(X)} & \hat{F}(Y_1) \end{array}$$

in which the columns are injections with dense image. The rows are nuclear by virtue of each factoring through a map from a dual nuclear Fréchet space to a nuclear Fréchet space as in [\(23.56\)](#).

In this way we are led to perhaps a better choice for $t\text{Vect}$. Namely, its objects are dense inclusions $\check{Y} \rightarrow \hat{Y}$ from a dual nuclear Fréchet space to a nuclear Fréchet space and its morphisms are diagrams [\(23.57\)](#). We would also need to assume that the colimit of [\(23.51\)](#) and the limit of [\(23.53\)](#) are the spaces given by the functor F into this choice of $t\text{Vect}$. This is the approach taken in [\[Wed\]](#).

⁷⁷As written in the notes, we assumed that M is real analytic. However, we should only assume that M is smooth and use the Laplace operator to define a notion of “analytic” functions.

⁷⁸assumed nuclear by our choice of $t\text{Vect}$, but we do not use that assumption in this argument

23.11 Duality

There is an involution $*$: $\text{Bord}_n(\mathcal{F}) \rightarrow \text{Bord}_n(\mathcal{F})^{\text{op}}$ as follows. If Y is an object in $\text{Bord}_n(\mathcal{F})$, then Y^* is the same closed $(n-1)$ -manifold, the same germ of an n -manifold, the same background field, but the opposite coorientation of Y inside the germ. Similarly, if $X: Y_0 \rightarrow Y_1$ is a bordism, then $X^*: Y_1^* \rightarrow Y_0^*$ reverses the decomposition of ∂X into incoming and outgoing submanifolds.

Remark 23.58. This involution plays the role of duality in a symmetric monoidal category, as in [Definition 23.78](#), but since $\text{Bord}_n(\mathcal{F})$ is only a *semicategory* ([Remark 23.13\(1\)](#)) one needs to give more care to this point.

Kontsevich–Segal prove in [\[KS, §3\]](#) that there are isomorphisms

$$(23.59) \quad \begin{aligned} \check{F}(Y^*) &\cong \hat{F}(Y)^* \\ \hat{F}(Y^*) &\cong \check{F}(Y)^* \end{aligned}$$

Furthermore, for a bordism $X: Y_0 \rightarrow Y_1$ the image of $X^*: Y_1^* \rightarrow Y_0^*$ is dual to [\(23.57\)](#):

$$(23.60) \quad \begin{array}{ccc} \hat{F}(Y_0^*) & \xleftarrow{\hat{F}(X^*)} & \hat{F}(Y_1^*) \\ \uparrow & \swarrow \text{dashed} & \uparrow \\ \check{F}(Y_0^*) & \xleftarrow{\check{F}(X^*)} & \check{F}(Y_1^*) \end{array} = \begin{array}{ccc} \hat{F}(Y_0)^* & \xleftarrow{\hat{F}(X)^*} & \hat{F}(Y_1)^* \\ \uparrow & \swarrow \text{dashed} & \uparrow \\ \check{F}(Y_0)^* & \xleftarrow{\check{F}(X)^*} & \check{F}(Y_1)^* \end{array}$$

23.12 Reflection positivity

As stated many times, unitarity is not built into [Axiom System 23.42](#). Here we tell the additional structure that encodes unitarity in Wick-rotated form, which is *reflection positivity*. We emphasize that *reflection* is a structure and *positivity* is a condition. We rely on the material in [§23.13.2](#).

Definition 23.61. A *reflection structure* on $\text{Bord}_n(\mathcal{F})$ is an involution $\bar{}: \text{Bord}_n(\mathcal{F}) \rightarrow \text{Bord}_n(\mathcal{F})$.

We denote the involution as $Y \mapsto \bar{Y}$ and $X \mapsto \bar{X}$. For the Wick rotation of a relativistic symmetry type there is a canonical choice; see [\[FH1, §4\]](#).

Example 23.62. If $\mathcal{F} = \{\text{Riemannian metrics}\}$, then the involution is the identity. If $\mathcal{F} = \{\text{orientations, Riemannian metrics}\}$, then the involution is orientation reversal. There are canonical choices of involution for spin structures, pin structures, etc.

Definition 23.63. Suppose $F: \text{Bord}_n(\mathcal{F}) \rightarrow t\text{Vect}$ is a field theory and $\text{Bord}_n(\mathcal{F})$ is equipped with a reflection structure. Then a *reflection structure* on F is equivariance data, where $t\text{Vect}$ is equipped with the involution of complex conjugation.

Recall from [Definition 23.84](#) that a *hermitian structure* on an object Y in $\text{Bord}_n(\mathcal{F})$ is an isomorphism $h: \bar{Y} \rightarrow Y^*$ that satisfies a symmetry condition. An object Y may not admit a hermitian structure. For example, if $\mathcal{F}$ includes a Riemannian metric, then a hermitian structure on an object Y is an isometry of the metric in the germ of Y that reverses the coorientation, and not every Riemannian metric admits such a reflection symmetry. If Y has a hermitian structure, then so too does $F(Y)$, assuming that F has a reflection structure. A hermitian structure on $F(Y)$ is a hermitian metric on the underlying vector space.

Definition 23.64. A reflection structure on a theory F is *positive* if the hermitian inner product on $F(Y)$ is positive definite for all hermitian objects Y in $\text{Bord}_n(\mathcal{F})$.

If this is the case, then for any hermitian object Y we can complete $F(Y)$ in this inner product to form a Hilbert space $F^{\text{Hilb}}(Y)$. It sits in a rigging

$$(23.65) \quad \check{F}(Y) \hookrightarrow F^{\text{Hilb}}(Y) \hookrightarrow \hat{F}(Y)$$

in which each map is a dense injection.

23.13 Mathematical background

23.13.1 Topological vector spaces. The following are some basic definitions. There are many texts that cover this material, among them [RS, Ru, Tr]. We take the ground field to be $\mathbb{C}$.

Definition 23.66. A *topological vector space* is a vector space V equipped with a topology such that every point of V is closed and the operations of vector addition $V \times V \rightarrow V$ and scalar multiplication $\mathbb{C} \times V \rightarrow V$ are continuous.

A finite dimensional vector space has a unique compatible topology. One need not include the closedness of points in the definition. It is a theorem that with this definition every topological vector space is Hausdorff.

Definition 23.67. Let V be a topological vector space.

- (1) V is *locally convex* if there exists a local basis for the topology at $0 \in V$ consisting of convex open sets.
- (2) V is *metrizable* if it is the underlying topological space of a metric space.
- (3) V is a *Fréchet space* if it is locally convex, metrizable, and complete.
- (4) V is *normable* if its topology is given by a norm.

We review the definition of a norm presently.

The topology of a locally convex topological vector space can be specified by seminorms.

Definition 23.68. Let V be a vector space. A *seminorm* $\rho: V \rightarrow [0, \infty)$ is a function that satisfies

$$(23.69) \quad \begin{aligned} \rho(\xi_1 + \xi_2) &\leq \rho(\xi_1) + \rho(\xi_2), & \xi_1, \xi_2 &\in V; \\ \rho(\lambda\xi) &= |\lambda|\rho(\xi), & \xi &\in V, \lambda \in \mathbb{C}. \end{aligned}$$

Furthermore, ρ is a norm if

$$(23.70) \quad \rho(\xi) = 0 \text{ iff } \xi = 0, \quad \xi \in V.$$

A family $\{\rho_\alpha\}_{\alpha \in A}$ of seminorms *separates points* if

$$(23.71) \quad \rho_\alpha(\xi) = 0 \text{ for all } \alpha \in A \text{ implies } \xi = 0, \quad \xi \in V.$$

A separating family of seminorms $\{\rho_\alpha\}_{\alpha \in A}$ defines a locally convex topology on V , the weakest topology for which each ρ_α is continuous. This topology satisfies the conditions of [Definition 23.66](#); in particular, it is Hausdorff.

Theorem 23.72. *Let V be a locally convex topological vector space. Then V is metrizable if and only if its topology is generated by a countable family of seminorms.*

The natural maps $T: V \rightarrow W$ between topological vector spaces are those which are *linear* and *continuous*. In particular, we have the following.

Definition 23.73. Let V be a topological vector space. The *dual space* V^* is the space of continuous linear functions $V \rightarrow \mathbb{C}$.

The dual space has a natural *strong* topology. Quite generally, if V, W are topological vector spaces and $\text{Hom}(V, W)$ is the vector space of continuous linear maps, then it has a topology generated by a basis of neighborhoods $\mathcal{U}(B, U)$ of zero, where $B \subset V$ is bounded, $U \subset W$ is open, and $\mathcal{U}(B, U)$ consists of continuous linear maps $T: V \rightarrow W$ such that $T(B) \subset U$.

23.13.2 Duality and hermitian objects. For the reader's convenience we reprint some material from [\[FHI, Appendix B\]](#).

Definition 23.74. Let $\mathcal{C}$ be a category.

- (1) An *involution* of $\mathcal{C}$ is a pair (τ, η) of a functor $\tau: \mathcal{C} \rightarrow \mathcal{C}$ and a natural isomorphism $\eta: \text{id}_{\mathcal{C}} \rightarrow \tau^2$ such that for any $x \in \mathcal{C}$ we have $\tau\eta_x = \eta_{\tau x}$ as morphisms $\tau x \rightarrow \tau^3 x$.
- (2) A *fixed point* of τ is a pair (x, θ) of an object $x \in \mathcal{C}$ and an isomorphism $x \xrightarrow{\theta} \tau x$ such that $\tau\theta \circ \theta = \eta_x$ as morphisms $x \rightarrow \tau^2 x$.

If $\mathcal{C}$ is a symmetric monoidal category, then the involution τ is required to be a symmetric monoidal functor: for $x, y \in \mathcal{C}$ there is given an isomorphism $\tau x \otimes \tau y \xrightarrow{\cong} \tau(x \otimes y)$ and these isomorphisms are compatible with the symmetry and with η .

Example 23.75. Let $\mathcal{C} = \text{Vect}_{\mathbb{C}}$ be the category of complex vector spaces and linear maps. Define $\tau: \mathcal{C} \rightarrow \mathcal{C}$ to be the functor that takes complex vector spaces and linear maps to their complex conjugates. (The complex conjugate vector space is the same underlying real vector space with the sign of multiplication by $\sqrt{-1} \in \mathbb{C}$ reversed; the complex conjugate of a linear map is the same map of sets.) Then there is a canonical identification of τ^2 with $\text{id}_{\mathcal{C}}$. A fixed point is a complex vector space with a real structure.

Definition 23.76. Let $\mathcal{B}, \mathcal{C}$ be categories with involutions and $F: \mathcal{B} \rightarrow \mathcal{C}$ a functor. Then *equivariance data* for F is an isomorphism $\phi: F\tau_{\mathcal{B}} \xrightarrow{\cong} \tau_{\mathcal{C}}F$ of functors $\mathcal{B} \rightarrow \mathcal{C}$ such that for every object $x \in \mathcal{B}$ the diagram

$$(23.77) \quad \begin{array}{ccc} Fx & \xrightarrow{F\eta_{\mathcal{B}}} & F\tau_{\mathcal{B}}^2 x \\ & \searrow \eta_x & \downarrow \phi^2 \\ & & \tau_{\mathcal{C}}^2 Fx \end{array}$$

commutes.

There are additional compatibilities for a symmetric monoidal functor between symmetric monoidal categories; we do not spell them out. We often loosely say that “ F is an equivariant functor”, but it is important to remember that equivariance is data+condition, not simply a condition.

Next, we review duality in a symmetric monoidal category. Let $\mathcal{C}$ be a symmetric monoidal category and $x \in \mathcal{C}$. Denote the tensor unit by $1 \in \mathcal{C}$.

Definition 23.78. Let x be an object in a symmetric monoidal category $\mathcal{C}$. *Duality data* for x is a triple $(x^\vee, c, e)$ consisting of an object $x^\vee \in \mathcal{C}$ together with morphisms $c: 1 \rightarrow x \otimes x^\vee$ and $e: x^\vee \otimes x \rightarrow 1$ such that the compositions

$$(23.79) \quad \begin{array}{ccccc} x & \xrightarrow{c \otimes \text{id}} & x \otimes x^\vee \otimes x & \xrightarrow{\text{id} \otimes e} & x \\ x^\vee & \xrightarrow{\text{id} \otimes c} & x^\vee \otimes x \otimes x^\vee & \xrightarrow{e \otimes \text{id}} & x^\vee \end{array}$$

are identity maps. If $x_0 \xrightarrow{f} x_1$ is a morphism, then the dual morphism is the composition

$$(23.80) \quad f^\vee: x_1^\vee \xrightarrow{\text{id} \otimes c_{x_0}} x_1^\vee \otimes x_0 \otimes x_0^\vee \xrightarrow{\text{id} \otimes f \otimes \text{id}} x_1^\vee \otimes x_1 \otimes x_0^\vee \xrightarrow{e_{x_1} \otimes \text{id}} x_0^\vee$$

The morphism c is called *coevaluation* and e is called *evaluation*. We say that $x^\vee$ is “the” dual to x since any two triples of duality data are uniquely isomorphic. Assuming all objects have duals, we can make choices of duality data for all objects at once and so obtain a duality involution δ on $\mathcal{C}$, but δ does not satisfy Definition 23.74 since the direction of morphisms is reversed (23.80); in other words, δ is a functor to the *opposite* category.

Definition 23.81. Let $\mathcal{C}$ be a category.

- (1) A *twisted involution* of $\mathcal{C}$ is a pair (δ, η) of a functor $\delta: \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ and a natural isomorphism $\eta: \text{id}_{\mathcal{C}} \rightarrow \delta^{\text{op}} \circ \delta$ such that for any $x \in \mathcal{C}$ we have $\delta \eta_x \circ \eta_{\delta x} = \text{id}_{\delta x}$.
- (2) A *fixed point* of δ is a pair (x, θ) of an object $x \in \mathcal{C}$ and an isomorphism $x \xrightarrow{\theta} \delta x$ such that $\delta \theta \circ \eta_x = \theta$ as morphisms $x \rightarrow \delta x$.

Example 23.82. For $\mathcal{C} = f\text{Vect}_{\mathbb{C}}$ the category of finite dimensional complex vector spaces, the duality involution $\delta: \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ maps a vector space V to its dual V^* and a linear map $f: V \rightarrow W$ to $f^*: W^* \rightarrow V^*$. A fixed point of δ is a vector space V equipped with a nondegenerate symmetric bilinear form. A fixed point for the composite of duality and complex conjugation (Example 23.75) is a complex vector space V with a nondegenerate hermitian form.

Remark 23.83. There is a higher categorical context for Definition 23.81. Let Cat denote the 2-category of categories. There is an involution $\alpha: \text{Cat} \rightarrow \text{Cat}$ that sends a category $\mathcal{C}$ to its opposite $\mathcal{C}^{\text{op}}$. (There is an extra categorical layer over Definition 23.74: there is a triple (α, η_1, η_2) of data and a single condition.) A twisted involution in the sense of Definition 23.81 is fixed point data for α .

Definition 23.84. Let (τ, η) be an involution on a symmetric monoidal category $\mathcal{C}$. A *hermitian structure* on an object $x \in \mathcal{C}$ is an isomorphism $h: \tau x \rightarrow x^\vee$ such that the composition

$$(23.85) \quad \tau x \cong \tau((x^\vee)^\vee) \xrightarrow{\tau(h^\vee)} \tau((\tau x)^\vee) \cong \tau^2(x^\vee) \xrightarrow{\eta^{-1}} x^\vee$$

is equal to h .

Observe that if $F: \mathcal{B} \rightarrow \mathcal{C}$ is an equivariant symmetric monoidal functor between symmetric monoidal categories with involution, as in Definition 23.76, then the image of a hermitian structure on an object $b \in \mathcal{B}$ is a hermitian structure on Fb .

Lecture 24: Classical bordism

The first semester of the course ended with Segal’s axioms (Axiom System 23.14) for Wick-rotated quantum field theory. *Topological* field theories are a special case, as we will make precise in a few lectures. Another route is to regard a topological field theory as a categorical version of a classical bordism invariant. We take that route in the next lecture. In this lecture we review some salient points of classical bordism theory, as developed by Pontrjagin, Thom, and other topologists in the middle of the last century.

We begin with the definition of a bordism and the equivalence relation that it determines. Equivalence classes are abelian groups, and we compute them in low dimensions. Then we state some general theorems about these bordism groups. There is a variant for oriented manifolds, which we introduce, and then we state some basic theorems about oriented bordism. An orientation is an example of a tangential structure, and there are bordism groups for arbitrary tangential structures. As a first link to field theory, we illustrate that tangential structures are equivalent to *locally constant* sheaves of the type that define fields (§18.1). Finally, we introduce Milnor’s *bordism categories* as a “categorification” of bordism groups.

Some of this material was introduced more quickly in Section 23.2.

24.1 Bordism groups

We reprise Definition 23.7.

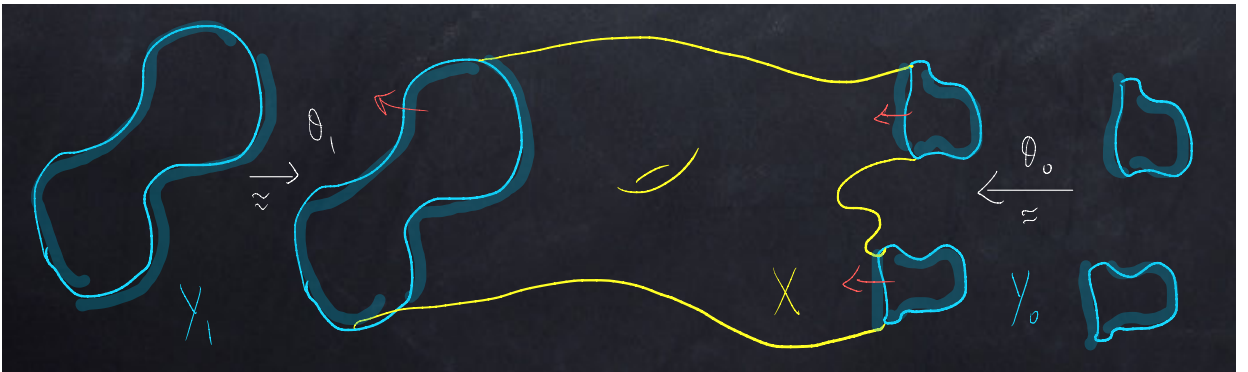


FIGURE 77. A bordism $X: Y_0 \rightarrow Y_1$

Definition 24.1. Fix $d \in \mathbb{Z}^{\geq 0}$. Let Y_0, Y_1 be closed d -dimensional manifolds. A *bordism* $(X, p, \theta_0, \theta_1)$ from Y_0 to Y_1 consists of a compact $(d+1)$ -dimensional manifold X with boundary, a partition

$p: \partial X \rightarrow \{0, 1\}$ of its boundary, and disjoint embeddings

$$(24.2) \quad \theta_0: [0, +1) \times Y_0 \longrightarrow X$$

$$(24.3) \quad \theta_1: (-1, 0] \times Y_1 \longrightarrow X$$

such that $\theta_i(0, Y_i) = (\partial X)_i$, $i = 0, 1$, where $(\partial X)_i = p^{-1}(i)$.

We usually do not specify all of the data and simply denote a bordism as $X: Y_0 \rightarrow Y_1$.

Definition 24.4. Let Y_0, Y_1 be closed d -dimensional manifolds. Then Y_0 is *bordant* to Y_1 if there exists a bordism $X: Y_0 \rightarrow Y_1$.

Proposition 24.5. *Bordism is an equivalence relation.*

We leave the proof to the reader. The main point that bears thought is the gluing of bordisms that enters the proof of transitivity: one needs to smooth the space obtained by naive gluing. Later in the lecture we modify [Definition 24.1](#) by replacing 1-sided collars with 2-sided collars. This obviates the need to smooth.

Let Ω_d denote the set of equivalence classes. The operation of disjoint union passes to equivalence classes.

Lemma 24.6. $(\Omega_d, \sqcup)$ is an abelian group in which each element has order 2.

Proof. First, observe that the empty set is a manifold of any dimension; as a manifold of dimension d we denote it by $\emptyset^d$. Then $\emptyset^d$ is an identity element for disjoint union. Also, for any closed d -manifold Y there is a canonical bordism $X = [0, 1] \times Y: \emptyset^d \rightarrow Y^{\sqcup 2}$. $\square$

Next, we determine the bordism groups in low dimensions.

Theorem 24.7.

- (1) $\Omega_0 \cong \mathbb{Z}/2\mathbb{Z}$ with generator the bordism class of a point.
- (2) $\Omega_1 = 0$.
- (3) $\Omega_2 \cong \mathbb{Z}/2\mathbb{Z}$ with generator the bordism class of the real projective plane $\mathbb{RP}^2$.

Proof. Every compact 0-manifold is a finite set of points. A basic theorem in differential topology asserts that the boundary of a compact 1-manifold with boundary has an even number of points. Now (1) follows.

The same classification theorem asserts that every compact 1-manifold is a finite disjoint union of circles. Since a circle bounds a 2-manifold, for example the 2-disk, every compact 1-manifold is *null bordant*.

Use the classification of compact surfaces, which we can take to be connected. An oriented surface is a finite connect sum of tori, and each of these bounds (a handlebody). An unoriented surface is a finite connect sum of real projective planes. We claim that $\mathbb{RP}^2$ does *not* bound. For if $\mathbb{RP}^2 = \partial X$ for a compact 3-manifold X with boundary, then consider the double $D = X \cup_{\mathbb{RP}^2} X$, which is a closed 3-manifold. Its Euler number

$$(24.8) \quad \text{Euler}(D) = \text{Euler}(X) + \text{Euler}(X) - \text{Euler}(\mathbb{RP}^2) = 2 \text{Euler}(X) - 1$$

is odd. But that contradicts the fact that the Euler number of a closed odd dimensional manifold vanishes. Hence $\mathbb{RP}^2$ does not bound.

The connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2$ does bound. One argument is that, in general, a connected sum is bordant to a disjoint union, a statement whose precise formulation we leave to the reader. A more direct argument in this case begins with the recognition that $\mathbb{RP}^2 \# \mathbb{RP}^2$ is diffeomorphic to a Klein bottle K . A Klein bottle is the total space of a fiber bundle $K \rightarrow S^1$ with fiber S^1 , constructed by taking the cylinder $[0, 1] \times S^1$ and gluing the two boundary circles via a reflection that identifies them. Then K is the boundary of the total space of a fiber bundle with fiber D^2 , obtained by “filling in” the fibers of $K \rightarrow S^1$. (Glue the ends of $[0, 1] \times D^2$.) $\square$

Remark 24.9. The proof shows that the mod 2 Euler number

$$(24.10) \quad \text{Euler}_2: \Omega_2 \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

is an isomorphism. The map (24.10) is an example of a *bordism invariant*.

Form the graded abelian group

$$(24.11) \quad \Omega_\bullet = \bigoplus_{d \in \mathbb{Z}^{\geq 0}} \Omega_d.$$

Cartesian product passes to bordism classes to give a map⁷⁹

$$(24.12) \quad \Omega_{d_1} \times \Omega_{d_2} \longrightarrow \Omega_{d_1+d_2}.$$

Then $(\Omega, \sqcup, \times)$ is a commutative ring. In his thesis [T], Thom determined the structure of this *bordism ring*.

Theorem 24.13 (Thom). *There is an isomorphism*

$$(24.14) \quad \Omega_\bullet \cong \mathbb{Z}/2\mathbb{Z}[x_2, x_3, x_5, x_6, x_8, \dots],$$

where the infinite polynomial ring on the right hand side has a single generator in every nonnegative degree not of the form $2^i - 1$.

Furthermore, for k even we can take the generator to be the bordism class of real projective space:

$$(24.15) \quad x_k = [\mathbb{RP}^k].$$

For k odd, write $k + 1 = 2^r(2s + 1)$ and set $m = 2^r - 1$ and $\ell = 2^r s$. Then

$$(24.16) \quad x_k = [Y],$$

where Y is the *Dold manifold* obtained from $S^m \times \mathbb{CP}^\ell$ as the quotient by the free involution that is the antipodal map on S^m and complex conjugation on $\mathbb{CP}^\ell$. Observe that Y is the total space of a fiber bundle

$$(24.17) \quad \mathbb{CP}^\ell \longrightarrow Y \longrightarrow \mathbb{RP}^m$$

⁷⁹Whereas disjoint union is used in bordism categories and topological field theory from the beginning, Cartesian product does not enter the definitions. It is used to define *dimensional reduction*.

24.2 Oriented bordism

Recall that the boundary of an oriented manifold with boundary inherits a canonical orientation.⁸⁰ Then to define an *oriented bordism*, stipulate in [Definition 24.1](#) that the manifolds Y_0, Y_1 carry orientations, that the Cartesian products in the domains of (24.2) and (24.3) carry the product orientation, and that θ_0, θ_1 are orientation-preserving. Let $\tilde{\Omega}_d$ be the set of equivalence classes, which is a commutative monoid under disjoint union. Then $(\tilde{\Omega}_d, \sqcup)$ is an abelian group: the negative of $[Y]$ is the bordism class $[-Y]$ of the orientation-reversed manifold.

Remark 24.18. The partition $p: \partial X \rightarrow \{0, 1\}$ of the boundary of an oriented bordism does not interact with the orientation. More generally, this partition into incoming and outgoing boundary components does not interact with any tangential structure.

In low dimensions we have

Theorem 24.19.

- (1) $\tilde{\Omega}_0 \cong \mathbb{Z}$ with generator a point.
- (2) $\tilde{\Omega}_1 = 0$.
- (3) $\tilde{\Omega}_2 = 0$.
- (4) $\tilde{\Omega}_3 = 0$.
- (5) $\tilde{\Omega}_4 \cong \mathbb{Z}$ with generator the complex projective plane $\mathbb{CP}^2$.
- (6) $\tilde{\Omega}_5 \cong \mathbb{Z}/2\mathbb{Z}$ with generator the Dold manifold Y that is the total space of a fiber bundle

$$(24.20) \quad \mathbb{CP}^2 \longrightarrow Y \longrightarrow \mathbb{RP}^1.$$

The proof of (1) is based on the basic theorem that underlies oriented intersection theory: the boundary of a compact oriented 1-manifold with boundary has the same number of positively oriented points as it does negatively oriented points. The proofs of (2) and (3) follow from the classification of oriented 1- and 2-manifolds. We do not comment on the proofs of (4), (5), and (6).

As in (24.11) there is a bordism ring

$$(24.21) \quad \tilde{\Omega}_\bullet = \bigoplus_{d \in \mathbb{Z}^{\geq 0}} \tilde{\Omega}_d.$$

The following theorem gives some general structure results about this graded commutative ring.

Theorem 24.22.

- (1) *There is an isomorphism*

$$(24.23) \quad \tilde{\Omega}_\bullet / \text{torsion} \cong \mathbb{Z}[y_4, y_8, y_{12}, \dots],$$

where the generators $y_{4\ell} = [\mathbb{CP}^{2\ell}]$ are the bordism classes of complex projective spaces.

- (2) *All torsion in $\tilde{\Omega}_\bullet$ is of order 2.*

⁸⁰The acronym ‘ONF’ simultaneously stands for ‘Outward Normal First’ and ‘One Never Forgets’.

24.3 Tangential structures and collars

An orientation is an example of a tangential structure, which is a geometric structure “on” the tangent bundle of a smooth manifold. Here are more examples of a tangential structure on a manifold M .

Example 24.24.

- (1) a spin structure on M
- (2) a continuous map $M \rightarrow \{A, B, C\}$
- (3) an orientation and a principal SU_2 -bundle $P \rightarrow M$ uncorrelated to the tangent bundle

The general definition of a tangential structure uses classifying spaces. Recall that for $n \in \mathbb{Z}^{>0}$ there is a classifying space $B\mathrm{GL}_n\mathbb{R}$ and a universal rank n real vector bundle $E_n \rightarrow B\mathrm{GL}_n\mathbb{R}$ so that for every n -dimensional manifold M there exist classifying maps

$$(24.25) \quad \begin{array}{ccc} TM & \xrightarrow{\gamma} & E_n \\ \downarrow & & \downarrow \\ M & \xrightarrow{\bar{\gamma}} & B\mathrm{GL}_n\mathbb{R} \end{array}$$

The diagram is a pullback and γ is linear on fibers.

Definition 24.26. Fix a nonnegative integer n .

- (1) An n -dimensional tangential structure is a topological space $\mathcal{X}_n$ and a fibration

$$(24.27) \quad \pi_n: \mathcal{X}_n \longrightarrow B\mathrm{GL}_n\mathbb{R}.$$

- (2) Let M be a smooth n -dimensional manifold and $\bar{\gamma}: M \rightarrow B\mathrm{GL}_n\mathbb{R}$ a classifying map for its tangent bundle. Then an $\mathcal{X}_n$ -structure on M is a lift

$$(24.28) \quad \begin{array}{ccc} & \mathcal{X}_n & \\ & \downarrow \pi_n & \\ M & \xrightarrow{\bar{\gamma}} & B\mathrm{GL}_n\mathbb{R} \end{array}$$

A symmetry type $\lambda_n: G_n \rightarrow \mathrm{GL}_n\mathbb{R}$ in the sense of [Definition 22.6](#) induces a tangential structure $B\lambda_n: BG_n \rightarrow B\mathrm{GL}_n\mathbb{R}$. [Example 24.24\(1\)](#) is the case $\mathrm{Spin}_n \rightarrow \mathrm{GL}_n\mathbb{R}$ and [Example 24.24\(3\)](#) is the case $\mathrm{GL}_n\mathbb{R} \times \mathrm{SU}_2 \rightarrow \mathrm{GL}_n\mathbb{R}$; [Example 24.24\(2\)](#) is not of this type. (Observe that BG_n is connected whereas [Example 24.24\(2\)](#) is the tangential structure $B\mathrm{GL}_n\mathbb{R} \times \{A, B, C\} \rightarrow B\mathrm{GL}_n\mathbb{R}$.)

Tangential structures were introduced into bordism theory by Dan Lashof [[La](#)]; see [[Sto](#), §2] for an exposition. Here the dimension d of [Definition 24.1](#) and the dimension n of [Definition 24.26](#) are related by

$$(24.29) \quad n = d + 1.$$

It is convenient to equip closed $d = (n - 1)$ -dimensional manifolds with collars.

Definition 24.30.

- (1) Let Y be a $(n-1)$ -manifold. A *collar structure* $(\tilde{Y}, \iota)$ on Y is an n -manifold $\tilde{Y}$ together with an embedding $\iota: Y \hookrightarrow \tilde{Y}$ such that there exists a diffeomorphism $\tilde{Y} \xrightarrow{\sim} (-1, 1) \times Y$ that maps $\iota(Y) \subset \tilde{Y}$ to $\{0\} \times Y \subset (-1, 1) \times Y$.
- (2) A *directed collar structure* is in addition an orientation o of the normal real line bundle

$$(24.31) \quad \nu_{\iota(Y) \subset \tilde{Y}} \longrightarrow \iota(Y).$$

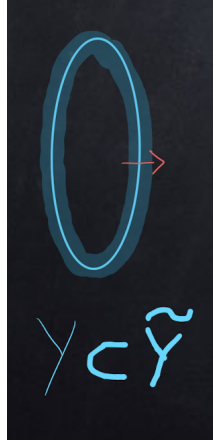


FIGURE 78. A closed $(n-1)$ -manifold Y with a directed collar structure $Y \subset \tilde{Y}$

We simply use ‘ Y ’ to denote a (*directed*) *collared manifold*, and we identify Y with its image $\iota(Y)$. The diffeomorphism to a Cartesian product is not part of the data; it is a condition. The *coorientation* o is a datum that is not usually part of the structure of this *thickening* of Y , but it is crucial for us. In **Figure 78** it is depicted as a red arrow, which can be called an *arrow of time*.⁸¹

Now we indicate the modification of **Definition 24.1** to include a tangential structure. We leave the reader to define a collared structure on a manifold with boundary. (It is a collaring of the boundary.)

Definition 24.32. Fix $d \in \mathbb{Z}^{\geq 0}$ and an $n = (d+1)$ -dimensional tangential structure $\mathcal{X}_n \rightarrow BGL_n \mathbb{R}$. Let $Y_0 \subset \tilde{Y}_0, Y_1 \subset \tilde{Y}_1$ be directed collared closed d -dimensional manifolds, and suppose $\tilde{Y}_0, \tilde{Y}_1$ are equipped with an $\mathcal{X}_n$ -structure. A *bordism* $(X, p, \theta_0, \theta_1)$ from Y_0 to Y_1 consists of a collared compact n -dimensional manifold $X \subset \tilde{X}$ with boundary, an $\mathcal{X}_n$ -structure on $\tilde{X}$, a partition $p: \partial X \rightarrow \{0, 1\}$ of the boundary of X , and disjoint embeddings

$$(24.33) \quad \theta_0: \tilde{Y}_0 \longrightarrow \tilde{X}$$

$$(24.34) \quad \theta_1: \tilde{Y}_1 \longrightarrow \tilde{X}$$

such that $\theta_i(Y_i) = (\partial X)_i$, $i = 0, 1$, where $(\partial X)_i = p^{-1}(i)$; the maps θ_0, θ_1 preserve $\mathcal{X}_n$ -structures; and the image of the coorientation of $Y_0 \subset \tilde{Y}_0$ under θ_0 points into X whereas the image of the coorientation of $Y_1 \subset \tilde{Y}_1$ under θ_1 points out of X .

⁸¹When we come to bordism categories, the $(n-1)$ -manifold Y represents space and the n -manifold $\tilde{Y}$ represents a Wick-rotated spacetime. One can imagine it carries a Lorentz metric, so the transverse direction to $Y \subset \tilde{Y}$ is real (non-Wick rotated) time.

We remark that “preservation” of $\mathcal{X}_n$ -structures is often data as well as a condition. The reader should contemplate how the collars facilitate the gluing necessary to prove that $\mathcal{X}_n$ -bordism is an equivalence relation. The bordism group of $\mathcal{X}_n$ -manifolds is denoted $\Omega_d(\mathcal{X}_{d+1})$.

Remark 24.35. One needn’t use the collaring in this topological context. Rather, it is usual to use an “infinitesimal collaring”, namely a stabilization $\underline{\mathbb{R}} \oplus TY$ of the tangent bundle to Y , obtained by direct sum with a trivial line bundle $\underline{\mathbb{R}} \rightarrow Y$ of rank one. The advantages of the collarings are: (1) they facilitate smooth gluing, and (2) they are necessary for nontopological theories, so provide a consistent framework across all theories.

Remark 24.36. An n -dimensional tangential structure induces a tangential structure in all dimensions $m < n$ by pullback:

$$(24.37) \quad \begin{array}{ccc} \mathcal{X}_m & \dashrightarrow & \mathcal{X}_n \\ \pi_m \downarrow & & \downarrow \pi_n \\ BGL_m \mathbb{R} & \longrightarrow & BGL_n \mathbb{R} \end{array}$$

There is a notion of a *stable* tangential structure, which is a collection of n -dimensional tangential structures for all $n \in \mathbb{Z}^{>0}$ with a commutative diagram

$$(24.38) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{X}_{n-1} & \longrightarrow & \mathcal{X}_n & \longrightarrow & \cdots \\ & & \pi_{n-1} \downarrow & & \downarrow \pi_n & & \\ \cdots & \longrightarrow & BGL_{n-1} \mathbb{R} & \longrightarrow & BGL_n \mathbb{R} & \longrightarrow & \cdots \end{array}$$

Orientations and spin structures are stable, for example.

24.4 Locally constant sheaves of fields and tangential structures

Recall the definition of fields from [Section 18.1](#). Fix a dimension $n \in \mathbb{Z}^{>0}$, let $\mathbf{Man}_n$ be the category of smooth n -manifolds and local diffeomorphisms, and suppose

$$(24.39) \quad \mathcal{F}: \mathbf{Man}_n^{\text{op}} \longrightarrow \mathbf{Set}_{\Delta}$$

is a sheaf. Our task is to characterize sheaves of *topological* fields among all sheaves and relate them to tangential structures.

Example 24.40. Examples of topological fields: orientations, spin structures, principal G -bundles for a finite group G , maps to a discrete space. Examples of nontopological fields: Riemannian metrics, G -connections (G a positive dimensional Lie group), maps to a smooth manifold of positive dimension.

We begin with a standard definition.

Definition 24.41. Suppose S is a topological space, $\mathbf{Open}(S)$ is the category of open subsets and inclusions, and $\mathcal{F}: \mathbf{Open}(S)^{\text{op}} \rightarrow \mathbf{Set}$ is a sheaf on S .

- (1) The *stalk* $\mathcal{F}_s$ of $\mathcal{F}$ at $s \in S$ is the direct limit

$$(24.42) \quad \mathcal{F}_s = \varinjlim_{U \ni s} \mathcal{F}(U)$$

of values of $\mathcal{F}$ over open sets that contain s .

- (2) $\mathcal{F}$ is *locally constant* if for all $s \in S$ there exists an open neighborhood $U \subset S$ such that the restriction map

$$(24.43) \quad \mathcal{F}(U) \longrightarrow \mathcal{F}_{s'}$$

is an isomorphism for all $s' \in U$.

Our situation has a variant in the domain and the codomain. The codomain of (24.39) is the category of simplicial sets, not sets, and for those we replace the requirement that (24.43) be an isomorphism of sets by the requirement that it be a weak equivalence of simplicial sets.⁸² As for the domain, $\mathbf{Man}_n$, we make the following definition.

Definition 24.44. Let $\mathcal{F}: \mathbf{Man}_n^{\text{op}} \rightarrow \mathbf{Set}_\Delta$ be a simplicial sheaf. The *stalk* of $\mathcal{F}$ is the colimit

$$(24.45) \quad \mathcal{F}_0 = \varinjlim_{r \rightarrow 0} \mathcal{F}(B_r),$$

where $B_r \in \mathbb{A}^n$ is the open ball of radius r with center 0 in affine n -space.

Regarding a simplicial set as a sequence of sets, this is a sequence of colimits of sets. Observe that the group of diffeomorphisms $f: B_r \rightarrow B_r$ such that $f(0) = 0$ acts on $\mathcal{F}(B_r)$, so the group of *germs* at 0 of these diffeomorphisms acts on the stalk $\mathcal{F}_0$.

Definition 24.46. A simplicial sheaf $\mathcal{F}: \mathbf{Man}_n^{\text{op}} \rightarrow \mathbf{Set}_\Delta$ is *locally constant* if the restriction map

$$(24.47) \quad \mathcal{F}(B_r) \longrightarrow \mathcal{F}_0$$

is a weak equivalence for all $r > 0$.

The condition only needs to hold for sufficiently small r , and then it holds for all r .

We sketch an argument that locally constant simplicial sheaves on $\mathbf{Man}_n$ are equivalent to n -dimensional tangential structures. Let $\mathcal{F}: \mathbf{Man}_n^{\text{op}} \rightarrow \mathbf{Set}_\Delta$ be locally constant. Identify the simplicial set $\mathcal{F}_0$ as a topological space.⁸³ The local constancy condition implies that there is a “homotopical action” of the group $\text{GL}_n \mathbb{R}$ on $\mathcal{F}_0$. This is equivalent to a fibration $\pi: \mathcal{X}_n \rightarrow \text{BGL}_n \mathbb{R}$. Conversely, given such a fibration we define a locally constant sheaf whose sections on a smooth n -manifold M are lifts in the diagram (24.28).

⁸²A map of simplicial sets is a weak equivalence if its geometric realization is a weak homotopy equivalence of topological spaces. For introductions to simplicial sets, see [Fr, R11] and the references therein.

⁸³Simplicial sets are a model (in a technical sense) for topological spaces.

24.5 Bordism categories

We elaborate and slightly modify [Definition 23.11](#).

The basic idea is to promote bordisms from a condition ([Definition 24.4](#)) to data. To obtain a category, and in particular to have an associative composition law, we define morphisms to be *equivalence classes* of bordisms.⁸⁴ We use the language of tangential structures; by the preceding, we could use locally constant sheaves on $\mathbf{Man}_n$ instead.

Definition 24.48. Fix $n \in \mathbb{Z}^{>0}$ and let $\pi_n: \mathcal{X}_n \rightarrow BGL_n\mathbb{R}$ be an n -dimensional tangential structure. The *bordism category* $\mathrm{Bord}_{\langle n-1, n \rangle}(\mathcal{X}_n)$ has objects closed $(n-1)$ -manifolds Y with directed collar structure and $\mathcal{X}_n$ -structure. A morphism $X: Y_0 \rightarrow Y_1$ is a diffeomorphism class of bordisms from Y_0 to Y_1 . It has a symmetric monoidal structure under disjoint union.

Milnor [\[Mi2\]](#) introduced bordism categories in his proof of the h-cobordism theorem. We'll have more to say about them in the next lecture.

Lecture 25: Topological field theory

In this lecture we introduce *topological field theory* as a “categorification” of a bordism invariant. I put ‘categorification’ in quotes since it, like quantization, is an art rather than a functor: there is no systematic procedure to “categorify” a mathematical structure. In the next lecture we define topological field theory as a special case of Wick-rotated quantum field theory.⁸⁵ These two different points of view complement each other. The relationship of topological field theory to bordism theory is sharper for *invertible* field theories, as we will see later in the course.

We begin with a brief discussion of bordism invariants. They play an important role in the topology of manifolds, but we will not pursue that here. We then give the definition of a topological field theory. In this lecture we do not produce any examples—there will be plenty soon—but rather we explain some general features. First, we pick up the discussion in [§24.5](#) and elaborate on bordism categories, in particular exploring the manifestation of diffeomorphisms in them. Then we introduce the concept of *duality* in symmetric monoidal categories, an important finiteness condition. Every object in a bordism category is dualizable, and this implies strong constraints on a topological field theory. It also implies that the category of topological field theories (with fixed domain and codomain) is a *groupoid*. This leads to a *space* of topological field theories.

⁸⁴This is familiar in the definition of the fundamental groupoid $\pi_{\leq 1}S$ of a topological space S : the points of S are objects of $\pi_{\leq 1}S$ and a morphism $s_0 \rightarrow s_1$ between points is a *homotopy class* of paths from s_0 to s_1 .

⁸⁵We purposely drop the word ‘quantum’ in ‘topological field theory’. Indeed, (exponentiated) classical field theories with imaginary time fit the Segal axioms for a Wick-rotated field theory, and are special among them: they are invertible.

25.1 Bordism invariants

Fix $d \in \mathbb{Z}^{\geq 0}$. Fix too a $(d+1)$ -dimensional tangential structure $\pi_{d+1}: \mathcal{X}_{d+1} \rightarrow BGL_{d+1}\mathbb{R}$, as in [Definition 24.26](#). There is a bordism group $\Omega_d(\mathcal{X}_{d+1})$ of d -dimensional (collared) manifolds equipped with an $\mathcal{X}_{d+1}$ -structure; see [Definition 24.32](#).⁸⁶

Definition 25.1. Let A be an abelian group. A d -dimensional A -valued bordism invariant of $\mathcal{X}_d$ -manifolds is a homomorphism of abelian groups

$$(25.2) \quad f: \Omega_d(\mathcal{X}_{d+1}) \longrightarrow A.$$

Recall from [Remark 24.36](#) that a $(d+1)$ -dimensional tangential structure induces a d -dimensional tangential structure.

Remark 25.3. A bordism invariant has a domain and a codomain. The domain is determined by two pieces of data: a dimension d and a tangential structure π_{d+1} . The codomain is a choice of abelian group. If we fix the domain and codomain, then we can speak of the abelian group of bordism invariants of a given type: we can compose two bordism invariants using the abelian group law of the codomain. A topological field theory has analogous choices of domain and codomain. Indeed, so too does a more general field theory—either relativistic or Wick-rotated.

We have already encountered an example of a bordism invariant; see [Remark 24.9](#). Here are more examples.

Example 25.4. Fix a positive integer $k \in \mathbb{Z}^{\geq 0}$. Then the *signature* is an integer-valued bordism invariant

$$(25.5) \quad \text{Sign}: \Omega_{4k}(\text{orientation}) \longrightarrow \mathbb{Z}.$$

It is defined as follows. The middle rational cohomology of a closed oriented $4k$ -manifold Y carries a symmetric bilinear form

$$(25.6) \quad b_Y: H^{2k}(Y; \mathbb{Q}) \times H^{2k}(Y; \mathbb{Q}) \xrightarrow{\smile} H^{4k}(Y; \mathbb{Q}) \xrightarrow{[Y]} \mathbb{Q},$$

where the second arrow in (25.6) is evaluation against the fundamental class. Poincaré duality implies that the form b_Y is nondegenerate. Such a bilinear form has two invariants $p, n \in \mathbb{Z}^{\geq 0}$. One can define p as the dimension of a maximal subspace of $H^{2k}(Y; \mathbb{Q})$ on which b_Y is positive definite; the number n defined by replacing ‘positive’ $\rightarrow$ ‘negative’. It is a lemma in linear algebra that these integers are well-defined. Note $p+n = \dim H^{2k}(Y; \mathbb{Q})$. Alternatively, there exists a basis $e_1, e_2, \dots$ of $H^{2k}(Y; \mathbb{Q})$ that diagonalizes b_Y :

$$(25.7) \quad b_Y(e_i, e_j) = \pm \delta_{ij} = \begin{cases} \pm 1, & i = j; \\ 0, & i \neq j. \end{cases}$$

⁸⁶We neglected to spell that out in the current version of [Lecture 24](#).

Then p is the number of basis elements e_i for which the sign is $+$, and n is the number of basis elements e_i for which the sign is $-$. The signature of Y is the difference:

$$(25.8) \quad \text{Sign } Y = p - n.$$

It is a basic theorem of Thom that the signature of a closed oriented manifold is a bordism invariant.

Remark 25.9. A bordism invariant is usually touted to be a *global* invariant, as opposed to other invariants that are more manifestly local. In differential topology one emphasizes that all smooth manifolds are locally diffeomorphic, so the subject is about global properties and invariants, of which one is the signature. Nonetheless, we will see from our study of invertible field theory that bordism invariants are *local* in a strong sense.

Example 25.10. The mod 2 Euler number (24.10) of a 2-dimensional closed manifold is a $\mathbb{Z}/2\mathbb{Z}$ -valued bordism invariant. It is manifestly local as Euler's formula illustrates. Namely, suppose Y is a closed 2-manifold equipped with a triangulation. Then

$$(25.11) \quad \text{Euler}_2(Y) = V - E + F \pmod{2},$$

where V is the number of vertices, E is the number of edges, and F is the number of faces in the triangulation.

Remark 25.12. The integer Euler invariant is a bordism invariant of a different kind; see [Rei].

25.2 Topological field theory

The categorification we now undertake transforms:

$$(25.13) \quad \begin{array}{l} \text{abelian group} \rightsquigarrow \text{symmetric monoidal category} \\ \text{homomorphism of abelian groups} \rightsquigarrow \text{symmetric monoidal functor} \\ \Omega_d(\mathcal{X}_{d+1}) \rightsquigarrow \text{Bord}_{\langle n-1, n \rangle}(\mathcal{X}_n) \\ A \rightsquigarrow \mathcal{C} = \text{Vect}_{\mathbb{F}}, \text{Ab}, \text{Mod}_R, \dots \end{array}$$

Note the switch in emphasis from the dimension d of the closed manifolds that made up a bordism group to the dimension $n = d + 1$ of the bordisms between them, which comprise morphisms in the bordism category. The abelian group A , which is the codomain of a bordism invariant, categorifies to a symmetric monoidal category, such as the category of vector spaces over a field $\mathbb{F}$, the category of abelian groups, the category of modules over a ring R , etc.

Remark 25.14. At this point the reader should review the definitions of a symmetric monoidal category, a symmetric monoidal functor, and a natural transformation of symmetric monoidal functors. These may be found in §22.3; more details are in [F4].

The replacements (25.13) lead to the following categorification of Definition 25.1.

Definition 25.15. Fix $n \in \mathbb{Z}^{\geq 0}$ and an n -dimensional tangential structure $\pi_n: \mathcal{X}_n \rightarrow BGL_n \mathbb{R}$. Let $\mathcal{C}$ be a symmetric monoidal category. A $\mathcal{C}$ -valued topological field theory of $\mathcal{X}_n$ -manifolds is a homomorphism

$$(25.16) \quad F: \text{Bord}_{\langle n-1, n \rangle}(\mathcal{X}_n) \longrightarrow \mathcal{C}.$$

The term ‘homomorphism’, used here in place of ‘symmetric monoidal structure’, follows standard usage in algebra.

Remark 25.17. **Remark 25.3** applies: a topological field theory has a domain determined by the same data as determines the domain of a bordism invariant, and a categorified choice of codomain. The domain and codomain determine the ‘type’ of topological field theory.

A $\mathcal{C}$ -valued topological field theory assigns to closed manifolds

$$(25.18) \quad \begin{aligned} Y^{n-1} &\rightsquigarrow \text{an object } y \in \mathcal{C} \\ X^n &\rightsquigarrow \text{an element of } \Omega\mathcal{C} = \text{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}}) \end{aligned}$$

In a general symmetric monoidal category, $\Omega\mathcal{C} = \text{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$ is a commutative monoid. In a symmetric monoidal category that is linear over a field $\mathbb{F}$, it is a commutative algebra over $\mathbb{F}$. It is in this abelian group or commutative algebra that a topological field theory assigns invariants of n -dimensional $\mathcal{X}_n$ -manifolds. Of course, to a general bordism $X: Y_0 \rightarrow Y_1$ a topological field theory assigns an element in $\text{Hom}_{\mathcal{C}}(y_0, y_1)$ in the obvious notation.

25.3 More on bordism categories

We continue with fixed $n \in \mathbb{Z}^{\geq 0}$ and an n -dimensional tangential structure $\pi_n: \mathcal{X}_n \rightarrow BGL_n \mathbb{R}$. The bordism category $\mathcal{B} = \text{Bord}_{\langle n-1, n \rangle}(\mathcal{X}_n)$ was introduced in **Definition 24.48**. An object Y is a closed directed collared $(n-1)$ -manifold equipped with an $\mathcal{X}_n$ -structure (on the collar). A morphism is a diffeomorphism class of bordisms between such manifolds. The symmetric monoidal structure is disjoint union of objects and bordisms. The empty set, viewed as a closed $(n-1)$ -manifold $\emptyset^{n-1}$, is the tensor unit object of $\mathcal{B}$.

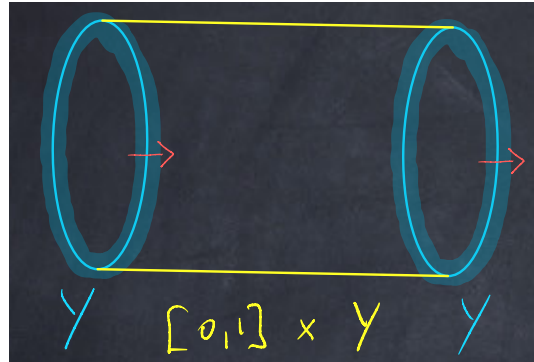


FIGURE 79. The identity morphism $\text{id}_Y: Y \rightarrow Y$

Our first observation is that for any object Y , the identity map is represented by the bordism $[0, 1] \times Y$ with incoming boundary $\{0\} \times Y$ and outgoing boundary $\{1\} \times Y$. Note that Y is equipped with a collaring, and so too $[0, 1] \times Y$ is equipped with a collaring. See [Figure 79](#) for a depiction of the identity morphism.

To emphasize the discrete algebraic nature of a bordism category, we give an algebraic model in the simplest case. Here by ‘algebraic model’ we mean an *equivalent* category; the description ‘algebraic’ is not rigorously defined, but we only mean that the equivalent category has an algebraic description. We leave the reader to give a complete such description, since we only indicate part of the structure.

Example 25.19. Let $\mathcal{B} = \text{Bord}_{\langle 0,1 \rangle}$. We indicate parts of an equivalent category $\mathcal{B}'$. An object of $\mathcal{B}$ is a compact 0-manifold, which is a finite set of points. Its cardinality is a complete invariant. Hence we let the set of objects of $\mathcal{B}'$ be the set $\mathbb{Z}^{\geq 0}$. The commutative monoidal structure on $\mathbb{Z}^{\geq 0}$ is part of a symmetric monoidal structure on $\mathcal{B}'$.

Next, observe that $\text{Hom}_{\mathcal{B}}(\text{pt}^{\sqcup n_0}, \text{pt}^{\sqcup n_1}) = \emptyset$ if $n_1 - n_0$ is odd, where $n_0, n_1 \in \mathbb{Z}^{\geq 0}$, and so we stipulate the same for the equivalent category $\mathcal{B}'$:

$$(25.20) \quad \text{Hom}_{\mathcal{B}'}(n_0, n_1) = 0, \quad n_1 - n_0 \text{ odd.}$$

The commutative monoid $\text{Hom}_{\mathcal{B}}(\emptyset^0, \emptyset^0)$ consists of diffeomorphism classes of closed 1-manifolds. Such manifolds are a finite disjoint union of circles. Hence $\text{Hom}_{\mathcal{B}}(\emptyset^{n-1}, \emptyset^{(n-1)}) \cong \mathbb{Z}^{\geq 0}$, and we stipulate the same for $\mathcal{B}'$:

$$(25.21) \quad \text{Hom}_{\mathcal{B}'}(0, 0) = \mathbb{Z}^{\geq 0}.$$

Similarly, we have

$$(25.22) \quad \text{Hom}_{\mathcal{B}'}(1, 1) = \mathbb{Z}^{\geq 0};$$

The morphism $k \in \mathbb{Z}^{\geq 0}$ is represented by the bordism depicted in [Figure 80](#); the underlying 1-

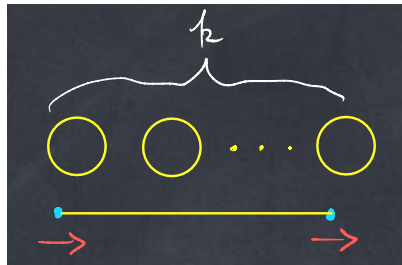
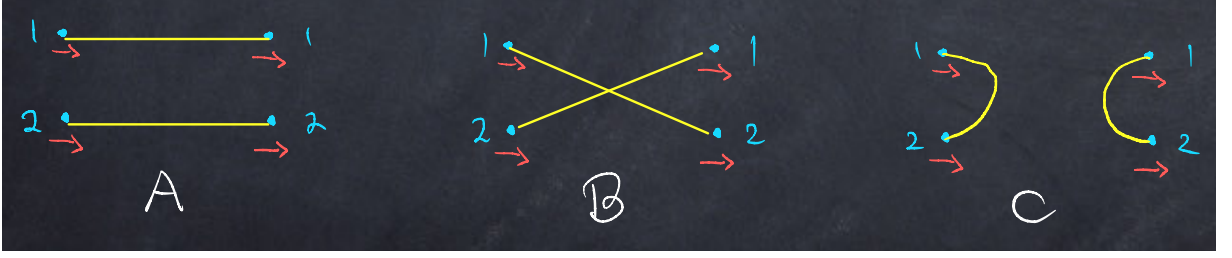


FIGURE 80. The morphism $k \in \text{Hom}_{\mathcal{B}'}(1, 1)$

manifold with boundary is the disjoint union of a closed interval with k circles. The morphisms from 2 points to 2 points are more interesting. There are 3 basic morphisms, depicted in [Figure 81](#); the first is the identity map. Any of these can tensored (disjoint unioned) with a finite number of

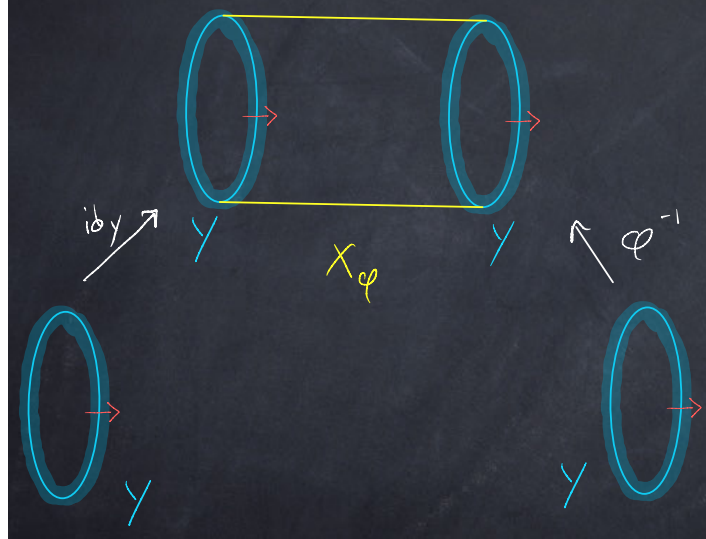
FIGURE 81. Three morphisms in $\text{Hom}_{B'}(2, 2)$

circles. So

$$(25.23) \quad \text{Hom}_{B'}(2, 2) = \{A, B, C\} \times \mathbb{Z}^{\geq 0},$$

where A, B, C are labels for the three morphisms depicted in [Figure 81](#).

Remark 25.24. In a category, the morphisms between two objects are a set; there is no topology. One can imagine a richer situation in which morphism *sets* are enhanced to morphism *spaces*. Such a structure is called a *topological category*, or an ∞ -category. There exist ∞ versions of bordism categories [\[CaSc\]](#). In that world a single circle is replaced by the universal space of circles, whose underlying homotopy type is the classifying space $B\text{Diff}(S^1) \simeq BO_2$. Similarly, the single point that represents k circles in [\(25.21\)](#) and [\(25.22\)](#) is replaced by $B(\text{Sym}_k \ltimes (\text{O}_2)^{\times k})$. The enhancement of topological field theories to homomorphisms of ∞ -categories—which in some ways is a generalization to evaluation in families of bordisms—leads to rich adventures and brings in diffeomorphism groups of smooth manifolds in a more profound manner.

FIGURE 82. The bordism $X_{\varphi}: Y \rightarrow Y$ associated to a diffeomorphism $\varphi: Y \rightarrow Y$

Return now to the general topological bordism category $\mathcal{B} = \text{Bord}_{\langle n-1, n \rangle}(\mathcal{X}_n)$. Then for any object Y there is a map

$$(25.25) \quad \text{Diff}(Y) \longrightarrow \text{Hom}_{\mathcal{B}}(Y, Y).$$

(Keep in mind that a diffeomorphism is a diffeomorphism of a directed collared manifold with $\mathcal{X}_n$ -structure.) Namely, to the diffeomorphism $\varphi: Y \rightarrow Y$ we attach the morphism represented by the bordism in [Figure 82](#). Diffeomorphisms φ that lead to diffeomorphic bordisms X_φ represent the same morphism. In particular, isotopic diffeomorphisms lead to the same morphism, so (25.25) factors through

$$(25.26) \quad \pi_0 \text{Diff}(Y) \longrightarrow \text{Hom}_{\mathcal{B}}(Y, Y).$$

In fact, *pseudoisotopic* diffeomorphisms lead to the same morphism, a point already made in [\[Mi2\]](#).

25.4 Duality

The following expresses a finite condition on objects in a general symmetric monoidal category. It is an abstraction of the dual vector space construction in the category $C = \text{Vect}_{\mathbb{F}}$ of vector spaces over a field $\mathbb{F}$ with symmetric monoidal structure the tensor product of vector spaces. But whereas the dual vector space exists as a vector space for any vector space, it only exists as a dual in the sense of the following for *finite dimensional* vector spaces.

Definition 25.27. Let C be a symmetric monoidal category and $y \in C$.

- (i) *Duality data* for y is a triple of data $(y^\vee, c, e)$ in which $y^\vee$ is an object of C and c, e are morphisms

$$(25.28) \quad \begin{aligned} c: 1_C &\longrightarrow y \otimes y^\vee \\ e: y^\vee \otimes y &\longrightarrow 1_C \end{aligned}$$

We require that the compositions

$$(25.29) \quad \begin{aligned} y &\xrightarrow{c \otimes \text{id}_y} y \otimes y^\vee \otimes y \xrightarrow{\text{id}_y \otimes e} y \\ y^\vee &\xrightarrow{\text{id}_{y^\vee} \otimes c} y^\vee \otimes y \otimes y^\vee \xrightarrow{e \otimes \text{id}_{y^\vee}} y^\vee \end{aligned}$$

be identity maps. If duality data exists for y , we say that y is *dualizable*.

- (ii) A *morphism of duality data* $(y^\vee, c, e) \rightarrow (\tilde{y}^\vee, \tilde{c}, \tilde{e})$ is a morphism $y^\vee \xrightarrow{f} \tilde{y}^\vee$ such that the diagrams

$$(25.30) \quad \begin{array}{ccc} & & y \otimes y^\vee \\ & \nearrow c & \downarrow \text{id}_y \otimes f \\ 1_C & & y \otimes \tilde{y}^\vee \\ & \searrow \tilde{c} & \end{array}$$

and

$$(25.31) \quad \begin{array}{ccc} y^\vee \otimes y & \xrightarrow{e} & 1_C \\ \downarrow f \otimes \text{id}_y & & \uparrow \tilde{e} \\ \tilde{y}^\vee \otimes y & & \end{array}$$

commute.

The morphisms c is called *coevaluation* and the morphism e is called *evaluation*. Let $\mathcal{D}_y$ be the category of duality data for y .

We now express the uniqueness of duality data. As duality data is an object in a category, as defined in Definition 25.27, we cannot say there is a unique object. Rather, here we have the strongest form of uniqueness possible in a category: duality data is unique up to unique isomorphism.

Definition 25.32. Let C be a category.

- (i) If for each pair $y_0, y_1 \in C$ the hom-set $C(y_0, y_1)$ is either empty or contains a unique element, we say that C is a *discrete groupoid*.
- (ii) If for each pair $y_0, y_1 \in C$ the hom-set $C(y_0, y_1)$ has a unique element, we say that C is *contractible*.

A discrete groupoid is equivalent to a set. A contractible groupoid is equivalent to a category with one object and one morphism, the categorical analog of a point.

Proposition 25.33. Let C be a symmetric monoidal category and $y \in C$. Then the category $\mathcal{D}_y$ of duality data for y is either empty or is contractible.

The proof is a homework problem. An object $y \in C$ is *dualizable* if $\mathcal{D}_y$ is contractible (nonempty).

A morphism between dualizable objects has a dual.

Definition 25.34. Let $y_0, y_1 \in C$ be dualizable objects in a symmetric monoidal category and $f: y_0 \rightarrow y_1$ a morphism. The *dual morphism* $f^\vee: y_1^\vee \rightarrow y_0^\vee$ is the composition

$$(25.35) \quad y_1^\vee \xrightarrow{\text{id}_{y_1^\vee} \otimes c_0} y_1^\vee \otimes y_0 \otimes y_0^\vee \xrightarrow{\text{id}_{y_1^\vee} \otimes f \otimes \text{id}_{y_0^\vee}} y_1^\vee \otimes y_1 \otimes y_0^\vee \xrightarrow{e_1 \otimes \text{id}_{y_0^\vee}} y_0^\vee$$

In the definition we choose duality data $(y_0^\vee, c_0, e_0)$, $(y_1^\vee, c_1, e_1)$ for y_0, y_1 .

We leave the reader to (1) check that this definition agrees with that of a dual linear map for $C = \text{Vect}_{\mathbb{F}}$, and (2) spell out the consequence of Proposition 25.33 for the dual morphism.

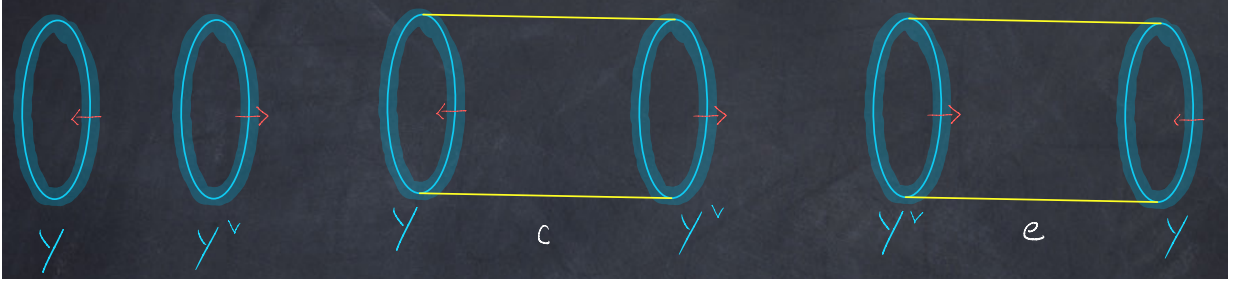
25.5 Duality in bordism categories; spaces of topological field theories

Theorem 25.36. Every object in a bordism category $\text{Bord}_{\langle n-1, n \rangle}(\mathcal{X}_n)$ is dualizable.

The data for a picture proof is contained in Figure 83. Pictured is an object Y of the bordism category, as specified in Definition 24.48. Its dual $Y^\vee$ is the same bordism with the reversed directed collar (Definition 24.30). The manifold with boundary that underlies the coevaluation c and evaluation e is the cylinder $[0, 1] \times Y$; the crucial point is the coorientation of Y inside the various collar neighborhoods of the boundary components. These pictures prove the following.

Since symmetric monoidal functors take duality data to duality data, we conclude the following strong finiteness constraint on a topological field theory.

Corollary 25.37. Let $F: \text{Bord}_{\langle n-1, n \rangle}(\mathcal{X}_n) \rightarrow \mathcal{C}$ be a topological field theory with codomain a symmetric monoidal category $\mathcal{C}$. Then for all $Y \in \text{Bord}_{\langle n-1, n \rangle}(\mathcal{X}_n)$, the image $F(Y) \in \mathcal{C}$ is dualizable.

FIGURE 83. Duality data for the object $Y \in \text{Bord}_{\langle n-1, n \rangle}(\mathcal{X}_n)$

So, for example, if $\mathcal{C} = \text{Vect}_{\mathbb{F}}$ is the category of vector spaces, then the vector space assigned to a closed $(n-1)$ -manifold is finite dimensional.

Remark 25.38. There are variations of the definition of a topological field theory that get around this finiteness constraint.

Here is another general consequence of dualizability, whose proof is also left as a homework problem.

Proposition 25.39. *Let B, C be symmetric monoidal categories, let $F, G: B \rightarrow C$ be symmetric monoidal functors, and suppose $\eta: F \rightarrow G$ is a symmetric monoidal natural transformation. Then if $y \in B$ dualizable, $\eta(y): F(y) \rightarrow G(y)$ is invertible.*

Since every object in a bordism category is dualizable ([Theorem 25.36](#)), we conclude the following.

Corollary 25.40. *Let $F, G: \text{Bord}_{\langle n-1, n \rangle}(\mathcal{X}_n) \rightarrow \mathcal{C}$ be topological field theories (with the same domain and codomain). Then every natural transformation $\eta: F \rightarrow G$ is invertible. Therefore, the category $\text{Hom}^{\otimes}(\text{Bord}_{\langle n-1, n \rangle}(\mathcal{X}_n), \mathcal{C})$ of topological field theories with fixed domain and codomain is a groupoid.*

Remark 25.41. A groupoid is equivalent to a space in a sense to be described in a later lecture. The construction of *spaces* of quantum field theories of a fixed type (meaning a fixed dimension, fixed background fields, and a fixed codomain) is a problem of great interest. For *topological* field theories this construction is apparent.

Lecture 26: TFT as a Wick-rotated QFT; finite gauge theory

The general field theory, as defined by [Axiom System 23.42](#), is difficult to construct and there is a paucity of general examples. Topological field theories, introduced in the previous lecture, are a special case with a richly developed mathematical theory and numerous examples and applications. We begin in this lecture with the question: When is a Wick-rotated quantum field theory (QFT) a topological field theory (TFT)? Recall that a field theory is a linear representation of a bordism category, and the bordism category is determined by two discrete choices: a dimension n and a collection $\mathcal{F}$ of background fields. So the question quickly becomes: When is a field topological? We

already answered this in §24.4: when it is a *locally constant* sheaf. This leads to a characterization of TFTs among Wick-rotated QFTs (Definition 26.1). We single out special features of topological theories among all theories.

In the second part of the lecture we give examples of topological field theories. The first is finite gauge theory. We focus mostly on the 2-dimensional theory, though it exists and is useful in any dimension. We give the construction in the next lecture. We then give other examples in 2 dimensions related to the Euler number. In fact, finite gauge theories lie in this more general class of theories.

26.1 Characterizing topological field theories

Recall the discussion in §24.4 of *locally constant* simplicial sheaves on $\mathbf{Man}_n$. We use it to characterize topological field theories among all Wick-rotated quantum field theories.

Definition 26.1. Fix $n \in \mathbb{Z}^{>0}$ and let $\mathcal{F}: \mathbf{Man}_n^{\text{op}} \rightarrow \mathbf{Set}_\Delta$ be a collection of background fields. A field theory

$$(26.2) \quad F: \text{Bord}_{\langle n-1, n \rangle}(\mathcal{F}) \longrightarrow t\text{Vect}$$

is *topological* if there exists a locally constant simplicial sheaf $\mathcal{F}': \mathbf{Man}_n^{\text{op}} \rightarrow \mathbf{Set}_\Delta$, a fiber bundle⁸⁷ of sheaves $p: \mathcal{F} \rightarrow \mathcal{F}'$, and a factorization

$$(26.3) \quad \begin{array}{ccc} \text{Bord}_{\langle n-1, n \rangle}(\mathcal{F}) & \xrightarrow{F} & t\text{Vect} \\ & \searrow p & \nearrow F' \\ & \text{Bord}_{\langle n-1, n \rangle}(\mathcal{F}') & \end{array}$$

We show below that a topological field theory factors through the category Vect of *finite dimensional* vector spaces and linear maps.

Example 26.4. A typical example of a fiber bundle $p: \mathcal{F} \rightarrow \mathcal{F}'$ of the type in Definition 26.1 is the map

$$(26.5) \quad \begin{array}{c} \{\text{Riemannian metrics, orientations}\} \\ \downarrow p \\ \{\text{orientations}\} \end{array}$$

that forgets some fields so that the remaining fields form a locally constant sheaf.

Let F be an n -dimensional topological field theory

$$(26.6) \quad F: \text{Bord}_{\langle n-1, n \rangle}(\mathcal{F}) \longrightarrow t\text{Vect}$$

over a locally constant sheaf $\mathcal{F}$. The local constancy of $\mathcal{F}$ has strong consequences.

⁸⁷This means that for every n -manifold M , the map of simplicial sets (spaces) $p(M): \mathcal{F}(M) \rightarrow \mathcal{F}'(M)$ is a fiber bundle of topological spaces.

- (1) Let Y be an object in the bordism category $\text{Bord}_{\langle n-1, n \rangle}(\mathcal{F})$. So Y is a closed $(n-1)$ -manifold embedded in a germ U of an n -manifold, the submanifold $Y \subset U$ is cooriented, and there is a choice $\phi \in \mathcal{F}(U)$ of background field. (For expositional purposes think of U as a representative of the germ—an n -manifold for which $Y \subset U$ is a tubular neighborhood.) The image $p(Y) \in \text{Bord}_{\langle n-1, n \rangle}(\mathcal{F}')$ is the same data with the background field $p(U)(\phi) \in \mathcal{F}'(U)$. By [Theorem 25.36](#), this image $p(Y)$ is dualizable, from which $F'p(Y) = F(Y)$ is also dualizable. Hence $F(Y)$ is a finite dimensional vector space.⁸⁸
- (2) In a field theory that includes a Riemannian metric, if Y is an object in the bordism category embedded in the product germ, then there is a Hamiltonian operator H_Y ; see [§23.4](#). In a topological theory, we have $H_Y = 0$ for all Y . In fact, the stronger statement is that the bordism

$$(26.7) \quad [0, 1] \times Y : Y \longrightarrow Y$$

is the identity morphism $\text{id}_Y : Y \rightarrow Y$ in $\text{Bord}_{\langle n-1, n \rangle}(\mathcal{F})$. So if $\mathcal{F}$ is locally constant, then $\text{Bord}_{\langle n-1, n \rangle}(\mathcal{F})$ is a category, not merely a semicategory.

- (3) The partition function $F(X) \in \mathbb{C}$ on a closed n -manifold X is diffeomorphism invariant. This is merely a special case of the general statement that if X, X' are closed n -manifolds equipped with background fields $\phi \in \mathcal{F}(X)$, $\phi' \in \mathcal{F}'(X)$, and if $\varphi : X' \rightarrow X$ is a diffeomorphism such that $\varphi^*(\phi) = \phi'$, then $F(X') = F(X)$. The relevant observation is that in the topological case, the torsor of such diffeomorphisms is much larger than in the nontopological case: e.g., compare orientation-preserving diffeomorphisms to isometries.
- (4) Let Y be an object in $\text{Bord}_{\langle n-1, n \rangle}(\mathcal{F})$. Then the action of $\text{Diff}(Y)$ on $F(Y)$ factors through the mapping class group $\pi_0 \text{Diff}(Y)$. (The symbol ‘ Y ’ includes background fields, and the diffeomorphism group is required to preserve them.) This follows from the discussion at the end of [§25.3](#).
- (5) In a general Wick-rotated field theory, observables on an n -manifold X form a factorization algebra $\mathcal{A}$; see [§23.5](#). Recall that if $\mathcal{U} \subset X$ is an open set whose closure $\overline{\mathcal{U}}$ is a smooth manifold with boundary, then $\mathcal{A}(\mathcal{U})$ is defined to be the vector space $F(\partial \overline{\mathcal{U}})$. Furthermore, for $x \in X$ we define the vector space $\mathcal{O}_x$ of observables at x as an inverse limit [\(23.27\)](#) of observables in open balls which contain x in their interior. If $\mathcal{F}$ is locally constant, then the inverse limit is realized at any stage, so $\mathcal{O}_x$ is the value of F on a sphere surrounding x : the link of the point x in the manifold X . The factorization algebra structure reduces to the more homotopical structure of an E_n -algebra.

Remark 26.8. In (5), we have not made explicit the background fields on spheres that link the point $x \in X$. That background field need not extend over the point x ; it may only be defined on the deleted ball about x . This allows for “twisted sectors” of observables at x .

26.2 Example: finite gauge theory

Let G be a finite group. Gauge theory with gauge group G is surely the simplest possible field theory. It is topological—in fact, there are no nontrivial background fields. Furthermore, finite

⁸⁸One needs to make precise the category of topological vector spaces to prove that every dualizable object is a finite dimensional vector space.

gauge theory exists in all dimensions. This theory is reviewed extensively in any treatment of topological field theory. One of the first references is [DW], who also give a twisted version. For an early mathematical treatment, see [FQ]. There are many expositions of this theory. We indicate a construction of finite gauge theory in the next lecture. Here we simply introduce it without justifying our statements.

Fix $n \in \mathbb{Z}^{>0}$. Let $\mathcal{F}: \mathbf{Man}_n^{\text{op}} \rightarrow \mathbf{Set}$ be the set-valued sheaf that assigns the singleton $\{*\}$ to each n -manifold. (This is the background field for what is oft called an ‘unoriented theory’. Since $\mathcal{F}$ is trivial, we write $\text{Bord}_{\langle n-1, n \rangle} = \text{Bord}_{\langle n-1, n \rangle}(\mathcal{F})$.) Let G be a finite group. Then finite gauge theory is a homomorphism

$$(26.9) \quad F_G: \text{Bord}_{\langle n-1, n \rangle} \longrightarrow \text{Vect}_{\mathbb{C}}.$$

In fact, the theory factors through the category of rational vector spaces, but convenient bases that we use below exist only over the complex numbers. (They diagonalize certain linear maps.)

Remark 26.10. Why is F_G called a ‘gauge theory’? That is because it has a construction that sums over principal G -bundles. More precisely, for every topological space S , let $\text{Bun}_G(S)$ denote the groupoid whose objects are principal G -bundles $P \rightarrow S$ and whose morphisms are isomorphisms of principal G -bundles that cover id_S :

$$(26.11) \quad \begin{array}{ccc} P & \xrightarrow{\cong} & P' \\ & \searrow & \swarrow \\ & S & \end{array}$$

(A principal G -bundle for a finite group G is also known as a ‘Galois cover’ or ‘regular covering space’ or ‘normal covering space’.) If S is compact, then $\text{Bun}_G(S)$ is a *finite* groupoid: it has a finite set of equivalence classes of objects, and each object has a finite automorphism group. The theory F_G is constructed by “quantizing” these groupoids, which for an n -manifold amounts to counting principal G -bundles.

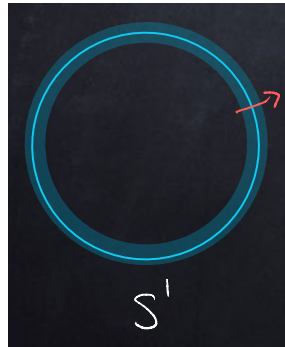


FIGURE 84. The circle as an object in $\text{Bord}_{\langle 1, 2 \rangle}$: it carries a directed collar structure

For the remainder of the lecture specialize to $n = 2$, so that

$$(26.12) \quad F_G: \text{Bord}_{\langle 1, 2 \rangle} \longrightarrow \text{Vect}_{\mathbb{C}}.$$

Any closed 1-manifold is a finite disjoint union of circles, and for a single circle (Figure 84) we obtain the vector space

$$(26.13) \quad A := F_G(S^1) = \text{Map}(G; \mathbb{C})^G$$

of *central* complex-valued functions on G . In (26.13) the group G acts by conjugation, and the notation indicates the invariant functions. These are called *central functions* on the group. If $\rho: G \rightarrow W$ is a linear representation of G on a complex vector space W , then the character

$$(26.14) \quad \chi(g) = \text{trace } \rho(g)$$

is an example of a character function. The characters $\chi_1, \dots, \chi_k$ of *irreducible* representations form a basis of A . Since F_G is a symmetric monoidal functor, disjoint unions map to tensor products:

$$(26.15) \quad F_G((S^1)^{\sqcup N}) = A^{\otimes N}.$$

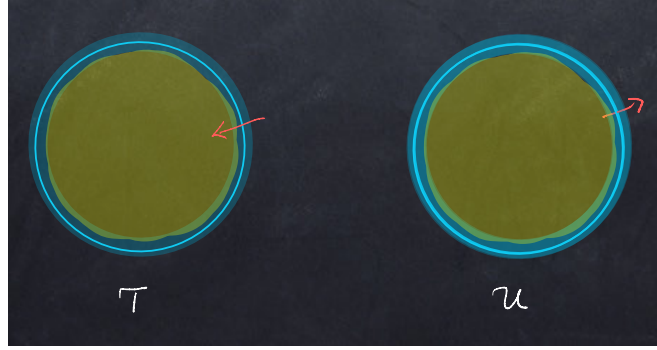


FIGURE 85. Morphisms $\tau: S^1 \rightarrow \emptyset^1$ and $u: \emptyset^1 \rightarrow S^1$ in $\text{Bord}_{\langle 1,2 \rangle}$

Consider the bordisms in Figure 85. They represent morphisms $\tau: S^1 \rightarrow \emptyset^1$ and $u: \emptyset^1 \rightarrow S^1$ in $\text{Bord}_{\langle 1,2 \rangle}$. Under F_G they map to a linear map $\text{trace}: A \rightarrow \mathbb{C}$ and a linear map $\mathbb{C} \rightarrow A$; the latter amounts to an element $1 \in A$. The morphism represented by the bordism depicted in Figure 86 evaluates under F_G to a linear map

$$(26.16) \quad m: A \otimes A \longrightarrow A.$$

The diffeomorphism of compositions of bordisms depicted in Figure 79 leads to the associativity equality on the multiplication map (26.16). We leave the reader to construct a similar diffeomorphism to prove that $1 \in A$ is a unit for the multiplication. Furthermore, the bordism in Figure 86 admits a diffeomorphism that exchanges the two incoming circles; evaluation on F_G proves that the multiplication is commutative. Finally, $\text{trace} = F_G(\tau)$ is a trace on the algebra: $\text{trace}(m(\chi_1, \chi_2)) = \text{trace}(m(\chi_2, \chi_1))$ for all $\chi_1, \chi_2 \in A$. Furthermore, the symmetric bilinear map

$$(26.17) \quad (\chi_1, \chi_2) \rightarrow \text{trace}(m(\chi_1, \chi_2))$$

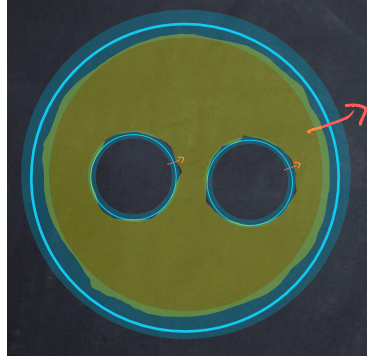
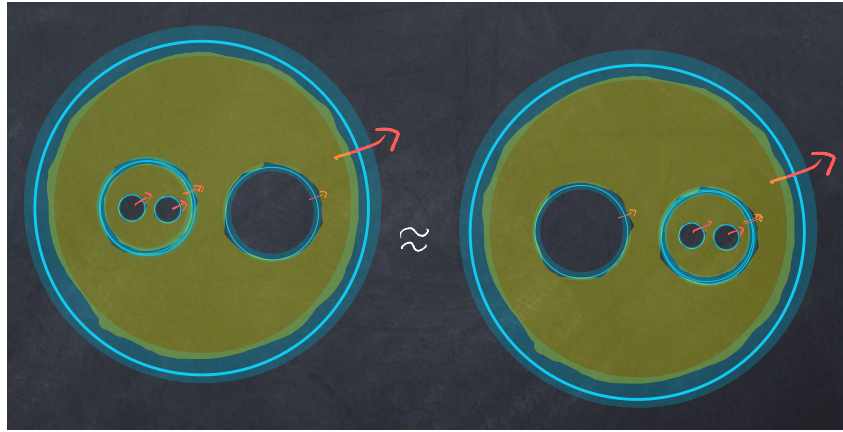
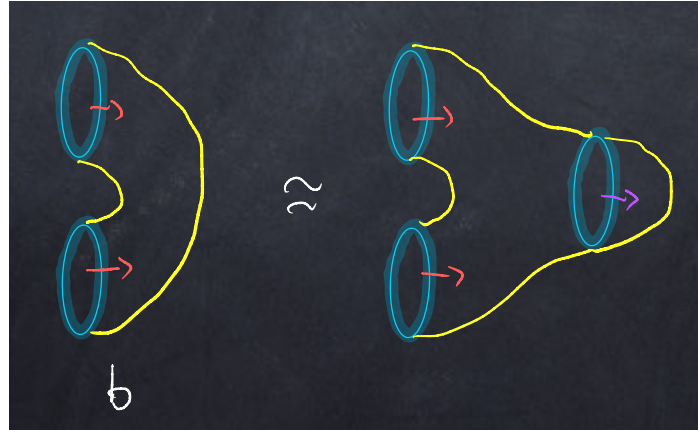

 FIGURE 86. A morphism $S^1 \sqcup S^1 \rightarrow S^1$ in $\text{Bord}_{(1,2)}$


FIGURE 87. The diffeomorphism that proves associativity


 FIGURE 88. The evaluation in the duality data of S^1

is associated to the linear map $F_G(b)$ obtained by applying F_G to the evaluation bordism b that is part of duality data for the circle; see [Figure 83](#) and [Figure 88](#). It follows that (26.17) is a *nondegenerate* symmetric bilinear form. In short, A with this structure is a *commutative Frobenius algebra*.

In the next lecture we will indicate how to compute that

$$(26.18) \quad m(\chi_1, \chi_2) = \chi_1 * \chi_2$$

is the convolution product on the group G :

$$(26.19) \quad (\chi_1 * \chi_2)(g) = \sum_{g_1 g_2 = g} \chi_1(g_1) \chi_2(g_2), \quad g \in G;$$

that the identity element is the character of trivial 1-dimensional representation:

$$(26.20) \quad 1(g) = \begin{cases} 1, & g = e; \\ 0, & g \neq e; \end{cases}$$

and that the trace is suitably normalized evaluation at the identity element of the group:

$$(26.21) \quad \text{trace}(\chi) = \frac{\chi(e)}{\#G}.$$

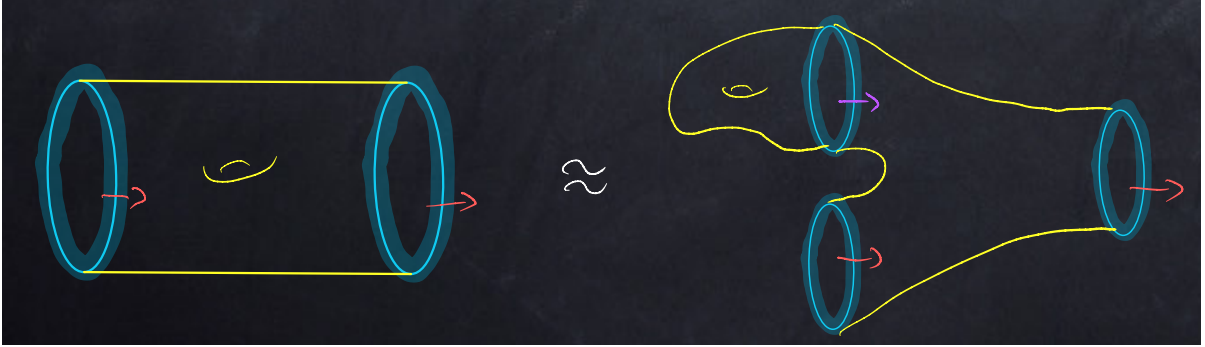


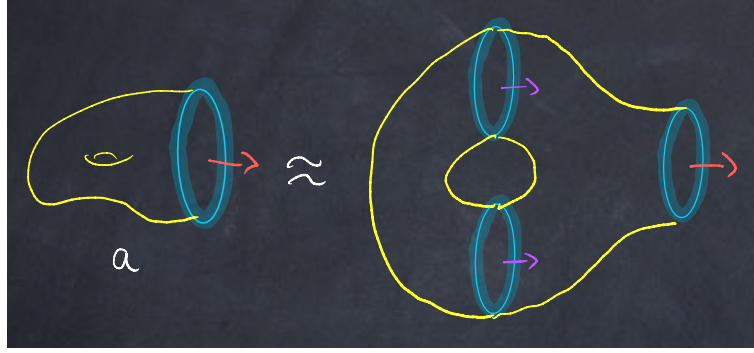
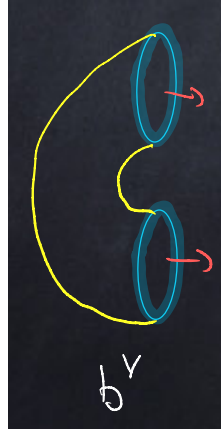
FIGURE 89. A bordism $\alpha: S^1 \rightarrow S^1$

This is the complete data one can extract from bordisms of genus 0. The essential piece of new information in nonzero genus is obtained from the bordism $\alpha: S^1 \rightarrow S^1$ illustrated in [Figure 89](#). The diffeomorphism in that figure makes clear that $F_G(\alpha)$ is multiplication by an element in A . That element is the image $F_G(a)$ of a morphism $a: \emptyset^1 \rightarrow S^1$ in $\text{Bord}_{\langle 1,2 \rangle}$, and is computed in [Figure 90](#) as coevaluation $b^\vee$ ([Definition 12.3](#)) followed by multiplication. The *Frobenius relations* tell that the characters $\{\chi_i\}$ of irreducible representations form an orthonormal basis of A with respect to the pairing (26.17). Hence the image $F_G(b^\vee)$ of coevaluation is the element

$$(26.22) \quad \sum_i \chi_i \otimes \chi_i \in A \otimes A,$$

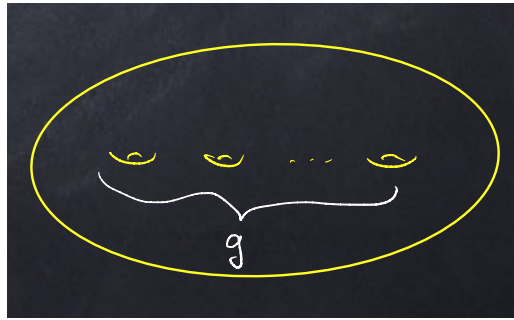
and from [Example 12.2](#) we deduce that

$$(26.23) \quad F_G(a) = \sum_i \chi_i * \chi_i.$$

FIGURE 90. Computation of $a \in A$ FIGURE 91. The coevaluation in the duality data of S^1

Let X_g be a closed oriented surface of genus g ; see [Figure 92](#). It is a morphism $X_g: \emptyset^1 \rightarrow \emptyset^1$ in $\text{Bord}_{(1,2)}$, and it can be expressed as a composition of $(g-1)$ copies of the bordism α in [Figure 89](#) preceded by coevaluation $b^\vee$ and followed by evaluation b . It follows that its *partition function* is

$$(26.24) \quad F_G(X_g) = \text{trace}(F_G(a)^{g-1}) = \sum_i \left(\frac{\chi_i(e)}{\#G} \right)^{2g-2}.$$

FIGURE 92. A closed connected oriented surface of genus g

It is interesting to compute the value of this theory on unorientable surfaces, such as the Möbius band, but we defer that for now.

26.3 Example: Euler theories

We briefly discuss two examples: an Euler theory and the “direct sum” of Euler theories.

Example 26.25. Fix a field $\mathbb{F}$ and a unit (invertible=nonzero element) $\lambda \in \mathbb{F}^\times$. Let $\text{Line}_{\mathbb{F}}$ denote the groupoid whose objects are 1-dimensional $\mathbb{F}$ -vector spaces and whose morphisms are invertible maps between these $\mathbb{F}$ -lines. The theory

$$(26.26) \quad F_\lambda: \text{Bord}_{\langle 1,2 \rangle} \longrightarrow \text{Line}_{\mathbb{F}}$$

assigns the trivial line

$$(26.27) \quad F_\lambda(S^1) = \mathbb{F}$$

to a single circle—hence also to a finite disjoint union of circles—and to a bordism $X: Y_0 \rightarrow Y_1$ in $\text{Bord}_{\langle 1,2 \rangle}$ it assigns multiplication by

$$(26.28) \quad F(X: Y_0 \rightarrow Y_1) = \lambda^{\text{Euler}(X)},$$

where $\text{Euler}(X) \in \mathbb{Z}$ is the Euler number of the surface that underlies the bordism. We leave the reader to verify that this defines a homomorphism (26.26). This is an example of an *invertible field theory*.

Remark 26.29. The partition function of a closed 2-manifold is a bordism invariant if $\lambda = \pm 1$, but in general it is not a bordism invariant.

Example 26.30. Fix a positive integer $k \in \mathbb{Z}^{\geq 0}$, a field $\mathbb{F}$, and units $\lambda_1, \dots, \lambda_k \in \mathbb{F}^\times$. Then there is a (noninvertible if $k > 1$) theory

$$(26.31) \quad F: \text{Bord}_{\langle 1,2 \rangle} \longrightarrow \text{Vect}_{\mathbb{F}}$$

that assigns

$$(26.32) \quad F(S^1) = (\mathbb{F})^{\oplus k}$$

and whose partition function on a closed surface X is

$$(26.33) \quad F(X) = \sum_i (\lambda_i)^{\text{Euler}(X)}.$$

The 2-dimensional finite gauge theory discussed in §26.2 is of this type.

Lecture 27: The finite path integral

The path integral [FeHi] was introduced by Feynman as a new way to pass from the classical mechanics of a particle to its quantum mechanics. Feynman later applied it to quantum field theory, and since it has been a basic tool in the field. Here, we discuss integration over “finite fields”. That is, we consider the integral over a sheaf of fields whose value on a compact manifold is a finite set or finite higher groupoid. (The classifying space is a π -finite space.) In this finite situation, there is an evident extension to higher codimension and so higher category numbers of mathematical objects: in place of a finite sum of numbers we encounter finite sums of vector spaces, of categories, of higher categories, etc. Quantization below the top dimension as a finite “path” integral was introduced in [F9] and it was further discussed in [FHLT]. Its rigorous implementation is in terms of *semiadditivity* and *ambidexterity* in higher categories; see [HeLu] for a leisurely account.

We begin in this lecture in the general context of Wick-rotated field theories that are not necessarily topological. We consider a fiber bundle $\pi: \hat{\mathcal{F}} \rightarrow \mathcal{F}$ of fields and describe the path integral as integrating out over the fibers of π , which is descent from a field theory over $\hat{\mathcal{F}}$ to a field theory over $\mathcal{F}$. Of course, one needs a measure or equivalent to perform the integration=descent=push forward. (For example, pushforwards in cohomology are performed using a relative orientation.) Note that the remaining background fields in $\mathcal{F}$ need not be finite. For example, we can sum over spin structures in quantum field theories that are not topological.

A key point in the discussion is that quantization in codimension one, which produces a vector space of states by quantizing a theory on space cross time, is usually done via canonical or geometric quantization. In the finite case we can reinterpret this as a finite integral over the category of (topological) vector spaces. This is the crucial step toward extending to higher codimensions.

We introduce the modern theory of integration over categories that are semiadditive. We then illustrate using 2-dimensional finite gauge theory.

A rigorous development of the finite path integral for 1-dimensional topological field theories appears in [Har1]. The general case is the subject of a forthcoming paper of Scheimbauer–Walde [Sch].

27.1 The finite path integral in Wick-rotated QFT

Fix a dimension n , a sheaf $\hat{\mathcal{F}}: \mathbf{Man}_n^{\text{op}} \rightarrow \mathbf{Set}_\Delta$ of fields, and a field theory

$$(27.1) \quad \hat{F}: \text{Bord}_{\langle n-1, n \rangle}(\hat{\mathcal{F}}) \longrightarrow t\text{Vect}.$$

Now let $\mathcal{F}: \mathbf{Man}_n^{\text{op}} \rightarrow \mathbf{Set}_\Delta$ be another sheaf of fields, and suppose $\pi: \hat{\mathcal{F}} \rightarrow \mathcal{F}$ is a fiber bundle of sheaves. This means that for each n -manifold M , the global sections $\pi(M): \hat{\mathcal{F}}(M) \rightarrow \mathcal{F}(M)$ form a fiber bundle of simplicial sets (or spaces).

Example 27.2. A typical example is

$$(27.3) \quad \begin{array}{c} \hat{\mathcal{F}} = \{\text{Riemannian metrics}, T\} \\ \downarrow \pi \\ \mathcal{F} = \{\text{Riemannian metrics}\} \end{array}$$

where T is a fixed smooth manifold and the space $\widehat{\mathcal{F}}(M)$ of global sections on an n -manifold M is the Cartesian product $\text{Riem}(M) \times \text{Map}(M, T)$ of the space of Riemannian metrics on M and the space of smooth maps⁸⁹ $M \rightarrow T$; the map π is projection onto the first factor.

In general, we want to “integrate out over the fibers of π ” to obtain a new field theory F over $\mathcal{F}$:

$$(27.4) \quad \begin{array}{ccc} \text{Bord}_{\langle n-1, n \rangle}(\widehat{\mathcal{F}}) & \xrightarrow{\widehat{F}} & t\text{Vect} \\ & \searrow \pi & \nearrow F \\ & \text{Bord}_{\langle n-1, n \rangle}(\mathcal{F}) & \end{array}$$

This is the general path integral. We examine it on closed manifolds and leave the reader to work out the formal discussion on bordisms.

Let X be a closed n -manifold and let $\hat{\phi} \in \widehat{\mathcal{F}}(X)$ be a section of the sheaf $\widehat{\mathcal{F}}$ over X . (In [Example 27.2](#) this is a Riemannian metric on X and a smooth map $X \rightarrow T$.) Then $\widehat{F}(X, \hat{\phi}) \in \mathbb{C}$ is the partition function. For $\phi \in \mathcal{F}(X)$ we want to define

$$(27.5) \quad F(X, \phi) = \int_{\hat{\phi} \in \widehat{\mathcal{F}}(X)_\phi} \widehat{F}(X, \hat{\phi}),$$

where the integral is over the fiber $\widehat{\mathcal{F}}(X)_\phi$ over ϕ of the fiber bundle

$$(27.6) \quad \begin{array}{c} \widehat{\mathcal{F}}(X) \\ \downarrow \pi(X) \\ \mathcal{F}(X) \end{array}$$

(In [Example 27.2](#) the fiber is $\text{Map}(X, T)$, independently of the Riemannian metric ϕ .)

Conclusion 27.7. *To carry out this “pushforward” of theories, we need additional data: a measure on the fibers of $\pi(X)$, coherently for all X . This must be supplemented by measure-like data on closed $(n-1)$ -manifolds and on bordisms between them, and the coherence applies to this data (which must also be local). In some topological situations, such as in [\[FHT2\]](#), the coherent measures are replaced by coherent orientations.*

To gain insight into the nature of this measure data, consider a closed $(n-1)$ -manifold⁹⁰ Y equipped with a field $\hat{\phi} \in \widehat{\mathcal{F}}(Y)$. Then $\widehat{F}(Y, \hat{\phi})$ is a topological vector space. Formally we would like to define

$$(27.8) \quad F(Y, \phi) = \int_{\hat{\phi} \in \widehat{\mathcal{F}}(Y)_\phi} \widehat{F}(Y, \hat{\phi}).$$

⁸⁹This is sometimes called a “ σ -model field”: integration in this case is called the σ -model with target T .

⁹⁰Recall from [§24.3](#) that Y is equipped with a collar and the notation ‘ $\widehat{\mathcal{F}}(Y)$ ’ means a section of the sheaf $\widehat{\mathcal{F}}$ over the n -dimensional collar.

But several comments are in order. First and foremost, this is an integral of topological vector spaces, and the result should be a topological vector space. In a unitary theory, we can replace these topological vector spaces by Hilbert spaces, and if $\mathcal{F}(Y)_\phi$ is equipped with a measure, and the Hilbert spaces $\hat{F}(Y, \hat{\phi})$ fit to a vector bundle over $\mathcal{F}(Y)_\phi$, then we can consider the Hilbert space of L^2 sections of this vector bundle. However, this is not the correct definition of $F(Y, \phi)$, as we know from geometric quantization: we must specify a *polarization* and take the subspace of L^2 sections adapted to the polarizations. This was discussed for free theories in §6.4 and §19.1.

One case in which the integral expression (27.8) for the quantum state space is more plausible is when the integral is over a finite set $S = \hat{\mathcal{F}}(Y)_\phi$; see [F9, Introduction]. Assume as well that the theory $\hat{F}$ is unitary, so that we can take its state spaces $\hat{F}(Y, \hat{\phi})$ to be Hilbert spaces. In other words, on the fiber of $\pi(Y): \hat{\mathcal{F}}(Y) \rightarrow \mathcal{F}(Y)$ over $\phi \in \mathcal{F}(Y)$, the theory $\hat{F}$ produces a bundle $\mathcal{H} \rightarrow S$ of Hilbert spaces over a finite set S . The correct quantization is the Hilbert space of L^2 sections. For this to make sense we need a measure on S , which is simply a function $\mu: S \rightarrow \mathbb{R}^{>0}$. Then the Hilbert space of L^2 sections of $\mathcal{H} \rightarrow S$ is

$$(27.9) \quad \bigoplus_{s \in S} \mu(s) \mathcal{H}_s = \int_S d\mu(s) \mathcal{H}_s,$$

where the product of a positive number $\mu(s)$ and a Hilbert space $\mathcal{H}_s$ is the same underlying complex vector space with hermitian inner product multiplied by $\mu(s)$. The right hand side of (27.9) is a formal expression meant to evoke the left hand side.

Remark 27.10. Notice the need for the measure μ , as discussed in [Conclusion 27.7](#).

Remark 27.11. There is a natural measure on a finite set S : counting measure, which is the constant function $\mu: S \rightarrow \mathbb{R}^{>0}$ with value 1.

We turn now to the categorical underpinnings of the finite path integral, which we note do not use unitarity.

27.2 Integrals in categories

As stated earlier, the relevant notions are *ambidexterity* or *semiadditivity*; see [Har1] for an exposition and further developments. These notions lead to an integration theory of maps from suitable finite spaces into higher categories. We only touch the surface here.

Definition 27.12. Let $\mathcal{C}$ be a category.

- (1) $\mathcal{C}$ is *(-1)-semiadditive* if it is pointed, i.e., there is an object $0 \in \mathcal{C}$ such that for any object $c \in \mathcal{C}$ there exist unique morphisms $0 \rightarrow c$ and $c \rightarrow 0$.
- (2) $\mathcal{C}$ is *0-semiadditive* if it is *(-1)-semiadditive* and if for all finite sets $S = \{c_i\}$ of objects of $\mathcal{C}$, the natural map

$$(27.13) \quad \operatorname{colim}_S \{c_i\} \longrightarrow \lim_S \{c_i\}$$

is an isomorphism.

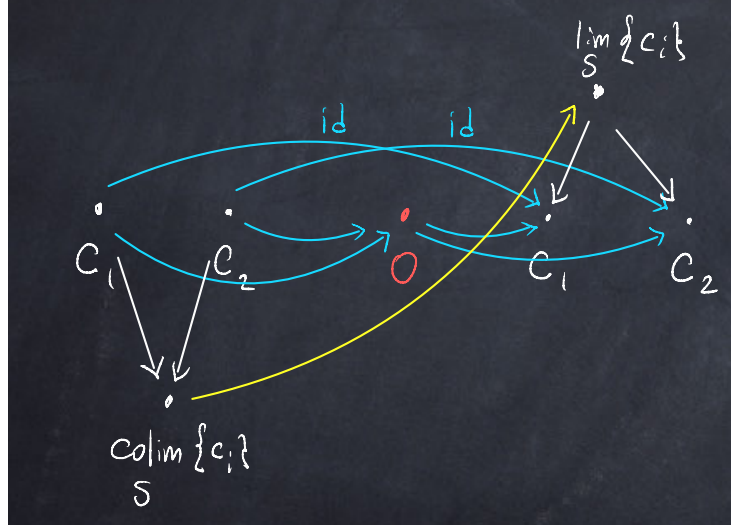


FIGURE 93. The natural map from the colimit to the limit

The definition of the map (27.13), illustrated in Figure 93, uses the existence of a zero object: it is built from the identity matrix on the index set S , where off diagonal maps are compositions $c_i \rightarrow 0 \rightarrow c_j$, $i \neq j$. For a 0-semiadditive category, the colimit=limit can be defined to be the integral of a finite set of objects.

Example 27.14. For the category $\mathcal{C} = \text{Vect}_{\mathbb{F}}$ of vector spaces over a field $\mathbb{F}$, the coproduct is the direct sum

$$(27.15) \quad \text{colim}_S \{V_i\} = \bigoplus_S V_i$$

and the product is the direct product

$$(27.16) \quad \lim_S \{V_i\} = \prod_S V_i.$$

The map from direct sum to direct product is an isomorphism if S is finite. Hence $\text{Vect}_{\mathbb{F}}$ is 0-semiadditive.

Example 27.17. The category $\mathcal{C} = \text{Top}$ of topological spaces and continuous maps is not (-1) -semiadditive. There is an initial object—the empty set—and a final object—any singleton (set of cardinality one)—but they are not equal.

Example 27.18. Let $\mathcal{C} = \text{Top}_*$ be the category of pointed topological spaces and basepoint preserving continuous maps. It is (-1) -semiadditive: any singleton is a zero object. However, it is not 0-semiadditive. Namely, for a collection $S = \{(X_i, x_i)\}$ of pointed topological spaces, the coproduct is the one point union (wedge)

$$(27.19) \quad \text{colim}_S \{(X_i, x_i)\} = \bigvee_S (X_i, x_i)$$

whereas the product is the Cartesian product

$$(27.20) \quad \lim_S \{(X_i, x_i)\} = \left(\bigtimes_S X_i, \bigtimes_S x_i \right).$$

For a category $\mathcal{C}$ there is a definition of 1-semiadditivity similar to [Definition 27.12\(2\)](#); it involves colimits and limits over diagrams indexed by a finite groupoid. We illustrate using the category of vector spaces.

Example 27.21. Let G be a finite group, and let $*//G$ be the groupoid consisting of a single object with automorphism group G . (The geometric realization of this groupoid is Milnor's construction of the classifying space BG .) A diagram in $\mathcal{C} = \text{Vect}_{\mathbb{F}}$ indexed by $*//G$ is a linear representation of G on a vector space V . The colimit is the space of coinvariants

$$(27.22) \quad \text{colim}_{*//G} V = V_G,$$

defined as the quotient of V by the subspace generated by vectors $g \cdot v - v$ for all $g \in G$, $v \in V$. The limit is the subspace of invariant vectors

$$(27.23) \quad \lim_{*//G} V = V^G.$$

The natural map $V_G \rightarrow V^G$ from coinvariants to invariants is induced from the linear map

$$(27.24) \quad \begin{aligned} V &\longrightarrow V^G \\ v &\longmapsto \sum_{g \in G} g \cdot v \end{aligned}$$

If $\mathbb{F}$ is a field of characteristic zero, then this is an isomorphism for all linear representations. In that case $\text{Vect}_{\mathbb{F}}$ is 1-semiadditive.

Example 27.25. Let $\mathcal{G}$ be a finite groupoid. The finiteness means that the set $\pi_0 \mathcal{G}$ of equivalence classes⁹¹ of objects is finite, and for all $x \in \mathcal{G}$ the automorphism group $\pi_1(\mathcal{G}, x) = \text{Aut}(x)$ is a finite group. A functor $f: \mathcal{G} \rightarrow \text{Vect}_{\mathbb{F}}$ is a vector bundle $\mathcal{E} \rightarrow \mathcal{G}$: a vector space for each object in $\mathcal{G}$ and isomorphisms of fibers for each morphism in $\mathcal{G}$. The integral of f is computed as the colimit=limit, as in [Example 27.21](#). The canonical way to express the result is as the vector space of sections of $\mathcal{E} \rightarrow \mathcal{G}$; this vector space realizes the limit of f . Noncanonically, choose an object in each equivalence class and write

$$(27.26) \quad \int_{\mathcal{G}} f = \bigoplus_{[x] \in \pi_0 \mathcal{G}} f(x)^{\text{Aut } x}.$$

⁹¹The equivalence relation: objects $x, y \in \mathcal{G}$ are equivalent if there exists a morphism $x \rightarrow y$.

27.3 Finite gauge theory

This theory was introduced in [Section 26.2](#). It can be constructed from the trivial theory using the finite path integral of [§22.3](#). Fix n and fix a finite group G . Define the sheaf of “classical fields” by $\widehat{\mathcal{F}}(M) = \text{Bun}_G(M)$ for all n -manifolds M . Let $\widehat{F} = \mathbb{1}$ be the tensor unit theory. Then finite gauge theory

$$(27.27) \quad F: \text{Bord}_{\langle n-1, n \rangle} \longrightarrow \text{Vect}_{\mathbb{C}}$$

is defined by integrating over Bun_G . We use the discussion in [§27.2](#) to carry out the finite path integral in dimension $n - 1$. That discussion can be extended to dimension n as well, but here we simply view integration of numerical functions over Bun_G as integration against a measure, and so we need a measure on Bun_G that is the generalization of counting measure ([Remark 27.11](#)) to finite groupoids. More generally, if $\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1)$ is a finite groupoid— $\mathcal{G}^0$ is the set of objects and $\mathcal{G}^1$ is the set of morphisms—then the measure is the function $\mu: \mathcal{G}^0 \rightarrow \mathbb{R}^{>0}$ defined by

$$(27.28) \quad \mu(x) = \frac{1}{\# \text{Aut}(x)}.$$

See [\[BaDo2, We2\]](#) for more discussion.

Let Y be a closed $(n - 1)$ -manifold. Then for any object in $\text{Bun}_G(Y)$, i.e., for any principal G -bundle $Q \rightarrow Y$, The value of the tensor unit theory is the vector space of scalars:

$$(27.29) \quad \mathbb{1}(Y, Q \rightarrow Y) = \mathbb{C}.$$

So the finite path integral over Bun_G , as in [Example 27.25](#), evaluates to the vector space

$$(27.30) \quad F(Y) = \int_{\text{Bun}_G(Y)} \mathbb{C} = \text{Map}(\text{Bun}_G(Y), \mathbb{C}).$$

Next, let $X: Y_0 \rightarrow Y_1$ be a bordism. Then restriction to the incoming and outgoing boundaries produces a *correspondence* (or *span*)

$$(27.31) \quad \begin{array}{ccc} & \text{Bun}_G(X) & \\ p_0 \swarrow & & \searrow p_1 \\ \text{Bun}_G(Y_0) & & \text{Bun}_G(Y_1) \end{array}$$

The finite path integral linearizes this diagram to a linear map

$$(27.32) \quad F(X): \text{Map}(\text{Bun}_G(Y_0), \mathbb{C}) \longrightarrow \text{Map}(\text{Bun}_G(Y_1), \mathbb{C}).$$

Namely, given a function f on $\text{Bun}_G(Y_0)$, first pull it back to a function $(p_0)^*f$ on $\text{Bun}_G(X)$ and then push it forward to a function on $\text{Bun}_G(Y_1)$:

$$(27.33) \quad F(X) = (p_1)_* \circ (p_0)^*.$$

The fibers of p_1 are finite groupoids, and the pushforward $(p_1)_*$ is defined as in [Example 27.25](#). The compatibility of (27.33) with compositions and with the symmetric monoidal structure follows from properties of Bun_G . Namely, first if $X: Y_0 \rightarrow Y_1$ and $X': Y_1 \rightarrow Y_2$ are bordisms, then the central diamond in the diagram

$$(27.34) \quad \begin{array}{ccccc} & & \text{Bun}_G(X' \circ X) & & \\ & \swarrow & & \searrow & \\ & \text{Bun}_G(X) & & \text{Bun}_G(X') & \\ & \swarrow & \searrow & \swarrow & \searrow \\ \text{Bun}_G(Y_0) & & \text{Bun}_G(Y_1) & & \text{Bun}_G(Y_2) \end{array}$$

is a pullback square. Then for disjoint unions we have equivalences

$$(27.35) \quad \begin{aligned} \text{Bun}_G(Y' \sqcup Y'') &\simeq \text{Bun}_G(Y') \times \text{Bun}_G(Y'') \\ \text{Bun}_G(X' \sqcup X'') &\simeq \text{Bun}_G(X') \times \text{Bun}_G(X'') \end{aligned}$$

Computations in finite gauge theory are facilitated using the following.

Lemma 27.36. *Let M be a path connected space, and fix a basepoint $m_0 \in M$. Then there is an equivalence*

$$(27.37) \quad \text{Bun}_G(M) \simeq \text{Hom}(\pi_1(M, m_0), G) // G,$$

where G acts by conjugation.

The map from left to right is *monodromy* or *holonomy*. Namely, suppose $P \rightarrow M$ is a principal G -bundle. Choose a basepoint $p_0 \in P$ in the fiber over m_0 . Then any loop γ in M based at m_0 lifts to a path in P that begins at p_0 and ends at a point $p(\gamma)$ in the fiber at m_0 . Write $p(\gamma) = p_0 \cdot \rho(\gamma)$ for $\rho(\gamma) \in G$. Then $\rho(\gamma)$ depends only on the homotopy class of γ , and it defines a homomorphism $\pi_1(M, m_0) \rightarrow G$. Different choices of basepoint p_0 lead to conjugate homomorphisms.

Applying the “groupoid counting measure” (27.28) to (27.37), we deduce that for any closed n -manifold X , the partition function of finite gauge theory is

$$(27.38) \quad F(X) = \frac{\# \text{Hom}(\pi_1(M, m_0), G)}{\#G}.$$

For $n = 2$ we told an alternative formula in (26.24). The equality of these formulas is a classical result that counts representations of the fundamental group of a surface into a finite group; see [Sn] and the references therein.

Continuing with $n = 2$, for the circle (27.37) specializes to

$$(27.39) \quad \mathrm{Bun}_G(S^1) \simeq G//G,$$

where G acts on itself by conjugation. It follows immediately that $F(S^1)$ is the vector space of class functions on G , as in (26.13). Following (27.31), the pair of pants bordism $X: S^1 \sqcup S^1 \rightarrow S^1$ (Figure 86) produces the correspondence

$$(27.40) \quad \begin{array}{ccc} & \mathrm{Bun}_G(X) & \\ p_0 \swarrow & & \searrow p_1 \\ G//G \times G//G & & G//G \end{array}$$

The map p_0 is an equivalence, and so the correspondence reduces to a map

$$(27.41) \quad m: G//G \times G//G \longrightarrow G//G$$

that is induced by multiplication on the group. So the bordism evaluates to the pushforward of multiplication, which is convolution (26.18). We leave the reader to work out the details of these computations and other computations in §26.2. (See also [F9].)

Lecture 28: The 1-dimensional cobordism hypothesis

The *cobordism hypothesis* was formulated in 1995 by Baez–Dolan [BaDo1]. It was proved by Hopkins–Lurie in dimension two and by Lurie [Lu1] in all dimensions, though a proof with complete details has yet to appear. Ayala–Francis [AF] have another approach to this theorem. In this lecture we discuss the one-dimensional case of the theorem, which in its simplest version does not involve higher categories. There is an “infinity version” that is more powerful; see [Lu1, §4.2] and [Har2]. In the next lecture we introduce extended topological field theories and the general cobordism hypothesis.

One important feature of the cobordism hypothesis is that it identifies the homotopy type of a *space* of field theories. It is of great interest to determine moduli spaces of quantum field theories of a fixed type in general; here, in the topological situation, we obtain a complete answer.

The crucial property of topological field theories identified by Baez–Dolan is *dualizability*. We discussed duality in symmetric monoidal categories in §25.4 and dualizability in bordism categories in §25.5. The reader is well-advised to review these notions.

28.1 Antecedent in classical bordism

A 0-dimensional manifold is a discrete set of points that is finite or countable; only finite sets are *compact* in the discrete topology. An orientation on a discrete set is a function to $\{\pm 1\}$. The disjoint union $\text{pt}_+ \sqcup \text{pt}_-$ of a positively oriented point and a negatively oriented point bounds an oriented closed interval. This leads to an isomorphism⁹²

$$(28.1) \quad \Omega_0(BSO_1) \xrightarrow{\cong} \mathbb{Z}$$

that maps a compact oriented 0-manifold to the integral of the orientation (as a function to $\{\pm 1\}$). In other words, the bordism group $\Omega_0(BSO_1)$ is free on a single generator. An alternative formulation is in terms of bordism invariants.

Proposition 28.2. *Let M be a commutative monoid, and denote by $M^\sim \subset M$ the subgroup of units. Then the map*

$$(28.3) \quad \begin{aligned} \phi: \text{Hom}(\Omega_0(BSO_1), M) &\longrightarrow M^\sim \\ f &\longmapsto f(\text{pt}_+) \end{aligned}$$

is an isomorphism of abelian groups.

There are several statements embedded in [Proposition 28.2](#). Namely, for an M -valued bordism invariant f :

- (1) $f(\text{pt}_+)$ is invertible.
- (2) $f(\text{pt}_+)$ determines f .
- (3) Any $m \in M^\sim$ determines an M -valued bordism invariant f_m .
- (4) The collection of M -valued bordism invariants is an abelian group.

The cobordism hypothesis is a categorification and generalization of [Proposition 28.2](#).

28.2 Cobordism hypothesis: 1-dimensional case

As mentioned above, we only treat the discrete case, i.e., field theories with values in a (discrete) category. Let $\mathcal{C}$ be a symmetric monoidal category. It has a subcategory⁹³ $\mathcal{C}^{\text{fd}} \subset \mathcal{C}$ of dualizable objects. It is the *full* subcategory in the sense that its Hom-set between dualizable objects is canonically isomorphic to the corresponding Hom-set in $\mathcal{C}$. Let $(\mathcal{C}^{\text{fd}})^\sim \subset \mathcal{C}^{\text{fd}}$ be the underlying groupoid, obtained from $\mathcal{C}^{\text{fd}}$ by omitting all noninvertible morphisms. So we have a sequence of containments

$$(28.4) \quad \mathcal{C} \supset \mathcal{C}^{\text{fd}} \supset (\mathcal{C}^{\text{fd}})^\sim$$

⁹²This isomorphism is [Theorem 24.19\(1\)](#). Here we adopt a notation for bordism groups consistent with previous usage, for example [\(25.2\)](#).

⁹³One can think ‘fd’ stands for ‘fully dualizable’ or ‘finite dimensional’. The ‘fully’ in ‘fully dualizable’ refers to the higher categorical case, as discussed in the next lecture. ‘Finite dimensional’ refers to the category of vector spaces: a vector space is dualizable iff it is finite dimensional.

Theorem 28.5. *The map*

$$(28.6) \quad \begin{aligned} \Phi: \operatorname{Hom}(\operatorname{Bord}_{\langle 0,1 \rangle}(BSO_1), \mathcal{C}) &\longrightarrow (\mathcal{C}^{\text{fd}})^{\sim} \\ F &\longmapsto F(\text{pt}_+) \end{aligned}$$

is an equivalence of groupoids.

As above, there are several statements embedded in this 1-dimensional cobordism hypothesis. For a $\mathcal{C}$ -valued topological field theory F :

- (1) $F(\text{pt}_+)$ is invertible.
- (2) $F(\text{pt}_+)$ determines F .
- (3) Any $c \in (\mathcal{C}^{\text{fd}})^{\sim}$ determines a $\mathcal{C}$ -valued topological field theory F_c .
- (4) The collection of $\mathcal{C}$ -valued topological field theories is a groupoid.

The last statement is [Corollary 25.40](#); the crucial step in the proof is [Proposition 25.39](#). The latter states that if $\eta: F \rightarrow G$ is a natural transformation of functors $\mathcal{B} \rightarrow \mathcal{C}$, and if $y \in \mathcal{B}$ is a dualizable object, then $\eta(y): F(y) \rightarrow G(y)$ is invertible:⁹⁴

$$(28.8) \quad \eta(y)^{-1} = \eta(y^\vee)^\vee.$$

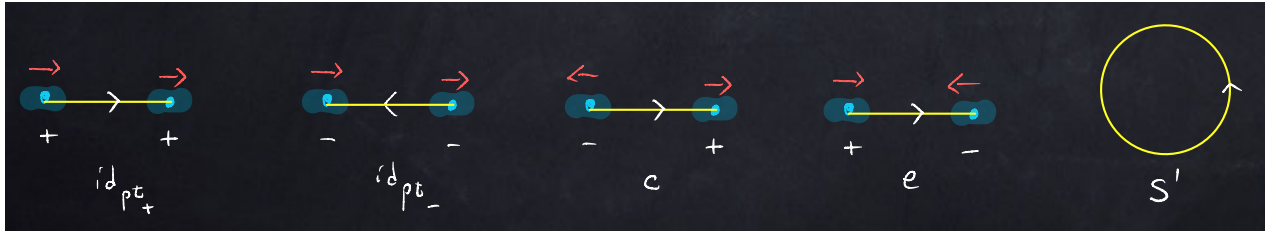


FIGURE 94. The five connected oriented bordisms in $\operatorname{Bord}_{\langle 0,1 \rangle}(BSO_1)$

In the remainder of this lecture we sketch a proof of [Theorem 28.5](#). It relies on the classification of closed 0-manifolds and compact 1-manifolds with boundary [\[Mi3\]](#). Note that if Y_0, Y_1 are closed 0-manifolds that are diffeomorphic, then the set of diffeomorphisms $Y_0 \rightarrow Y_1$ is a torsor for the group of permutations (of, say, Y_0). A connected compact 1-manifold with boundary is diffeomorphic to a circle or a closed interval, which immediately leads to the classification of connected morphisms in $\operatorname{Bord}_{\langle 0,1 \rangle}(BSO_1)$, as illustrated in [Figure 94](#): every connected oriented bordism is diffeomorphic to one of the five illustrations in that figure. (The white arrows indicate orientation; the red arrows indicate the arrow of time.)

⁹⁴Formula (28.8) uses the dual of a morphism between dualizable objects: if y_0, y_1 are dualizable objects in a symmetric monoidal category with duality data $(y_i^\vee, c_i, e_i)$, and if $f: y_0 \rightarrow y_1$ is a morphism, then the dual of f is the morphism

$$(28.7) \quad f^\vee: y_1^\vee \xrightarrow{1c_0} y_1^\vee \otimes y_0 \otimes y_0^\vee \xrightarrow{1f1} y_1^\vee \otimes y_1 \otimes y_0^\vee \xrightarrow{e_11} y_0^\vee$$

where ‘1’ indicates the identity morphism.

Proof of Theorem 28.5. By a basic theorem in category theory, we must show that Φ is fully faithful and essentially surjective.

First, let $F, G: \text{Bord}_{\langle 0,1 \rangle}(BSO_1) \rightarrow \mathcal{C}$ be field theories, let $\eta_1, \eta_2: F \rightarrow G$ be isomorphisms, and suppose $\eta_1(\text{pt}_+) = \eta_2(\text{pt}_+)$. Since $\text{pt}_- = \text{pt}_+^\vee$, then according to (28.8) we have $\eta(\text{pt}_-) = (\eta(\text{pt}_+))^\vee$ for any natural isomorphism η . It follows that $\eta_1(\text{pt}_-) = \eta_2(\text{pt}_-)$. Since any compact oriented 0-manifold Y is a finite disjoint union of copies of pt_+ and pt_- , it follows that $\eta_1(Y) = \eta_2(Y)$ for all Y , hence $\eta_1 = \eta_2$. This shows that Φ is faithful.

To show that Φ is full, suppose given F, G and an isomorphism $f: F(\text{pt}_+) \rightarrow G(\text{pt}_+)$. We must construct $\eta: F \rightarrow G$ such that $\eta(\text{pt}_+) = f$. First, define $\eta(\text{pt}_+) = f$ and $\eta(\text{pt}_-) = (f^\vee)^{-1}$. Extend using the monoidal structure in $\mathcal{C}$ to define $\eta(Y)$ for all compact oriented 0-manifolds Y . This uses the statement given before the proof that any such Y is diffeomorphic to $(\text{pt}_+)^{\amalg n_+} \amalg (\text{pt}_-)^{\amalg n_-}$ for unique $n_+, n_- \in \mathbb{Z}^{\geq 0}$. Also, the diffeomorphism is determined up to permutation, but because of coherence the resulting map $\eta(Y)$ is independent of the chosen diffeomorphism. It remains to show that η is a natural isomorphism, so to verify the usual commutative diagram for each morphism in $\text{Bord}_{\langle 0,1 \rangle}(BSO_1)$. It suffices to consider connected bordisms, i.e., each of the five morphisms in Figure 94. The first two are identity maps, for which the verification is trivial. The commutativity of the diagram

$$(28.9) \quad \begin{array}{ccc} & & F(-)F(+) \\ & \nearrow^{F(c)} & \downarrow (f^\vee)^{-1}f \\ 1 & & \\ & \searrow_{G(c)} & \\ & & G(-)G(+) \end{array}$$

for coevaluation c follows from the commutativity of

$$(28.10) \quad \begin{array}{ccccc} 1 & \xrightarrow{F(c)} & F(-)F(+) & \xrightarrow{1f} & F(-)G(+) \\ \downarrow G(c) & & \downarrow 11G(c) & & \uparrow 1G(e)1 \\ G(-)G(+) & \xrightarrow{F(c)11} & F(-)F(+)G(-)G(+) & \xrightarrow{1f11} & F(-)G(+)G(-)G(+) \end{array}$$

In these diagrams we use ‘+’ and ‘−’ for ‘ pt_+ ’ and ‘ pt_- ’, and also denote identity maps as ‘1’. The argument for evaluation e is similar, and that for the circle follows since the circle is $e \circ \sigma \circ c$ for σ the symmetry. Notice that the naturality diagram for the circle S^1 as a 1-morphism asserts the equality $F(S^1) = G(S^1)$ of partition functions.

Finally, we must show that Φ is essentially surjective. Given $y \in \mathcal{C}$ dualizable, we must⁹⁵ construct a field theory F with $F(\text{pt}_+) = y$. Let $(y^\vee, c_y, e_y)$ be duality data for y . Define $F(\text{pt}_+) = y$, $F(\text{pt}_-) = y^\vee$, and

$$(28.11) \quad F((\text{pt}_+)^{\amalg n_+} \amalg (\text{pt}_-)^{\amalg n_-}) = y^{\otimes n_+} \otimes (y^\vee)^{\otimes n_-}.$$

⁹⁵In fact, we only need construct F with $F(\text{pt}_+) \cong y$, but we will construct F such that the stronger condition $F(\text{pt}_+) = y$ holds.

Any compact oriented 0-manifold Y is diffeomorphic to some $(\text{pt}_+)^{\sqcup n_+} \amalg (\text{pt}_-)^{\sqcup n_-}$, and again by coherence the choice of diffeomorphism does not matter. Now any oriented bordism $X: Y_0 \rightarrow Y_1$ is diffeomorphic to a disjoint union of the bordisms in Figure 94, and for these standard bordisms we define $F(c) = c_y$, $F(e) = e_y$, and $F(S^1) = e_y \circ \sigma \circ c_y$; the first two bordisms in the figure are identity maps, which necessarily map to identity maps. Define F to map X to the corresponding tensor product of c , e , and $e \circ \sigma \circ c$. It remains to check that F is a functor, i.e., that compositions map to compositions. When composing in $\text{Bord}_{\langle 0,1 \rangle}(BSO_1)$ the only nontrivial compositions are those indicated in Figure 95. The relations (25.29) show that the first two compositions are consistent under F . The last composition is what we use to define $F(S^1)$. $\square$

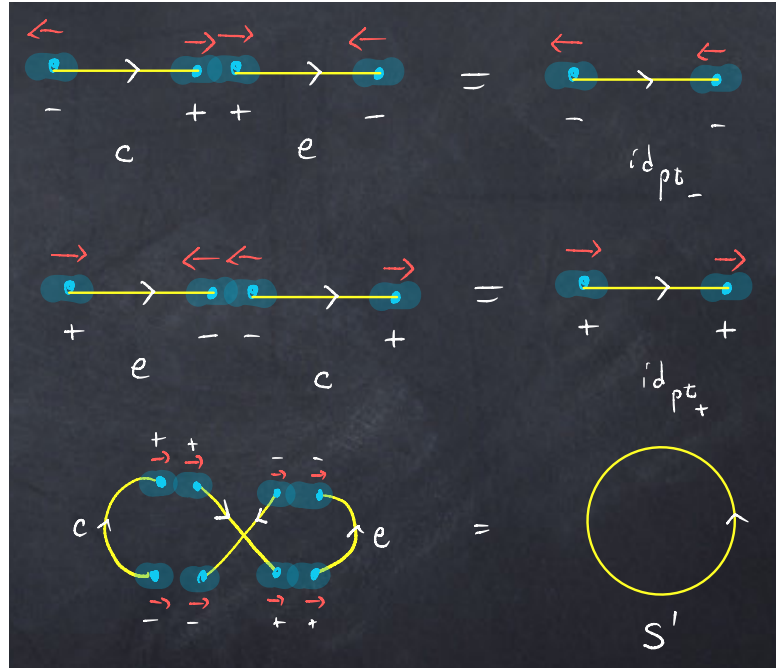


FIGURE 95. Nontrivial compositions in $\text{Bord}_{\langle 0,1 \rangle}(BSO_1)$

Remark 28.12. This last construction shows how one uses duality data to compute the values of 1-dimensional topological field theory from the value on a point: the coevaluation and evaluation tells the values on Morse handles, and any 1-manifold is built by successively attaching Morse handles. This simple idea is behind the higher dimensional case as well, but one needs higher duality data to compute higher dimensional Morse handles.

Lecture 29: Extended locality and the cobordism hypothesis

Physicists often say that the two pillars of quantum field theory are locality and unitarity. Field theories—topological and more general—as discussed so far have “one level” of locality. For

example, the partition function of a closed manifold can be computed by cutting the manifold along a codimension one submanifold, thereby factoring the partition function through a vector space. *Extended locality* allows to cut twice transversely, and so factor the vector space through a category. In principle, extended locality can be defined for any quantum field theory, but in practice the notion is most developed for topological field theories. Continued cutting leads naturally to higher categories. A *fully local* theory allows to cut maximally, essentially expressing the manifold as glued from balls. The *cobordism hypothesis* asserts that in the fully local topological case, the partition function—indeed, the entire theory—can be reconstructed from its values on balls. Extended locality and the cobordism hypothesis are the subject of this lecture.

There are many references on extended topological field theory and the cobordism hypothesis. Extended field theory was in the air in the early 1990's; the first papers on the topic are [F9, L2]. The cobordism hypothesis was conjectured by Baez–Dolan in [BaDo1]. The paper by Lurie [Lu1] contains much useful exposition as well as a (sketch) proof of the cobordism hypothesis. Introductory surveys include [Be, F9, Te]. See also [AF].

29.1 Extended locality in topological field theory

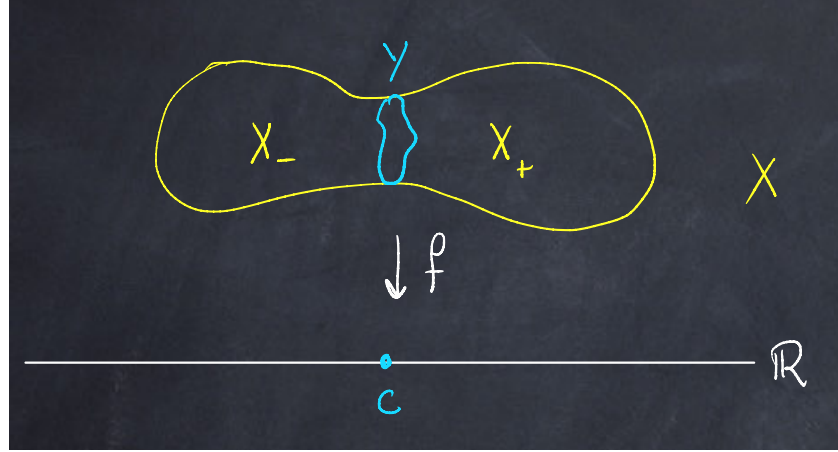


FIGURE 96. Decomposition of X as a composition of bordisms

Let $F: \text{Bord}_{\langle n-1, n \rangle}(\mathcal{X}_n) \rightarrow f\text{Vect}_{\mathbb{C}}$ be a topological field theory of some dimension $n \in \mathbb{Z}^{>0}$ on manifolds with tangential structure $\mathcal{X}_n \rightarrow B\text{GL}_n \mathbb{R}$. Suppose X is a closed $\mathcal{X}_n$ -manifold. The partition function $F(X) \in \mathbb{C}$ exhibits some level of locality. Namely, suppose $f: X \rightarrow \mathbb{R}$ is a smooth (Morse) function and $c \in \mathbb{R}$ is a regular value such that $f^{-1}(c) \subset X$ is nonempty. Set

$$\begin{aligned}
 Y &= f^{-1}(c) \\
 X_- &= f^{-1}((-\infty, c]): \emptyset^{n-1} \longrightarrow Y \\
 X_+ &:= f^{-1}([c, \infty)): Y \longrightarrow \emptyset^{n-1}
 \end{aligned}
 \tag{29.1}$$

(See Figure 96.) Then X , regarded as a bordism $\emptyset^{n-1} \rightarrow \emptyset^{n-1}$, can be written as a composition $X = X_+ \circ X_-$, and so

$$F(X) = F(X_+) \circ F(X_-).$$

(29.2)

This exhibits some locality: the numerical invariant $F(X)$ is factored through vector-valued invariants of smaller submanifolds.

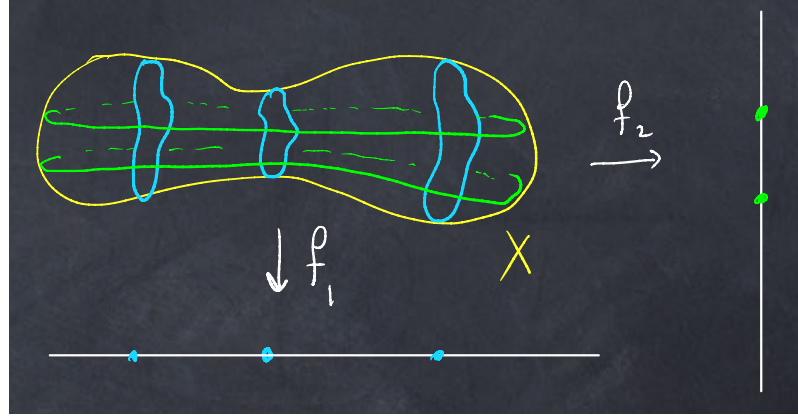


FIGURE 97. Cutting X in two directions

Now consider two (Morse) functions $f_1, f_2: X \rightarrow \mathbb{R}$, and suppose we cut simultaneously using regular values of each function, as in [Figure 97](#). This expresses X as a union of manifolds with corners. If $n = 2$, then by choosing enough critical values we can ensure that the interior of each of these manifolds is diffeomorphic to a ball. To keep track of this decomposition, we need two composition laws, one for each of the two “directions” encoded in the functions f_1, f_2 . To achieve full locality in n dimensions for any n we need to cut using n functions and use an algebraic structure that has n composition laws. An n -category is the desired algebraic structure. (A ‘1-category’ is an ordinary category.)

29.2 Higher categories

A category is an algebraic structure defined by the following data:⁹⁶ a pair of sets⁹⁷ C^0, C^1 ; a map $C^0 \rightarrow C^1$; two maps $C^1 \rightarrow C^0$; and a map $C^1 \times_{C^0} C^1 \rightarrow C^1$. There are several axioms satisfied by this data. I list the data in this form to demystify ‘category’, which is a standard algebraic structure. Now, for any $n \in \mathbb{Z}^{>0}$, there is an algebraic structure called an n -category, and one can easily imagine that the specification of its data and conditions gets more and more complicated as n increases. Indeed, it is only since the mid 1990’s that there is a robust mathematical theory of higher categories, and that theory is still under active development. In fact, there are variations (n -categories and n -uple categories), each of which has its advantages, both conceptual and practical. An n -category has a set C^0 of objects and sets $C^1, \dots, C^n$ of morphisms; elements of C^k are called

⁹⁶ C^0 is the set of objects, C^1 is the set of morphisms, the map $C^0 \rightarrow C^1$ assigns an identity morphism to each object, the maps $C^1 \rightarrow C^0$ tell the source and target of each morphism, the fiber product $C^1 \times_{C^0} C^1$ is the set of composable pairs of morphisms, and the map $C^1 \times_{C^0} C^1 \rightarrow C^1$ is the composition of composable morphisms.

⁹⁷One needs to allow larger collections than sets in many examples, but we will not discuss the underlying set theory. See [\[Mac, Ri2\]](#) for an introduction to categories.

k-morphisms. For example, a 2-morphism α is depicted in the diagram:

$$(29.3) \quad \begin{array}{ccc} & f & \\ y_0 & \begin{array}{c} \searrow \\ \Downarrow \alpha \\ \nearrow \end{array} & y_1 \\ & f' & \end{array}$$

In this diagram y_0 and y_1 are objects, f and f' are 1-morphisms $y_0 \rightarrow y_1$, and α is a 2-morphism $f \rightarrow f'$. Manifolds with corners do not have this shape; one must “blow up” the vertices y_0 and y_1 and remember that the resulting edges came from vertices. See [FT1] for a description and [CaSc] for a complete treatment of the bordism category as an n -category.

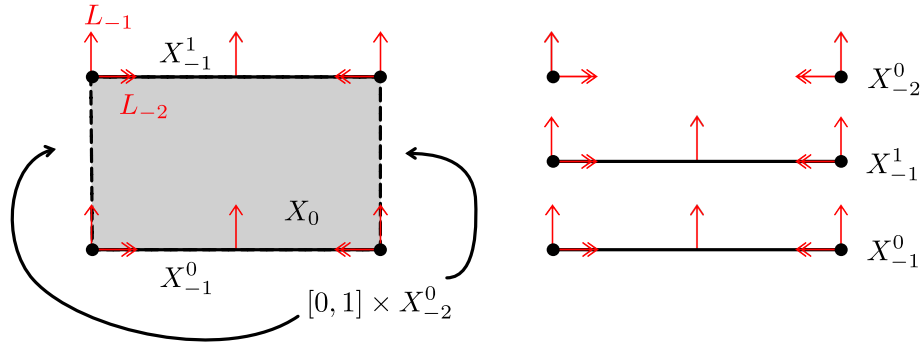
Remark 29.4. Recall that a groupoid is a category in which every morphism is invertible. An n -category in which there exists an integer $0 \leq m \leq n$ such that every k -morphism for $k > m$ is invertible is called an (n, m) -category. An n -category in this language is an (n, n) -category. A (1-)groupoid is a $(1, 0)$ -category. An n -groupoid is an $(n, 0)$ -category. There is also a notion of an ∞ -category in which k -morphisms for all $k \in \mathbb{Z}^{>0}$ are defined. Then an (∞, n) -category is one in which every k -morphism for $k > n$ is invertible. This is a topological variant of an n -category: the collection of n -morphisms between fixed lower degree morphisms and objects form a *space* rather than a set. There are many models of (∞, n) -categories; see [Be] and the references therein. A fully local topological field theory is a homomorphism of symmetric monoidal (∞, n) -categories. In this lecture we simply use ‘ n -category’, which can be taken to mean ‘ (∞, n) -category’.

Remark 29.5. There is a notion of a symmetric monoidal n -category, or more generally of a symmetric monoidal (n, m) -category (including $n = \infty$). We use this notion but do not engage with its algebraic underpinnings; see [Lu2, §2] for the $(\infty, 1)$ case and [CaSc, Part 1] for the (∞, m) -case.

We give some examples of n -categories.

Example 29.6. Let S be a topological space. The simplest algebraic invariant is $\pi_0 S$, the set of path components of S . These are equivalence classes of points of S : two points $y_0, y_1 \in S$ are equivalent if there exists a continuous path $y_0 \rightarrow y_1$. The next simplest invariant is $\pi_{\leq 1} S$, the *fundamental groupoid*. Here the set of objects is S and a morphism $y_0 \rightarrow y_1$ is an equivalence class of paths: two paths $y_0 \rightarrow y_1$ are equivalent if they are homotopic. (The homotopy is required to fix the endpoints y_0, y_1 .) Note that paths are reversible, and the composition of the reverse of a path and a path is homotopic to the path that sits at a point, which acts as an identity morphism in $\pi_{\leq 1} S$. The next invariant is $\pi_{\leq 2} S$, the *fundamental 2-groupoid* of S . Now the set of morphisms is S , the set of 1-morphisms is the set of continuous maps $[0, 1] \rightarrow S$, and the set of 2-morphisms $\gamma \rightarrow \gamma'$ between paths $\gamma, \gamma': y_0 \rightarrow y_1$ is an equivalence class of homotopies $\gamma \rightarrow \gamma'$: two homotopies $\gamma \rightarrow \gamma'$ are equivalent if they are homotopic. Observe that a homotopy $\gamma \rightarrow \gamma'$ is a continuous map $[0, 1] \times [0, 1] \rightarrow S$ that obeys constancy conditions along two edges of the domain square: one edge is the constant map y_0 and the other edge is the constant map y_1 . In this way a 2-morphism has the shape (29.3). One can continue and construct higher groupoids.

Example 29.7. The bordism 2-category $\text{Bord}_{\langle 0,1,2 \rangle}$ has the same objects as the bordism category $\text{Bord}_{\langle 0,1 \rangle}$, namely compact 0-manifolds. A 1-morphism is a bordism between 0-manifolds. In

FIGURE 98. A 2-morphism in $\text{Bord}_{(0,1,2)}$

$\text{Bord}_{(0,1)}$ one identifies diffeomorphic bordisms; in $\text{Bord}_{(0,1,2)}$ there are no such identifications. A 2-morphism is a 2-manifold with corners, and there are constancy conditions along some of the edges. **Definition 14.24** depicts a 2-morphism

$$(29.8) \quad \begin{array}{ccc} & X_{-1}^0 & \\ \curvearrowright & \Downarrow X & \curvearrowleft \\ X_{-2}^0 & & \emptyset^1 \\ & X_{-1}^1 & \end{array}$$

See [FT1, Appendix A] for an explanation of the notation. There is a symmetric monoidal structure given by disjoint union. There are higher bordism categories of manifolds with a specified tangential structure, as in §24.5.

Example 29.9. Let $\mathbb{F}$ be a field. There is a 2-category $\text{Alg}_{\mathbb{F}}$, oft called the *Morita 2-category*, and this is a common codomain for a once-extended topological field theory. The objects are algebras. A 1-morphism $A_0 \rightarrow A_1$ is an (A_1, A_0) -bimodule. Composition of 1-morphisms is defined as a tensor product of bimodules over an algebra: the composition $A_0 \xrightarrow{B} A_1 \xrightarrow{B'} A_2$ is the (A_2, A_0) -bimodule $B' \otimes_{A_1} B$. A 2-morphism

$$(29.10) \quad \begin{array}{ccc} & B & \\ \curvearrowright & \Downarrow f & \curvearrowleft \\ A_0 & & A_1 \\ & B' & \end{array}$$

is a linear map $f: B \rightarrow B'$ that preserves the (A_1, A_0) -bimodule structures. There is a symmetric monoidal structure given by tensoring over $\mathbb{F}$.

Example 29.11. Another common codomain is the 2-category $\text{Cat}_{\mathbb{F}}$ whose objects are linear categories, whose 1-morphisms are linear functors between linear categories, and whose 2-morphisms are natural transformations. One must carefully specify what types of categories one considers, especially to define the symmetric monoidal structure as a tensor product of categories; see [FT1, §1.2] and the references therein. There is a homomorphism $\text{Alg}_{\mathbb{F}} \rightarrow \text{Cat}_{\mathbb{F}}$ that assigns to an algebra A the category of left A -modules.

Example 29.9 is an example of what we might call the *Morita principle*; see [BaDo1, §5] where it goes by the name *suspension*. Namely, the 2-category $\text{Alg}_{\mathbb{F}}$ is based on the 1-category $\text{Vect}_{\mathbb{F}}$ and becomes a 2-category by leveraging the algebra structure. Objects and 1-morphisms in $\text{Alg}_{\mathbb{F}}$ are objects of $\text{Vect}_{\mathbb{F}}$ endowed with additional structure. 2-morphisms in $\text{Alg}_{\mathbb{F}}$ are 1-morphisms in $\text{Vect}_{\mathbb{F}}$. This is a special case of a general construction. Let $\mathcal{C}$ be a symmetric monoidal n -category. Then there is an $(n+1)$ -category $\text{Alg}(\mathcal{C})$ whose objects are *algebra objects* in $\mathcal{C}$, whose 1-morphisms are objects in $\mathcal{C}$ endowed with a *bimodule structure*, whose 2-morphisms are 1-morphisms in $\mathcal{C}$, etc. There are also higher versions $\text{Alg}_k(\mathcal{C})$, $k \in \mathbb{Z}^{>0}$, defined using the E_k operad. (Notation: $\text{Alg}_1(\mathcal{C}) = \text{Alg}(\mathcal{C})$.) Each level of algebra structure increases the category number, so $\text{Alg}_k(\mathcal{C})$ is an $(n+k)$ -category.

Example 29.12. The 3-category $\text{Alg}(\text{Cat})$ has objects that are *tensor categories*. The 4-category $\text{Alg}_2(\text{Cat})$ has objects that are *braided tensor categories*. For the algebraic theory of these categories, especially in the presence of finiteness conditions, see [EGNO].

29.3 Extended topological field theory

We give the following definition without defining its terms in detail.

Definition 29.13. Fix positive integers $m < n$ and fix an n -dimensional tangential structure $\mathcal{X}_n \rightarrow B\text{GL}_n\mathbb{R}$. Let $\mathcal{C}$ be an $(n-m)$ -category.

- (1) An $(n-m-1)$ times extended n -dimensional topological field theory is a symmetric monoidal functor

$$(29.14) \quad F: \text{Bord}_{\langle m, m+1, \dots, n \rangle}(\mathcal{X}_n) \longrightarrow \mathcal{C}$$

- (2) F is *fully local* if $m = 0$.

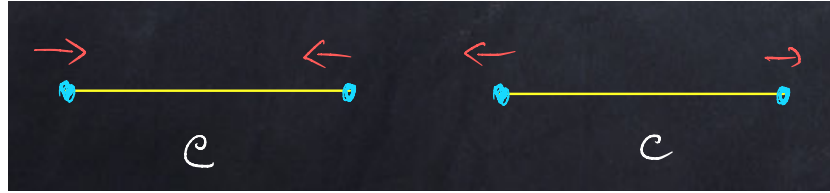


FIGURE 99. Evaluation and coevaluation 1-morphisms

As an example, 2-dimensional finite gauge theory for a finite group G (§26.2) extends to a fully local theory

$$(29.15) \quad F: \text{Bord}_{\langle 0, 1, 2 \rangle} \longrightarrow \text{Cat}_{\mathbb{C}}.$$

We indicate some values of (29.15). For example, $F(\text{pt})$ is the linear category $\text{Rep}(G)$ of finite dimensional complex linear representations of G . Note that this is the category of left modules over the group algebra $\mathbb{C}[G]$, so one can replace the codomain in (29.15) by $\text{Alg}_{\mathbb{C}}$ and send $\text{pt} \mapsto \mathbb{C}[G]$.

The point pt is self-dual in the (unoriented) bordism category; the duality data of evaluation e and coevaluation c are depicted in [Figure 99](#). The theory F evaluates on e to the functor

$$(29.16) \quad \begin{aligned} \text{Rep}(G) \otimes \text{Rep}(G) &\longrightarrow \text{Vect}_{\mathbb{C}} \\ W_1 \otimes W_2 &\longmapsto (W_1 \otimes W_2)^G \end{aligned}$$

that computes the invariant subspace of the tensor product. To evaluate the coevaluation, in each equivalence class of irreducible representation of G fix a representative. In this way we obtain a finite set $\{W_i\}$ of irreducible representations. Then coevaluation c evaluates under F to the functor $\text{Vect}_{\mathbb{C}} \rightarrow \text{Rep}(G) \otimes \text{Rep}(G)$ that sends $\mathbb{C}$ to $\bigoplus (W_i \otimes W_i)$

29.4 The cobordism hypothesis

The cobordism hypothesis for 1-dimensional topological field theories is [Theorem 28.5](#). It is for 1-dimensional theories of oriented manifolds. In the higher dimensional case the orientation is generalized to a framing.

Definition 29.17. Fix a positive integer n . A *framing* of an n -dimensional smooth manifold M is an isomorphism $\underline{\mathbb{R}}^n \xrightarrow{\cong} TM$ of its tangent bundle with the trivial bundle of rank n .

A framing is also called a *parallelism*, and a manifold that admits a framing is called *parallelizable*. We also use the term ‘ n -framing’ to avoid ambiguity.

Remark 29.18. In the bordism category $\text{Bord}_n = \text{Bord}_{\langle 0,1,\dots,n \rangle}$, a manifold of dimension $< n$ is embedded in a (germ of an) n -manifold, and so a framing is an n -framing. For example, if $n = 3$ then a topological field theory typically attaches a vector space to a closed 2-manifold Y . In a 3-framed theory, the 3-dimensional germ of Y is 3-framed, which amounts to a 3-framing of the stabilized tangent bundle $\underline{\mathbb{R}} \oplus TY \rightarrow Y$. Every orientable 2-manifold Y admits a 3-framing in this sense, but few admit a 2-framing.

Remark 29.19. A 1-framing of a 1-manifold determines an orientation; the space of 1-framing refinements of an oriented 1-manifold is contractible.

Let $\mathcal{C}$ be a symmetric monoidal n -category (or (∞, n) -category). In [§25.4](#) we discussed dualizability of an object, and in [§25.5](#) we pointed out that every object in a bordism category is dualizable. In an n -category there are higher notions of k -dualizability, $1 \leq k \leq n$, and the term ‘fully dualizable’ is used synonymously with ‘ n -dualizable’. See [\[Lu1, §2.3\]](#) for an introduction. As in [\(28.4\)](#) there is a full subcategory $\mathcal{C}^{\text{fd}} \subset \mathcal{C}$ whose objects are fully dualizable and whose morphisms are also maximally dualizable=adjunctable. Let $(\mathcal{C}^{\text{fd}})^{\sim} \subset \mathcal{C}^{\text{fd}} \subset \mathcal{C}$ the the underlying n -groupoid. Write $\text{Bord}_n(\text{fr})$ for the bordism category of n -framed manifolds. The generalization of [Theorem 28.5](#) is the cobordism hypothesis for framed manifolds.

Theorem 29.20. *The map*

$$(29.21) \quad \begin{aligned} \Phi: \text{Hom}(\text{Bord}_n(\text{fr}), \mathcal{C}) &\longrightarrow (\mathcal{C}^{\text{fd}})^{\sim} \\ F &\longmapsto F(\text{pt}_+) \end{aligned}$$

is an equivalence of groupoids.

Here pt_+ is the origin in $\mathbb{A}^n$ with framing the standard basis of $\mathbb{R}^n$. (The corresponding object in $\text{Bord}_n(\text{fr})$ includes the embedding of the origin in a germ of a neighborhood in $\mathbb{A}^n$.)

There is an extension of the cobordism hypothesis for theories on the bordism category $\text{Bord}_n(\mathcal{X}_n)$, where $\mathcal{X}_n \rightarrow B\text{GL}_n\mathbb{R}$ is a tangential structure (Definition 24.26). We view $\mathcal{X}_n$ as parametrizing the standard point (origin in $\mathbb{A}^n$) equipped with an $\mathcal{X}_n$ -structure on a germ of a neighborhood. Roughly speaking, the theory is determined by its value on all of these points, i.e., by a map from $\mathcal{X}_n$ to these values. For a concrete picture, we use the equivalence between spaces and higher groupoids—to be discussed in a future lecture—to identify $(\mathcal{C}^{\text{fd}})^{\sim}$ as a topological space $\mathcal{S}_{\mathcal{C}}$. One consequence of Theorem 29.20 is that $\mathcal{S}_{\mathcal{C}}$ carries an action of the Lie group $\text{GL}_n\mathbb{R}$. Use the mixing construction and the map $\mathcal{X}_n \rightarrow B\text{GL}_n\mathbb{R}$ to construct a fiber bundle over $\mathcal{X}_n$ with fiber $\mathcal{S}_{\mathcal{C}}$. Then the values of a fully extended theory F on all points is a section of this fiber bundle. It is this section that determines the theory. See [Lu1, §2.4] for more details.

Example 29.22. Consider 3-dimensional topological field theories with values in the 3-category $\text{Alg}(\text{Cat})$; the latter is introduced in Example 29.12. Let G be a finite group. Then the fully local finite gauge theory

$$(29.23) \quad F_G: \text{Bord}_3 \longrightarrow \text{Alg}(\text{Cat})$$

can be constructed using the finite path integral (Lecture 27). The value on the point is the colimit (= limit) of the constant functor $*//G \rightarrow \text{Alg}(\text{Cat})$ with value Vect . That turns out to be the categorified group algebra $\text{Vect}[G]$. This is the tensor category whose objects are vector bundles $W \rightarrow G$. The monoidal product of bundles $W' \rightarrow G$ and $W'' \rightarrow G$ is the vector bundle with fiber

$$(29.24) \quad (W' * W'')_g = \bigoplus_{g'g''=g} W'_{g'} \otimes W''_{g''}$$

We will say more about this theory in the next lecture. There is another 3-dimensional theory whose value on a point is the tensor category $\text{Rep}(G)$ of finite dimensional representations of G . As far as we know, it does not have a semiclassical construction. It can be constructed using the cobordism hypothesis Theorem 29.20. To obtain a functor out of Bord_3 we must specify the section of $\mathcal{S}_{\mathcal{C}} \rightarrow B\text{GL}_3\mathbb{R}$ discussed above, which amounts to $\text{GL}_3\mathbb{R}$ -invariance data on $\text{Rep}(G)$. See [FT2, §3B] for more details and for the relationship between the two theories.

Lecture 30: Applications of topological field theory; 3d finite gauge theory

We begin by sketching some uses of topological field theory (TFT) in both mathematics and physics. Our account is neither exhaustive nor thorough. In the second part of the lecture we take up 3-dimensional finite gauge theory; the general finite gauge theory was introduced in §27.3 and the 2-dimensional theory was elaborated a bit in §29.3. The 3-dimensional theory is a special case of *Chern–Simons theory*, which we discuss in a future lecture.

30.1 Applications of TFT in mathematics

Recall that in algebraic topology one extracts *algebraic* invariants from topological spaces. There results an interplay between the two: one uses the algebraic invariants to prove theorems about spaces, and conversely one uses topology to prove theorems in algebra. So, for example, homology or the fundamental group apply to prove that there is no retraction $D^2 \rightarrow S^1$, and the structure of graphs applies to prove that every subgroup of a free group is free; see [H], for example. Topological field theory belongs to *differential*, rather than algebraic, topology: it is part of the study of smooth manifolds. The cobordism hypothesis (Theorem 29.20) is a deep theorem about smooth manifolds and their diffeomorphism groups. I feel that its applications to differential topology have not yet been fully explored. Here is one small indication of its power.

Example 30.1. Suppose that for some $n \in \mathbb{Z}^{>0}$ and some tangential structure $\mathcal{X}_n \rightarrow BGL_n \mathbb{R}$ we have a topological field theory

$$(30.2) \quad F: \text{Bord}_{\langle n-1, n \rangle}(\mathcal{X}_n) \longrightarrow \text{Vect}_{\mathbb{F}}$$

for some field $\mathbb{F}$. Then if Y is a closed $(n-1)$ -manifold equipped with an $\mathcal{X}_n$ -structure, the field theory F produces a linear representation (see (4) in §26.1)

$$(30.3) \quad \pi_0 \text{Diff } Y \longrightarrow \text{Aut } F(Y)$$

Suppose we have an extended topological field theory in $n = 3$ dimensions

$$(30.4) \quad F: \text{Bord}_{(1,2,3)}(\mathcal{X}_3) \longrightarrow \text{Cat}_{\mathbb{F}}$$

with values in linear categories over $\mathbb{F}$. Then we can extend (30.3) to surfaces with boundary. First, let $C = F(S^1)$ be the $\mathbb{F}$ -linear category attached to the circle. Let Y be the 2-sphere S^2 minus k open balls with disjoint closures for some $k \in \mathbb{Z}^{>0}$. Regard Y as a bordism $Y: (S^1)^{\sqcup k} \rightarrow \emptyset^1$. Then

$$(30.5) \quad F(Y): C^{\otimes k} \longrightarrow \text{Vect}_{\mathbb{F}}$$

Let $c \in C$ be an object, and consider the vector space $F(Y)(c)$ obtained by evaluating (30.5) on $c \otimes \cdots \otimes c$. Then we obtain a linear representation

$$(30.6) \quad \pi_0 \text{Diff } Y \longrightarrow \text{Aut } F(Y)(c)$$

The group $\pi_0 \text{Diff } Y$ is essentially the braid group. In this way 3-dimensional extended TFTs produce linear representations of the braid group.

Theories of type (30.4) were constructed by Reshetikhin–Turaev [RT2], based in part on Witten’s Chern–Simons theories [W3]. These theories lead to representations (30.6), which also have descriptions using closely related 2-dimensional conformal field theories. The linearity of the braid group [Big] was proved using similar representations [L3].

The preceding is an indication of an application of TFT to diffeomorphism groups. There have also been applications to algebra. For example, an application of 2-dimensional gauge theory reproves a classical formula of Frobenius and Schur; see [Sn]. Another application, this time using derived 2-dimensional field theories, is to reprove the Deligne conjecture about Hochschild homology. Moving to higher dimensions and higher algebra, consider a fully extended 3-dimensional theory

$$(30.7) \quad F: \text{Bord}_3 \longrightarrow \text{Alg}(\text{Cat})$$

of the type considered in Example 29.22. (We might also consider theories with a background orientation.) We study a particular example of such theories below in §30.3. The value $C = F(\text{pt})$ is a tensor category that enjoys stringent finiteness properties: it is 3-dualizable. One class of 3-dualizable tensor categories are *fusion categories*; see [EGNO, DSS]. One application of the field theory (30.7) is to prove Radford’s theorem in the theory of fusion categories; see [DSS, §3.3]. Another is to characterize fusion categories in terms of dualizability in the 3-category $\text{Alg}(\text{Cat})$; see [FT1]. There is current research on higher fusion categories that is intimately tied to topological field theory.

The illustrations mentioned largely involve reproofs. There are many other examples of TFT in mathematics that have forged new directions, especially in low dimensional topology, symplectic topology, and geometric representation theory. I do not attempt a survey here.

30.2 Applications of TFT in physics

Topological field theories were introduced in the late 1980’s in papers of Witten. The first [W7] carves out a “topological sector” of a nontopological field theory. That theory is supersymmetric, and the mechanism to construct a topological field theory is called “topological twisting”. We explore it in the second part of the semester. Witten’s paper [W3] on the Jones polynomial introduced a 3-dimensional topological field theory of a somewhat different type. This laid the groundwork for the many developments that followed. By now topological field theory appears in many guises on its own and as a tool to explore nontopological quantum field theories. Here are a few examples.

- (1) *A playground for structures in QFT.* The mathematical basis of topological field theory is much more developed than that of quantum field theories in general, particularly for theories with extended locality, including fully local theories and the cobordism hypothesis. This makes topological theories an ideal arena in which to explore features of general quantum field theories. These include extended locality and extended unitarity, the path integral, boundaries and defects, etc.
- (2) *Low energy approximations to gapped theories.* It is common lore that a gapped quantum system can be approximated at low energies by a topological field theory. This seems to be true in many circumstances, but not in general. One modification is that the low energy theory is only expected to be topological as a *projective* theory, not necessarily as a *linear* theory. But there are other caveats as well. In any case, when applicable the technology of topological field theories can be brought to bear on large scale questions.

- (3) *Classification of gapped phases.* If one assumes a topological low energy approximation to a gapped quantum theory, then that low energy theory is expected to capture all large scale features of the quantum theory, including its deformation class. The latter is called its *phase*. The cobordism hypothesis determines the homotopy type of the space of TFTs of a fixed dimension and with a fixed tangential structure, and this can be used to explore questions about gapped phases.
- (4) *Classification of gapped boundaries and defects.* Along these lines, when a gapped theory has a low energy topological approximation one can use it to investigate the existence and uniqueness of gapped boundary theories and gapped defects. (Boundary theories and defects are discussed in the next lecture.)
- (5) *Symmetry in quantum field theory.* Quantum field theory allows an expansion of the usual notions of symmetry in mathematics. There is a framework in which topological field theories act as “algebras” of symmetries, and defects in topological field theories play the role of elements of an algebra; see [FMT] and the references therein.

30.3 Three-dimensional finite gauge theory

Let G be a finite group. The 3-dimensional (pure) gauge theory

$$(30.8) \quad F_G: \text{Bord}_3 \longrightarrow \text{Alg}(\text{Cat}_{\mathbb{C}})$$

was mentioned in [Example 29.22](#). One construction is via the finite path integral ([Lecture 27](#)), as was illustrated for the truncation $\text{Bord}_{\langle 2,3 \rangle} \rightarrow \text{Vect}_{\mathbb{C}}$ in [§27.3](#). Here we indicate a bit more about the theory in higher codimension, i.e., on 0- and 1-manifolds.

The finite path integrals in this theory are computed as a colimit=limit in the 3-category $\text{Alg}(\text{Cat}_{\mathbb{C}})$. Rather than compute these rigorously as limits, we simply state some shortcuts; see [FHLT, §§3,8] and [FMT, Appendix A] for more details. Observe that in terms of the paradigm in [§27.1](#), the theory $\hat{F}$ that we are integrating is the tensor unit theory $\mathbb{1}$. Suppose $\mathcal{G}$ is a finite groupoid and we want to integrate $\mathbb{1}$ over $\mathcal{G}$. (In finite gauge theory, $\mathcal{G}$ is the finite groupoid of principal G -bundles on a compact manifold; see [§27.3](#).) Of course, the answer depends on the codomain of the field theory and the level we are at. If we are after a complex vector space, then the answer is the vector space $\text{Map}(\mathcal{G}, \mathbb{C})$ of complex-valued functions on $\mathcal{G}$, as in [Example 27.25](#). These are functions $f: \mathcal{G}^0 \rightarrow \mathbb{C}$ of objects such that $f(x_0) = f(x_1)$ for every morphism $x_0 \rightarrow x_1$ in $\mathcal{G}^1$. If instead we are after a complex linear category, then the answer is the linear category $\text{Vect}(\mathcal{G})$ of vector bundles on $\mathcal{G}$. These are vector bundles $W \rightarrow \mathcal{G}^0$ together with coherent lifts of morphisms $x_0 \rightarrow x_1$ to linear isomorphisms $W_{x_0} \rightarrow W_{x_1}$ of the fibers. If instead we are after a complex algebra, then it is the path algebra $\mathbb{C}[\mathcal{G}^1]$. As a vector space, it is the free vector space $\mathbb{C}\langle \mathcal{G}^1 \rangle$ on the morphism set $\mathcal{G}^1$, and the product of two morphisms is their composition if they are composable and is zero otherwise. Observe that $\text{Vect}(\mathcal{G})$ is the category of left $\mathbb{C}[\mathcal{G}^1]$ -modules. Finally, if instead we are after a tensor category, then the answer is the categorified path algebra $\text{Vect}[\mathcal{G}^1]$.

After these generalities, turning to the finite gauge theory [\(30.8\)](#), to quantize the point observe that $\text{Bun}_G(\text{pt}) \simeq *//G$. The tensor category obtained from the finite path integral is the categorified

group algebra

$$(30.9) \quad F_G(\text{pt}) = \text{Vect}[G]$$

We can interpret its objects as vector bundles over the group G , and the tensor product is convolution (29.24) of vector bundles. It is also of interest to compute the value of F_G on a circle. The circle is a bordism $S^1: \emptyset^0 \rightarrow \emptyset^0$, and so $F_G(S^1)$ is a 1-morphism $\mathbb{1} \rightarrow \mathbb{1}$ in $\text{Alg}(\text{Cat}_{\mathbb{C}})$. That is, $F_G(S^1)$ is a $(\text{Vect}_{\mathbb{C}}, \text{Vect}_{\mathbb{C}})$ -bimodule in $\text{Cat}_{\mathbb{C}}$, i.e., a linear category. Now as in (27.39) we have $\text{Bun}_G(S^1) \simeq G//G$, where G acts on itself by conjugation, and so

$$(30.10) \quad F_G(S^1) = \text{Vect}(G//G) = \text{Vect}_G(G)$$

is the category of G -equivariant vector bundles over G . This is a finite semisimple category. The simple objects are irreducible vector bundles supported on a single conjugacy class. At each element $x \in G$ in the conjugacy class, the fiber of the bundle is an irreducible representation of the centralizer subgroup $Z(x) \subset G$.

Remark 30.11. As an illustration of the principle (2) in §30.2 above, recall the discrete quantum mechanical system introduced in §5.5: the toric code. On a closed 2-manifold Y equipped with an embedded graph $\Lambda \subset Y$, the Hilbert space of the model is isomorphic to the tensor product of two dimensional Hilbert spaces over the set of edges of Λ and there is a Hamiltonian. The lowest energy eigenspace of the Hamiltonian is independent of Λ and can be identified with the vector space of complex-valued functions on the set of equivalence classes of double covers of Y . This is precisely the vector space $F_G(Y)$ for $G = \mu_2$. The toric code is a model with 2 space and 1 time dimension, so we might expect a field theoretic approximation to be 3-dimensional. The description of the low energy eigenspace is an indication that the low energy approximation is the 3-dimensional theory F_G .

There is a twisted variant of finite gauge theory. As a warmup we begin with the 2-dimensional theory. A twisting is a cocycle for a cohomology class in $H^2(BG; \mathbb{C}^\times) \cong H^3(BG; \mathbb{Z})$. Heuristically, if not rigorously, we write $H^2(BG; \mathbb{C}^\times) \cong H^1(BG; \text{Line})$ as cohomology with coefficients the groupoid of lines (with $\pi_0 = 0$ and $\pi_1 = \mathbb{C}^\times$), which indicates that we can represent the cohomology class as a multiplicative map from G to Line . This is precisely a central extension of groups

$$(30.12) \quad 1 \longrightarrow \mathbb{C}^\times \longrightarrow G^\tau \longrightarrow G \longrightarrow 1$$

As in (29.15) we can take the codomain of the theory to be the 2-category $\text{Cat}_{\mathbb{C}}$ of $\mathbb{C}$ -linear categories. Then the value of the theory on a point is the category $\text{Rep}^\tau(G)$ of complex linear representations of G^τ for which the center $\mathbb{C}^\times$ acts by scalar multiplication. If the codomain is the Morita 2-category $\text{Alg}_{\mathbb{C}}$, then the value on a point is the twisted group algebra $\mathbb{C}^\tau[G]$. The elements of this algebra may be regarded as sections of the complex line bundle over G that is associated to the extension (30.12). These answers may be derived from the finite path integral using a variant of the shortcuts outlined at the beginning of this section. Now the theory $\hat{F}$ that we are integrating

is a nontrivial invertible field theory constructed from the extension (30.12). It produces a central extension of the finite groupoid $\text{Bun}_G(\text{pt}) \simeq *//G$, which is precisely the extension (30.12) over the set G of morphisms in $*//G$. Therefore, in place of the path algebra we obtain the twisted path algebra.

Similar reasoning applies to twisted variants of the 3-dimensional theory (30.8). Now the twisting is a cocycle for a cohomology class in $H^3(BG; \mathbb{C}^\times) \cong H^4(BG; \mathbb{Z})$. (The twisting is a *level* in the language of Chern–Simons theory; indeed, this is an example of a Chern–Simons theory.) Inspired by the 2-dimensional case, write $H^3(BG; \mathbb{C}^\times) \cong H^2(BG; \text{Line})$ and represent a twisting as a vector bundle $K \rightarrow G \times G$ that satisfies a cocycle condition: so K is a 2-cocycle on G with values in lines. This is the approach in [FHLT, §4], which we quote here.

Definition 30.13. A *2-cocycle* on G with values in Line is a pair (K^τ, θ^τ) consisting of a line bundle $K^\tau \rightarrow G \times G$, for each triple $x, y, z \in G$ an isomorphism

$$(30.14) \quad \theta_{x,y,z}: K_{y,z} \otimes K_{xy,z}^{-1} \otimes K_{x,yz} \otimes K_{x,y}^{-1} \longrightarrow \mathbb{C}$$

and a cocycle condition

$$(30.15) \quad \theta_{y,z,w} \theta_{xy,z,w}^{-1} \theta_{x,yz,w} \theta_{x,y,zw}^{-1} \theta_{x,y,z} = 1, \quad x, y, z, w \in G.$$

for each quadruple of elements of G .

Proposition 30.16. For G finite the cohomology group $H^4(BG; \mathbb{Z})$ is the set of isomorphism classes of 2-cocycles (K^τ, θ^τ) on G with values in Line .

Proof. Since $H^4(BG; \mathbb{R}) = 0$, the Bockstein homomorphism $H^3(BG; \mathbb{C}^\times) \rightarrow H^4(BG; \mathbb{Z})$ from the exponential sequence of coefficients is an isomorphism.⁹⁸ Given (K^τ, θ^τ) choose $k_{x,y} \in K_{x,y}$ nonzero. Then

$$(30.17) \quad \omega_{x,y,z} = \theta_{x,y,z}(k_{y,z} k_{xy,z}^{-1} k_{x,yz} k_{x,y}^{-1})$$

is a 3-cocycle with values in $\mathbb{C}^\times$. A routine check shows that the resulting element of $H^3(BG; \mathbb{C}^\times)$ is independent of $\{k_{x,y}\}$ and of the representative (K^τ, θ^τ) in an equivalence class. Also, the resulting map from equivalence classes of 2-cocycles to $H^3(BG; \mathbb{C}^\times)$ is an injective homomorphism. Surjectivity is also immediate: if $\omega_{x,y,z}$ is a cocycle, set $K_{x,y} = \mathbb{C}$ and $\theta_{x,y,z} = \omega_{x,y,z}$. $\square$

The value of the twisted gauge theory on a point is then a twisted version $\text{Vect}^\tau[G]$ of the categorified group algebra. The underlying linear category is the same, but the monoidal product (29.24) is replaced by

$$(30.18) \quad (W' *^\tau W'')_x = \bigoplus_{x'x''=x} K_{x'x''} \otimes W'_{x'} \otimes W''_{x''}$$

Similarly, the value of the theory on a circle is a twisted version $\text{Vect}_G^\tau(G)$ of the category of G -equivariant vector bundles on G . We refer to [FHLT, F9] for details and for interpretations both as a Drinfeld center and as a twisted Drinfeld double.

⁹⁸Here $\mathbb{C}^\times$ has either the discrete topology or the continuous topology.

Lecture 31: Boundary theories and defects

Boundaries and defects are important structures in quantum field theory. One might study a theory on a particular manifold with boundary, and so ask for data on that boundary. More universally, and more locally, one can have a boundary *theory* for a given (bulk) quantum field theory. A boundary theory can then be put (painted/colored) onto any boundary. Formally, one obtains a representation of a bordism category that includes colored boundaries. Boundaries are a special case of more general defects, which can be supported on stratified submanifolds of any dimension. In this lecture we introduce boundaries and defects in topological field theory.

We begin with classical nontopological instances of boundaries and defects. In classical field theory one has a hyperbolic wave equation, and such equations can have local boundary conditions in space. For elliptic partial differential operators there is a notion of a local elliptic boundary *condition* [Ho], and we give some simple examples. We also point out that there are elliptic operators for which no such boundary condition exists. (This has an echo in topological field theory.)

A familiar example of a classical defect is in the coupled system of an electromagnetic field and a charged particle. If the particle has a fixed trajectory, then one can solve for the electromagnetic field in its presence, and in this case the charged particle is considered a defect.

We then turn to boundaries and boundary theories in topological field theory. Boundary theories are a special case of domain walls, which are (in general noninvertible) maps $\delta: F_0 \rightarrow F_1$ between theories; in the case of boundaries one of F_0 and F_1 is the tensor unit theory $\mathbb{1}$. So there are two kinds of boundary theory, which we term *left* ($F_0 = \mathbb{1}$) and *right* ($F_1 = \mathbb{1}$) by analogy with left and right modules over an algebra. For fully local theories domain walls are determined by their value on a point—this is a form of the cobordism hypothesis.

Defects have support on a submanifold, which can be stratified. They can be global or local; the latter are the generalization of boundary theories. The cobordism hypothesis produces an integration from local defects to global defects, though we do not develop that here. Instead, we illustrate the notion of a defect in 3-dimensional field theories of Reshetikhin–Turaev type [RT2]. These defects were used by Witten [W3] to produce invariants of knots and links, such as the Jones polynomial, from topological field theory.

31.1 Local boundary conditions for partial differential operators

Consider first the 2-dimensional wave equation

$$(31.1) \quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0,$$

where $u: \mathbb{R}_t \times [0, 1]_x \rightarrow \mathbb{R}$ is a smooth function. This is a hyperbolic equation, naturally associated to the Lorentz metric on spacetime $\mathbb{R} \times [0, 1]$. Since space has a boundary, for (31.1) to be well-posed we must impose a boundary condition in space. A typical boundary condition asserts that

$$(31.2) \quad u(t, 0) = u(t, 1) = 1$$

for all $t \in \mathbb{R}$. (Alternatively, we might impose $\frac{\partial u}{\partial t}(t, 0) = \frac{\partial u}{\partial t}(t, 1) = 0$.) One can also consider boundaries in time, so for example functions $u: [0, \infty) \times [0, 1] \rightarrow \mathbb{R}$ that solve (31.1). In that case one needs to impose a boundary condition at $t = 0$, which might be a Cauchy initial value of the function u .

Consider now an elliptic equation, say the 2-dimensional Laplace equation

$$(31.3) \quad -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

for a function $u: D \rightarrow \mathbb{R}$ on the unit disk $D \subset \mathbb{A}_{x,y}^2$ in the standard affine plane. The Laplace operator is naturally associated to the Euclidean metric on D . A typical boundary condition, called a *Dirichlet* condition, fixes a function $f: \partial D \rightarrow \mathbb{R}$ and imposes

$$(31.4) \quad u|_{\partial D} = f.$$

One can imagine, instead of a boundary *condition* on u , positing boundary *data* and imposing a coupling between u and the boundary data. For example, one might have a variable function $v: \partial D \rightarrow \mathbb{R}$ and consider a coupled system of (31.3) and the equations

$$(31.5) \quad \begin{aligned} \frac{d^2 v}{d\theta^2} + v^2 &= 0 \\ u|_{\partial D} &= v \end{aligned}$$

where θ is a coordinate on ∂D . This is a better classical analog of a quantum boundary *theory*.

We remark that *local* boundary conditions, such as (31.4), do not exist for every linear elliptic differential operator. The simplest case of nonexistence is for the $\bar{\partial}$ operator

$$(31.6) \quad \frac{\partial \phi}{\partial x} + \sqrt{-1} \frac{\partial \phi}{\partial y} = 0,$$

where $\phi: D \rightarrow \mathbb{C}$ is a complex-valued function. This is a special case of nonexistence for Dirac operators, as was proved by Atiyah–Bott [AB1].

31.2 A classical defect

Let $\mathbb{M} = \mathbb{M}_{t,x,y,z}^4$ be standard Minkowski spacetime, and let $F \in \Omega_{\mathbb{M}}^2$ be an electromagnetic field. (See Section 19.4 and Section 14.1 as background for this discussion.) We consider a particle of rest mass $m_0 \in \mathbb{R}^{>0}$ and charge $q \in \mathbb{R}$ moving in this electromagnetic field along a worldline $C \subset \mathbb{M}$. This is a coupled system, described by a lagrangian density that combines (19.22) and (14.2) together with an interaction term. Let $A \in \Omega_{\mathbb{M}}^1 / \Omega_{\mathbb{M}, \text{closed}}^1$, as in (19.19), and set $F = dA$. Instead of a lagrangian density we write the action

$$(31.7) \quad S = -\frac{1}{2} \int_{\mathbb{M}} dA \wedge *dA - m_0 c \int_C d\mu_C - q \int_C A,$$

where despite the notation we perform the integral over a bounded region in $\mathbb{M}$ and its intersection with C . Observe that the last term requires an orientation of C , which is defined using the time-orientation of $\mathbb{M}$.

The action (31.7) describes a coupled system, and we can use it in various ways. For example, we can treat A as a background field and so solve for the motion of a charged particle in the presence of a fixed electromagnetic field; this gives the Lorentz force law. Conversely, we can fix C and solve for the electromagnetic field induced by a charged particle. This is the point of view in which the charged particle is a *defect* in Maxwell theory.

Remark 31.8. In general, there may be other fields propagating on the support of a defect. We illustrated that for boundaries in (31.5).

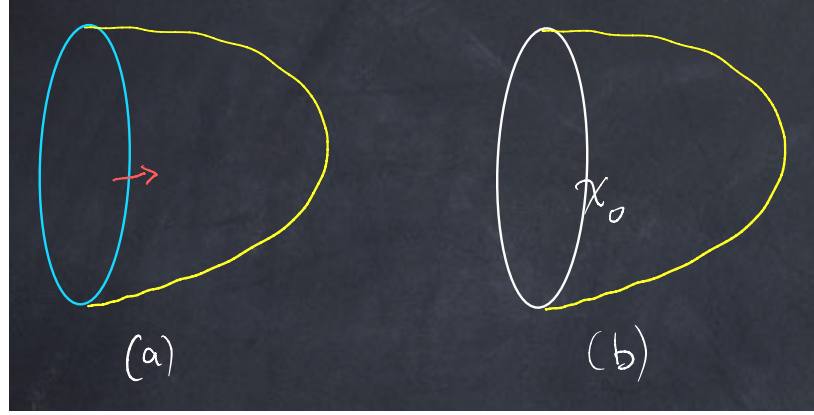


FIGURE 100. (a) a bordism $S^1 \rightarrow \emptyset^1$ and (b) a 2-manifold with boundary state

31.3 Boundaries and domain walls in topological field theory

We begin with an example. Let G be a finite group and let

$$(31.9) \quad F_G: \text{Bord}_2 \rightarrow \text{Cat}_{\mathbb{C}}$$

be the associated 2-dimensional finite gauge theory (§26.2). Recall that $F(S^1)$ is the vector space $A = \text{Map}(G, \mathbb{C})^G$ of complex-valued central functions on G . The bordism in Figure 100(a) is a map $D: S^1 \rightarrow \emptyset^1$; it evaluates to the linear functional

$$(31.10) \quad \begin{aligned} A &\longrightarrow \mathbb{C} \\ \chi &\longmapsto \frac{\chi(e)}{\#G} \end{aligned}$$

So we can insert a *global* boundary condition on ∂D labeled by a vector $\chi_0 \in A$. From the point of view of an extended bordism category, the resulting bordism depicted in Figure 100(b) is a map $\emptyset^1 \rightarrow \emptyset^1$, and its value under an extended F_G is a number, namely $\chi_0(e)/\#G$.

Remark 31.11. In classical PDE one needs a bit of analysis to see that (31.4) closes off the disk in the sense that the resulting partial differential operator (31.3)+(31.4) behaves equivalently to an elliptic partial differential operator on a closed manifold. In Wick-rotated quantum field theory, this closing—or coning—off of the boundary is accomplished by fiat.

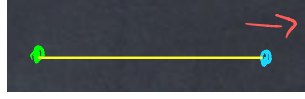


FIGURE 101. A boundary theory

A more local and universal construction is of a *boundary theory*. This allows to “color” parts of boundaries in a bordism with this theory. The bordism in Figure 101 is a map $\emptyset^0 \rightarrow \text{pt}$; the boundary theory, depicted in green, cones off the boundary component on which it is supported. So the value of F_G on this bordism is a map $\text{Vect}_{\mathbb{C}} \rightarrow F_G(\text{pt})$, or equivalently an object in the category $F_G(\text{pt}) = \text{Rep}(G)$. Let ρ be the representation that defines this boundary theory. A variant of the cobordism hypothesis tells that the value ρ determines the entire boundary theory. For example, Figure 102 depicts a bordism $\emptyset^1 \rightarrow S^1$ in an extended bordism category Bord_2^∂ in which one can color boundaries (according to rules: see [FT1]) with the boundary theory. It evaluates to a vector in $F_G(S^1) = A$, which is the character $\chi(\rho)$ of the representation ρ . Characters of representations live in a lattice in A , in fact, in a positive cone in the lattice. Contrast with the boundary state $\chi_0 \in A$ in Figure 100(b), which can be any vector in A . More locality imposes *integrality* and, in this case, *positivity* as well.⁹⁹ As another illustration, the bordism $\emptyset^1 \rightarrow \emptyset^1$ in Figure 103 evaluates to the rational number $\dim \rho / \#G$.

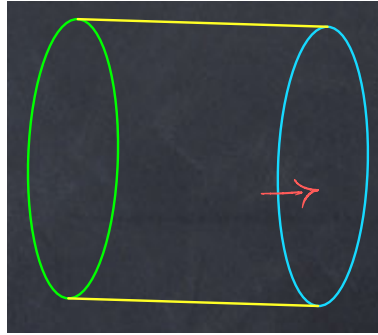


FIGURE 102. A bordism $\emptyset^1 \rightarrow S^1$ in the bordism category Bord_2^∂

Remark 31.12. The boundary theory is defined as an extension of (31.9) to a homomorphism $\text{Bord}_2^\partial \rightarrow \text{Cat}_{\mathbb{C}}$. The cobordism hypothesis asserts that the extension is determined by its value on Figure 101.

A boundary is a special case of a *domain wall*. Figure 104 depicts two theories F_1, F_2 and a domain wall from F_1 to F_2 . The direction of the domain wall is not indicated in the figure.

Remark 31.13.

⁹⁹In derived settings we would not see positivity, only integrality.

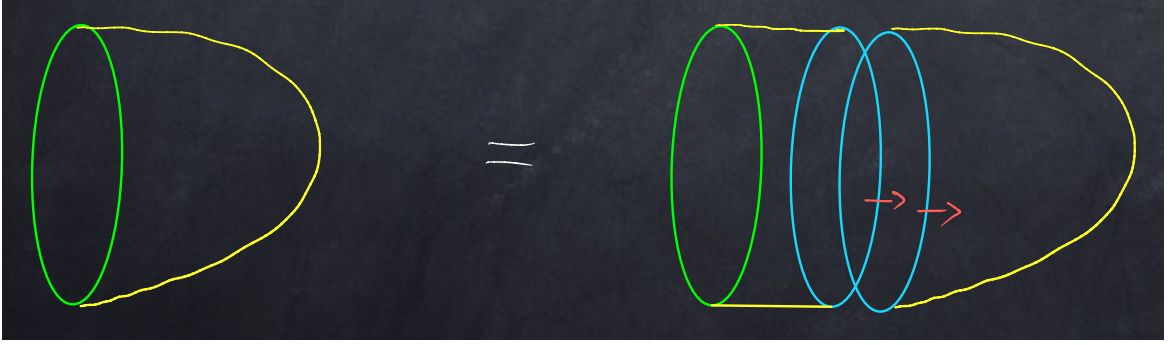
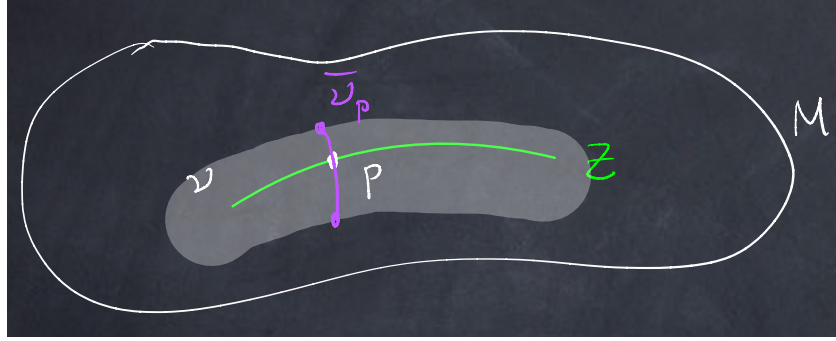

 FIGURE 103. Another bordism in Bord_2^∂

 FIGURE 104. A domain wall from F_1 to F_2

- (1) [Corollary 25.40](#) asserts that a naive map of field theories is invertible, and the same assertion holds for extended field theories. In general, a domain wall between theories $F_1, F_2: \text{Bord}_n(\mathcal{X}_n) \rightarrow \mathcal{C}$ is *not* a natural transformation of these functors, but rather is a natural transformation of the *truncated* functors $\text{Bord}_{n-1}(\mathcal{X}_n) \rightarrow \mathcal{C}$. Such natural transformations need not be invertible. (Think through the case $n = 1$ with codomain $\mathcal{C} = \text{Vect.}$) We can factor these truncations through an $(\infty, n-1)$ subcategory of $\mathcal{C}$. Note that the tangential structure of the truncation is still n -dimensional, not $(n-1)$ -dimensional.
- (2) If both F_1 and F_2 equal the tensor unit theory $\mathbb{1}$, then a domain wall is an ordinary $(n-1)$ -dimensional topological field theory, except that the tangential structure is n -dimensional. If one of F_1 and F_2 equals $\mathbb{1}$, then a domain wall reduces to a boundary theory. This makes clear that there are two types of boundary theory.
- (3) There is an analogy in which F_1 and F_2 are algebras and the domain wall is an (F_1, F_2) -bimodule. If $F_2 = \mathbb{1}$ we obtain a left F_1 -module, which we naturally call a *left* boundary theory for F_1 . If $F_1 = \mathbb{1}$ we obtain a right F_2 -module, which we naturally call a *right* boundary theory for F_2 . The nomenclature emphasizes the position of the bulk theory relative to the boundary theory, not the other way around.
- (4) The cobordism hypothesis implies that the theories F_1, F_2 and domain wall $F_1 \rightarrow F_2$ are determined by objects $c_1, c_2 \in \mathcal{C}$ and a 1-morphism $f: c_1 \rightarrow c_2$. The objects c_1, c_2 must be n -dualizable and the morphism f must be $(n-1)$ -dualizable.
- (5) One can relax the finiteness conditions and only require that c_1, c_2 be $(n-1)$ -dualizable. This is a powerful generalization.

31.4 Defects in topological field theory

Boundaries and domain walls are special cases of more general defects. We only touch the surface here; see [\[FMT\]](#) and the references therein for more discussion.

FIGURE 105. A submanifold $Z \subset M$ and a tubular neighborhood

Let $F: \text{Bord}_n(\mathcal{X}_n) \rightarrow \mathcal{C}$ be a fully local topological field theory, and let M be a closed manifold of dimension $\leq n$. (The discussion applies to manifolds with boundaries and corners—more general bordisms—but for simplicity of exposition and notation we only consider closed manifolds.) Let $\nu \subset M$ be an open tubular neighborhood of $Z \subset M$ such that its closure is the total space of a fiber bundle $\bar{\nu} \rightarrow Z$ with fiber a closed ℓ -dimensional disk, where $\ell = \text{codim}_M Z$. The boundary $\partial \bar{\nu}_p$ of the fiber at $p \in Z$ is called the *link* of $Z \subset M$ at p . It is diffeomorphic to $S^{\ell-1}$, but not canonically so. A *normal framing* of $Z \subset M$ provides a diffeomorphism. The first part of the following is a generalization of a boundary theory; the second part is a generalization of the global state in Figure 100(b).

Definition 31.14.

- (1) A *local defect* at $p \in Z$ is a morphism

$$(31.15) \quad \delta_p \in \text{Hom}(1, F(\partial \bar{\nu}_p)).$$

- (2) Assume $Z \subset M$ is a closed submanifold. Then a *global defect* with support Z is a morphism

$$(31.16) \quad \delta_Z \in \text{Hom}(1, F(\partial \bar{\nu})).$$

In (31.16) the submanifold $\partial \bar{\nu} \subset M$ has codimension one. If $\dim M = n$, then $F(\partial \bar{\nu})$ is a vector space and δ_Z is a vector in it. If $\dim M < n$, then one has a higher categorical analog. One can treat $M \setminus \nu$ as a bordism from $\partial \bar{\nu}$ to the empty manifold. The composition of its evaluation under F with δ_Z is a morphism $1 \rightarrow 1$ at the appropriate categorical level. In this way a global defect supported on Z gives a value of the same type as if Z is not there. Given a local defect at each $p \in Z$, i.e., a section of a bundle of local defects over Z , a version of the cobordism hypothesis can be used to integrate to a global defect on Z ; see [FMT, §2.5].

As an example, consider an extended 3-dimensional theory

$$(31.17) \quad F: \text{Bord}_{\langle 1,2,3 \rangle}(\mathcal{X}_3) \longrightarrow \text{Cat}_{\mathbb{C}}$$

We implicitly allow a tangential structure. We study this theory on a 3-manifold M with a defect supported on an embedded link Z , as in Figure 106. Assume $Z \subset M$ is normally framed. Then

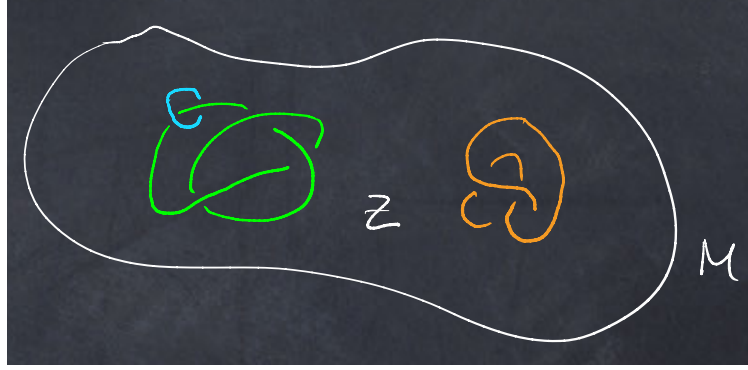


FIGURE 106. A link in a 3-manifold

$\partial\bar{v} \approx S^1 \times S^1$ on each component. Consider a global defect supported on Z . Since we only need this identification up to isotopy (Example 26.4(4)), we only need a basis of $H_1(\partial\bar{v})$. Note that we already have a distinguished element up to sign: a generator of $\ker(H_1(\partial\bar{v}) \rightarrow H_1(\bar{v}))$. Then

$$(31.18) \quad F(\partial\bar{v}) \cong F(S^1 \times S^1) =: V$$

and the global defect is given by a vector in V . Turning to local defects, the link of a point $p \in Z$ is a circle, and the normal framing identifies it with the standard circle S^1 . Set $C = F(S^1)$. Assume that the linear category C is finite semisimple, and let $S(C)$ be the finite set of isomorphism classes of simple objects. Any object is a finite sum of simple objects, so the isomorphism class of a local defect lies in $\mathbb{Z}^{\geq 0}\langle S(C) \rangle$. In terms of C we have $V \cong \mathbb{C}\langle S(C) \rangle$. There is an (easy in the normally framed case) integration process that produces a global defect from a local defect, which in this case is simply the inclusion $\mathbb{Z}^{\geq 0}\langle S(C) \rangle \hookrightarrow \mathbb{C}\langle S(C) \rangle$. So we see again that extended locality leads to integrality and positivity.

Example 31.19. Reshetikhin–Turaev theories [RT2] provide examples. For Chern–Simons theory [W3] there is a *semiclassical* manifestation of defects: Wilson lines. Let G be the gauge group of the Chern–Simons theory. If G is simply connected, then there is a surjection

$$(31.20) \quad R(G) \longrightarrow \mathbb{Z}\langle S(C_{G,\lambda}) \rangle,$$

where $C_{G,\lambda}$ is the (modular tensor) category attached to G and a “level” λ that determines the theory. Typically one uses representations of G to identify local defects in the quantum theory.

Lecture 32: Introduction to differential cohomology

The prehistory of differential cohomology begins in the early 1970’s with the work of Chern and Simons [CS] on secondary invariants for principal bundles with connection. Soon after Cheeger and Simons [ChS] introduced *differential characters* to codify and explore properties of these secondary

invariants. Differential cohomology is from one perspective a cochain theory of differential characters. This has the advantage that it is local on a smooth manifold. Another source of differential cohomology is *Deligne cohomology* [De3] in algebraic geometry. The modern theory on smooth manifolds uses a more homotopic perspective [BNV, ADH]. In this lecture we introduce differential cohomology as a warmup to invertible field theories.

We begin with a brief discussion of primary and secondary invariants. It is the latter that leads to differential cohomology. These invariants are differential geometric, as opposed to topological, so they depend on local data on a manifold, as encoded in a differential form. A second warmup is a description of the abelian Lie group of isomorphism classes of line bundles with connection. The picture developed here applies to all differential cohomology groups.

Next, we introduce differential characters and the basic commutative diagram (32.10). This diagram is not a pullback, nor are differential characters local in the sense of sheaves. The cochain theory that rectifies these problems is the subject of §32.4. Finally, we very briefly indicate a way toward generalized differential cohomology and the modern theory in terms of sheaves on **Man**.

Our brief discussion does not do justice to the many geometric models for different flavors of differential cohomology, nor to the applications in geometry and physics, to which we allude in Remark 32.16.

32.1 Primary and secondary invariants in topology and differential geometry

A *primary invariant* in this context is a discrete invariant of topological data that is invariant under some kind of deformation. For example, a continuous map $F: S^2 \rightarrow S^2$ has a degree $\deg F \in \mathbb{Z}$ that is invariant under homotopies.¹⁰⁰ It is a famous theorem of Hopf that the degree is a complete invariant: two maps $F_0, F_1: S^2 \rightarrow S^2$ are homotopic iff $\deg F_0 = \deg F_1$.

A *secondary invariant* is an invariant of differential geometric data. It takes real values, typically, but is only well-defined modulo integers. It is not invariant under deformations, and it does depend on the differential geometric data, not just on topological data. There is a related primary topological invariant. For example, fix an orientation of S^2 and a 2-form $\omega \in \Omega_{S^2}^2$ such that $\int_{S^2} \omega = 1$. With this data we construct a secondary invariant of smooth maps $f: S^1 \rightarrow S^2$. Namely, extend f to a smooth map $F: D^2 \rightarrow S^2$ and compute

$$(32.1) \quad \int_{D^2} F^* \omega \pmod{\mathbb{Z}}.$$

The expression is well-defined, since for any two extensions the difference is a primary integer-valued invariant: the degree of a map $S^2 \rightarrow S^2$. The secondary invariant is not invariant under homotopies of f , and it depends on the choice of ω .

There is a similar invariant for maps into an oriented torus $S^1 \times S^1$, once we fix a 2-form of integral one. But now a smooth map $S^1 \rightarrow S^1 \times S^1$ may not have an extension over a disk D^2 , nor over any oriented surface. So there is no definition of the form (32.1). This pushes us to look for a more direct definition. Indeed, even when extensions exist such direct definitions are useful and informative. In this case, let $P \rightarrow S^1 \times S^1$ be a principal $\mathbb{R}/\mathbb{Z}$ -bundle of degree one, and choose a

¹⁰⁰It is defined if we independently identify the domain S^2 with the codomain S^2 ; if not, then we assume an orientation on both.

connection whose curvature is ω . Then the secondary invariant of a map $f: S^1 \rightarrow S^1 \times S^1$ is the holonomy of the pullback connection on $f^*P \rightarrow S^1$.

32.2 Line bundles with connection

The preceding motivates us to study the space of $\mathbb{R}/\mathbb{Z}$ -connections.¹⁰¹ Now $\mathbb{R}/\mathbb{Z}$ -connections are the objects of a groupoid—the morphisms are equivalences of connections—and there is a set of equivalence classes. We can topologize it into a space, and even better into an infinite dimensional Lie group; see [BSS], [DGRW, §6] for this structure in a more general situation. Here the group structure is tensoring of principal $\mathbb{R}/\mathbb{Z}$ -bundles with connection; the tensor product uses the fact that multiplication $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ is a homomorphism of groups. A thorough understanding of this space pays dividends for the more general constructions that follow.

Let $\mathcal{L}_\nabla(M)$ denote the abelian Lie group of isomorphism classes of $\mathbb{R}/\mathbb{Z}$ -connections. (The notation suggests line bundles with connection; see footnote ¹⁰¹.) There are homomorphisms

$$(32.2) \quad \begin{aligned} \mathcal{L}_\nabla(M) &\longrightarrow \Omega_M^2 \\ \mathcal{L}_\nabla(M) &\longrightarrow H^2(M; \mathbb{Z}) \end{aligned}$$

The first assigns the curvature to a connection; the second is the equivalence class of the underlying principal $\mathbb{R}/\mathbb{Z}$ -bundle. Neither homomorphism is injective, and the first is not surjective. Rather, the image of the first is the subgroup $\Omega_{M;\mathbb{Z}}^2 \subset \Omega_M^2$ of *closed* 2-forms with integral periods. The kernel is the group of isomorphism classes of flat $\mathbb{R}/\mathbb{Z}$ -connections: there is a group extension of abelian groups

$$(32.3) \quad 0 \longrightarrow H^1(M; \mathbb{R}/\mathbb{Z}) \longrightarrow \mathcal{L}_\nabla(M) \longrightarrow \Omega_{M;\mathbb{Z}}^2 \longrightarrow 0$$

Observe that $\Omega_{M;\mathbb{Z}}^2$ is a union of affine spaces over the vector space $d\Omega_M^1$ of exact 2-forms; the affine spaces are indexed by the de Rham cohomology class in the free quotient of the second integral cohomology $\text{Free } H^2(M; \mathbb{Z}) \subset H^2(M; \mathbb{R})$. The kernel is a finite dimensional abelian Lie group that sits in a group extension

$$(32.4) \quad 0 \longrightarrow \frac{H^1(M; \mathbb{R})}{H^1(M; \mathbb{Z})} \longrightarrow H^1(M; \mathbb{R}/\mathbb{Z}) \longrightarrow \text{Tors } H^2(M; \mathbb{Z}) \longrightarrow 0$$

The torsion subgroup and free quotient of $H^2(M; \mathbb{Z})$ combine differently in the group extension that contains the second homomorphism in (32.2):

$$(32.5) \quad 0 \longrightarrow \frac{\Omega_M^1}{\Omega_{M;\mathbb{Z}}^1} \longrightarrow \mathcal{L}_\nabla(M) \longrightarrow H^2(M; \mathbb{Z}) \longrightarrow 0$$

The kernel is the abelian group of connections on the product $\mathbb{R}/\mathbb{Z}$ -bundle $M \times \mathbb{R}/\mathbb{Z} \xrightarrow{\text{proj}_1} M$.

¹⁰¹The group $\mathbb{R}/\mathbb{Z}$ acts on $\mathbb{C}$ by sending $x \pmod{1}$ to multiplication by $\exp(2\pi i x)$. Thus a principal $\mathbb{R}/\mathbb{Z}$ -bundle with connection has an associated hermitian line bundle with covariant derivative. There is an inverse construction, and this justifies the title of the section.

Remark 32.6.

- (1) An abelian Lie group A has simple topology. Its group of components $\pi_0 A$ is a (discrete) abelian group. Its fundamental group $\pi_1 A$ is also a discrete abelian group. The homotopy groups $\pi_q A$, $q > 1$, vanish. This is true for $\mathcal{L}_\nabla(M)$, as can be seen from these decompositions, from which we compute

$$(32.7) \quad \begin{aligned} \pi_0 \mathcal{L}_\nabla(M) &= H^2(M; \mathbb{Z}) \\ \pi_1 \mathcal{L}_\nabla(M) &= H^1(M; \mathbb{Z}) \end{aligned}$$

- (2) In fact, with some finiteness conditions an abelian Lie group is a “twisted product” of a vector space, a discrete abelian group, and a torus. For $\mathcal{L}_\nabla(M)$, the vector space is the kernel of (32.5), the torus is the kernel of (32.4), and the discrete abelian group is $H^2(M; \mathbb{Z})$.
- (3) The following picture of the infinite dimensional abelian Lie group $\mathcal{L}_\nabla(M)$ emerges. Two connections are connected by a continuous path iff the underlying bundles are isomorphic, hence the abelian group of components of $\mathcal{L}_\nabla(M)$ is the abelian group $H^2(M; \mathbb{Z})$ of isomorphism classes of principal $\mathbb{R}/\mathbb{Z}$ -bundles. The curvature map restricted to a component is surjective onto an affine translate of the exact 1-forms $d\Omega_M^1$. That map is a principal torus bundle for the torus $H^1(M; \mathbb{R}) / H^1(M; \mathbb{Z})$ of isomorphism classes of flat connection on trivializable principal $\mathbb{R}/\mathbb{Z}$ -bundles.

32.3 Differential characters

The homomorphisms (32.2) have a compatibility expressed in the commutative diagram

$$(32.8) \quad \begin{array}{ccc} \mathcal{L}_\nabla(M) & \longrightarrow & \Omega_{M, \text{closed}}^2 \\ \downarrow & & \downarrow \\ H^2(M; \mathbb{Z}) & \longrightarrow & H^2(M; \mathbb{R}) \end{array}$$

In other words, the real image of the first Chern class of the principal $\mathbb{R}/\mathbb{Z}$ -bundle underlying a connection is the de Rham cohomology class of the curvature. It is important to note that (32.8) is *not* a pullback diagram. For example, if $M = S^1$ then (32.8) is the diagram

$$(32.9) \quad \begin{array}{ccc} \mathbb{R}/\mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

which is decidedly not a pullback. The isomorphism $\mathcal{L}_\nabla(S^1) \xrightarrow{\cong} \mathbb{R}/\mathbb{Z}$ is the holonomy map, defined after choosing an orientation of S^1 . We come back to this missing piece shortly.

The diagram suggests that there is a sequence of abelian Lie groups $\check{H}^q(M)$ attached to a smooth manifold that fit into commutative diagrams

$$(32.10) \quad \begin{array}{ccc} \check{H}^q(M) & \longrightarrow & \Omega_{M,\text{closed}}^q \\ \downarrow & & \downarrow \\ H^q(M; \mathbb{Z}) & \longrightarrow & H^q(M; \mathbb{R}) \end{array}$$

The underlying abelian groups were introduced by Cheeger–Simons [ChS].¹⁰²

Definition 32.11. Let M be a smooth manifold and $q \in \mathbb{Z}^{>0}$. A *differential character* of degree q is a homomorphism $f: Z_{q-1}(M) \rightarrow \mathbb{R}/\mathbb{Z}$ such that

$$(32.12) \quad f(\partial c) = \int_c \omega \pmod{1}, \quad c \in C_q(M),$$

for some differential form $\omega \in \Omega_M^q$.

Here we use smooth singular cycles and chains. The differential form ω is uniquely determined by f , and the map $f \mapsto \omega$ is the top horizontal map in (32.10). The left vertical map takes a bit more effort to define. The function f in a differential character can be regarded as a secondary invariant of the type discussed in §32.1.

Example 32.13. We leave the reader to construct an isomorphism

$$(32.14) \quad \check{H}^1(M) \xrightarrow{\cong} \text{Map}(M, \mathbb{R}/\mathbb{Z})$$

and an isomorphism

$$(32.15) \quad \check{H}^2(M) \xrightarrow{\cong} \mathcal{L}_{\nabla}(M)$$

It is not at all obvious from Definition 32.11, but there is an abelian Lie group structure on $\check{H}^q(M)$. The assertions in Remark 32.6 hold with an appropriate shift of degrees.

The commutative diagram (32.10) tells that the groups $\check{H}^q(M)$ of differential characters marry the local information of a differential form with a global integrality as encoded in integral cohomology. The compatibility between local and global rests of the de Rham theorem, which expresses the cohomological meaning of a closed differential form. The presence of integral cohomology gives more than integrality: there is torsion as well. Anticipating the local constructions in the next section, we use the term *differential cohomology* for the group of differential characters.

Remark 32.16.

¹⁰²They used a grading downshifted by one.

- (1) Integrality and torsion is part of *any* cohomology theory, as is passage to real coefficients, so there are generalized differential cohomology groups that fit into diagrams analogous to (32.10). These generalizations are useful in both geometry and physics.
- (2) One can combine primary and secondary invariants in differential cohomology theory, as we indicate below. Differential K -theory also provides a setting for geometric index theory; see [FL] and the references therein. There are also purely topological applications. For example, there is an Euler class of a real vector bundle in differential cohomology, and it is a more delicate invariant than the usual topological Euler class.
- (3) The marriage of local information and global integrality occurs in electromagnetism. The classical electric current on spacetime is a closed differential form that encodes both charge density and the flow of electric current. But in semiclassical electromagnetism, which includes Dirac charge quantization, the total electric charge is required to satisfy integrality constraints. The electric charge as a real number is global cohomological information derived from the closed differential form.¹⁰³ Therefore, the marriage of local differential forms and global integrality makes differential cohomology a suitable home for currents, and therefore for electromagnetic fields as well. The underlying topological cohomology theory need not be standard Eilenberg-MacLane cohomology; more exotic choices appear in string theory, for example.
- (4) Secondary invariants appear in field theory, and these are best expressed in differential cohomology, including with exotic underlying topological cohomology theories [F10].

32.4 The local theory: differential cochains

The assignment $M \mapsto \check{H}^q(M)$ is *not* local in the sense of Definition 18.11, i.e., it is not a sheaf on **Man**. In both geometry and physics one would like to refine this to local objects. For ordinary cohomology groups, the local refinements are groups of cochains, or objects in a groupoid of geometric representatives of cohomology classes. There are similar refinements of differential cohomology groups.

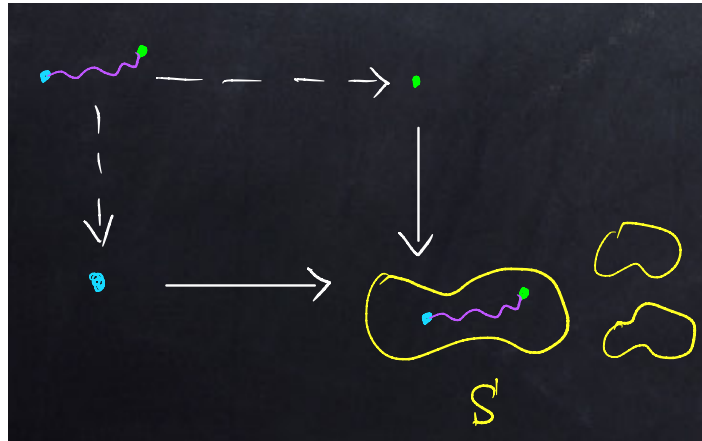


FIGURE 107. A homotopy pullback

¹⁰³The current has compact support on spatial slices, and it is this compactly supported cohomology class that encodes charge.

The basic realization is that while (32.10) is not a pullback diagram, it is a *homotopy pullback*. As a basic example, consider the diagram in Figure 107. Here we have a topological space S and two maps of a point into S . The strict pullback—in the category of topological spaces and continuous maps—is the empty set unless the images of the two points coincide, in which case it is a single point. By contrast, the homotopy pullback is the empty set if the images lie in distinct path components, but is nonvanishing otherwise. In fact, one model for the homotopy pullback is the space of continuous paths between the image points:

$$(32.17) \quad \begin{array}{ccc} P_{x,y}(S) & \longrightarrow & \text{pt} \\ \downarrow & & \downarrow y \\ \text{pt} & \xrightarrow{x} & S \end{array}$$

There is a diagram of sets obtained by applying π_0 to (32.17). For example, if $x = y$ one obtains the commutative diagram

$$(32.18) \quad \begin{array}{ccc} \pi_1(S, x) & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \pi_0 S \end{array}$$

If $\pi_1(S, x) \neq 1$, then this is not a pullback diagram. So a homotopy pullback retains extra levels of information. In this case there is one level of extra information: a choice of homotopy.

The basic idea, then, is that (32.10) is the induced commutative diagram on π_0 of a homotopy pullback. This is the starting point for [HS], which treats both ordinary differential cohomology and for generalized differential cohomology. Now we indicate the homotopy pullback underlying (32.10) in a cochain model, as indicated in [DF2, §6.3] and developed in [HS, §3.2]; see also [FN, Appendix A].

As a first step, we need a cochain model for the three abelian groups in (32.8) that define the pullback. For $H^2(M; \mathbb{Z})$ and $H^2(M; \mathbb{R})$ we use smooth singular cochains, but the choice is less clear for $\Omega_{M, \text{closed}}^2$. For each $q \in \mathbb{Z}^{>0}$, define a cochain complex $\Omega(q)^\bullet_M$ as

$$(32.19) \quad \cdots \longrightarrow 0 \longrightarrow \Omega_M^q \xrightarrow{d} \Omega_M^{q+1} \xrightarrow{d} \Omega_M^{q+2} \xrightarrow{d} \cdots$$

where Ω_M^q is in degree q . If we view this as a complex of sheaves on **Man**, in the sense of §18.1, then by the Poincaré lemma its cohomology is the sheaf $\Omega_{M, \text{closed}}^q$, viewed as a complex concentrated in degree q . This is precisely the northeast corner of (32.10). Now form the pullback

$$(32.20) \quad \begin{array}{ccc} \check{C}(q)^\bullet(M) & \dashrightarrow & \Omega(q)^\bullet_M \\ \downarrow & & \downarrow \\ C^q(M; \mathbb{Z}) & \longrightarrow & C^q(M; \mathbb{R}) \end{array}$$

in the category of cochain complexes. Then

$$(32.21) \quad \check{C}(q)^p(M) = \begin{cases} C^p(M; \mathbb{Z}) \times C^{p-1}(M; \mathbb{R}) \times \Omega^p(M), & p \geq q; \\ C^p(M; \mathbb{Z}) \times C^{p-1}(M; \mathbb{R}), & p < q, \end{cases}$$

with differential acting on $(c, h, \omega) \in \check{C}(q)^p(M)$, $p \geq q$, or on $(c, h) \in \check{C}(q)^p(M)$, $p < q$, given by

$$(32.22) \quad \begin{aligned} d(c, h, \omega) &= (\delta c, \omega - c - \delta h, d\omega), & p \geq q; \\ d(c, h) &= \begin{cases} (\delta c, -c - \delta h, 0), & p = q - 1; \\ (\delta c, -c - \delta h), & p < q - 1. \end{cases} \end{aligned}$$

The cohomology of the cochain complex $\check{C}(q)^\bullet$ is

$$(32.23) \quad \check{H}(q)^p(M) \cong \begin{cases} H^p(M; \mathbb{Z}), & p > q; \\ H^p(M; \mathbb{R}/\mathbb{Z}), & p < q, \end{cases}$$

and the diagonal group $\check{H}(q)^q(M)$ is the *differential cohomology*. The diagonal groups

$$(32.24) \quad \begin{aligned} \check{C}^q(M) &:= \check{C}(q)^q(M) \\ \check{Z}^q(M) &:= \check{Z}(q)^q(M) \\ \check{H}^q(M) &:= \check{H}(q)^q(M), \end{aligned}$$

are the new ingredient; here $\check{Z}(q)^p(M) \subset \check{C}(q)^p(M)$ is the subgroup of cocycles. The group $\check{H}^q(M)$ is isomorphic to the group of Cheeger–Simons differential characters on M ([Definition 32.11](#)).

One crucial point is that $\check{C}^q(-)$ and $\check{Z}^q(-)$ are sheaves on **Man**, so they are the useful local objects in geometry and physics. One can define products of differential cocycles, compatible with the cup product on smooth singular cochains and the wedge product on differential forms. Similarly, there is an integration theory on oriented manifolds, compatible with pushforward (slant product) on smooth singular cochains and with integration of differential forms. In other words, there is a differential and integral calculus of differential cochains that refines the usual calculus of differential forms.

32.5 Other models

Hopkins–Singer [[HS](#)] give another model of geometric objects that represent $\check{H}^q(M)$ but do not form a cochain complex. Let K be an Eilenberg–MacLane space with a single nonzero homotopy group $\pi_q K = \mathbb{Z}$. A (topological) cohomology class in $H^q(M; \mathbb{Z})$ can be represented by a map $c: M \rightarrow K$; the homotopy class of c is the associated cohomology class. Fix a cocycle $\iota \in Z^q(K; \mathbb{R})$ that represents the image in real cohomology of the tautological class in $H^q(K; \mathbb{Z})$. Then a *differential function* is a triple (c, h, ω) in which $c: M \rightarrow K$ is a continuous map, $\omega \in \Omega_M^q$, and $h \in C^{q-1}(M; \mathbb{R})$ satisfy

$$(32.25) \quad \delta h = \omega - c^* \iota.$$

(Compare (32.22).) This model is amenable to generalizations to other cohomology theories.

The modern theory of differential cohomology is set in the context of sheaves on the category **Man** of smooth manifolds. One can execute the homotopy pullback in that category for arbitrary cohomology theories. In addition, one can define and compute the differential cohomology of sheaves that are not representable, i.e., do not come from smooth manifolds. We refer to [BNV, ADH] and the references therein for more.

Lecture 33: Invertible field theories

In the next few lectures we turn to the second special class of field theories: *invertible field theories*. These were first introduced in [FM2, §5.5]. In physics they have some interest as theories in their own right, and they also appear in association with general field theories. In mathematics they appear in topology, differential geometry, and index theory.

In any symmetric monoidal category there is a distinguished class of invertible objects, and it is in this context that we begin. The definition of an invertible field theory (IFT) is a special case. Note that an IFT may or may not be topological. After describing some special properties of IFTs, we conclude the lecture with several examples.

33.1 Warmup

Definition 33.1. Let C be a symmetric monoidal category and $x \in C$. Then *invertibility data* for x is a pair (x', θ) consisting of $x' \in C$ and an isomorphism $\theta: 1_C \rightarrow x \otimes x'$. If invertibility data exists, then we say that x is *invertible*.

There is a category of invertibility data, and it is a contractible groupoid (Definition 25.32). So an inverse to x , if it exists, is unique up to unique isomorphism. We denote any choice of inverse as x^{-1} . Note that the set of invertible objects is closed under the tensor product and it contains the unit object 1_C .

Remark 33.2. An object $x \in C$ is invertible if and only if the functor $x \otimes -: C \rightarrow C$ is an equivalence.

Example 33.3. Let $C = \text{Vect}_k$ be the category of vector spaces over a field k with symmetric monoidal structure the tensor product. Then $V \in \text{Vect}_k$ is invertible if and only if $\dim V = 1$. In this case we say V is a *line*.

Lemma 33.4. Let C be a symmetric monoidal category.

- (i) If $x \in C$ is invertible, then x is dualizable and x^{-1} is a dual object.
- (ii) If C is a symmetric monoidal groupoid and $x \in C$ is dualizable, then x is invertible with inverse $x^\vee$.

We leave the proof to the reader.

Definition 33.5. A *Picard groupoid* is a symmetric monoidal category in which all objects and morphisms are invertible.

Example 33.6. Given a field k , there is a Picard groupoid Line_k whose objects are k -lines and whose morphisms are isomorphisms of k -lines. Given a space X , there are Picard groupoids $\text{Line}_{\mathbb{R}}(X)$ and $\text{Line}_{\mathbb{C}}(X)$ of line bundles over X .

Definition 33.7. Let C be a symmetric monoidal category. An *underlying Picard groupoid* is a pair $(C^\times, i)$ consisting of a Picard groupoid $C^\times$ and a functor $i: C^\times \rightarrow C$ that satisfies the universal property: If D is any Picard groupoid and $j: D \rightarrow C$ a symmetric monoidal functor, then there exists a unique $\tilde{j}: D \rightarrow C^\times$ such that the diagram

$$(33.8) \quad \begin{array}{ccc} C^\times & \xrightarrow{i} & C \\ & \nwarrow \tilde{j} & \nearrow j \\ & D & \end{array}$$

commutes.

We obtain $C^\times$ from C by discarding all non-invertible objects and non-invertible morphisms.

Example 33.6 generalizes from spaces X to stacks X . In the particular case $X = */G$ for a finite group G , we obtain the symmetric monoidal category of, say, complex representations of G . The invertible objects are the *characters* of G , that is, the 1-dimensional representations. A character is a homomorphism $G \rightarrow \text{GL}_1$; an N -dimensional representation for any $N \in \mathbb{Z}^{>0}$ is a homomorphism $G \rightarrow \text{GL}_N$. In general, GL_N is nonabelian whereas GL_1 is abelian. This abelianness leads to a cohomological interpretation of characters: the abelian group of isomorphism classes of characters is $H^1(G; \mathbb{C}^\times)$.

Remark 33.9.

- (1) This cohomological interpretation extends to the abelian group of isomorphism classes of complex line bundles on a space or stack X ; it is the sheaf cohomology group $H^1(X; \underline{\mathbb{C}^\times})$. If X is a (reasonable) space, this is isomorphic to $H^2(X; \mathbb{Z})$.
- (2) If M is a smooth manifold, then the Picard groupoid of hermitian line bundles with connections has set of equivalence classes the differential cohomology group $\check{H}^2(M)$.
- (3) The cohomological interpretation of invertible field theories is the subject of the next lecture.

33.2 Invertible field theories

Fix a positive integer n and a simplicial sheaf $\mathcal{F}: \mathbf{Man}_n^{\text{op}} \rightarrow \mathbf{Set}_\Delta$ of background fields. Recall (**Axiom System 23.42**) that a Wick-rotated field theory is a homomorphism $F: \text{Bord}_{\langle n-1, n \rangle}(\mathcal{F}) \rightarrow t\text{Vect}$. The collection of field theories of this type (dimension n , background fields $\mathcal{F}$) is a symmetric monoidal category

$$(33.10) \quad \text{FT}(n, \mathcal{F}) = \text{Hom}(\text{Bord}_{\langle n-1, n \rangle}(\mathcal{F}), t\text{Vect}).$$

Definition 33.11. An *invertible field theory* (IFT) is an invertible object in $\text{FT}(n, \mathcal{F})$. The Picard category of invertible field theories is denoted $\text{IFT}(n, \mathcal{F})$.

Lemma 33.12. *Let $\alpha: \text{Bord}_{\langle n-1, n \rangle}(\mathcal{F}) \rightarrow t\text{Vect}$ be an invertible field theory. Then α factors*

$$(33.13) \quad \begin{array}{ccc} \text{Bord}_{\langle n-1, n \rangle}(\mathcal{F}) & \xrightarrow{\alpha} & t\text{Vect} \\ & \searrow & \nearrow \\ & \text{Line} & \end{array}$$

We leave the proof to the reader.

Invertible field theories enjoy many special properties, as we now enumerate.

- (1) The state spaces of an IFT are all 1-dimensional, as is clear from [Lemma 33.12](#). In other words, on each space Y there is a unique vacuum and no other states. Especially, then, in light of our slogan (8.1) that quantum theory is projective, we might conclude that invertible field theories have no interest. Not true! One immediate comment is that even projectively there are nontrivial *families* of IFTs. In fact, we will see several ways in which IFTs are useful in general field theory.
- (2) As already mentioned, there is a cohomological interpretation—in general, in *differential* cohomology—that we take up in the next lecture.
- (3) Invertibility is also defined for extended field theories, whether topological or not.
- (4) Invertible field theories are amenable to special techniques, such as homotopy theory, and as such provide a nice playground for general concepts in field theory.
- (5) The Picard groupoid of invertible field theories acts on the symmetric monoidal category of field theories by multiplication: $\text{IFT}(n, \mathcal{F}) \circ \text{FT}(n, \mathcal{F})$. Field theories in the same orbit have the same physical properties; this is another form of the slogan that quantum theory is projective.
- (6) As we will discuss in a lecture soon, there is a notion of a projective field theory, and the obstruction to a linear lift is a *shifted* invertible field theory, called the *anomaly* of the projective theory.
- (7) IFTs also show up in physics as “topological terms” or as “changes of renormalization scheme”.
- (8) It is common lore that a gapped field theory that has a unique vacuum has a low energy approximation by an invertible field theory. Applied to a family F of theories, the resulting family F' of IFTs should capture the homotopical information in the family F . In particular, the deformation type (called the *phase*) is captured by F' .
- (9) There are mathematical applications of IFTs as well, in topology, differential geometry, and in index theory. Many of these are underdeveloped.

33.3 Examples of IFTs

We work up in dimension. Some examples are topological (the sheaf $\mathcal{F}$ is locally constant), and others are not.

Example 33.14. Fix an integer $k \in \mathbb{Z}$. Then there is an invertible field theory

$$(33.15) \quad \alpha_k \in \text{IFT}(1, B\text{SO}_1 \times B_{\nabla} \mathbb{C}^\times).$$

The sheaf of background fields consists of a 1-dimensional orientation and a principal $\mathbb{C}^\times$ -bundle with connection. Let pt_+ be the point with its positive orientation. Then

$$(33.16) \quad \alpha_k(\text{pt}_+, P \rightarrow \text{pt}_+) = P \times_{\mathbb{C}^\times} \mathbb{C}$$

is the line associated to the $\mathbb{C}^\times$ -torsor P by the character of $\mathbb{C}^\times$ in which $\lambda \in \mathbb{C}^\times$ acts as multiplication by λ^k . Denote that line as $L_P^{\otimes k}$. If we replace pt_+ by pt_- then we obtain $L_P^{\otimes(-k)}$.

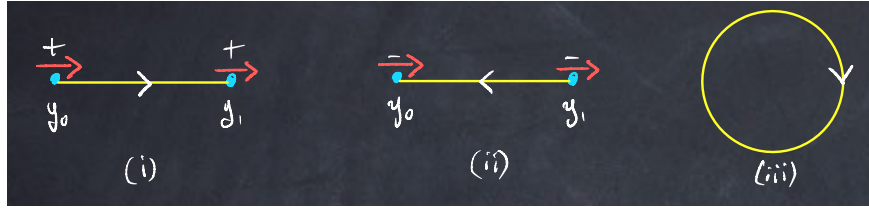


FIGURE 108. Some 1-morphisms in the bordism category

Next, consider the morphisms depicted in Figure 108, which come equipped with a principal $\mathbb{C}^\times$ -bundle with connection. (The white arrows indicate orientation; the red arrows are arrows of time.) There is an induced line bundle with covariant derivative. Then α_k evaluates on the first bordism (i) to the parallel transport $L_{y_0}^{\otimes k} \rightarrow L_{y_1}^{\otimes k}$, on the second bordism (ii) to the parallel transport $L_{y_0}^{\otimes(-k)} \rightarrow L_{y_1}^{\otimes(-k)}$, and on the third bordism (iii) to the k^{th} power of the holonomy of the principal $\mathbb{C}^\times$ -bundle. Observe that each of these is invertible.

Field theories in general, and invertible field theories in particular, can be evaluated in parametrized families. For example, suppose given a smooth family of positively oriented points parametrized by a smooth manifold M . As a family of objects in $\text{Bord}_{(0,1)}(BSO_1 \times_{B\mathbb{C}^\times})$ it comes equipped with a principal $\mathbb{C}^\times$ -bundle $P \rightarrow M$ with connection. The value of α_k on this smooth family of objects is a smooth family of lines, i.e., the line bundle over M associated to $P \rightarrow M$ via the character $\lambda \mapsto \lambda^k$. Significantly, this line bundle comes equipped with a covariant derivative.¹⁰⁴ The curvature is k times the curvature of $P \rightarrow M$. As another example, let $\pi: X \rightarrow M$ be a fiber bundle whose fibers are circles, suppose the relative tangent bundle $T(X/M) \rightarrow X$ is oriented, and suppose given a principal $\mathbb{C}^\times$ -bundle $P \rightarrow X$ with connection. The value of α_k on this family of circles is the smooth function $M \rightarrow \mathbb{C}^\times$ whose value at $m \in M$ is the holonomy of the restriction of $P|_{\pi^{-1}(m)} \rightarrow \pi^{-1}(m)$.

One can imagine a universal object that encodes the theory α_k . It lives over a classifying object $B_{\mathbb{C}^\times}$ of principal $\mathbb{C}^\times$ -bundles with connection and is a differential cohomology object that represents a class in $\check{H}^2(B_{\mathbb{C}^\times})$ whose underlying topological object that represents k times a generator in $H^2(B\mathbb{C}^\times; \mathbb{Z})$. One can make mathematical sense of this [FH2].

The theory α_k is *not* topological.

Example 33.17. There is an important *topological* IFT $\alpha \in \text{IFT}(2, B\text{Spin})$ called the *Arf theory*. Its partition function on a closed 2-manifold X with spin structure σ is the exponentiated Arf

¹⁰⁴We expect such “differential data” in any field theory evaluated in families.

invariant¹⁰⁵

$$(33.18) \quad \alpha(X, \sigma) = (-1)^{\text{Arf}(\sigma)}.$$

The (exponentiated) Arf invariant is a spin bordism invariant

$$(33.19) \quad \Omega_2(B\text{Spin}) \xrightarrow{\cong} \mu_2$$

One consequence of the homotopy theoretic setting for invertible field theories is that any bordism invariant is the partition function of a *fully local* IFT. This strong locality of bordism invariants is perhaps surprising. The codomain of this fully local IFT requires gradings. Thus for the two spin circles we have

$$(33.20) \quad \begin{aligned} \alpha(S^1, \sigma_{\text{bounding}}) &= \mathbb{C} \\ \alpha(S^1, \sigma_{\text{nonbounding}}) &= \Pi \end{aligned}$$

where Π is the odd line. The value of the Arf theory on a spin point is the complex Clifford algebra $\text{Cliff}_1^{\mathbb{C}}$, if we take the codomain to be the Picard 2-groupoid $\text{Alg}_{\mathbb{C}}^{\times}$ of complex invertible super algebras. (A superalgebra is invertible iff it is central simple in the graded sense.) This Picard groupoid has homotopy groups

$$(33.21) \quad \begin{aligned} \pi_2 \text{Alg}^{\times} &\cong \mathbb{C}^{\times} \\ \pi_1 \text{Alg}^{\times} &\cong \mathbb{Z}/2\mathbb{Z} \\ \pi_0 \text{Alg}^{\times} &\cong \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

See [A4, F12, Gu, DeGu] for more about this theory and its cousins, including applications.

Example 33.22. Let G be a Lie group with finitely many components, and suppose λ is a cocycle for a class in $H^4(BG; \mathbb{Z})$. This class is a characteristic class of principal G -bundles. We are interested in principal G -bundles with connection.¹⁰⁶ This data determines an invertible field theory

$$(33.23) \quad \alpha = \alpha_{(G, \lambda)} \in \text{IFT}(3, B\text{SO}_3 \times_{B_{\nabla}} G)$$

which in physics is called *classical Chern–Simons theory*. The partition function of a G -connection over an oriented closed 3-manifold is the exponentiated Chern–Simons invariant [CS] of the connection. For more on this IFT, see [F7], for example.

A particular example, which physicists call “gravitational Chern–Simons theory”, is a 3-dimensional IFT with background fields an orientation and a Riemannian metric. Chern–Simons use it in [CS, §6] to construct an obstruction to conformal immersions of 3-dimensional Riemannian manifold into $\mathbb{E}^4$. (In fact, their argument is best understood in terms of a variation of this theory for spin manifolds.) This theory also plays an important role in [W3].

¹⁰⁵A spin structure induces a quadratic function $q: H_1(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}$. The abelian group $H_1(X; \mathbb{Z}/2\mathbb{Z})$ is finite, and the Arf invariant is the value of q that occurs most often.

¹⁰⁶In fact, the class lifts uniquely to a differential cohomology class in $\check{H}^4(B_{\nabla} G)$, a *differential characteristic class*.

Example 33.24. There is a related, but distinct, 3-dimensional invertible field theory

$$(33.25) \quad \alpha \in \text{IFT}(3, B\text{Spin}_3 \times \text{Riem})$$

whose partition function on a closed oriented Riemannian spin 3-manifold is $\exp(2\pi i \eta/4)$, where η is the appropriate Atiyah–Patodi–Singer invariant [APS1]. This IFT is closely related [APS2, §4] to the gravitational Chern–Simons IFT in Example 33.22, and the two are often mistakenly conflated. The theory (33.25) occurs in geometric index theory and in the anomaly theory of spinor fields in 2 dimensions, among other places. The value on a family of closed Riemannian spin 2-manifolds is the $\mathbb{Z}/2\mathbb{Z}$ -graded Pfaffian line bundle of the associated family of Dirac operators. We remark that this IFT can be located in differential KO -theory.

Observe that neither Example 33.22 nor Example 33.24 is a topological field theory.

Lecture 34: Invertible field theories and homotopy theory

A quantum field theory is a homomorphism of symmetric monoidal categories, and it is invertible if it factors through the maximal Picard subgroupoid of the codomain (Lemma 33.12). This means that it also factors through a Picard groupoid quotient of the domain, and so becomes a homomorphism of Picard groupoids. This is equivalent to a map of infinite loop spaces, or equivalently of *spectra* in stable homotopy theory. These moves locate the theory of IFTs in homotopy theory, and it opens up a whole new array of techniques. Furthermore, for *topological* IFTs one can explicitly identify the Picard groupoid quotient of the domain bordism category, and this leads to detailed information about such theories.

This link between IFTs and homotopy theory is at least implicit in [FHT2], and is made more explicit in subsequent works. There are different accounts in the literature, such as [FH1, §5] and [F13, Lecture 6]. We defer to these references for many more details of the brief sketch we give in this lecture.

A basic step is the equivalence between spaces and (higher) groupoids, which is where we begin the lecture. This is refined to an equivalence between infinite loop spaces and Picard groupoids. This understood, we can quickly translate an invertible field theory as a map of spectra.

The domain of a topological field theory is a bordism *category*; in the case of an invertible topological field theory this is replaced by a quotient bordism *spectrum*. The precise version is a Madsen–Tillmann spectrum; see [GMTW]. We illustrate this quotient process in the 1-dimensional case, where there is a cute explicit argument.

Continuing with *topological* IFTs, we observe that they may be made fully local, and furthermore there is a distinguished choice for a codomain spectrum, characterized by the universal property that the partition function determines the theory. That partition function is a bordism invariant, though perhaps not of the classical kind, and it is striking that any such bordism invariant is the partition function of some topological IFT. This brings strong locality into classical bordism theory.

34.1 Groupoids and spaces

Let S be a topological space. The most fundamental invariant of S is the set $\pi_0 S$ of path components. Namely, define an equivalence relation on S by $s_0 \sim s_1$ if there exists a continuous path from s_0 to s_1 ; then $\pi_0 S$ is the set of equivalence classes. We can categorify this construction (compare §24.5). Let S be the set of objects of a category $\pi_{\leq 1} S$ in which a morphism $s_0 \rightarrow s_1$ is a homotopy class of continuous paths from s_0 to s_1 . Then $\pi_{\leq 1} S$ is a groupoid, called the *fundamental groupoid* of S . One can categorify further by regarding continuous paths as data, so as 1-morphisms in a 2-groupoid $\pi_{\leq 2} S$ in which a 2-morphism between paths $\gamma_0, \gamma_1: s_0 \rightarrow s_1$ is a homotopy class of pointed homotopies from γ_0 to γ_1 . Continuing, we extract from S higher groupoids and even an ∞ -groupoid.

There is a converse construction of a space from a groupoid or higher groupoid, as explained in [Mi4], [Se4], [Se5, Appendix A]. One first forms a simplicial set whose k -simplices are compositions of k morphisms in the groupoid. Then there is a geometric realization of a simplicial set that represents each element of the k^{th} space as a geometric k -simplex. We defer to the references for details. There is also a geometric realization of higher groupoids, and the *homotopy hypothesis* of Grothendieck is the assertion that topological spaces and infinity groupoids are equivalent. For a modern treatment, see [Lu3, §3.6].

A general category also has a geometric realization; it factors through a quotient groupoid that effectively inverts all morphisms.

Definition 34.1. Let C be a category. A *groupoid completion* $(|C|, i)$ of C is a groupoid $|C|$ and a functor $i: C \rightarrow |C|$ that satisfies the following universal property: If $\mathcal{G}$ is a groupoid and $f: C \rightarrow \mathcal{G}$ a functor, then there exists a unique groupoid map $\tilde{f}: |C| \rightarrow \mathcal{G}$ which makes the diagram

$$(34.2) \quad \begin{array}{ccc} C & \xrightarrow{i} & |C| \\ & \searrow f & \swarrow \tilde{f} \\ & \mathcal{G} & \end{array}$$

commute.

Intuitively, $|C|$ is obtained from C by “inverting all of the arrows”, much in the same way that the group completion of a monoid is constructed. The groupoid completion $(|C|, i)$ is unique up to unique isomorphism, and the map i is an isomorphism $i_0: C_0 \rightarrow |C|_0$ on objects. There is a generalization to higher categories as well.

Recall Definition 33.5 of a Picard groupoid. A Picard groupoid $\mathcal{G}$ has three invariants: the abelian group $\pi_0 \mathcal{G}$ of isomorphism classes of objects, the abelian group $\pi_1 \mathcal{G} = \text{Aut}(1)$ of automorphisms of the unit object, and the k -invariant

$$(34.3) \quad k: \pi_0 \mathcal{G} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \pi_1 \mathcal{G}$$

defined on an object $x \in \mathcal{G}$ as the composition

$$(34.4) \quad k(x): 1 \xrightarrow{-\otimes(x \otimes x)} x \otimes x \xrightarrow{\sigma_x} x \otimes x \xrightarrow{-\otimes(x \otimes x)^{-1}} 1$$

where $\sigma_x: x \otimes x \rightarrow x \otimes x$ is the symmetry. The geometric realization of a Picard groupoid is an *infinite loop space* with corresponding π_0, π_1 and all other homotopy groups vanishing.

Remark 34.5. An infinite loop space is equivalent to a *spectrum* in the sense of algebraic topology. Spectra are the basic objects in stable homotopy theory, and there is a large literature on them. For a quick introduction, see [FH1, §6.1].

Of particular interest to us is the groupoid completion of (higher) bordism categories. Since a bordism category is symmetric monoidal, its groupoid completion is Picard. To illustrate, we work out a simple example in low dimensions, taken almost verbatim from [F13, §6.3].

Theorem 34.6. *The groupoid completion of the oriented bordism category has*

$$(34.7) \quad \pi_0 |\mathrm{Bord}_1(BSO_1)| \cong \mathbb{Z}, \quad \pi_1 |\mathrm{Bord}_1(BSO_1)| \cong \mathbb{Z}/2\mathbb{Z},$$

with nontrivial k -invariant.

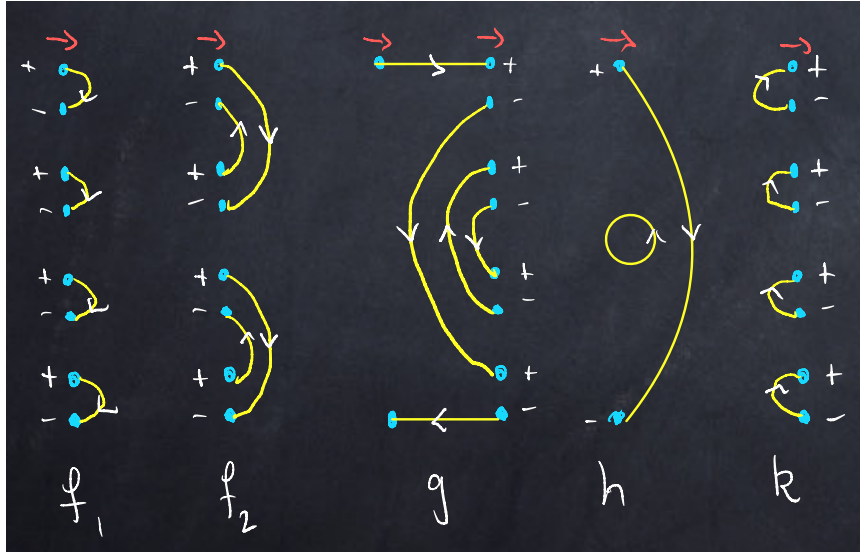


FIGURE 109. Some oriented 1-dimensional bordisms

Proof. The argument for π_0 is straightforward and amounts to the assertion $\Omega_0^{BSO} \cong \mathbb{Z}$.

Set $B = \mathrm{Bord}_1(BSO_1)$. To compute $\pi_1 B$ we argue as follows. First, the tensor unit $1_B = \emptyset^0$ is the empty 0-manifold, so $\mathrm{End}(1_B) = B(1_B, 1_B)$ consists of diffeomorphism classes of closed oriented 1-manifolds. By the classification of 1-manifolds there is an isomorphism of commutative monoids $\mathrm{End}(1_B) \cong \mathbb{Z}^{\geq 0}$ which counts the number of (circle) components of a bordism X . Let X_n denote the disjoint union of n oriented circles. Note that the circle has a unique orientation up to orientation-preserving diffeomorphism. Then using the bordisms defined in Figure 109 we have

$$(34.8) \quad f_1 \circ g = f_2 \circ g = h$$

as morphisms in B . In the groupoid completion $|B|$ we can compose on the right with the inverse of $i(g)$ to conclude that $i(f_1) = i(f_2)$, where $i: B \rightarrow |B|$. That implies that in $|B|$ we have

$$(34.9) \quad i(f_1) \circ i(k) = i(f_2) \circ i(k),$$

hence $i(X_4) = i(X_2)$. Cancel to conclude $i(X_2) = i(\emptyset^0) = 1_{|B|}$. To rule out the possibility that $i(X_1)$ is also the tensor unit, observe that there is an invertible 1-dimensional topological field theory of oriented manifolds that sends pt_+ to the odd line, and so maps the oriented circle X_1 to -1 .

It remains to show that every morphism $\emptyset^0 \rightarrow \emptyset^0$ in $|B|$ is equivalent to a union of circles and their formal inverses. Observe first that the inverse of the “right elbow” is the “left elbow” union a circle, since their composition in one order is $i(X_2)$, which is equivalent to the identity map. Next, any morphism $\emptyset^0 \rightarrow \emptyset^0$ in $|B|$ is the composition of a finite number of morphisms $Y_{2k} \rightarrow Y_{2\ell}$ and inverses of such morphisms, where Y_n is the 0-manifold consisting of n points. Furthermore, each such morphism is the disjoint union of circles, identities, right elbows, left elbows, and their inverses. Identities are self-inverse and the elbows are each others inverse, up to a circle, hence carrying out the compositions of elbows and identities we obtain a union of circles and their inverses, as desired.

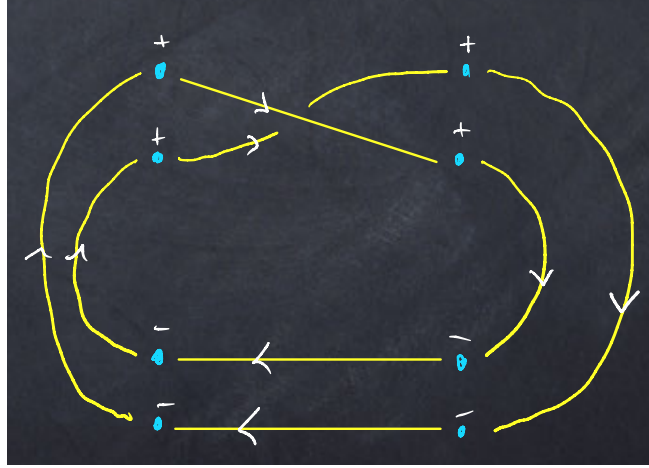


FIGURE 110. The k -invariant of $|Bord_{(0,1)}(BSO_1)|$

Figure 110 illustrates the computation of the k -invariant (34.3) of $|B|$. The nontrivial element of $\pi_0|B^{BSO}|$ is represented by pt_+ , and the top part of the diagram is the symmetry $\sigma: \text{pt}_+ \amalg \text{pt}_+ \rightarrow \text{pt}_+ \amalg \text{pt}_+$ in B^{BSO} . The inverse of $\text{pt}_+ \amalg \text{pt}_+$ is $\text{pt}_- \amalg \text{pt}_-$, and we tensor with the identity map on the inverse. That tensor product is the disjoint union of the top and bottom four strands. The left and right ends of Figure 110 implement the isomorphism $1_{|B|} \cong \text{pt}_+ \amalg \text{pt}_+ \amalg \text{pt}_- \amalg \text{pt}_-$. The result is the oriented circle X_1 , which is the generator of $\pi_1|B|$. $\square$

34.2 Invertible field theories as infinite loop maps

We resume the general discussion of invertible field theories from §33.2. Fix a positive integer n and a simplicial sheaf $\mathcal{F}: \mathbf{Man}_n^{\text{op}} \rightarrow \mathbf{Set}_\Delta$ of background fields. Suppose $\alpha: \text{Bord}_{\langle n-1, n \rangle}(\mathcal{F}) \rightarrow$

$stVect$ is an invertible field theory with values in the category of super topological vector spaces. Then by [Lemma 33.12](#) α factors through the maximal subgroupoid of $tVect$:

$$(34.10) \quad \begin{array}{ccc} \mathrm{Bord}_{\langle n-1, n \rangle}(\mathcal{F}) & \xrightarrow{\alpha} & tVect \\ & \searrow & \uparrow \\ & & sLine \end{array}$$

Now, since the category $sLine$ of super lines is a (Picard) groupoid, the dotted map factors through the quotient groupoid of the domain:

$$(34.11) \quad \begin{array}{ccc} \mathrm{Bord}_{\langle n-1, n \rangle}(\mathcal{F}) & \xrightarrow{\alpha} & tVect \\ \downarrow & \searrow & \uparrow \\ |\mathrm{Bord}_{\langle n-1, n \rangle}(\mathcal{F})| & \dashrightarrow & sLine \end{array}$$

Since α is a homomorphism of symmetric monoidal categories, after taking geometric realizations the map

$$(34.12) \quad \tilde{\alpha}: |\mathrm{Bord}_{\langle n-1, n \rangle}(\mathcal{F})| \longrightarrow sLine$$

is an infinite loop map of infinite loop spaces, or equivalently a map of spectra.

Remark 34.13. The Picard groupoid $sLine$ of super lines has $\pi_0 = \mathbb{Z}/2\mathbb{Z}$, $\pi_1 = \mathbb{C}^\times$, and nonzero k -invariant (from the Koszul sign rule). It is instructive to see the 1-dimensional topological IFT mentioned after [\(34.9\)](#) as a map of infinite loop spaces $|\mathrm{Bord}_1(BSO_1)| \rightarrow |sLine|$. The map is surjective on π_0 and nonzero on π_1 .

The next step in the theory is an identification of the Picard groupoid quotient $|\mathrm{Bord}_{\langle n-1, n \rangle}(\mathcal{F})|$ for locally constant sheaves $\mathcal{F}$. Namely, the associated spectrum is that constructed by Madsen–Tillmann. There is a version of this result for ordinary bordism categories as well as for extended (higher) bordism categories. It is natural, then, in this topological case to immediately pass to fully local IVTs. Then one needs a codomain spectrum that extends $sLine$, and there is a universal choice. We defer to the accounts listed in the introduction to this lecture and to the references therein.

In the nontopological case, when $\mathcal{F}$ is not locally constant, IFTs can be studied using generalized differential cohomology. This theory awaits further development; see [\[FN, Appendix B\]](#) for some comments in this direction.

Lecture 35: Anomalies and projectivity

Recall our slogans [\(8.1\)](#) and [\(8.2\)](#):

QUANTUM THEORY IS PROJECTIVE. QUANTIZATION IS LINEAR.

with a corollary

THE ANOMALY OF A QUANTUM THEORY EXPRESSES ITS PROJECTIVITY.

In §8 we discussed projectivity in quantum mechanics. Whereas one usually specifies a quantum system by a pair $(\mathcal{H}, U)$ of a Hilbert space and a unitary one-parameter group of operators on $\mathcal{H}$, one can (and should) replace the linear space $\mathcal{H}$ by a projective space $\mathbb{P}$ and so replace the linear representation U by a projective representation of $\mathbb{R}$. (The projective space $\mathbb{P}$ determines a $*$ -algebra $\mathcal{A}$, and the projective representation is generated by a self-adjoint element of $\mathcal{A}$, the Hamiltonian.) In this lecture we take up this idea in quantum field theory. The expository paper [F14] tells the story in greater detail. In this lecture we introduce the main idea and then illustrate with the simplest example, taken largely verbatim from [FHT3].

35.1 Two QM warmups

Consider first a family of projective quantum mechanical systems parametrized by a space (or stack) S . The data defining the family includes a fiber bundle $\mathbb{P} \rightarrow S$ of projective spaces with appropriate metric data. A linearization is a vector bundle $E \rightarrow S$ and an isometry $\mathbb{P} \xrightarrow{\cong} \mathbb{P}E$ of the projective bundle with the projectivization of the linear bundle. By local triviality of fiber bundles, linearizations exist locally. Hence cover S by open sets U_α on which we have chosen linearizations, where α runs over some indexing set. On intersections $U_\alpha \cap U_\beta$ the two linearizations compare by a hermitian line bundle

$$(35.1) \quad L_{\alpha\beta} \longrightarrow U_\alpha \cap U_\beta$$

and on triple intersections there is a trivialization of the triple tensor product

$$(35.2) \quad \mathbb{C} \xrightarrow{\cong} L_{\beta\gamma} \otimes L_{\alpha\gamma}^{-1} \otimes L_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma.$$

The data (35.1) and (35.2) defines a *gerbe* [Hi]; its equivalence class lies in the sheaf cohomology group $H^2(S; \mathbb{T})$. (The sheaf $\mathbb{T}$ has sections that are continuous—or smooth, depending on the context—functions with values in $\mathbb{T}$.) A global linearization is a trivialization of this gerbe, which may or may not exist.

Observation: A gerbe over S is a *once shifted* or *once categorified* family of *invertible* quantum mechanical systems. Note that a family of invertible QM systems over S is in particular a hermitian line bundle over S , and its equivalence class lies in $H^1(S; \mathbb{T})$.

As a second warmup, consider a single projective QM system with a (finite) group G of symmetries.¹⁰⁷ The data, excluding the unitary evolution, is a projective space—finite or infinite

¹⁰⁷This is a special case of the previous discussion in which S is the classifying stack $*//G$ of the group. The group G need not be finite in this discussion; it can be any Lie group.

dimensional—that carries a unitary projective representation of G . This is the final vertical arrow in the diagram

$$(35.3) \quad \begin{array}{ccccc} \mathbb{T} & \longrightarrow & \mathbf{U} & \longrightarrow & \mathbf{PU} \\ \parallel & & \uparrow & & \uparrow \\ \mathbb{T} & \longrightarrow & \tilde{G} & \longrightarrow & G \end{array}$$

Here $\mathbf{PU}$ is the projective unitary group, and the top row fits the unitary group into a central extension, which already appears in (8.53). The bottom row is the pullback central extension of G ; the centrally extended group carries a unitary lift of the projective representation of G .

Example 35.4. There is a 1-dimensional¹⁰⁸ complex projective representation of $G = \mathrm{SO}_3$ —the spin representation—which is the action of SO_3 on $\mathbb{CP}^1 \simeq S^2$ by rotations. It has a linear unitary lift to the standard representation of $\tilde{G} = \mathrm{U}_2$ on $\mathbb{C}^2$.

A *linearization* of the projective action is a lift of $G \rightarrow \mathbf{PU}$ to a homomorphism $G \rightarrow \mathbf{U}$. A linearization is equivalent to a *splitting* of the induced central extension:

$$(35.5) \quad \begin{array}{ccccc} \mathbb{T} & \longrightarrow & \mathbf{U} & \longrightarrow & \mathbf{PU} \\ \parallel & & \uparrow & \nearrow & \uparrow \\ \mathbb{T} & \longrightarrow & \tilde{G} & \longrightarrow & G \end{array}$$

The *obstruction* to a linearization is measured by a homomorphism α from G to a classifying space for the center $\mathbb{T}$:

$$(35.6) \quad \begin{array}{ccccccc} \mathbb{T} & \longrightarrow & \mathbf{U} & \longrightarrow & \mathbf{PU} & \longrightarrow & B\mathbb{T} \\ \parallel & & \uparrow & & \uparrow & \nearrow \alpha & \\ \mathbb{T} & \longrightarrow & \tilde{G} & \xrightarrow{\pi} & G & & \end{array}$$

One geometric model for $B\mathbb{T}$ is the category of $\mathbb{T}$ -torsors and isomorphisms of $\mathbb{T}$ -torsors. The category $B\mathbb{T}$ is a groupoid—all morphisms are invertible—and furthermore it carries a symmetric tensor product. It is in this sense that we have a homomorphism $\alpha: G \rightarrow B\mathbb{T}$: for each $g \in G$ there is a $\mathbb{T}$ -torsor L_g ; for each pair $g_1, g_2 \in G$ there is an isomorphism $L_{g_1} \otimes L_{g_2} \rightarrow L_{g_1 g_2}$; and for each triple $g_1, g_2, g_3 \in G$ there is a compatibility (associativity, cocycle condition) among these isomorphisms. This is precisely the data (35.1) and (35.2) of a gerbe over the stack $S = BG$. Said differently, a homomorphism $G \rightarrow B\mathbb{T}$ is a central extension of G by $\mathbb{T}$. The torsor L_g is $\pi^{-1}(g)$ in the central extension (35.6). The homomorphism $\alpha: G \rightarrow B\mathbb{T}$ is the *projectivity* of the projective representation; it measures the deviation from linearizability. Here it appears as the obstruction to linearizing a projective QM theory with symmetry. Observe that for a finite group G , the equivalence class of the central extension lies in $H^2(G; \mathbb{T}) = \mathrm{Ext}(G, \mathbb{T})$. This is *once shifted* or *once categorified* from $H^1(G; \mathbb{T}) = \mathrm{Hom}(G, \mathbb{T})$, which is the abelian group of unitary characters of G . (Similar statements for arbitrary Lie groups use an appropriate version of group cohomology [Se6].)

¹⁰⁸The underlying projective space of the representation is 1-dimensional.

35.2 Anomalies in Wick-rotated quantum field theory

Fix a positive integer n and a simplicial sheaf $\mathcal{F}: \mathbf{Man}_n^{\text{op}} \rightarrow \mathbf{Set}_\Delta$ of background fields. A *linear* Wick-rotated QFT ([Axiom System 23.42](#)) is a homomorphism

$$(35.7) \quad F: \text{Bord}_{\langle n-1, n \rangle}(\mathcal{F}) \longrightarrow t\text{Vect}.$$

To implement our slogan that quantum theory is projective, we want to construct a symmetric monoidal category $t\text{Proj}$ so that we can define a *projective* Wick-rotated QFT as a homomorphism

$$(35.8) \quad \overline{F}: \text{Bord}_{\langle n-1, n \rangle}(\mathcal{F}) \longrightarrow t\text{Proj}.$$

Then we will ask for the obstruction to linearizing a projective theory $\overline{F}$, which in our warmup [§35.1](#) is a gerbe or a central extension of a group (α in [\(35.6\)](#)).

Warmup: Suppose S is a set equipped with the action of a group G . The naive quotient is the set S/G of orbits of the G -action. But it flouts the core ethos of categorical thinking to identify elements of S to form a quotient. Rather, one should remember the group element that effects the identification. This leads to the action groupoid $S//G$ whose set of objects is S and whose set of morphisms is $S \times G$. Lesson: The quotient of a set by a group is a 1-category.

Just as the projective general linear group PGL is the quotient of the linear group GL by $\mathbb{C}^\times$, we formally expect $t\text{Proj}$ to be the quotient of $t\text{Vect}$ by the action of the Picard groupoid Line , and so the previous paragraph suggests that this quotient $t\text{Proj}$ is a 2-category. The Picard groupoid Line has a realization as $B\mathbb{C}^\times$, the groupoid with a single object whose automorphism group is $\mathbb{C}^\times$. Then for each $V', V'' \in t\text{Vect}$ the group $\mathbb{C}^\times$ acts on $\text{Hom}(V', V'')$ by multiplication. The action is free on nonzero maps, but it is not free overall. So we build a 2-category by adjoining a 2-morphism to $t\text{Vect}$ for each $T \in \text{Hom}(V', V'')$ and each $\lambda \in \mathbb{C}^\times$; this 2-morphism maps $T \mapsto \lambda T$.

Remark 35.9. Suppose X is a pointed topological space and $Y = \Omega X$ is the based loop space of X . Define the “projectivization” of Y to be the space of paths of X that emanate from the basepoint. It is this construction that we can imitate in higher categories: if $\widetilde{\mathcal{C}}$ is a symmetric monoidal $(n+1)$ -category and $\mathcal{C} = \Omega \widetilde{\mathcal{C}}$ is the endomorphism category of the unit, then we define $\mathbb{P}\mathcal{C}$ to be a category of arrows $1 \rightarrow x$ in $\widetilde{\mathcal{C}}$. This is explained a bit more in [\[F14, Appendix A\]](#) and is developed in detail in [\[vD\]](#). Appropriate categories of arrows are constructed in [\[JFS\]](#), and in this way a projective theory becomes a linear theory relative to an invertible theory [\[F11\]](#).

This discussion leads to a diagram analogous to [\(35.6\)](#):

$$(35.10) \quad \begin{array}{ccccccc} \text{Line} & \longrightarrow & \text{Vect} & \longrightarrow & \text{Proj} & \longrightarrow & B\text{Line} \\ \parallel & & \uparrow \widetilde{F} & & \uparrow \overline{F} & \nearrow \alpha & \\ \text{Line} & \longrightarrow & \widetilde{\text{Bord}}_{\langle n-1, n \rangle}(\mathcal{F}) & \longrightarrow & \text{Bord}_{\langle n-1, n \rangle}(\mathcal{F}) & & \end{array}$$

Here $\overline{F}$ is a projective field theory, as in [\(35.8\)](#), there is an induced central extension of the bordism category and a linear field theory on that centrally extended bordism category. The obstruction to

linearizing $\overline{F}$, or equivalently to splitting the central extension in the bottom row, is the map α . As a map out of a bordism category it is a field theory, in fact an n -dimensional field theory. Since the codomain $B\text{Line}$ is a (higher) Picard groupoid, the field theory α is invertible. But the codomain is not the usual Picard 1-groupoid Line , as in (33.13), but rather is its classifying Picard 2-groupoid. Hence we call α a *once shifted* or *once categorified* n -dimensional invertible field theory. In this context it is called the *anomaly theory* of $\overline{F}$. If X is a closed n -manifold, then $\alpha(X)$ is a complex line and $\overline{F}(X): \mathbb{1} \rightarrow \alpha(X)$ is an element of the line. If Y is a closed $(n-1)$ -manifold, then $\alpha(Y)$ is a gerbe and $\tilde{F}(Y)$ can be interpreted as a projective space whose projectivity is $\alpha(Y)$.

Remark 35.11. There are many important extensions of this basic construction. For example, we can and should use super vector spaces in the codomain. Then, in addition to projective spaces, there are other “twistings” of the notion of a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space. For example, the grading may be by a $\mathbb{Z}/2\mathbb{Z}$ -torsor rather than by the group $\mathbb{Z}/2\mathbb{Z}$, as in Remark 10.4. Or, we can replace $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces by $\mathbb{Z}/2\mathbb{Z}$ -graded modules over the complex Clifford algebra $\text{Cl}_1^{\mathbb{C}}$. This form of “twisting” or “anomaly” occurs in the example below. All of these twistings mix in interesting ways in families, and they occur in topology in the form of twistings of K -theory [FHT1].

Remark 35.12. The invertible anomaly theory typically extends down to a fully local once shifted n -dimensional invertible theory. In good cases it is also the truncation of an $(n+1)$ -dimensional invertible theory $\tilde{\alpha}$. In that case the *anomalous* theory $\overline{F}$ appears as an n -dimensional boundary theory of $\tilde{\alpha}$.

35.3 A simple example

In the next lecture we discuss an interesting anomalous 3-dimensional topological field theory: Chern–Simons theory. Here, to illustrate the ideas, we describe a simple, but illuminating, anomalous theory: the single free fermionic field in 1-dimension. This quantum mechanical system turns out to be topological and projective; its anomaly extends to an interesting 2-dimensional field theory. Our exposition is taken almost verbatim from [FHT3].

The Wick-rotated QM system we consider is 1-dimensional with background fields a Riemannian metric and Spin structure. We describe it first as a classical system, and for clarity we consider the more general case of N fermions for $N \in \mathbb{Z}^{>0}$. Let W be an N -dimensional real inner product space. A spin structure on a spin Riemannian 1-manifold X is given as a double cover $\tilde{X} \rightarrow X$. A spinor field ψ on X is a function¹⁰⁹ $\tilde{\psi}: \tilde{X} \rightarrow \Pi W$ that is odd under the deck transformation of $\tilde{X} \rightarrow X$. Here ΠW is the parity-reversal of W . The lagrangian density that describes this system is

$$(35.13) \quad L = \frac{1}{2} \langle \psi, D_X \psi \rangle d\mu_X,$$

where μ_X is the Riemannian measure on X and D_X is the Dirac operator on X . The latter is $D_X \psi = \partial \tilde{\psi}$ for ∂ the positively oriented unit norm vector field on $\tilde{X}$.

Compute canonical quantization on $Y = \text{pt}$ as follows. First, solve the classical equations of motion on $\mathbb{R} \times Y$ to obtain a symplectic vector space, which in this case is odd. Here the

¹⁰⁹We have not discussed odd, or fermionic, classical fields in these lectures; see [F5] for an exposition.

classical equation asserts that ψ is locally constant, so the vector space of solutions is canonically isomorphic to ΠW . After complexification, classical observables form the algebra $\text{Sym}^\bullet \Pi W_{\mathbb{C}}^*$. This is the finite dimensional exterior algebra of $W_{\mathbb{C}}^*$ with the $\mathbb{Z}/2\mathbb{Z}$ -grading that makes $\wedge^1 W_{\mathbb{C}}^* = W_{\mathbb{C}}^*$ odd. Second, form the *deformation quantization* using the Poisson bracket derived from the lagrangian (35.13). The result is the Clifford algebra $\text{Cliff}(W_{\mathbb{C}}^*)$ for the complexified dual inner product; the Clifford algebra carries its usual $\mathbb{Z}/2\mathbb{Z}$ -grading. Third, quantum states are defined by producing an irreducible $\mathbb{Z}/2\mathbb{Z}$ -graded module M over $\text{Cliff}(W_{\mathbb{C}}^*)$. Here is where the parity of N shows up. If N is even, then the complex algebra $\text{Cliff}(W_{\mathbb{C}}^*)$ of observables is the full endomorphism algebra of M . In this case M is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space of dimension

$$(35.14) \quad \dim M = 2^{N/2-1} \mid 2^{N/2-1};$$

the notation ‘ $b|f$ ’ indicates the dimensions of the even (Bose) and odd (Fermi) homogeneous subspaces of a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space. For N odd the action of $\text{Cliff}(W_{\mathbb{C}}^*)$ on M has a nontrivial *commutant*:¹¹⁰ the subalgebra of $\text{End}(M)$ consisting of operators that commute with every element of $\text{Cliff}(W_{\mathbb{C}}^*)$ is isomorphic to $\text{Cl}_1^{\mathbb{C}}$, the complex Clifford algebra on a single odd generator δ .

Example 35.15 ($N = 1$). Let $W = \mathbb{R}$ with its standard inner product, so $\text{Cliff}(W_{\mathbb{C}}^*) = \text{Cl}_1^{\mathbb{C}}$ is the complex superalgebra generated by an odd element γ . An irreducible module is $M = \mathbb{C}^{1|1}$ with $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The commutant is generated by $\delta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Remark 35.16.

- (1) One must consider the state space as a module over the commutant. Explanation: In this case the algebra of complex observables A is the commutant of the commutant B inside the full endomorphism algebra $\text{End}(M)$, so is identified as $\text{End}_B(M)$, the algebra of endomorphisms of M as a B -module. Typically, one must complete the module M to obtain a Hilbert space of states, but in this case M is finite dimensional and no completion is necessary.
- (2) That the state space M is a module over $\text{Cl}_1^{\mathbb{C}}$ signals that the theory is *anomalous*; this form of anomaly is described in [Remark 35.11](#).

One can do much more with this example. First, the path integral over the circle can be done explicitly. Even without that, for N odd one can identify a full 2-dimensional invertible anomaly theory (called $\tilde{\alpha}$ in [Remark 35.12](#)) whose value on a spin point is $\text{Cl}_1^{\mathbb{C}}$. This is a topological theory of order 2, called the *Arf theory*. Then the quantum theory of N quantum mechanical fermions is a topological boundary theory of the Arf theory. See [\[FHT3\]](#) for details.

Lecture 36: Three-dimensional Chern–Simons theory

Recall that there is a basic dichotomy of quantum systems: gapped vs. gapless. We discussed this for quantum mechanical systems in §5.2.2 and more generally for relativistic quantum systems

¹¹⁰As always in super algebra, the commutant is taken in the $\mathbb{Z}/2\mathbb{Z}$ -graded sense: odd elements a_1, a_2 commute if $a_2 a_1 = -a_1 a_2$.

in §14.3. (We also gave an example of a gapped *discrete* quantum mechanical system in §5.3, though not a general discussion of this dichotomy in the discrete context.) There is a standard lore that at low energies a gapped quantum theory is well-approximated by a topological field theory. For a gapped relativistic system, this approximation occurs in the Wick-rotated theory, and for Wick-rotated quantum field theories should be realized as a singular limit. Now this lore must be modified: one only expects the approximating theory to be *projectively* topological.¹¹¹ In this lecture we discuss the poster child example: 3-dimensional Yang–Mills theory with a Chern–Simons term. The associated topological field theory is what is known in the mathematics literature as Chern–Simons theory.

The introduction of 3-dimensional Chern–Simons theory by Witten [W3] was a game-changer in the geometric approach to quantum field theory. It has a rich structure and is amenable to computations from many points of view. Witten expressed the Jones polynomial invariants of knots and links as built from correlation functions in a 3-dimensional topological field theory. Furthermore, he introduced new topological invariants of 3-manifolds and of links in arbitrary 3-manifolds. Witten’s starting point is the secondary Chern–Simons invariant of a connection in differential geometry, and from there he uses the non mathematically rigorous path integral. Contemporaneous work by Reshetikhin–Turaev [RT1, RT2] has a much different mathematical starting point—quantum groups—from which they rigorously construct the knot and 3-manifold invariants Witten found. The two approaches are linked by extended topological field theory, as introduced in Lecture 29.

Our exposition here emphasizes the structural aspects as they relate to general quantum field theory. Thus we begin with a discussion of the Chern–Simons term in the relativistic setting. The simplest case is a classical abelian gauge theory, in which it is a *mass term*: it converts a gapless free theory to a gapped free theory. One can see this explicitly from the representation theory, as we show. The singular limit that leads to the low energy theory is invertible as a relativistic theory; we see much more when we pass to Wick-rotated theory. We remark that a nondegenerate Chern–Simons term creates a mass in gauge theory with general gauge group.

The Wick-rotated Yang–Mills + Chern–Simons theory has background fields a Riemannian metric and orientation. Witten’s claim is that there is a singular limit in which the Yang–Mills term is scaled away and one is left with a well-defined theory that is *projectively* topological. We show how the introduction of a p_1 -structure leads to a *linear* theory that is topological. This is the theory that is called ‘Chern–Simons theory’ in topology.

We then describe how this 3-dimensional topological field theory—what we term a ‘(2,3)’-theory—extends to a (1,2,3)-theory. In this extension a linear category is assigned to S^1 , and it is this category that makes contact with the Reshetikhin–Turaev approach. In fact, Reshetikhin–Turaev construct this (1,2,3)-theory precisely [Tu]. We close with a brief discussion of observables, including line defects, which brings us to Witten’s description of the Jones polynomial.

We conclude with mathematical background on p_1 -structures.

¹¹¹In fact, we expect further modifications of this lore.

36.1 The Chern–Simons term in relativistic gauge theory

The Chern–Simons term as a mass for a gauge field originates in [DJT], as far as we know. The following is drawn from the solution to Problem FP4 in [W4].

Let $\mathbb{M}^3$ be standard 3-dimensional Minkowski spacetime, and fix real constants e, λ with $e > 0$. We write a classical field theory with field in $\Omega_{\mathbb{M}^3}^1/d\Omega_{\mathbb{M}^3}^0$, i.e., the field α is a 1-form modulo the equivalence $\alpha \sim \alpha + df$ for a function f . The lagrangian density is

$$(36.1) \quad L = -\frac{1}{4e^2} d\alpha \wedge *d\alpha + \lambda \alpha \wedge d\alpha.$$

Here $*$ is the Hodge $*$ operator in Lorentz signature; it maps the 2-form $d\alpha$ to a 1-form twisted by the orientation double cover, hence the first term in (36.1) is a density on $\mathbb{M}^3$. The second term is a 3-form. This means that whereas the first term does not require an orientation to integrate, the second term—the Chern–Simons term—does require an orientation. From (36.1) we compute the Euler–Lagrange equation

$$(36.2) \quad d * d\alpha = 4\lambda e^2 d\alpha.$$

Therefore, using the freedom to shift $\alpha \rightarrow \alpha + df$, we can impose the stronger equation

$$(36.3) \quad *d\alpha = 4\lambda e^2 \alpha.$$

This is a Poincaré invariant linear differential equation, so the space of solutions is a representation of the Poincaré group. We proceed to determine that representation. Note that if $\lambda = 0$ then we are reduced to the discussion around (19.26).

Introduce standard coordinates $x = (x^0, x^1, x^2)$ on $\mathbb{M}^3$ so that the Lorentz metric is $(dx^0)^2 - (dx^1)^2 - (dx^2)^2$. Write $\alpha(x) = \alpha_0 dx^0 + \alpha_1(x) dx^1 + \alpha_2(x) dx^2$, and Fourier transform as in (18.27). Then (36.3) becomes the linear matrix equation

$$(36.4) \quad 4\lambda e^2 \begin{pmatrix} \hat{\alpha}_0 \\ \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{pmatrix} = i \begin{pmatrix} 0 & -p_2 & p_1 \\ -p_2 & 0 & p_0 \\ p_1 & -p_0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\alpha}_0 \\ \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{pmatrix}$$

where $(p_0, p_1, p_2) \in V^* = (\mathbb{R}^{1,2})^*$ is a vector in the vector space dual to translations. Let I be the identity matrix and let A be the square matrix in (36.4). Then $\hat{\alpha}_i = 0$ unless $\det(4\lambda e^2 I - iA) = 0$, which is the condition

$$(36.5) \quad p_0^2 - p_1^2 - p_2^2 = (4\lambda e^2)^2.$$

In other words, the Fourier transform of α is supported on the mass shell $\mathcal{O}_m \subset V^*$ with mass $m = 4|\lambda|e^2$ if $\lambda \neq 0$; if $\lambda = 0$, then it is supported on the dual null cone in V^* .

Further analysis pins down the equivariant line bundle over $\mathcal{O}_m$ cut out by (36.4). (We assume $\lambda \neq 0$ and leave the $\lambda = 0$ case to the reader.) If $\lambda > 0$ consider the point $(p_0, p_1, p_2) = (4\lambda e^2, 0, 0)$.

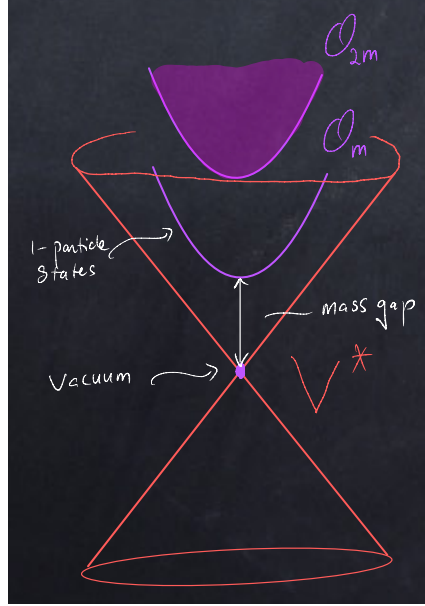


FIGURE 111. Free relativistic gauge theory with a Chern–Simons term

Then the solution space to (36.4) is the line spanned by $(0, 1, -i)$, and a rotation by θ in the (p_1, p_2) -plane acts as multiplication by $e^{i\theta}$. Thus the representation of the little group Spin_2 has spin 1. If $\lambda < 0$ then some signs reverse and the representation of the little group has spin -1 .

In summary: The Chern–Simons term with coefficient λ gives a mass $4|\lambda|e^2$ to the gauge field α . The 1-particle states form an irreducible massive representation of the Poincaré group with spin ± 1 according to the sign of λ . As in (19.10), the full quantum theory has a unique vacuum state and a mass gap $4|\lambda|e^2$ to the rest of the spectrum; see Figure 111. One can contemplate the singular limit $e \rightarrow \infty$. All of the nonzero spectrum is pushed to infinite mass, and one is left only with the vacuum state: the infinite dimensional Hilbert space of the theory has a singular limit which is the vacuum line. We now turn to the same singular limit in the Wick-rotated theory, where what is left behind is richer than merely a 1-dimensional vector space.

36.2 The singular limit of the Wick-rotated theory

In the language of Definition 23.34, the Euclidean symmetry type of the theory is the inclusion map $\text{SO}_3 \rightarrow \text{O}_3$. In the language of Definition 23.38, the background fields are

$$(36.6) \quad \mathcal{F} = \{\text{Riemannian metrics, orientations}\}.$$

We consider the theory for an arbitrary gauge group.

Let G be a compact Lie group. A general level is a class $\lambda \in H^4(BG; \mathbb{Z})$, or better a cocycle that represents the cohomology class. If G is a connected Lie group, then $H^4(BG; \mathbb{Z})$ is torsionfree and the level is determined by its image in the vector space $H^4(BG; \mathbb{R})$. That is canonically the vector space of symmetric G -invariant bilinear forms $\langle -, - \rangle$ on the Lie algebra $\mathfrak{g}$. Those forms in the image satisfy an integrality constraint. If A is a connection on a principal G -bundle $P \rightarrow X$

over a smooth manifold X , so $A \in \Omega_P^1(\mathfrak{g})$, then the Chern–Simons 3-form on P is

$$(36.7) \quad \Gamma(A) = \langle A \wedge F_A \rangle - \frac{1}{6} \langle A \wedge [A \wedge A] \rangle.$$

Here $F_A \in \Omega_X^2(\mathfrak{g}_P)$ is the curvature of the connection, a 2-form on the base X with values in the adjoint bundle of Lie algebras $\mathfrak{g}_P \rightarrow X$. The classic paper [CS] is the primary source; see also [F7, F8] for more expository material. The lagrangian for the Wick-rotated Yang–Mills + Chern–Simons theory is the expression

$$(36.8) \quad L = \frac{1}{4e^2} F_A \wedge *F_A + \Gamma(A).$$

We remark that the integral of $\Gamma(A)$ over a closed oriented 3-manifold is only defined modulo $2\pi\mathbb{Z}$, but that is good enough to define an exponentiated action.

There are three claims implicit in [W3].

Claim 36.9.

- (1) *The lagrangian (36.8) determines a family of Wick-rotated field theories (Axiom System 23.42) parametrized by $e \in \mathbb{R}^{>0}$.*
- (2) *The singular limit as $e \rightarrow \infty$ exists and defines a field theory*

$$(36.10) \quad F: \text{Bord}_{\langle 2,3 \rangle}(\mathcal{F}) \longrightarrow t\text{Vect}.$$

- (3) *The underlying projective theory is topological. More precisely, it factors to a theory*

$$(36.11) \quad \overline{F}: \text{Bord}_{\langle 2,3 \rangle}(\mathcal{F}') \longrightarrow f\text{Proj}$$

where

$$(36.12) \quad \mathcal{F}' = \{\text{orientations}\}$$

and the codomain is, roughly, a category of finite dimensional projective spaces.

The projective theory $\overline{F} = \overline{F}_{(G,\lambda)}$ is the first form of *topological* Chern–Simons theory: the sheaf $\mathcal{F}'$ is locally constant (Definition 24.46). The linear theory F in (36.10) has background fields (36.6): both a Riemannian metric and an orientation. But as we will see, the dependence of F on the Riemannian metric is very mild.

Remark 36.13.

- (1) We can think of $\overline{F}$ as defined by a path integral over connections A using the lagrangian $\Gamma(A)$. Since this lagrangian does not depend on the metric, it stands to reason that neither does the theory $\overline{F}$. But—as path integrals go—this one is even more badly defined than the path integral of (36.8): the kinetic term—the norm square of the curvature—in the Euclidean path integral acts as an exponential damping factor. In that sense the singular limit description is more plausible. It may be that a rigorous construction of this theory and its singular limit is tractable [W5].

- (2) In this example, the lore discussed at the beginning of the lecture works: this gapped quantum system has a singular limit that is a projectively topological field theory. One can view this as a low energy approximation to the quantum theory.
- (3) As mentioned earlier, Reshetikhin–Turaev have a rigorous algebraic and topological construction of $\overline{F}$ and the variations to follow.

36.3 The (framing) anomaly

The *projectivity* of a projective theory is measured by an invertible theory α , called the *anomaly*, which for 3-dimensional quantum Chern–Simons theory is an invertible 4-dimensional theory

$$(36.14) \quad \alpha: \text{Bord}_{\langle 3,4 \rangle}(\mathcal{F}') \longrightarrow \text{Line}$$

called the “framing anomaly”. In particular, for a closed oriented 3-manifold X , the value of the theory α is a complex line $\alpha(X)$. The *anomalous theory* $\overline{F}'$, which is another form of the projective theory, evaluates on X to an element of the line $\overline{F}'(X): \mathbb{C} \rightarrow \alpha(X)$. Chern–Simons theory has a *central charge* $c = c(\lambda) \in \mathbb{Q}$ which is a simple function of the level. For example, if $G = \text{SU}_2$ then the level $\lambda \in H^4(\text{BSU}_2; \mathbb{Z}) \cong \mathbb{Z}$ is an integer k and the central charge is

$$(36.15) \quad c = 3k/(k+2).$$

The partition function of the anomaly theory on a closed oriented 4-manifold W is

$$(36.16) \quad \alpha(W) = e^{2\pi i c \text{Sign}(W)/8} = e^{2\pi i c p_1(W)/24} \in \mu_N,$$

where $\text{Sign}(W)$ is the signature of W . The positive integer N is determined by the denominator of c , and the entire theory (36.14) factors through $\mu_N \subset \mathbb{C}^\times$. For example, the complex line $\alpha(X)$ can be refined to a μ_N -torsor. Thus either $\overline{F}'(X) = 0$ or $\overline{F}'(X)$ lies in a μ_N -torsor which is embedded in $\mathbb{C}$. More concretely, if nonzero, the invariant $\overline{F}'(X)$ is not a single complex number but rather a collection of complex numbers that differ from each other by an N^{th} root of unity. This still provides useful information.

36.4 Linearization of the projective theory

The projectivization α is nonzero, as we can see from the partition function (36.16): there are closed oriented 4-manifolds of nonzero signature. (For example, $\text{Sign}(\mathbb{CP}^2) = 1$.) The idea, then, is to find a set of background fields $\mathcal{F}''$ and a map $\mathcal{F}'' \rightarrow \mathcal{F}'$ such that the lift of the anomaly theory (36.14) to $\text{Bord}_{\langle 3,4 \rangle}(\mathcal{F}'')$ is trivializable.¹¹² Then a trivialization lifts the projective theory $\overline{F}$ to a linear theory.

There are different choices possible for $\mathcal{F}''$. One is a framing of the tangent bundle. But a more effective choice is a p_1 -structure: a trivialization of the first Pontrjagin class; see §?? for an

¹¹²More properly, the theory (36.14) is truncated to a *once-categorified* 3-dimensional field theory, and it is this truncated theory that is trivialized. In particular, the background fields are sheaves on $\mathbf{Man}_3$, not on $\mathbf{Man}_4$.

introduction. In this terminology, an orientation is a trivialization of the first Stiefel–Whitney class, i.e., is a w_1 -structure. Set

$$(36.17) \quad \mathcal{F}'' = \{(w_1, p_1)\text{-structures}\}.$$

Then $\mathcal{F}''$ is a locally constant sheaf on $\mathbf{Man}_3$, so a field theory over $\mathcal{F}''$ is topological. The sheaves of background fields fit into the following diagram of fiber bundles, with ‘RM’=‘Riemannian metrics’:

$$(36.18) \quad \begin{array}{ccc} & \mathcal{F}''' & \\ \swarrow & & \searrow \\ \mathcal{F} & & \mathcal{F}'' \\ \searrow & & \swarrow \\ & \mathcal{F}' & \end{array} \quad \begin{array}{ccc} & \{\text{RM}, (w_1, p_1)\} & \\ \swarrow & & \searrow \\ \{\text{RM}, w_1\} & & (w_1, p_1) \\ \searrow & & \swarrow \\ & w_1 & \end{array}$$

Claim 36.19. *The pullback of $\overline{F}$ to $\text{Bord}_{\langle 2,3 \rangle}(\mathcal{F}'')$ lifts to a linear topological field theory*

$$(36.20) \quad Z: \text{Bord}_{\langle 2,3 \rangle}(\mathcal{F}'') \longrightarrow \text{Vect}.$$

In summary, we begin with Yang–Mills + Chern–Simons (36.8), take the singular limit in Claim 36.9, which is projectively topological, and then introduce an extra background field to lift to a linear theory.

Witten [W3, §2] gives a construction of Z that we interpret as follows. Introduce

$$(36.21) \quad \mathcal{F}''' = \{\text{Riemannian metrics}, (w_1, p_1)\text{-structures}\}$$

and the invertible field theory $\beta: \text{Bord}_{\langle 2,3 \rangle}(\mathcal{F}''') \rightarrow \text{Line}$ whose partition function on a closed oriented Riemannian 3-manifold X with p_1 -structure is

$$(36.22) \quad \beta(X) = e^{-2\pi i c \gamma(X)/24},$$

where $\gamma(X)$ is the integral over X of the Chern–Simons 3-form (36.7) of the Levi–Civita connection on the principal SO_3 -bundle of frames. The p_1 -structure enables us to interpret (36.7) as a differential form on X .

Claim 36.23. *The tensor product $F \otimes \beta$ factors from a theory over $\mathcal{F}'''$ to a theory over $\mathcal{F}''$.*

Notice that $F \otimes \beta$ has the same underlying projective theory as F since β is invertible. A colloquial form of Claim 36.19 is that the metric dependence of (36.10) is via an invertible field theory. The theory (36.20) is the claimed factorization of $F \otimes \beta$. It is this theory that is commonly called “Chern–Simons theory” in the mathematical literature.

36.5 The Reshetikhin–Turaev approach

In Witten’s construction the starting point is the classical Chern–Simons form (36.7) (as a supplement to Yang–Mills theory). Reshetikhin–Turaev [RT1, RT2] begin instead with algebraic data, which can be taken to be a *modular tensor category* $\mathcal{A}$. (Modular tensor categories were introduced by Moore–Seiberg [MoSe].) Briefly, $\mathcal{A}$ is a braided tensor category which has an additional ribbon structure, and it satisfies many conditions: it is finite semisimple abelian, and most importantly it satisfies a nondegeneracy condition. It takes some work to elaborate, so we defer to references such as [BK, EGNO].

Conjecture 36.24. *There is a canonical construction*

$$(36.25) \quad (G, \lambda) \rightsquigarrow \mathcal{A}_{(G, \lambda)}$$

that attaches a modular tensor category to a compact Lie group and level.

Special cases are known. These include finite groups, torus groups, and simply connected groups. But, to quote Bert Kostant, one doesn’t truly understand something for compact Lie groups if it is only known via classification or on a case-by-case basis. A shadow of $\mathcal{A}_{(G, \lambda)}$, known as the *fusion ring* or *Verlinde ring*, does have a uniform construction.

Here is an illustration of the *combinatorial* data that defines a modular tensor category.

Example 36.26. Let $G = \mathrm{SU}_2$ and take the level $k = 2$. In that case the central charge (36.15) is $c = 3/2$. The modular tensor category $\mathcal{A} = \mathcal{A}_{(\mathrm{SU}_2, 2)}$ has 3 simple objects $1, \sigma, \psi$. The aforementioned fusion ring has these objects as an additive basis, 1 is the identity element, and

$$(36.27) \quad \begin{aligned} \sigma^2 &= 1 + \psi \\ \sigma\psi &= \sigma \\ \psi^2 &= 1 \end{aligned}$$

The ribbon structure is encoded in a root of unity for each object, which for $1, \sigma, \psi$ is $1, \mu, -1$, respectively, where $\mu = e^{(2\pi i(3/16))}$. There is braiding data

$$(36.28) \quad R_1^{\sigma\sigma} = -\mu^{-1}, \quad R_1^{\psi\psi} = -1, \quad R_\sigma^{\psi\sigma} = R_\sigma^{\sigma\psi} = i, \quad R_\psi^{\sigma\sigma} = e^{(2\pi i)/16},$$

an S -matrix

$$(36.29) \quad S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix},$$

and associativity data that we do not list here.

There are explicit algorithms that compute invariants of 3-manifolds from this data. For example, if X is a Seifert fibered 3-manifold, then $Z(X)$ is computed from the matrix (36.29) and the roots of unity attached to simple objects; see [FG] for explicit computations. Algorithms for different presentations of general 3-manifolds exist in abundance; a very small sample is [Wa, KM, BHMV, L1].

The contrast between Witten’s starting point (the classical Chern–Simons invariant) and Reshetikhin–Turaev’s starting point (a modular tensor category) provokes the following.

Question 36.30. How are these two starting points related?

The resolution of this puzzle lies in the notion of an *extended field theory*, as discussed in Lecture 29. Here we work with an extension of (36.20). The domain of the extension is the bordism 2-category $\text{Bord}_{\langle 1,2,3 \rangle}(\mathcal{F}'')$ whose objects are closed 1-manifolds, 1-morphisms are 2-dimensional bordisms, and 2-morphisms are 3-dimensional bordisms with corners. The codomain is a 2-category Cat of linear categories, and the extension of (36.20) is a homomorphism

$$(36.31) \quad \tilde{Z}: \text{Bord}_{\langle 1,2,3 \rangle}(\mathcal{F}'') \longrightarrow \text{Cat}$$

This extension gives further locality: now state spaces $Z(Y)$ have one level of gluing law. We call (36.31) a $(1, 2, 3)$ -theory, and so (36.20) is a $(2, 3)$ -theory. One can go further, at least in principle if not in this particular case, to contemplate *fully local theories* or $(0, 1, 2, 3)$ -theories.

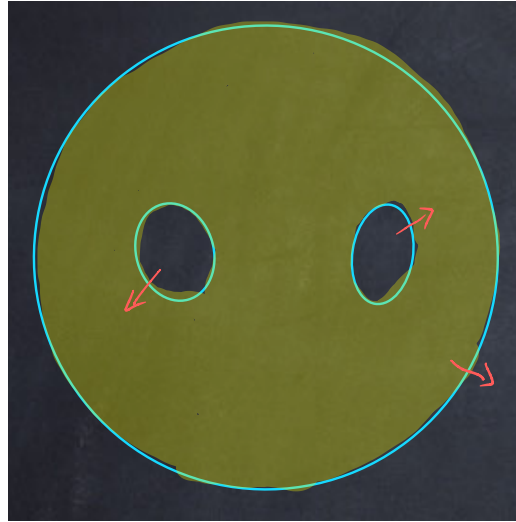


FIGURE 112. The pair of pants

The first key insight toward answering Question 36.30: $\mathcal{A} = \tilde{Z}(S^1)$ is the modular tensor category of Reshetikhin–Turaev. According to (36.31) we only have that $\tilde{Z}(S^1)$ is a linear category. One defines a tensor structure using the “pair of pants” bordism depicted in Figure 112. The braiding comes from the diffeomorphism that exchanges the two legs while fixing the waist. In modern language, $S^1 \in \text{Bord}_{\langle 1,2,3 \rangle}(\mathcal{F}'')$ is an E_2 -object, and so $\mathcal{A} = \tilde{Z}(S^1)$ is an E_2 -category, which is a braided tensor category. The ribbon structure is induced by the generator of $\pi_1 \text{Diff}(S^1) \cong \mathbb{Z}$. (This is the group of diffeomorphisms that preserves the (w_1, p_1) -structure.)

The second key insight is the *finite path integral*, as discussed in [Lecture 27](#) and [§30.3](#). Here one begins with the classical Chern–Simons invariant, understood as an extended *invertible* field theory, and then “integrates” over G -connections on S^1 . If G is a finite group, then this integration is a finite algebraic process, and it computes the modular tensor category $\mathcal{A}$ from the Chern–Simons functional. For a finite group one can take vanishing Chern–Simons functional, in which case [\(30.10\)](#) is the modular tensor category. See [\[F9\]](#) for the case of nonzero Chern–Simons functional.

36.6 Observables

Recall from remark (5) in [§26.1](#) that in an n -dimensional *topological* field theory Z , the vector space of observables $\mathcal{O}_x$ at a point $x \in X$ in an n -manifold is the value $Z(S^{n-1})$ of the theory on an $(n-1)$ -sphere. For 3-dimensional Chern–Simons theory, one can compute it by expressing the 2-sphere as a composition of bordisms

$$(36.32) \quad S^2: \emptyset^1 \xrightarrow{D'} S^1 \xrightarrow{D''} \emptyset^1$$

as depicted in [Figure 113](#); each of D', D'' is a 2-disk. Compose the pair of pants bordism in [Figure 112](#) with D' (disjoint union the identity map on S^1) to conclude that $\tilde{Z}(D')$ is the unit for the tensor structure of the modular tensor category $\mathcal{A} = \tilde{Z}(S^1)$; it is the special object 1. (We identify a linear functor $\text{Vect} \rightarrow \mathcal{A}$ with an object of $\mathcal{A}$) On the other hand, $\tilde{Z}(D''): \mathcal{A} \rightarrow \text{Vect}$ is the functor $x \mapsto \text{Hom}(1, x)$, which one can prove with a bit more knowledge about duality in this situation. It follows that $\tilde{Z}(S^2) = Z(S^2) = \text{Hom}_{\mathcal{A}}(1, 1)$. In a modular tensor category, $\text{Hom}(1, 1)$ is the vector space $\mathbb{C}$ of scalars. This means that the only observable at a point is the identity; there are no nontrivial correlation functions [\(23.24\)](#).

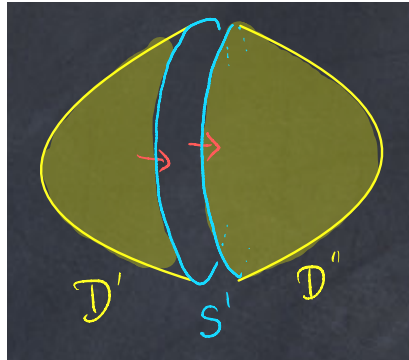


FIGURE 113. A decomposition of S^2

As introduced in [Lecture 31](#), there are more general “observables” or “operators” in a quantum field theory whose support is not a point, but rather a higher dimensional submanifold, or stratified submanifold. The general word for these are *defects*. In Chern–Simons theory $\tilde{Z}$, the interesting defects are 1-dimensional: so-called *line defects*.¹¹³ Just as the vector space of point defects in a topological field theory is computed as the value of the theory on the link of a point, so too the *category* of line defects at a point $x \in L \subset X$ on a curve in a 3-manifold is the value of the theory

¹¹³The support is a 1-dimensional submanifold, a *curve* in a 3-manifold.

on the link of $L \subset X$ at x ; see [Definition 31.14](#). If $L \subset X$ is normally framed, then this link can be identified as the standard circle S^1 , and the category of local line defects is the modular tensor category $\mathcal{A} = \tilde{Z}(S^1)$. One can specialize to $X = S^3$, and then $L \subset S^3$ is a knot or link. A bit more discussion is in [§31.4](#). From correlation functions of line defects in the theories $Z = Z_{(\mathrm{SU}_2, k)}$ one recovers the Jones polynomial, which was Witten's starting point [[W3](#), [A3](#), [FK](#)].

36.7 p_1 -structures

Fix a positive integer n and a space $B\mathrm{GL}_n\mathbb{R}$ that classifies rank n real vector bundles. Also fix a choice $E_n \rightarrow B\mathrm{GL}_n\mathbb{R}$ of universal bundle. Then if M is a smooth n -manifold, there is a contractible space of classifying diagrams

$$(36.33) \quad \begin{array}{ccc} TM & \longrightarrow & E_n \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & B\mathrm{GL}_n\mathbb{R} \end{array}$$

for its tangent bundle. An *orientation*, or w_1 -*structure*, on M is a lift of f to the total space of the double cover $B\mathrm{GL}_n^+\mathbb{R} \rightarrow B\mathrm{GL}_n\mathbb{R}$, which classifies rank n oriented real vector bundles:

$$(36.34) \quad \begin{array}{ccc} & B\mathrm{GL}_n^+\mathbb{R} & \\ & \nearrow & \downarrow 2:1 \\ M & \xrightarrow{f} & B\mathrm{GL}_n\mathbb{R} \end{array}$$

The set of lifts is either empty or is a torsor over $\mathrm{Map}(M, \mu_2) \cong H^0(M; \mu_2)$.

The first Pontrjagin class can be realized as a map

$$(36.35) \quad B\mathrm{GL}_n^+\mathbb{R} \xrightarrow{p_1} K(\mathbb{Z}, 4).$$

Let F_n be a space that realizes the homotopy fiber of (36.35).

Definition 36.36. Let M be an oriented n -manifold. A p_1 -*structure* on M is a lift

$$(36.37) \quad \begin{array}{ccc} & F_n & \\ & \nearrow & \downarrow \\ M & \longrightarrow & B\mathrm{GL}_n^+\mathbb{R} \xrightarrow{p_1} K(\mathbb{Z}, 4) \end{array}$$

of the classifying map of the oriented tangent bundle.

If M is a closed connected oriented 4-manifold, then a p_1 -structure exists iff $p_1(M) = 0$ iff $\mathrm{Sign}(M) = 0$. For a closed oriented 3-manifold M , a p_1 -structure always exists. The set of homotopy classes of lifts in (36.37) is a torsor over $H^3(M; \mathbb{Z})$.

[Definition 36.36](#) was introduced in [[BHMV](#), Appendix B]. For 3-manifolds, an alternative is Atiyah's *2-framings* [[A2](#)].

Lecture 37: Supersymmetric quantum mechanics I

In this lecture we begin our study of supersymmetry and its interplay with mathematics, which will occupy us for the rest of the semester. The first three lectures cover supersymmetric quantum mechanics, which is the 1-dimensional case.

We begin with a quick review of DOSO—data of states and observables—including both bosons and fermions. In this very general context, we say what it means to have an odd symmetry, also known as a supersymmetry. We emphasize that theories can have both bosons and fermions without having supersymmetry.

Then we focus on the case of quantum mechanics. First, we review some basics about super Hilbert spaces and adjoints of operators, as well as background on super Lie algebras and super Lie groups. We introduce the super Lie groups of symmetries of a supersymmetric quantum mechanical system. The next lecture discusses the basic examples.

37.1 Odd symmetries of states and observables

The data and axioms for states and observables were introduced in [Axiom System 1.1](#), with the crucial modification to include fermions in §10.1. We recall the basics here.

- (1) Homogeneous states form two real convex spaces $\mathcal{S}^0, \mathcal{S}^1$, whose elements are called *bosonic states* and *fermionic states*, respectively. The labels ‘0’ and ‘1’ lie in $\mathbb{Z}/2\mathbb{Z}$, but a more general picture has the labels in a $\mathbb{Z}/2\mathbb{Z}$ -torsor; in the latter case one cannot canonically apply the labels ‘bosonic’ and ‘fermionic’. A general state is a convex combination of a bosonic and fermionic state: a point in the join (10.1).
- (2) There is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $\mathcal{O} = \mathcal{O}^0 \oplus \mathcal{O}^1$ equipped with a real structure and a dense subspace $\mathcal{O}^\infty$ on which there is a super Lie algebra structure. Elements of $\mathcal{O}_{\mathbb{R}}^0$ are (*even*) *observables* and elements of $\mathcal{O}_{\mathbb{R}}^1$ are (*odd*) *observables*. One can apply Borel functions to even observables; see (10.2).
- (3) Only bosonic observables can be measured; see (10.3). Only bosonic smooth observables—those in $(\mathcal{O}_{\mathbb{R}}^\infty)^0$ —generate one-parameter groups.

A *symmetry* is an automorphism of this structure. Any symmetry preserves the $\mathbb{Z}/2\mathbb{Z}$ -grading on observables: it maps $\mathcal{O}^0 \rightarrow \mathcal{O}^0$ and $\mathcal{O}^1 \rightarrow \mathcal{O}^1$. However, it may map each convex space $\mathcal{S}^i \rightarrow \mathcal{S}^i$ or it may exchange them $\mathcal{S}^0 \leftrightarrow \mathcal{S}^1$. A *supersymmetric theory* is a theory with symmetries of the latter type.

Remark 37.1. A mechanical system consists of more than the data of states and observables: there is motion ([Definition 1.11](#)) or, more generally, a representation of a symmetry group that includes motion (see [Definition 14.12](#) for the relativistic case).

37.2 Super Hilbert spaces, super Lie algebras, and super Lie groups

Superalgebra is the subject of [Lecture 9](#). It is the foundation upon which we build quantum mechanical systems with fermionic states. To rest on firm foundations we apply the Koszul sign rule (9.11) religiously. We refer to [\[DM\]](#) for much more about superalgebra.

As in [Definition 9.31](#), a *super Hilbert space* is a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ equipped with a pairing $\langle -, - \rangle: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ under which $\mathcal{H}^0$ and $\mathcal{H}^1$ pair trivially, the restriction of $\langle -, - \rangle$ to $\mathcal{H}^0$ is a Hilbert space structure, and the restriction of $-i\langle -, - \rangle$ to $\mathcal{H}^1$ is a Hilbert space structure. Let $U(\mathcal{H})$ denote the group of even maps $\mathcal{H} \rightarrow \mathcal{H}$ that preserve the pairing $\langle -, - \rangle$. Then

$$(37.2) \quad U(\mathcal{H}) \cong U(\mathcal{H}^0) \times U(\mathcal{H}^1)$$

Let $T: \mathcal{H} \rightarrow \mathcal{H}$ be a bounded endomorphism, which we assume is homogeneous. Its adjoint $T^*: \mathcal{H} \rightarrow \mathcal{H}$ is characterized by the equation

$$(37.3) \quad \langle \overline{T^* \psi_1}, \psi_2 \rangle = (-1)^{|T||\psi_1|} \langle \overline{\psi_1}, T \psi_2 \rangle$$

for all $\psi_1, \psi_2 \in \mathcal{H}$, with ψ_1 homogeneous. The operator T is self- or skew-adjoint if $T^* = \pm T$.

Lemma 37.4.

- (1) *If T is odd self-adjoint, then $\text{spec } T \subset i^{1/2}\mathbb{R}$.*
- (2) *If T is odd skew-adjoint, then $\text{spec } T \subset i^{-1/2}\mathbb{R}$.*
- (3) *In either case a nonzero eigenvector is not homogeneous.*

Proof. If T is odd and either self- or skew-adjoint, then $T^2 = T \circ T$ is even skew-adjoint. Hence $\text{spec } T^2 \subset i\mathbb{R}$. Now $\text{spec } T^2$ is the square of $\text{spec } T$, so for (1) it suffices to show that if T is odd self-adjoint, then $\text{spec } T^2 \subset i\mathbb{R}^{\geq 0}$. Suppose that $-i\lambda$, $\lambda > 0$, is in the point spectrum of T^2 , i.e., there is a nonzero vector ψ such that $T^2\psi = -i\lambda$. Then the homogeneous components of ψ are eigenvectors, and one of them is nonzero. Suppose, then, that ψ is even. (The argument can be carried out for ψ odd with little change.) Then

$$(37.5) \quad \begin{aligned} 0 &< -i\langle T\psi, T\psi \rangle \\ &= -i\langle \psi, T^2\psi \rangle \\ &= -\lambda\|\psi\|^2 \end{aligned}$$

which is a contradiction. Suppose there is nonzero continuous spectrum that contains $-i\lambda_0$ for some $\lambda_0 > 0$. Let π_{T^2} denote the spectral measure, as in [Theorem 1.46](#). Choose $\delta < \lambda_0$, and let ψ be in the image of the projection $\pi_{T^2}(I)$ for $I = [-i(\lambda_0 - \delta), -i\lambda_0]$. As before, assume that ψ is an even vector. Then

$$(37.6) \quad \begin{aligned} 0 &< -i\langle T\psi, T\psi \rangle \\ &= -i\langle \psi, T^2\psi \rangle \\ &= -\lambda_0\|\psi\|^2 - i \left\langle \psi, \int_0^\delta \mu d\pi_{T^2}(-i(\lambda_0 - \mu))\psi \right\rangle. \end{aligned}$$

The integral is over μ . The absolute value of the last term is bounded above by $\delta\|\psi\|^2$, and so the entire last line is negative, again yielding a contradiction.

The argument for (2) is the similar; some signs change. We leave (3) to the reader. $\square$

A super Lie algebra was also defined in §9.2. It is a $\mathbb{Z}/2\mathbb{Z}$ graded vector space $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$ equipped with a skew-symmetric bilinear pairing

$$(37.7) \quad [-, -]: \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

that satisfies the Jacobi identity. As explained in [DM, §1.7] (“even rules”), computations in superalgebra are more accurately executed by working in the even part of $\mathbb{R}[\eta_1, \eta_2, \dots, \eta_N] \otimes \mathfrak{g}$, where η_i are odd and N is variable. More formally, a super Lie algebra is equivalent to a functor from a category of $\mathbb{Z}/2\mathbb{Z}$ -graded commutative algebras to the category of (ungraded) Lie algebras. For example, to work out the commutator of $\xi_1, \xi_2 \in \mathfrak{g}^1$, we compute

$$(37.8) \quad \begin{aligned} [\eta_1 \xi_1, \eta_2 \xi_2] &= -[\eta_2 \xi_2, \eta_1 \xi_1] \\ -\eta_1 \eta_2 [\xi_1, \xi_2] &= \eta_2 \eta_1 [\xi_2, \xi_1] \\ [\xi_1, \xi_2] &= [\xi_2, \xi_1] \end{aligned}$$

The first line is the skew-symmetry of the commutator in the Lie algebra $(\mathbb{R}[\eta_1, \eta_2] \otimes \mathfrak{g})^0$.

Lemma 37.9. *A super Lie algebra $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$ is equivalent to the following data: (1) a Lie algebra $\mathfrak{g}^0$, (2) a $\mathfrak{g}^0$ -module $\mathfrak{g}^1$, and (3) a $\mathfrak{g}^0$ -invariant symmetric bilinear pairing $\mathfrak{g}^1 \times \mathfrak{g}^1 \rightarrow \mathfrak{g}^0$. The pairing is required to satisfy*

$$(37.10) \quad [\xi, [\xi, \xi]] = 0, \quad \xi \in \mathfrak{g}^1.$$

Example 37.11. The free Lie algebra on a single odd generator D is

$$(37.12) \quad \mathfrak{g} = \mathbb{R}\langle [D, D] \rangle \oplus \mathbb{R}\langle D \rangle$$

By (37.10) the bracket of an odd element with an even element vanishes; the only nonzero bracket is of two nonzero odd elements.

A super Lie group can be defined as a group object in the category of supermanifolds, that is, if one develops a theory of supermanifolds. A more convenient approach for us is the following [DM, §3.8].

Definition 37.13. *A super Lie group consists of a Lie group G^0 , a super Lie algebra $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$, a G^0 -module structure on $\mathfrak{g}$, and a G^0 -module isomorphism $\text{Lie}(G^0) \xrightarrow{\cong} \mathfrak{g}^0$. Under this isomorphism the Lie bracket $\mathfrak{g}^0 \rightarrow \text{End } \mathfrak{g}^1$ is required to be the infinitesimal action of G^0 on $\mathfrak{g}^1$.*

This structure can be called a *Harish–Chandra pair*.

Example 37.14. Let M be a smooth manifold. There is a $\mathbb{Z}$ -graded Lie algebra

$$(37.15) \quad \begin{aligned} \mathfrak{g} &= \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \\ &= \mathcal{X}_M \oplus \mathcal{X}_M \oplus \mathbb{R}\langle d \rangle \end{aligned}$$

where $\mathcal{X}_M$ is the vector space of vector fields on M . It acts on the differential $\mathbb{Z}$ -graded algebra of differential forms $(\Omega_M^\bullet, d)$ by sending $\xi \in \mathfrak{g}^{-1}$ to contraction ι_ξ and $\xi \in \mathfrak{g}^0$ to the Lie derivative $\mathcal{L}_\xi$. The commutators in (37.15) express the fundamental equations of differential calculus. There is a corresponding $\mathbb{Z}$ -graded Lie group with $G^0 = \text{Diff } M$, the group of diffeomorphisms of M . (There are subtleties about the topology in this infinite dimensional example.) This $\mathbb{Z}$ -graded Lie algebra and $\mathbb{Z}$ -graded Lie group give rise to a $\mathbb{Z}/2\mathbb{Z}$ -graded Lie algebra and $\mathbb{Z}/2\mathbb{Z}$ -graded Lie group.

There is a super Lie group of unitary transformations of a super Hilbert space, and a corresponding notion of a unitary representation of an arbitrary super Lie group.

Definition 37.16. Let $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ be a super Hilbert space and let $(G^0, \mathfrak{g})$ be a super Lie group. A *unitary representation* of $(G^0, \mathfrak{g})$ on $\mathcal{H}$ is a pair of compatible homomorphisms $G^0 \rightarrow \text{U}(\mathcal{H})$ and $\mathfrak{g} \rightarrow \mathfrak{o}(\mathcal{H})$, where $\mathfrak{o}(\mathcal{H})$ is the super Lie algebra of skew-adjoint endomorphisms of $\mathcal{H}$. Crucially, we allow unbounded skew-adjoint linear maps.

To work with self-adjoint rather than skew-adjoint operators we make the following convention: $Q \in \mathfrak{g}$ is represented by the skew-adjoint operator $-i\hat{Q}$, where $\hat{Q}$ is self-adjoint. (In quantum mechanics we should write $Q \mapsto -(i/\hbar)\hat{Q}$, but we typically omit $\hbar$.) The sign choice is one of many interlocking sign choices; see [DF1].

37.3 Supersymmetric quantum mechanical systems

A super Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ gives rise to DOSO. (So too does a projective version, but for convenience we use a linear space.) The bosonic states $\mathcal{S}^0$ are formed from $\mathcal{H}^0$ in the usual way; for example, the space of pure bosonic states is the projective space $\mathbb{P}\mathcal{H}^0$. The fermionic states $\mathcal{S}^1$ are similarly formed from $\mathcal{H}^1$. The space of observables is $\mathcal{O} = \text{End } \mathcal{H} = \text{End}^0 \mathcal{H} \oplus \text{End}^1 \mathcal{H}$, graded as in Example 9.17(2). Measurement is via the spectral theorem, as in (1.28).

Motion (Definition 1.11) is induced from a unitary representation of $\mathbb{R}$ on $\mathcal{H}$. By (37.2) this is separately a unitary representation on $\mathcal{H}^0$ and on $\mathcal{H}^1$. The representation is generated by a (possibly unbounded) *even* self-adjoint operator H , the Hamiltonian. More generally, a theory has a Lie group $\mathcal{G}^0$ of symmetries that both fits into a group extension

$$(37.17) \quad 1 \longrightarrow K \longrightarrow \mathcal{G}^0 \longrightarrow \mathbb{R} \longrightarrow 1$$

and contains the group $\mathbb{R}$ of time translations as a normal subgroup; see (15.9). Therefore, $\mathcal{G}^0$ is the semidirect product

$$(37.18) \quad \mathcal{G}^0 = \mathbb{R} \ltimes K$$

The action of the group $\mathcal{G}^0$ on the affine time line $\mathbb{M}^1$ factors through $\mathbb{R}$, which acts by translations. The subgroup K acts trivially on $\mathbb{M}^1$; it is the group of internal symmetries. A quantum mechanical system is a unitary representation

$$(37.19) \quad \mathcal{G}^0 \longrightarrow \text{U}(\mathcal{H})$$

(There may be a larger group that maps to the two-component group of isometries of $\mathbb{M}^1$ that includes time-orientation reversing symmetries, and the latter usually act antiunitarily; recall [Lemma 11.3](#). This larger group is an extension that need not be a semidirect product.)

On the level of Lie algebras we have

$$(37.20) \quad \text{Lie}(\mathcal{G}^0) = \mathbb{R} \ltimes \text{Lie}(K).$$

Now we introduce supersymmetries.

Definition 37.21. An infinitesimal *odd symmetry* is an operator $\hat{Q} \in \text{End}^1 \mathcal{H}$ that intertwines the action of $\mathcal{G}^0$. A quantum mechanical system with odd symmetries is said to be *supersymmetric*.

As explained at the end of [§37.2](#), in our notation $\hat{Q}$ is odd *self-adjoint*.

Now we introduce the super Lie groups of interest in supersymmetric quantum mechanics. Let ¹¹⁴ $\mathbb{S}^*$ be a finite dimensional real inner product space with positive definite inner product

$$(37.22) \quad \Gamma: \mathbb{S}^* \times \mathbb{S}^* \longrightarrow \mathbb{R}$$

Define the super Lie algebra

$$(37.23) \quad \mathcal{L} = \mathbb{R}\langle P \rangle \oplus \mathbb{S}^*$$

in which the only nonzero bracket is

$$(37.24) \quad [Q, Q'] = -2\Gamma(Q, Q')P, \quad Q, Q' \in \mathbb{S}^*.$$

The even part of $\mathcal{L}$ acts on the affine time line $\mathbb{M}^1$ by infinitesimal translation, so $\mathcal{L}$ is an extension to a super Lie algebra of “translations”. Hence $Q \in \mathbb{S}^*$ is sometimes called an “odd translation”. Note that the orthogonal group $O(\mathbb{S}^*)$ acts on $\mathcal{L}$, trivially on $\mathcal{L}^0$, and the action preserves the Lie bracket [\(37.24\)](#).

A general super Lie group of symmetries in quantum mechanics has the following form.

Definition 37.25. A *quantum mechanical supersymmetry group* is a super Lie group $(\mathcal{G}^0, \mathfrak{g})$ in which $\mathcal{G}^0$ fits into a group extension [\(37.17\)](#), $\mathfrak{g}^1 = \mathbb{S}^*$ as above, and the Lie bracket on $\mathfrak{g}^1$ has $\mathbb{R}$ -component [\(37.24\)](#) relative to the splitting [\(37.20\)](#). A *supersymmetric quantum mechanical system* is given by a super Hilbert space and a unitary representation of $(\mathcal{G}^0, \mathfrak{g})$.

Let $K \subset \mathcal{G}^0$ be the group of internal symmetries, as in [\(37.17\)](#), and let $N \subset K$ be the kernel of the action $K \curvearrowright \mathbb{S}^*$. The quotient K/N is called the *R-symmetry* group. It acts effectively on $\mathbb{S}^*$ by orthogonal transformations, so can be identified with a subgroup of $O(\mathbb{S}^*)$. The Lie bracket on $\mathfrak{g}^1$ may have a component in the center of $\text{Lie}(K)$, as we will discuss in a future lecture.

¹¹⁴The choice of ‘ $\mathbb{S}^*$ ’ instead of ‘ $\mathbb{S}$ ’ is part of the larger story in [\[DF3\]](#). It looks obscure here.

Lecture 38: Supersymmetric quantum mechanics II

We continue our study of supersymmetric quantum mechanics. Recall [Definition 37.25](#), in which we define the general notion of a *quantum mechanical supersymmetry group*. The nature of the theory is controlled by the dimension of the vector space of odd translations, or supersymmetries.

We begin with an abstract study of the supersymmetry group whose odd translations have dimension 1. Then the Hamiltonian is the square of an odd self-adjoint operator, up to a factor. In this case, assuming that the Hamiltonian is a *Fredholm operator*, there is an *index*¹¹⁵ that measures the difference of dimensions of the homogeneous subspaces of the kernel of the Hamiltonian. We explain how this index is related to the partition function of the theory on a spin circle. In fact, there are two spin structures on the circle, and it is the *nonbounding*¹¹⁶ spin structure that computes the index.

Then we turn to supersymmetric extensions of the quantum mechanical system formed by a particle moving on a Riemannian manifold (§[6.1](#)). We focus on the case in which the dimension of the vector space of odd translations is 1 or 2. (In §[38.3](#) we point out general features of supersymmetric quantum mechanical systems with $s = 2$.) These are related to the Dirac operator and the de Rham complex, respectively. A variation in the latter case sets up the next lecture on supersymmetry and Morse theory. (A special case was analyzed in §[7.1](#).) We conclude with a brief resume of the cases with more supersymmetry, which relate to Kähler and hyperkähler manifolds.

38.1 Positivity and the index

Let

$$(38.1) \quad \mathfrak{g} = \mathbb{R}\langle P \rangle \oplus \mathbb{R}\langle Q \rangle$$

be the super Lie algebra whose only nonzero bracket is

$$(38.2) \quad [Q, Q] = -2P.$$

It is part of the super Lie group $\mathcal{G} = (\mathcal{G}^0, \mathfrak{g})$ with $\mathcal{G}^0 = \mathbb{R}$. Suppose given a unitary representation of $\mathcal{G}$ on a super Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$, i.e., a supersymmetric quantum mechanical system. Let $\hat{Q}$ be the odd self-adjoint operator such that Q is represented by $-i\hat{Q}$ on $\mathcal{H}$, and recall that the Hamiltonian is $H = -\hat{P}$, where $-i\hat{P}$ represents P on $\mathcal{H}$. Then ([38.2](#)) implies

$$(38.3) \quad H = -i\hat{Q}^2$$

and this in turn implies the positivity of the Hamiltonian.

¹¹⁵This index of a Fredholm operator is originally due to Fritz Noether [\[N\]](#). In this physics context it is often called the *Witten index*.

¹¹⁶That spin structure is sometimes called ‘trivial’, ‘periodic’, or ‘Ramond’, terms that require additional knowledge to decode. By contrast, the term ‘nonbounding’ is clear and unambiguous.

Proposition 38.4. *The Hamiltonian in a supersymmetric quantum mechanical system is nonnegative.*

Proof. The proposition follows immediately from Lemma 37.4(1). $\square$

According to Definition 5.7, the nonnegativity of the Hamiltonian implies that a supersymmetric quantum mechanical system has positive energy. Furthermore, it is *gapped* iff the Hamiltonian is a *Fredholm* operator.¹¹⁷ This is true iff the odd operator $\hat{Q}$ is Fredholm. Note that $\hat{Q}$ takes the form

$$(38.5) \quad \begin{pmatrix} 0 & \hat{Q}|_{\mathcal{H}^1} \\ \hat{Q}|_{\mathcal{H}^0} & 0 \end{pmatrix}$$

relative to the splitting $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$.

Definition 38.6. Assume that $\hat{Q}$ is Fredholm. Then the *index* of $\hat{Q}$ is

$$(38.7) \quad \begin{aligned} \text{index } \hat{Q} &= \dim \ker \hat{Q}|_{\mathcal{H}^0} - \dim \ker \hat{Q}|_{\mathcal{H}^1} \\ &= \dim_s(\ker \hat{Q}) \end{aligned}$$

This is the Fredholm index of the operator $\hat{Q}: \mathcal{H}^0 \rightarrow \mathcal{H}^1$.

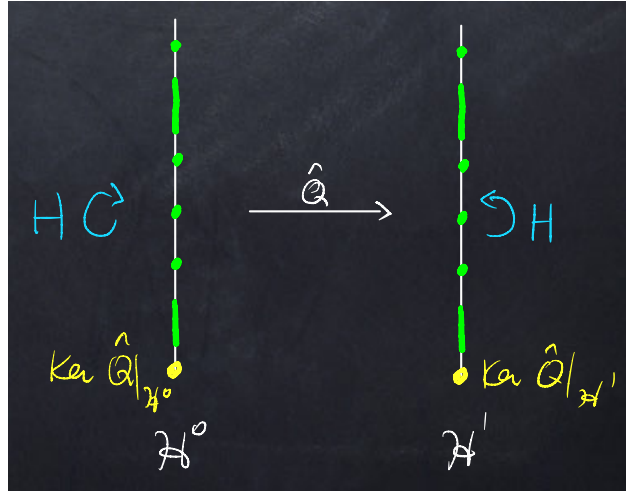


FIGURE 114. The spectrum of a supersymmetric Hamiltonian

The spectrum of the Hamiltonian H is depicted in Figure 114. The Fredholm assumption implies that there is a spectral gap above the kernel, and then both discrete and continuous spectrum are depicted above the gap. The map $\hat{Q}$ is an isomorphism of the spectral projection away from the kernel. (The isomorphism of discrete spectrum is Lemma 7.12.) If $f: \mathbb{R}^{>0} \rightarrow \mathbb{R}^{>0}$ is a continuous function, then we can form the self-adjoint operator $f(H)$. Suppose that the spectrum of H is discrete, that $f(0) = 1$, and that $f(H)$ is a trace class operator [RS, VI.6].

¹¹⁷A possibly unbounded operator is Fredholm if it has closed range and finite dimensional kernel and cokernel. A self-adjoint operator is Fredholm iff 0 lies in the discrete spectrum, that is, 0 is an isolated point of the spectrum and is an eigenvalue of finite multiplicity.

Lemma 38.8. *We have*

$$(38.9) \quad \text{index } \widehat{Q} = \text{Tr}_s(f(H)),$$

where Tr_s is the supertrace

$$(38.10) \quad \text{Tr}_s(T) = \text{Tr}(\epsilon T)$$

for $\epsilon: \mathcal{H} \rightarrow \mathcal{H}$ the grading operator

$$(38.11) \quad \epsilon = \begin{cases} +1, & \text{on } \mathcal{H}^0; \\ -1, & \text{on } \mathcal{H}^1. \end{cases}$$

The supertrace is the natural trace for operators on super vector spaces, as we show in the next section. Formulas like (38.9) are ubiquitous in the index theory literature; see [McS, AB2] for early examples. In quantum mechanics one typically considers the family of functions $f_\tau(\lambda) = e^{-\tau\lambda}$, $\tau \in \mathbb{R}^{>0}$. If the spectrum is discrete and enjoys sufficient growth, then $f_\tau(H)$ is trace class for all τ .

An important property of the Fredholm index is its *deformation invariance*, so long as one deforms through Fredholm operators. In terms of Figure 114 one can imagine the spectrum moving in a family, and as long as only discrete spectrum comes down to zero the Fredholm property is maintained. The Fredholm property is destroyed if continuous spectrum comes down to zero. (In the language of phase transitions, those of first-order preserve the index and those of higher order may change it.)

38.2 The categorical trace and spin circles

Let C be a symmetric monoidal category, and suppose $f: x \rightarrow x$ is a morphism of a dualizable object $x \in C$. Let $(x^\vee, c, e)$ be duality data for x (Definition 25.27).

Definition 38.12. The *categorical trace* of f is the composition

$$(38.13) \quad 1 \xrightarrow{c} x \otimes x^\vee \xrightarrow{f \otimes \text{id}} x \otimes x^\vee \xrightarrow{\sigma} x^\vee \otimes x \xrightarrow{e} 1$$

where σ is the symmetry. The *categorical dimension* of x is the categorical trace of id_x .

The trace lies in $\text{End}_C(1)$. If $C = (\text{Vect}, \otimes)$, then the categorical trace of an endomorphism of a finite dimensional vector space is the usual trace. For the category of super vector spaces, the Koszul sign rule leads to the following.

Lemma 38.14. *If V is a finite dimensional super vector space, and if $T: V \rightarrow V$ is an endomorphism, then the categorical trace of T is the super trace $\text{Tr}_s(T)$.*

The categorical trace of the identity map is the categorical dimension $\dim_s(V)$, which already occurred in (38.7).

Turning to a topological bordism category $\text{Bord}_{\langle n-1, n \rangle}(\mathcal{X}_n)$ (Definition 24.48) for manifolds with $\mathcal{X}_n$ -structure, the categorical dimension of an object Y has underlying manifold $S^1 \times Y$; see Figure 83. It is of interest to determine the tangential structure, which we do in one case.

Lemma 38.15. *In the spin bordism category, the categorical dimension of an object Y , i.e., of a closed spin manifold Y , is $S^1_{\text{nonbounding}} \times Y$.*

There are two spin structures on the circle, and $S^1_{\text{nonbounding}}$ is the spin circle that does not bound a spin disk.

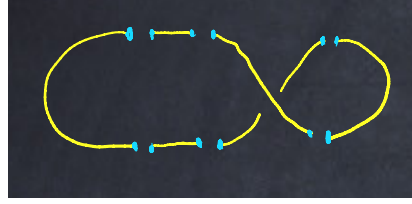


FIGURE 115. The categorical dimension of a spin point

Proof. It suffices to consider $n = 1$ and $Y = \text{pt}$ with its unique spin structure. The composition (38.13), which computes the categorical dimension of Y , is depicted in Figure 115. The spin frame bundle is a double cover of the depicted 1-manifold, and over an interval it is trivializable. Importantly, the spin manifold $Y^\vee$ is identified with the spin manifold Y ; the distinction in the bordism category is the coorientation of the normal bundle to the thickening, as in Figure 85. It follows that the composition depicted in Figure 115 has spin frame bundle the product double cover, which is the nonbounding spin structure. $\square$

Corollary 38.16. *Let $F: \text{Bord}_{\langle n-1, n \rangle}(\text{BSpin}_n) \rightarrow s\text{Vect}$ be a spin topological field theory, and suppose Y is a closed spin $(n-1)$ -manifold. Assume that F obeys spin-statistics in the sense that the image of the spin flip of Y is the grading operator of the $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $F(Y)$. Then*

$$(38.17) \quad \begin{aligned} \dim_s F(Y) &= F(S^1_{\text{nonbounding}} \times Y) \\ \dim F(Y) &= F(S^1_{\text{bounding}} \times Y) \end{aligned}$$

The second statement follows by observing that the bounding spin circle is the categorical trace of the *spin flip* automorphism of Y , which is the identity diffeomorphism covered by the action of the central element of the spin group on the spin frame bundle.

Now consider the Wick-rotated version of the supersymmetric quantum mechanical system in §38.1. As in §23.1, it is a linear representation of a bordism category of 0- and 1-manifolds, with background fields a spin structure and a Riemannian metric. The codomain is a category of $\mathbb{Z}/2\mathbb{Z}$ -graded topological vector spaces. Roughly speaking (see §23.7) a spin point maps to a Hilbert space $\mathcal{H}$, the spin flip of a spin point maps to the grading operator of $\mathcal{H}$, and an interval of length $\tau \in \mathbb{R}^{>0}$ —viewed as a bordism from a spin point to itself—maps to the contractive operator $e^{-\tau H/\hbar}$, where H is the Hamiltonian. Assume the operator $\hat{Q}$ is Fredholm. The partition function of a circle of length τ depends on the spin structure:

$$(38.18) \quad \begin{aligned} F(S^1_{\text{bounding}}(\tau)) &= \text{tr}(e^{-\tau H/\hbar}) \\ F(S^1_{\text{nonbounding}}(\tau)) &= \text{tr}_s(e^{-\tau H/\hbar}) = \text{index } \hat{Q} \end{aligned}$$

In particular, the partition function of a nonbounding spin circle is independent of the length τ .

Remark 38.19. This is the first example of a topological quantity embedded in a nontopological supersymmetric field theory. We will encounter several more.

38.3 Supersymmetric quantum mechanical systems with $s = 2$

We begin with some general remarks about the super Lie group in [Definition 37.25](#) for $\dim \mathbb{S}^* = 2$. The main point is to exploit the R -symmetry group, which here is an orthogonal group on a 2-dimensional inner product space. We use the notation of [§37.3](#).

Fix an orthonormal basis Q_1, Q_2 of $\mathbb{S}^*$. In this basis, the transformation $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a basis of the Lie algebra of the internal symmetry group K . It satisfies the commutation relations

$$(38.20) \quad \begin{aligned} [R, Q_1] &= Q_2 \\ [R, Q_2] &= -Q_1 \end{aligned}$$

In a unitary representation of this super Lie group, R is represented by a self-adjoint operator $\hat{R}$ and $e^{\theta \hat{R}}$, $0 \leq \theta \leq 2\pi$ is an action of SO_2 . This produces a $\mathbb{Z}$ -grading on the Hilbert space $\mathcal{H}$ underlying the representation. (More generally, for a projective representation we obtain a grading by a $\mathbb{Z}$ -torsor.) The $\mathbb{Z}$ -grading is given by the eigen-decomposition of $\hat{R}$.

Remark 38.21. The operator $e^{\pi \hat{R}}$ squares to the identity; its eigenspaces determine the quotient mod 2 grading of the $\mathbb{Z}$ -grading. In many systems this $\mathbb{Z}/2\mathbb{Z}$ -grading agree with the boson-fermion grading $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. For a general quantum mechanical supersymmetry group ([Definition 37.25](#)) we might ask that there be a distinguished element of order 2 in the internal symmetry group K that maps to the grading operator in a unitary representation. Then for the agreement of gradings just described, that element is required to be $e^{\pi R}$.

Introduce the *complex* basis of $\mathbb{S}^* \otimes \mathbb{C}$:

$$(38.22) \quad \begin{aligned} Q &= \frac{1}{\sqrt{2}}(Q_1 + iQ_2) \\ \bar{Q} &= \frac{1}{\sqrt{2}}(Q_1 - iQ_2) \end{aligned}$$

Then

$$(38.23) \quad \begin{aligned} [\hat{R}, \hat{Q}] &= \hat{Q} \\ [\hat{R}, \hat{\bar{Q}}] &= -\hat{\bar{Q}} \end{aligned}$$

This implies that $\hat{Q}$ raises the grading by 1 unit and $\hat{\bar{Q}}$ lowers the grading by 1 unit. As for the commutation relations in the abstract complex super Lie algebra, we have

$$(38.24) \quad \begin{aligned} [Q, Q] &= [\bar{Q}, \bar{Q}] = 0 \\ [Q, \bar{Q}] &= -2P \end{aligned}$$

Remark 38.25. The complex odd element Q of the super Lie algebra satisfies $Q^2 = \frac{1}{2}[Q, Q] = 0$. Quite generally, (necessarily) complex odd elements that square to zero are important in what follows. In a representation we also have $\hat{Q}^2 = 0$, i.e., $\hat{Q}$ acts as a differential. Its cohomology $H^\bullet(\hat{Q})$, a subquotient of $\mathcal{H}$, has special significance in theories. Relative to the grading, (38.23) implies that the differential $\hat{Q}$ has degree +1. Note that $\hat{\bar{Q}}$ is also a differential, but it has degree -1. It defines a chain complex and so a corresponding homology $H_\bullet(\hat{\bar{Q}})$.

So far we have discussed the structure if the internal symmetry group is $K = \text{SO}_2$. If K is enhanced to the complete orthogonal group O_2 , then the reflection

$$(38.26) \quad \begin{aligned} Q_1 &\longmapsto Q_1 \\ Q_2 &\longmapsto -Q_2 \end{aligned}$$

is included. This exchanges $Q \leftrightarrow \bar{Q}$ and also $R \leftrightarrow -R$. A theory with this enhanced symmetry has a $\mathbb{Z}$ -grading that is symmetric about 0. In addition, the implementation of the reflection (38.26) is an abstract Poincaré duality that induces an isomorphism of $H^\bullet(\hat{Q})$ and $H_\bullet(\hat{\bar{Q}})$.

38.4 Supersymmetric extensions of particle motion on a Riemannian manifold

One of the most basic quantum mechanical systems (§6.1) is a particle moving on a Riemannian manifold M . Here we describe the supersymmetric extensions with “ s supersymmetries”. More precisely, these systems are unitary representations of a super Lie group of the type constructed in §37.3 with $s = \dim \mathbb{S}^*$. We consider $s = 1, 2, 4, 8$. The systems with $s = 4$ and $s = 8$ require extra structure on M : a Kähler or hyperkähler structure, respectively.

Remark 38.27.

- (1) In general we only obtain a projective, or anomalous, theory. The anomaly theory is a 2-dimensional invertible field theory that in this case is topological. For $s = 1$ it is described in [F11, §5]; see also [FHT3].
- (2) There are also classical versions of these systems. The interplay between classical and quantum leads to important intuition and results.

38.4.1 $s = 1$. To avoid anomalies assume that $\dim M$ is even and that M is a spin manifold. (See [F11, FHT3] for the general case.) Then $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ is the Hilbert space of complex spinor fields on M and $\hat{Q} \in \mathcal{H}$ is the Dirac operator. Relative to the decomposition of $\mathcal{H}$ it takes the form

$$(38.28) \quad \hat{Q} = \begin{pmatrix} 0 & i^{1/2}D^* \\ i^{1/2}D & 0 \end{pmatrix},$$

where $D: \mathcal{H}^0 \rightarrow \mathcal{H}^1$ is the usual Dirac operator. The Hamiltonian (38.3) is the usual Dirac Laplacian

$$(38.29) \quad H = \begin{pmatrix} D^*D & \\ & DD^* \end{pmatrix}$$

If M is a closed manifold, then $\hat{Q}$ and H are Fredholm operators. In this case $\text{index } \hat{Q} = \text{index } D$ is the index of the Dirac operator. In the Wick-rotated theory this index appears as the partition function (38.18) of the theory on a nonbounding spin circle of any radius $\tau > 0$. The path integral formulation of the Wick-rotated theory leads to asymptotics as $\tau \rightarrow 0$, and from these asymptotics one recovers the Atiyah–Singer index formula [Ag, FW, W8].

38.4.2 $s = 2$. Let M be a smooth Riemannian manifold. The Hilbert space of the theory is the L^2 completion of the super vector space of complex differential forms

$$(38.30) \quad \Omega_M^{\text{even}}(\mathbb{C}) \oplus \Omega_M^{\text{odd}}(\mathbb{C})$$

The SO_2 -action of internal symmetries induces the usual $\mathbb{Z}$ -grading on differential forms. The odd differentials are

$$(38.31) \quad \begin{aligned} \hat{Q} &= i^{1/2} d \\ \hat{\bar{Q}} &= i^{1/2} d^* \end{aligned}$$

They are adjoint in the super sense (37.3). The Hamiltonian is the nonnegative Hodge laplacian

$$(38.32) \quad H = \frac{1}{2}(dd^* + d^*d)$$

To enhance the internal symmetry to an O_2 -action we must first shift the grading of differential forms to be symmetric about 0; if $n = \dim M$ is odd, then we obtain a grading by half-integers. We must also produce an involution

$$(38.33) \quad \Omega_M^q(\mathbb{C}) \longrightarrow \Omega_M^{n-q}(\mathbb{C})$$

for each q so that the involution exchanges d and d^* . In general, we only expect a projective representation of the symmetry group, so allow for central extensions. The Hodge $*$ operator does the job up to signs. We leave the reader to work out the details, and in particular to determine the conditions under which the symmetry can be implemented linearly (as opposed to projectively).

Suppose $h: M \rightarrow \mathbb{R}$ is a smooth real-valued function. Then we can conjugate $\hat{Q}$ by e^h to construct a new representation that has

$$(38.34) \quad \begin{aligned} \hat{Q} &= i^{1/2}(d + \epsilon(dh)) \\ \hat{\bar{Q}} &= i^{1/2}(d^* + \iota(dh)), \end{aligned}$$

where $\epsilon(\alpha)$ is exterior multiplication by the 1-form α and $\iota(\alpha)$ is interior product with α . Thus we obtain a family of supersymmetric QM systems parametrized by $\text{Map}(M, \mathbb{R})$. We take this up in the next lecture.

38.4.3 $s = 4$ and $s = 8$. We defer to [FKS] and the references therein for details and only give a very basic sketch here.

It is a theme of supersymmetric theories that more supersymmetry leads to geometric constraints. Here the constraints are on the “target” manifold M : theories with $s = 4$ require that M be Kähler and theories with $s = 8$ require that M be hyperkähler. Early references are [Z, AgF]; see [DF3, §3] for systematic constructions of these supersymmetric quantum mechanical models.

Here are some features of the $s = 4$ case, in which M is a Kähler manifold. First, there is a basis of the complexification $\mathbb{S}^* \otimes \mathbb{C}$ that maps, up to a multiple, to the first-order differential operators $\bar{\partial}, \bar{\partial}^*, \partial, \partial^*$ on complex differential forms. There is an internal symmetry group SO_4 —with appropriate modifications O_4 —and an $\mathrm{SO}_2 \times \mathrm{SO}_2$ subgroup gives the $(\mathbb{Z} \times \mathbb{Z})$ -grading into forms of type (p, q) . Furthermore, there is an SO_3 -subgroup that stabilizes the operator $\bar{\partial}$, and so it acts on $\bar{\partial}$ -cohomology. This leads to the Hodge identities and Lefschetz decomposition.

On a hyperkähler manifold there are similar, but more elaborate, decompositions; see [V]. These are apparent in the supersymmetric QM model [FKS].

Lecture 39: Supersymmetry and Morse theory

The title of this lecture is the title of Witten’s 1982 paper [W6]. This early work was a first application of supersymmetry to mathematics, and it attracted much attention from leading geometers, who wrote several works in response; a very small sample is [HeSj, Bot, For] as well as [Roe, Chapter 14]. The first and last contain rigorous proofs of Witten’s statements, the Bott paper is a delightful exposition of the subject, and Forman adapts Witten’s methods in a CW context. Witten quickly followed his initial work with an extension [W9] to the holomorphic context. We introduce Witten’s ideas in this lecture.

We begin with an exposition of some mathematical background, beginning with an algebraic version of the Morse inequalities and we also recall the Hodge theorem. Then we introduce Smale’s approach to Morse theory.

Turning to supersymmetric quantum mechanics, we introduce Witten’s theory, defined for a Riemannian manifold M equipped with a smooth real-valued function $h: M \rightarrow \mathbb{R}$. A special case ($M = \mathbb{R}$) was discussed in detail in §7.1, and it provides insight into the general case. We conclude with a summary of Witten’s arguments when h is Morse.

39.1 Mathematical background

We begin with an algebraic lemma. Fix $n \in \mathbb{Z}^{>0}$ and finite dimensional real or complex inner product spaces $C^0, \dots, C^n$. Assume given a differential $d: C^i \rightarrow C^{i+1}$, and let d^* be its adjoint:

$$(39.1) \quad 0 \longrightarrow C^0 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{d^*} \end{array} C^1 \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{d^*} \end{array} \cdots \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{d^*} \end{array} C^n \longrightarrow 0$$

Form the “Laplace operator” of degree 0:

$$(39.2) \quad \Delta = dd^* + d^*d.$$

Set

$$(39.3) \quad \begin{aligned} H^i &= \ker \Delta|_{C^i} \\ c_i &= \dim C^i \\ h_i &= \dim H^i \end{aligned}$$

Theorem 39.4.

(1) *For each i there is an orthogonal direct sum*

$$(39.5) \quad C^i = H^i \oplus \operatorname{im} d^* \oplus \operatorname{im} d.$$

(2) *There exist nonnegative integers $q_i \in \mathbb{Z}^{\geq 0}$ such that*

$$(39.6) \quad c_i = h_i + q_i + q_{i-1}.$$

Equivalently, for t an indeterminate we have

$$(39.7) \quad \sum c_i t^i = \sum h_i t^i + (1+t) \sum q_i t^i.$$

(3) *For each i we have the weak Morse inequality*

$$(39.8) \quad c_i \geq h_i.$$

(4) *For each positive integer k we have the strong Morse inequality*

$$(39.9) \quad \sum_{i=0}^k (-1)^{k-i} c_i \geq \sum_{i=0}^k (-1)^{k-i} h_i.$$

(5) *For the Euler number we have*

$$(39.10) \quad \sum_{i=0}^n (-1)^i c_i = \sum_{i=0}^n (-1)^i h_i.$$

(6) $\ker \Delta = \ker d \cap \ker d^*$.

(7) *The quotient map $\ker \Delta \rightarrow H^\bullet(C^\bullet, d)$ is an isomorphism.*

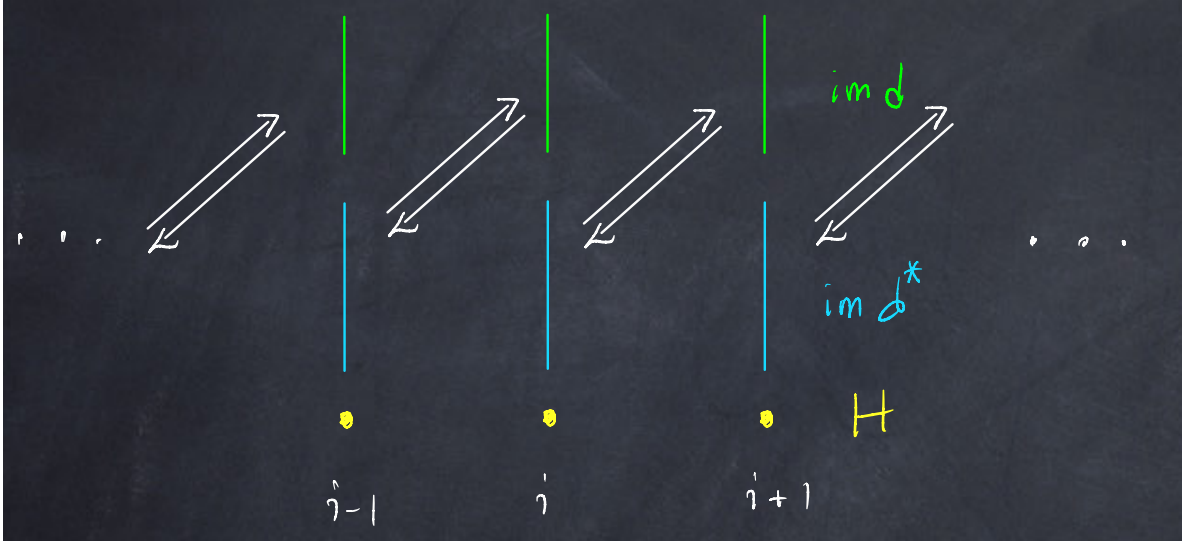


FIGURE 116. A depiction of the complexes (39.1) and (39.11)

We leave the pleasant proof to the reader. The situation is illustrated in the useful diagram [Figure 116](#).

There is an infinite dimensional version of [Theorem 39.4](#) for a compact Riemannian manifold (M, g) . In that case we take (39.1) to be the de Rham complex

$$(39.11) \quad 0 \longrightarrow \Omega_M^0 \xrightleftharpoons[d^*]{d} \Omega_M^1 \xrightleftharpoons[d^*]{d} \cdots \xrightleftharpoons[d^*]{d} \Omega_M^n \longrightarrow 0$$

together with the (formal) adjoint d^* of the Cartan differential d . Then (6) holds by a fairly easy argument. Assertions (1) and (7) also hold by the Hodge theorem [[War](#), Chapter 6]. (The rest of the assertions do not make sense if M has positive dimension, since the spaces of differential forms are infinite dimensional.) Whereas the proof of [Theorem 39.4](#) is finite dimensional linear algebra, the proof of the Hodge theorem uses the theory of elliptic differential operators.

Remark 39.12.

- (1) The de Rham theorem [[War](#), Chapter 5] asserts that the de Rham cohomology $H^\bullet(\Omega_M^\bullet, d) = H_{\text{dR}}^\bullet(M)$ is independent of the Riemannian metric g .
- (2) The space $\ker \Delta \subset \Omega_M^i$ changes as the metric g changes, but the dimension does not jump: the dimension is the Betti number h_i .
- (3) The *index* of the elliptic complex (39.11) in this context is the right hand side of (39.10), which is the Euler number of M . It is a deformation invariant.

In Witten's approach to Morse theory, the 1-parameter family of elliptic complexes (39.11) is extended to a 2-parameter family parametrized by a pair (g, h) of a Riemannian metric g and a smooth function $h: M \rightarrow \mathbb{R}$. When h is Morse, then we can derive a finite dimensional subcomplex (39.1) of the de Rham complex and use it to prove the basic results of Morse theory. This is in a sense dual to Smale's approach, which we rapidly review now, leaving the reader to enjoy the more leisurely treatment in [[Bot](#)].

We begin with a reminder.

Definition 39.13. Let M be a smooth manifold. A smooth function $h: M \rightarrow \mathbb{R}$ is a *Morse function* if for all $p \in M$ such that $dh_p = 0$, the second differential $d^2h_p: T_pM \times T_pM \rightarrow \mathbb{R}$ is a nondegenerate symmetric bilinear form.

It is the nondegeneracy that is the Morse condition. Recall that a point $p \in M$ with $dh_p = 0$ is a *critical point* of h . Let $\text{Crit}(h) \subset M$ denote the subset of critical points. The *index*

$$(39.14) \quad \lambda_p = \text{index}_p(h)$$

of a critical point p is the dimension of a maximal subspace on which d^2h_p is negative definite.

The following is the basic result of Morse theory on compact manifolds.

Theorem 39.15. Suppose M is a compact manifold and $h: M \rightarrow \mathbb{R}$ is Morse. For $i \in \mathbb{Z}^{\geq 0}$, let c_i be the number of critical points of h with index i . Then the Morse inequalities (39.8) and (39.9) hold, as does equation (39.10) for the Euler number of M .

The approach begun by Morse, and continued by Thom and Smale is to introduce a Riemannian metric g on M and to study integral curves of the negative gradient vector field $-\nabla h$. For M compact these integral curves exist for all time t , and they limit as $t \rightarrow -\infty$ to some $p \in \text{Crit}(h)$ and as $t \rightarrow \infty$ to some $p' \in \text{Crit}(h)$. We say the integral curve starts at p and ends at p' .

Proposition 39.16. The union W_p of integral curves that start at p is a cell of dimension λ_p . So there is a decomposition

$$(39.17) \quad M = \bigsqcup_{p \in \text{Crit}(h)} W_p.$$

Note that the closures of the cells W_p may not be nice, so that (39.17) is not necessarily a CW decomposition of M . There is a similar decomposition

$$(39.18) \quad M = \bigsqcup_{p' \in \text{Crit}(h)} W'_{p'}$$

of the ascending cells—the union of integral curves that end at p' .

Smale introduced the transversality condition

$$(39.19) \quad W_p \bar{\cap} W'_{p'}, \quad p, p' \in \text{Crit}(h)$$

This is a condition on the pair (g, h) of the Riemannian metric and the Morse function h . The intersection $W_p \cap W'_{p'}$ is the union of trajectories (integral curves) of the negative gradient flow that start at p and end at p' . In the transverse case, this intersection is a smooth submanifold of dimension $\lambda_p - \lambda_{p'}$.

Assuming the Smale transversality condition (39.19), introduce

$$(39.20) \quad C_\bullet = \mathbb{Z}\langle [p] : p \in \text{Crit}(h) \rangle,$$

where the degree of $[p]$ is the index λ_p . These is a $\mathbb{Z}^{\geq 0}$ -graded finitely generated free abelian group with the differential defined by

$$(39.21) \quad \begin{aligned} \partial: C_\lambda &\longrightarrow C_{\lambda-1} \\ [p] &\longmapsto n(p, q)[q] \end{aligned}$$

where $\lambda_p = \lambda$, $\lambda_q = \lambda - 1$, and $n(p, q)$ is the *signed* count of trajectories from p to q . (We refer to [KrMr, §2.2] for a treatment of the signs.)

Theorem 39.22. *The chain complex (39.20) computes the homology of M , i.e., there is an isomorphism $H_\bullet(C_\bullet) \cong H_\bullet(M; \mathbb{Z})$.*

We return to supersymmetric quantum mechanics and Witten's dual version of Smale's complex.

39.2 Witten's cochain complex

Let M be a smooth manifold equipped with a Riemannian metric g and a smooth function $h: M \rightarrow \mathbb{R}$. We construct a supersymmetric QM system of the type discussed in §38.3. Recalling the notation of that discussion, we have a complex basis $Q, \bar{Q}$ of the odd subspace of the super Lie algebra of symmetries, a basis P of the even subspace, and the commutation relations are

$$(39.23) \quad Q^2 = \bar{Q}^2 = 0, \quad [Q, \bar{Q}] = -2P.$$

The theory we construct is a perturbation (using h) of the system in §38.4.2. The Hilbert space of the theory is still the L^2 completion of the super vector space of complex differential forms

$$(39.24) \quad \Omega_M^{\text{even}}(\mathbb{C}) \oplus \Omega_M^{\text{odd}}(\mathbb{C})$$

The SO_2 -action of internal symmetries still induces the usual $\mathbb{Z}$ -grading on differential forms. However, the odd differentials now involve the function h :

$$(39.25) \quad \begin{aligned} \hat{Q} &= i^{1/2}(d + \epsilon(dh)) \\ \hat{\bar{Q}} &= i^{1/2}(d^* + \iota(dh)) \end{aligned}$$

where $\epsilon(dh)$ is exterior multiplication by the 1-form dh and $\iota(dh)$ is interior multiplication by dh , the latter defined using the Riemannian metric. We can also write

$$(39.26) \quad \hat{Q} = i^{1/2} e^{-h} \circ d \circ e^h$$

with a similar conjugation for the adjoint $\hat{\bar{Q}}$. The Hamiltonian is now

$$(39.27) \quad H = -\frac{i}{2}[\hat{Q}, \hat{\bar{Q}}] = \frac{1}{2}(dd^* + d^*d) + \frac{1}{2}|dh|^2 + T,$$

where T is the operator

$$(39.28) \quad T = \frac{\mathcal{L}_{\nabla h} + \mathcal{L}_{\nabla h}^*}{2}.$$

Here ∇h is the gradient vector field of h and $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ . The sum of this Lie derivative and its adjoint is an algebraic operator on differential forms, as we verify in the next section. The potential energy of the Hamiltonian is $\frac{1}{2}|dh|^2$, so the minima of potential energy are the critical points of h .

The formulas in (39.25) lead to Witten's deformation of the de Rham complex (39.11):

$$(39.29) \quad 0 \longrightarrow \Omega_M^0 \xrightleftharpoons[d^* + \iota(dh)]{d + \epsilon(dh)} \Omega_M^1 \xrightleftharpoons[d^* + \iota(dh)]{d + \epsilon(dh)} \cdots \xrightleftharpoons[d^* + \iota(dh)]{d + \epsilon(dh)} \Omega_M^n \longrightarrow 0$$

Example 39.30. An extended example of this supersymmetric QM system is discussed in §7.1, though without explicitly identifying the supersymmetry. The algebraic term T is so labeled in the expression (7.2) for the Hamiltonian. However, beware that in §7.1 we (deliberately) did not follow the sign conventions of superalgebra, so for example (7.4) should read

$$(39.31) \quad \begin{aligned} \hat{Q} &= i^{1/2} \left(+\frac{d}{dx} + h'(x) \right) \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \hat{\bar{Q}} &= i^{1/2} \left(-\frac{d}{dx} + h'(x) \right) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and the Hamiltonian $H = -\frac{i}{2}[\hat{Q}, \hat{\bar{Q}}]$ must be altered as well. Examples in that section include both Morse and non-Morse functions h . These examples provide much insight for the general case and for the arguments that follow.

Introduce a deformation parameter $s \in \mathbb{R}^{\geq 0}$. So (39.25) becomes

$$(39.32) \quad \begin{aligned} \hat{Q}_s &= i^{1/2} (d + s\epsilon(dh)) \\ \hat{\bar{Q}}_s &= i^{1/2} (d^* + s\iota(dh)) \end{aligned}$$

leading to the Hamiltonian

$$(39.33) \quad H_s = \frac{1}{2}(dd^* + d^*d) + \frac{1}{2}s^2|dh|^2 + sT$$

and to the deformed de Rham complex

$$(39.34) \quad 0 \longrightarrow \Omega_M^0 \xrightleftharpoons[d^* + s\iota(dh)]{d + s\epsilon(dh)} \Omega_M^1 \xrightleftharpoons[d^* + s\iota(dh)]{d + s\epsilon(dh)} \cdots \xrightleftharpoons[d^* + s\iota(dh)]{d + s\epsilon(dh)} \Omega_M^n \longrightarrow 0$$

The idea is to study the limit $s \rightarrow \infty$ in case h is a Morse function.

The local picture at a Morse critical point in dimension 1 can be gleaned from the analysis in §7.1.3. Observe in the first figure in Figure 16 that as $s \rightarrow \infty$ the positive spectrum is pushed to infinity while the kernel stays fixed. This widening spectral gap separates the shortrange behavior from the longrange topological features.¹¹⁸ For a bit more insight, replace (7.14) with the family of potentials

$$(39.35) \quad h_s(x) = \frac{1}{2}scx^2, \quad s \in \mathbb{R},$$

where $c \in \mathbb{R}^{\neq 0}$ is a constant. In the context of a critical point of a Morse function h on a 1-dimensional manifold, the constant c is the second derivative of h at the critical point. Observe two changes in Figure 16. First, the gap in the spectrum of the Hamiltonian H_s is proportional to $|sc|$. Second, and key for the general story, is that the kernel of H_s lies in $\mathcal{H}^0$ if $c > 0$ and in $\mathcal{H}^1$ if $c < 0$. In other words, the index of the critical point determines the parity of the kernel.

39.3 The algebraic operator T

Next, we analyze the operator (39.28). It is the sum of two first order differential operators, so ostensibly is also a first order differential operator, but in fact it is zeroth order, i.e., it is an algebraic operator on differential forms. One way to see this is to compute its leading (first order) symbol. The symbol of the Lie derivative operator $\mathcal{L}_\xi$ is the linear map¹¹⁹

$$(39.36) \quad \begin{aligned} T^*M &\longrightarrow \text{End}(\wedge^\bullet T^*M) \\ \alpha &\longmapsto \text{multiplication by } \iota(\xi)(\alpha) \end{aligned}$$

which is multiplication by the pairing of the vector ξ and the cotangent vector α . The symbol of the adjoint $\mathcal{L}_\xi$ is the negative of (39.36), and so the first order symbol of T vanishes.

We now compute a formula for T . The computation is made easier by knowing it is an algebraic operator. Let $\text{O}(M) \rightarrow M$ be the orthonormal frame bundle of the Riemannian manifold M , and assume $\dim M = n$. There are canonical horizontal vector fields $\partial_1, \dots, \partial_n$ on $\text{O}(M)$, the horizontal lifts (relative to the Levi-Civita connection) of the basis vectors in a frame. A differential form on M is a function $\omega: \text{O}(M) \rightarrow \wedge^\bullet(\mathbb{R}^n)^*$ that satisfies an O_n -equivariance condition. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$, and let $e^1, \dots, e^n$ be the dual basis of $(\mathbb{R}^n)^*$. Let ϵ^μ denote exterior multiplication by e^μ , and let ι^μ denote¹²⁰ interior multiplication by e_μ . Then

$$(39.37) \quad \begin{aligned} d &= \epsilon^\mu \partial_\mu \\ d^* &= -\iota^\mu \partial_\mu \end{aligned}$$

¹¹⁸We remark that similar limits—often termed *adiabatic limits*—apply fruitfully in other spectral analyses; see [W10, MM] for two of many examples.

¹¹⁹Choose a function f with $df = \alpha$ at a point, and compute the part of $\mathcal{L}_\xi(f\omega) = (d\iota(\xi) + \iota(\xi)d)(f\omega)$ that involves df .

¹²⁰It is really the interior product with e^μ using the metric, but since we use orthonormal bases it is the same.

Write the gradient vector field and differential of h as

$$(39.38) \quad \begin{aligned} \nabla h &= \sum_{\nu} \partial_{\nu} h e_{\nu} \\ dh &= \partial_{\nu} h e^{\nu} \end{aligned}$$

which is an O_n -equivariant function $O(M) \rightarrow \mathbb{R}^n$. Then compute on $O(M)$:

$$(39.39) \quad \begin{aligned} 2T &= \mathcal{L}_{\nabla h} + \mathcal{L}_{\nabla h}^* = (d\iota(\nabla h) + \iota(\nabla h)d) + (d^*\epsilon(dh) + \epsilon(dh)d) \\ &= (\epsilon^{\mu} \partial_{\mu} (\partial_{\nu} h \iota^{\nu}) + \partial_{\nu} h \iota^{\nu} \epsilon^{\mu} \partial_{\mu}) - (\iota^{\mu} \partial_{\mu} (\partial_{\nu} h \epsilon^{\nu}) + \partial_{\nu} h \epsilon^{\nu} \iota^{\mu} \partial_{\mu}) \\ &= \partial_{\mu} \partial_{\nu} h (\epsilon^{\mu} \iota^{\nu} - \iota^{\mu} \epsilon^{\nu}) \end{aligned}$$

Passage to the last line is simplified by knowing in advance that the answer is algebraic, so all terms with derivative operators cancel, but of course that can be verified directly. All the magic is in (39.39). The coefficient in the last line is a second derivative of h that in general uses the Riemannian structure in the form of the Levi-Civita connection, though at a critical point is independent of it. At any point $\partial_{\mu} \partial_{\nu} h$ is symmetric in μ, ν , so we can diagonalize it. The following easy computation is crucial in the argument.

Lemma 39.40. *Let $e^{\mu_1} \wedge \cdots \wedge e^{\mu_k}$ be a monomial in $\bigwedge^{\bullet}(\mathbb{R}^n)^*$. Then*

$$(39.41) \quad (\epsilon^{\mu} \iota^{\mu} - \iota^{\mu} \epsilon^{\mu})(e^{\mu_1} \wedge \cdots \wedge e^{\mu_k}) = \pm e^{\mu_1} \wedge \cdots \wedge e^{\mu_k},$$

where the sign is $+$ if $\mu_i = \mu$ for some $i \in \{1, \dots, k\}$ and is $-$ otherwise.

We leave the proof to the reader.

39.4 The model case

Take $M = \mathbb{E}^n$ to be standard Euclidean space, and suppose $h: \mathbb{E}^n \rightarrow \mathbb{R}$ is a nondegenerate quadratic function. Then there exist an orthonormal coordinate system $x^1, \dots, x^n$ and *nonzero* real numbers $c_1, \dots, c_n$ such that

$$(39.42) \quad h = \sum_i \frac{c_i}{2} (x^i)^2.$$

Now consider a differential form

$$(39.43) \quad \omega = e^{-q(x)/2} \omega_0,$$

where

$$(39.44) \quad q(x) = \sum_i |c_i| (x^i)^2$$

and ω_0 is a constant differential form on $\mathbb{E}^n$. Then from (39.27) we compute

$$(39.45) \quad H\omega = \left(\sum_i \frac{|c_i|}{2} \right) e^{-q(x)/2} \omega_0 + e^{-q(x)} T\omega_0$$

Now from (39.39) and Lemma 39.40, for a monomial $\omega_0 = dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}$ we have

$$(39.46) \quad T\omega_0 = \left(\sum_i \pm \frac{c_i}{2} \right) \omega_0,$$

where the sign is $+$ if $i \in \{\mu_1, \dots, \mu_k\}$ and is $-$ otherwise. It follows that $H\omega = 0$ iff

$$(39.47) \quad \omega_0 = \bigwedge_{i:c_i < 0} dx^i.$$

In particular, the degree of ω_0 is the number of negative coefficients in (39.42), which is the index of $0 \in \mathbb{E}^n$ as a critical point of h .

39.5 Morse functions on compact manifolds

Now assume that M is a compact n -dimensional manifold, equipped with a Riemannian metric g and a Morse function $h: M \rightarrow \mathbb{R}$. Recall the deformed Hamiltonian (39.33).

Claim 39.48.

- (1) As $s \rightarrow \infty$ the spectrum of H_s splits (39.34) into the orthogonal direct sum of a finite dimensional subcomplex (with differential $d_s = d + s\epsilon(dh)$)

$$(39.49) \quad C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^n$$

and an infinite dimensional subcomplex. The eigenvalues of H_s on (39.49) are bounded above by Ce^{-cs} for some $C, c > 0$.

- (2) As $s \rightarrow \infty$ the eigenfunctions of H_s decay exponentially away from $\text{Crit}(h)$. There exists a basis of C^i labeled by critical points $p \in \text{Crit}(h)$ with index $\lambda_p = i$. The support of the corresponding eigenfunction is exponentially small away from p .

Witten's argument for Claim 39.48 relies on instanton analysis using the semiclassical path integral, so essentially tunneling (§6.6) in quantum mechanics. We saw an example of this in §7.1.5. There are proofs of Claim 39.48 in [HeSj, Roe].

Lecture 40: Supersymmetry in higher dimensions

Clifford algebras and spin groups are essential background for the general study of supersymmetry, so we spend most of this lecture with a review of basic properties. (We previously gave an introduction to Clifford algebras in §9 and briefly to spin groups in §12.4.) The classic reference [ABS] remains invaluable; more details and proofs may be found in many references, such as [LM, De1]. We highlight some special properties in Lorentz signature that underlie the relativistic study of both fermions and supersymmetry. At the end of the lecture we introduce the notion of a *super relativistic symmetry type*, a generalization of Definition 15.13.

Throughout this lecture, fix $p, q \in \mathbb{Z}^{\geq 0}$ and set $n = p + q$.

40.1 Clifford algebras and spin groups

Let $\mathbb{R}^{p,q}$ denote the vector space $\mathbb{R}^n$ with standard basis $e_1, \dots, e_n$. Set

$$(40.1) \quad \theta_i = \begin{cases} +1, & 1 \leq i \leq p; \\ -1, & p+1 \leq i \leq n. \end{cases}$$

and define the nondegenerate real-valued symmetric bilinear form

$$(40.2) \quad \langle e_i, e_j \rangle = \theta_i \delta_{ij}, \quad 1 \leq i, j \leq n.$$

The symmetry group of this structure is the orthogonal group $O_{p,q}$. The maximal compact subgroup is $O_p \times O_q \subset O_{p,q}$, and the inclusion is a homotopy equivalence. Therefore, $\pi_0 O_{p,q} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if both p and q are nonzero, and if one of them vanishes then $\pi_0 O_{p,q} \cong \mathbb{Z}/2\mathbb{Z}$. The fundamental group of $O_{p,q}$ can also be read off from that of $O_p \times O_q$. The orthogonal group is a subgroup of the group of units of an associative unital algebra, the algebra of $n \times n$ real matrices:

$$(40.3) \quad O_{p,q} \subset M_n \mathbb{R}.$$

The Clifford algebra plays the same role for the spin group as does the matrix algebra for the orthogonal group. Note carefully our sign convention.

Definition 40.4. The *Clifford algebra* $Cl_{q,p}$ is the real associative unital algebra generated by $e_1, \dots, e_n$ subject to the relation

$$(40.5) \quad e_i e_j + e_j e_i = -2\langle e_i, e_j \rangle = -2\theta_i \delta_{ij}, \quad 1 \leq i, j \leq n.$$

Set $Cl_0 = \mathbb{R}$ and for $n \in \mathbb{Z}^{>0}$ we use the notation

$$(40.6) \quad \begin{aligned} Cl_{+n} &= Cl_{n,0} \\ Cl_{-n} &= Cl_{0,n} \end{aligned}$$

The Clifford algebra $\mathcal{C}\ell_{q,p}$ has an increasing filtration: an element is in the k^{th} filtered piece if it can be expressed as a linear combination of products of $\leq k$ generators. There is a compatible $\mathbb{Z}/2\mathbb{Z}$ -grading:

$$(40.7) \quad \mathcal{C}\ell_{q,p} = \mathcal{C}\ell_{q,p}^0 \oplus \mathcal{C}\ell_{q,p}^1,$$

and the even subspace $\mathcal{C}\ell_{p,q}^0$ is an ungraded algebra.

The Clifford algebra has two distinguished linear subspaces:

$$(40.8) \quad \begin{aligned} \mathbb{R}^n &\subset \mathcal{C}\ell_{q,p} & \mathbb{R}\langle e_i : 1 \leq i \leq n \rangle \\ \mathfrak{o}_{p,q} &\subset \mathcal{C}\ell_{q,p} & \mathbb{R}\langle \frac{1}{2}e_i e_j : i < j \rangle \end{aligned}$$

The subspace $\mathfrak{o}_{p,q}$ obtains a Lie algebra structure by the commutator in $\mathcal{C}\ell_{q,p}$. It is isomorphic to the Lie algebra of $\mathcal{O}_{p,q}$, embedded in the matrix algebra by (40.3), via the correspondence ($i < j$)

$$(40.9) \quad \frac{1}{2}e_i e_j \longleftrightarrow \begin{matrix} & i & j \\ i & \begin{pmatrix} & -1 \\ +1 & \end{pmatrix} \\ j & \end{matrix}$$

The orthogonal algebra $\mathfrak{o}_{p,q}$ acts on $\mathbb{R}^{p,q}$ by the commutator in the Clifford algebra.

Definition 40.10. The *spin group* is the group generated by the exponential of the orthogonal algebra and the group $\mu_2 = \{\pm 1\}$ in the even Clifford algebra:

$$(40.11) \quad \text{Spin}_{p,q} = \mu_2 \cdot \exp(\mathfrak{o}_{p,q}) \subset \mathcal{C}\ell_{q,p}^0.$$

Note that the spin group comes to us as embedded in the units of the Clifford algebra. Except in low dimensions ($p, q \leq 1$), there is an inclusion $\mu_2 \subset \exp(\mathfrak{o}_{p,q})$ —see (40.16) below for $\theta = 2\pi$ —and in that case the spin group is the exponential of a Lie algebra, which implies that it is a *connected* Lie group. In low dimensions it is not connected; see (40.23) below. The action

$$(40.12) \quad \begin{aligned} \text{Spin}_{p,q} &\longrightarrow \text{Aut}(\mathbb{R}^{p,q}) \\ g &\longmapsto (\xi \mapsto g\xi g^{-1}) \end{aligned}$$

defines a homomorphism

$$(40.13) \quad \text{Spin}_{p,q} \longrightarrow \mathcal{O}_{p,q}$$

which is a double cover onto the identity component.

Definition 40.14. The element $-1 \in \text{Spin}_{p,q} \subset \text{Cl}_{q,p}$ is the *central element* of the spin group.

It is the non-identity element in the kernel of the double cover (40.13).

Example 40.15. In $\text{Cl}_{\pm n}$, $n \geq 2$, we have

$$(40.16) \quad \exp\left(\frac{\theta}{2}e_1e_2\right) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}e_1e_2.$$

This element of the spin group acts on $\mathbb{R}^n$ as rotation through angle θ in the (e_1, e_2) -plane. The infinitesimal rotation that corresponds to $\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$ on the (e_1, e_2) -plane is commutator with $\frac{1}{2}e_1e_2$:

$$(40.17) \quad \begin{aligned} \left[\frac{1}{2}e_1e_2, e_1\right] &= -e_2 \\ \left[\frac{1}{2}e_1e_2, e_2\right] &= e_1 \end{aligned}$$

(Compare with (40.9).)

40.2 Structure theorems; spin representations

An algebra A determines a linear category ${}_A\text{Mod}$, its category of left A -modules. One is often more interested in that category than in the actual algebra, and different algebras may give equivalent categories of modules. For example, the algebra A of $(N \times N)$ -matrices has a left module of dimension N —column vectors—and any other module is isomorphic to a vector space tensor this module. In other words, the category of left A -modules is equivalent to the category of vector spaces, which is the category of left modules over the ground field. This idea is encoded in the Morita 2-category of algebras (Example 29.9). Here we give a concrete equivalence. We work with super algebras (§9.2).

Definition 40.18.

- (1) A super algebra A is *Morita trivial* if $A \cong \text{End}(W)$ for some super vector space W .
- (2) Super algebras A, A' are *Morita equivalent*, written $A \simeq_{\text{Mor}} A'$, if there exist super vector spaces W, W' and an isomorphism

$$(40.19) \quad A \otimes \text{End}(W) \cong A' \otimes \text{End}(W').$$

For example, the *complex* Clifford algebra $\text{Cl}_2^{\mathbb{C}} = \text{Cl}_{+2} \otimes \mathbb{C} \cong \text{Cl}_{-2} \otimes \mathbb{C}$ is Morita trivial: the map

$$(40.20) \quad e_1 \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

determines an isomorphism $\text{Cl}_2^{\mathbb{C}} \xrightarrow{\cong} \text{End}(\mathbb{C}^{1|1})$.

The following is a basic structure theorem for Clifford algebras over $\mathbb{R}$ and $\mathbb{C}$.

Theorem 40.21.

- (1) $\text{Cl}_n^{\mathbb{C}}$ is Morita trivial iff $n \in 2\mathbb{Z}$.
- (2) $\text{Cl}_{p,p}$ is Morita trivial for all $p \in \mathbb{Z}^{>0}$.
- (3) $\text{Cl}_{p,q} \simeq_{\text{Mor}} \text{Cl}_{+(p-q)}$.
- (4) Cl_n is Morita trivial iff $n \in 8\mathbb{Z}$.

We refer to [De1] for a proof of this and many more facts about Clifford algebras.

A left module over the matrix algebra $M_n\mathbb{R}$ restricts to a representation of the orthogonal group $O_{p,q}$, but “most” representations of $O_{p,q}$ do not extend to $M_n\mathbb{R}$ -modules. We say a representation is *standard* if it does. There is a unique (up to isomorphism) irreducible $M_n\mathbb{R}$ -module of dimension n , and any module is a direct sum of irreducibles. The situation is similar for representations of the spin group.

Definition 40.22. A representation $\mathbb{S}$ of $\text{Spin}_{p,q}$ is a *spin* (or *spinor* or *spinorial*) *representation* if it extends to a $\text{Cl}_{q,p}^0$ -module.

In particular, $\epsilon \mapsto -1$ in a spin representation, but that is hardly sufficient. Observe that $\mathbb{S}$ is an ungraded vector space. There may be one or two irreducible real spin representations up to isomorphism, depending on the Morita class of $\text{Cl}_{q,p}^0$, which by Theorem 40.21 only depends on $q - p \pmod{8}$. For complex spin representations, there are two if $n \equiv 0 \pmod{2}$ and there is a unique one if $n \equiv 1 \pmod{2}$.

40.3 Lorentz signature

In quantum field theory we encounter compact spin groups— $(p, q) = (n, 0)$ —and also Lorentz signature spin groups— $(p, q) = (1, n - 1)$. There are special properties of spin representations in Lorentz signature that we summarize here.

First, recall the special isomorphisms (12.24) in low dimensions:

n	Spin_n	$\text{Spin}_{1,n-1}$
1	μ_2	μ_2
2	$\mathbb{T}$	$\mathbb{R}^{>0} \times \mu_2$
3	SU_2	$\text{SL}_2 \mathbb{R}$
4	$\text{SU}_2 \times \text{SU}_2$	$\text{SL}_2 \mathbb{C}$
5	Sp_2	$\text{Sp}_{1,1}$
6	SU_4	$\text{SL}_2 \mathbb{H}$

These are used throughout the study of low dimensional supersymmetry.

Now specialize to Lorentz signature $(p, q) = (1, n - 1)$ and real spinorial representations. Recall that $P \subset \mathbb{R}^{1,n-1}$ is the cone of forward timelike vectors (Definition 12.17).

Theorem 40.24.

- (1) If $n \not\equiv 2 \pmod{4}$ then there is a unique irreducible spin representation S , up to isomorphism. If $n \equiv 2 \pmod{4}$ then there are two nonisomorphic irreducible spin representations S_1, S_2 , up to isomorphism.

- (2) Let $\mathbb{S}$ be a spinor representation of $\text{Spin}_{1,n-1}$. Then there exist real vector spaces W ($n \not\equiv 2 \pmod{4}$) or W_1, W_2 ($n \equiv 2 \pmod{4}$) and an isomorphism

$$(40.25) \quad \mathbb{S} \cong \begin{cases} S \otimes W, & n \not\equiv 2 \pmod{4}; \\ S_1 \otimes W_1 \oplus S_2 \otimes W_2, & n \equiv 2 \pmod{4}. \end{cases}$$

- (3) There is a contractible space of $\text{Spin}_{1,n-1}$ -equivariant symmetric pairings

$$(40.26) \quad \Gamma: \mathbb{S}^* \times \mathbb{S}^* \longrightarrow \mathbb{R}^{1,n-1}$$

that are positive definite in the sense that $\Gamma(Q, Q) \in \overline{P}$ for all $Q \in \mathbb{S}^*$.

- (4) For each choice of Γ , there is a compatible super $\text{Cl}_{n-1,1}$ -structure on $\mathbb{S} \oplus \mathbb{S}^*$.

Property (3) is a distinguishing feature of Lorentz signature: in no other signature does the vector representation appear in the symmetric square of *every* real spin representation. We defer to [De1, Chapter 6] for proofs, and [FH1, §9.2.4] for more information and an application to free spinor fields.

40.4 Super relativistic symmetry types

Recall the notion of an n -dimensional *relativistic symmetry type* in Definition 15.13: it is a Lie group G_n equipped with a homomorphism

$$(40.27) \quad \lambda_n: G_n \rightarrow \text{O}_{1,n-1}^\uparrow$$

whose image contains the identity component $\text{SO}_{1,n-1}^\uparrow \subset \text{O}_{1,n-1}^\uparrow$. There is an induced affine symmetry type $\tilde{\lambda}_n$, defined by pullback in the commutative diagram (15.14), reproduced here:

$$(40.28) \quad \begin{array}{ccccccc} & & & 1 & & 1 & \\ & & & \downarrow & & \downarrow & \\ & & & K & \xlongequal{\quad} & K & \\ & & & \downarrow & & \downarrow & \\ 1 & \longrightarrow & \mathbb{R}^{1,n-1} & \longrightarrow & \mathcal{G}_n & \longrightarrow & G_n \longrightarrow 1 \\ & & \parallel & & \downarrow \tilde{\lambda}_n & & \downarrow \lambda_n \\ 1 & \longrightarrow & \mathbb{R}^{1,n-1} & \longrightarrow & \mathcal{I}_{1,n-1}^\uparrow & \longrightarrow & \text{O}_{1,n-1}^\uparrow \longrightarrow 1 \end{array}$$

From the structure Theorem 21.27, for $n \geq 3$, we deduce a canonical homomorphism

$$(40.29) \quad \text{Spin}_{1,n-1} \longrightarrow G_n$$

such that the composition $\text{Spin}_{1,n-1} \rightarrow G_n \rightarrow \text{O}_{1,n-1}^\uparrow$ is a double cover of the identity component. The internal symmetry group is $K = \ker \lambda_n$, and there is a splitting of Lie algebras

$$(40.30) \quad \text{Lie } G_n \cong \mathfrak{o}_{1,n-1} \oplus \text{Lie } K.$$

Let $Z(K) \subset K$ be the center. We use all of this notation below in [Definition 40.31](#).

We now generalize relativistic symmetry types to include supersymmetries, as we did in [Definition 37.25](#) for the case of super quantum mechanics ($n=1$). A basic theorem of Haag–Lopusansky–Sohnius [\[HLS\]](#) classifies possible super Lie groups of symmetries of quantum mechanical systems. It is a generalization of the Coleman–Mandula theorem [\[CM\]](#), based on similar arguments. Our [Theorem 21.27](#) replaces the Coleman–Mandula theorem, and it is desirable to have an analog based on Wick rotation in the super case. Here, instead, we encapsulate the result in the following definition, in which $\mathbb{S}^*$ is a finite dimensional real vector space.

Definition 40.31. A *super relativistic symmetry type* is a relativistic symmetry type $\lambda: G_n \rightarrow \text{O}_{1,n-1}^\uparrow$ and a super Lie group $(\mathcal{G}_n, \text{Lie } \mathcal{G}_n \oplus \mathbb{S}^*)$ such that

- (i) the action of $\mathcal{G}_n$ on $\mathbb{S}^*$ factors through the quotient $\mathcal{G}_n \rightarrow G_n$ and the restriction of this action along $\text{Spin}_{1,n-1} \rightarrow G_n$ is a spin representation; and
- (ii) the Lie bracket

$$(40.32) \quad \mathbb{S}^* \times \mathbb{S}^* \longrightarrow \text{Lie } \mathcal{G}_n$$

factors through the inclusion

$$(40.33) \quad \text{Lie } Z \oplus \text{Lie } \mathbb{R}^{1,n-1} \hookrightarrow \text{Lie } \mathcal{G}_n,$$

where $Z \subset Z(K)$ is the subgroup of central elements that act trivially on $\mathbb{S}^*$. Relative to this direct sum, write

$$(40.34) \quad [Q_1, Q_2] = 2\zeta(Q_1, Q_2) - 2\Gamma(Q_1, Q_2), \quad Q_1, Q_2 \in \mathbb{S}^*.$$

Then the $\text{Lie } \mathbb{R}^{1,n-1}$ -component satisfies the following positivity condition for all $Q \in \mathbb{S}^*$:

$$(40.35) \quad \begin{aligned} \Gamma(Q, Q) &\in \overline{P} \\ \Gamma(Q, Q) &= 0 \quad \text{only for } Q = 0 \end{aligned}$$

We give examples in the next lecture. For now we simply make a few comments.

Remark 40.36.

- (1) The super Lie algebra $\text{Lie } \mathcal{G}_n \oplus \mathbb{S}^*$ has a subalgebra $\text{Lie } \mathbb{R}^{1,n-1} \oplus \mathbb{S}^*$ of *super translations* in which the only nonzero Lie bracket is

$$(40.37) \quad [Q_1, Q_2] = -2\Gamma(Q_1, Q_2), \quad Q_1, Q_2 \in \mathbb{S}^*.$$

- (2) Just as affine symmetry groups act on affine space, there is an affine space on which the super Lie group $(\mathcal{G}_n, \text{Lie } \mathcal{G}_n \oplus \mathbb{S}^*)$ acts: super Minkowski spacetime. It is a supermanifold. It is very useful in the study of supersymmetric field theories, particularly in low dimensions. We refer to [DM] for supermanifolds and to [DF3] for super Minkowski spacetime and the action of the super Lie group.
- (3) Define $N = \ker(K \curvearrowright \mathbb{S}^*)$ to be the kernel of the action of internal symmetries on $\mathbb{S}^*$. The quotient group $R = K/N$ acts effectively on $\mathbb{S}^*$. It is called the *R-symmetry group*.
- (4) Observe that $Z = Z(K) \cap N \subset K$. Also $\text{Lie } Z$ is central in the super Lie algebra $\text{Lie } \mathcal{G}_n \oplus \mathbb{S}^*$. (Recall the splitting Theorem 21.27(2).) Elements of $\text{Lie } Z$ are called *central charges*.
- (5) Physicists often use coded language to specify a super relativistic symmetry type. For example, ‘ $N = 2$ supersymmetry’ indicates that the representation $\text{Spin}_{1,n-1} \curvearrowright \mathbb{S}^*$ is twice the minimal real spinor representation. In dimensions $2 \pmod{4}$ there are two inequivalent irreducible real spinor representations, so phrases such as ‘ $N = 0|2$ supersymmetry’ are deployed to specify this representation. Nonetheless, often this language does not unambiguously describe the super relativistic symmetry type, so we advocate spelling out explicitly this important defining structure of a theory.

Lecture 41: More higher dimensional supersymmetry

At the end of Lecture 40 we gave a general definition (Definition 40.31) of a supersymmetric symmetry type. In the first half of this lecture we give some examples. The most basic example is based on a *super Poincaré group*, which is a super Lie group extension of an ordinary Poincaré group. Then we turn to specific examples to see how larger symmetry groups arise. In particular, we encounter nonzero central charges and a nontrivial *R*-symmetry group.

In the second part of this lecture we generalize the representation theory of relativistic symmetry types (Lecture 17) to supersymmetric symmetry types. We see that irreducible representations are *multiplets* that consist of a finite set of (nonsupersymmetric) particles. In an example with central charges we illustrate the *BPS bound* and *BPS representations*.

Our emphasis here is on general structure, focused on consequences of symmetry. In later lectures we describe some examples. We refer to [DF3] for much more about supersymmetry in general and in various dimensions.

41.1 Super Poincaré groups

We resume the notation in §40.4. Recall (Definition 13.21) that the Poincaré group is associated to the relativistic symmetry type $\lambda: \text{Spin}_{1,n-1} \rightarrow \text{O}_{1,n-1}^\uparrow$. The induced Lie group $\mathcal{G}_n = \mathcal{P}_n$ is then the Poincaré group.

Definition 41.1. Let $\text{Spin}_{1,n-1} \curvearrowright \mathbb{S}^*$ be a real spinor representation (Definition 40.22). The *super Poincaré group* $\mathcal{P}_n|_{\mathbb{S}^*}$ is the super Lie group $(\mathcal{P}_n, \text{Lie } \mathcal{P}_n \oplus \mathbb{S}^*)$.

The Lie algebra of the Poincaré group is $\text{Lie } \mathcal{P}_n \cong \mathfrak{o}_{1,n-1} \oplus \mathbb{R}^{1,n-1}$. The Lie bracket $\mathbb{S}^* \times \mathbb{S}^* \rightarrow \mathbb{R}^{1,n-1}$ is a contractible choice ([Theorem 40.24\(3\)](#)), which is why we have not made it explicit. The internal symmetry group K is the cyclic group μ_2 ; its Lie algebra is zero and so there are no central charges. The R -symmetry group is μ_2 ; the nonidentity element acts by -1 on $\mathbb{S}^*$.

41.2 Super relativistic symmetry types with $\dim \mathbb{S}^* = 2$

The minimal dimension of a real spinor representation of $\text{Spin}_{1,n-1}$ grows with n . One can see from ([40.23](#)) that 2-dimensional real spinor representations only exist for $n \leq 3$. Hence we begin with $n = 3$.

Recall the special isomorphism $\text{Spin}_{1,2} \cong \text{SL}_2\mathbb{R}$. Under that isomorphism the minimal spinor representation is the standard representation $\mathbb{R}^2$ of $\text{SL}_2\mathbb{R}$. For illustrative purposes—and despite the title of this section—set

$$(41.2) \quad \mathbb{S}^* = U \otimes \mathbb{R}^2$$

for U a finite dimensional real inner product space. Define the relativistic symmetry type

$$(41.3) \quad G_3 := \text{Spin}_{1,2} \times \text{O}(U) \longrightarrow \text{O}_{1,2}^\uparrow,$$

and the supersymmetric symmetry type by the linear action $G_3 \curvearrowright \mathbb{S}^*$; the factor $\text{O}(U)$ acts trivially. The internal symmetry group is $K = \text{O}(U)$, and it is equal to the R -symmetry group as well.

Remark 41.4. If $u = \dim U$, this is called “ $N = u$ supersymmetry” in three dimensions.

Let $\partial_0, \partial_1, \partial_2$ be the standard basis of $\mathbb{R}^{1,2}$, an orthonormal basis for which ∂_0 is timelike and ∂_1, ∂_2 are spacelike. Let Q_1, Q_2 be the standard basis of $\mathbb{R}^2$. Define

$$(41.5) \quad \begin{aligned} \Gamma(Q_1, Q_1) &= \partial_0 + \partial_1 \\ \Gamma(Q_2, Q_2) &= \partial_0 - \partial_1 \\ \Gamma(Q_1, Q_2) &= \partial_2 \end{aligned}$$

The resulting map $\Gamma: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^{1,2}$ is $\text{SL}_2\mathbb{R}$ -equivariant. This determines the Lie bracket of two vectors in $\mathbb{S}^*$ in case $U = \mathbb{R}$. For general U we have

$$(41.6) \quad [u_1 \otimes Q_i, u_2 \otimes Q_j] = -2\langle u_1, u_2 \rangle_U \Gamma(Q_i, Q_j), \quad i, j \in \{1, 2\}.$$

As a preliminary to discussing the *dimensional reduction* of this supersymmetric symmetry type to 2 dimensions, we make a few general remarks. First, if M is an affine space over a vector space V and $V' \subset V$ is a subspace, then the quotient M/V' of the affine space by this subspace of translations is an affine space over the quotient vector space V/V' . If M is a Minkowski spacetime and $V' \subset V$ is spacelike, then there is an induced Minkowskian structure on M/V' : identify V/V' with the orthogonal complement $(V')^\perp$ and use the induced Lorentz metric on $(V')^\perp$. If $\mathcal{G} \rightarrow \text{Aff}(M)$ is

a group of symmetries acting on M by affine transformations, then the subgroup of $\mathcal{G}$ consisting of symmetries whose differential preserves $V' \subset V$ acts by affine transformations on M/V' . This subgroup includes the normal subgroup $V \subset \mathcal{G}$ of translations of M ; the subgroup $V' \subset V$ acts trivially on M/V' , i.e., it is a subgroup of the internal symmetry group of the quotient.

These principles apply to $M = \mathbb{M}^n$ to reduce an n -dimensional relativistic or supersymmetric symmetry type to a symmetry type in lower dimensions.

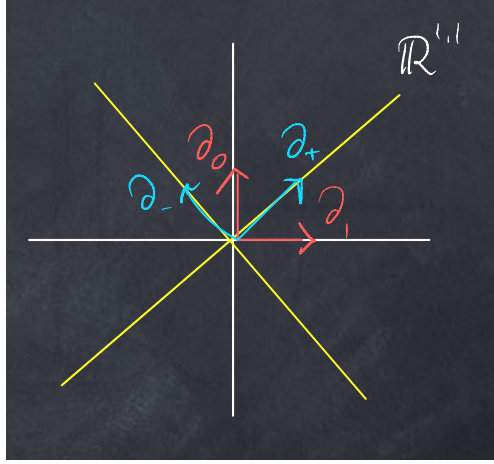


FIGURE 117. A basis of lightlike vectors in 2 dimensions

With these preliminaries understood we compute the reduction of (41.3) with $U = \mathbb{R}$ to a 2-dimensional supersymmetric symmetry type, reducing along the span of ∂_2 . The result is

$$(41.7) \quad G_2 = \text{Spin}_{1,1} \times (\mathbb{R} \times \mu_2) \longrightarrow \text{O}_{1,1}^\uparrow.$$

The $\mathbb{R}$ subgroup of the internal symmetry group comes from translations in the ∂_2 -direction; the other factor is $\mu_2 = \text{O}_1$. Let ζ be the generator of $\text{Lie}(\mathbb{R})$ that corresponds to ∂_2 , and write

$$(41.8) \quad \begin{aligned} Q_+ &= Q_1 \\ Q_- &= Q_2 \\ \partial_+ &= \partial_0 + \partial_1 \\ \partial_- &= \partial_0 - \partial_1 \end{aligned}$$

The vectors ∂_+, ∂_- are lightlike, as indicated in Figure 117. The bracketing relations (41.5) and (41.6) become

$$(41.9) \quad \begin{aligned} [Q_+, Q_+] &= -2\partial_+ \\ [Q_-, Q_-] &= -2\partial_- \\ [Q_+, Q_-] &= +2\zeta \end{aligned}$$

Observe that $\text{Spin}_{1,1} \cong \mu_2 \times \mathbb{R}^{>0} \subset G_2$ acts on $\mathbb{S}^* \cong \mathbb{R} \oplus \mathbb{R}$ as the sum of the weight $+1$ and weight -1 representations of $\mathbb{R}^{>0}$ tensor the sign representation of μ_2 , the other copy of $\mu_2 \subset G_2$ also acts as the sign representation, and $\mathbb{R} \subset G_2$ acts trivially. It follows that G_2 acts trivially on $[Q_+, Q_-]$, which is consistent with ζ being central in the Lie algebra $\text{Lie } \mathcal{G}_2 \oplus \mathbb{S}^*$.

Remark 41.10. There is a variation of the symmetry type (41.3) that includes a Pin group extension of $\text{Spin}_{1,2}$. In particular, it includes reflection in the $(0, 1)$ -plane. In the dimensional reduction, this reflection acts trivially on Minkowski spacetime, so it is an extra element of order two in the internal symmetry group. It acts nontrivially on $\mathbb{S}^*$, so it also contributes to the R -symmetry group.

The central charge ζ , as well as the last equation in (41.9), play an important role in certain 2-dimensional supersymmetric theories; for an expository account, see [F5, Lecture 4]. We illustrate this a bit at the end of the lecture.

41.3 Super relativistic symmetry types with $\dim \mathbb{S}^* = 8$

The maximal dimension in which there is a real spinor representation of dimension eight is $n = 6$. By (40.23), there is an isomorphism $\text{Spin}_{1,5} \cong \text{SL}_2\mathbb{H}$. Under that isomorphism the action $\text{SL}_2\mathbb{H} \curvearrowright \mathbb{H}^2 \cong \mathbb{R}^8 = \mathbb{S}^*$ is one of two inequivalent 8-dimensional irreducible spinor representations; the other is constructed by precomposing with the automorphism of quaternionic conjugation on $\text{SL}_2\mathbb{H}$.

The complexification of $\mathbb{S}^*$, which is a representation of the complex spin group $\text{Spin}_6(\mathbb{C}) \cong \text{SL}_4(\mathbb{C})$, can be written as follows. Let (S_0, ν) be the 4-dimensional standard complex vector $\mathbb{C}^4$ space equipped with a standard volume form ν . Let (W, ϵ) be the standard 2-dimensional complex vector space $\mathbb{C}^2$ with a standard volume form ϵ , and suppose $j: W \rightarrow W$ is an antilinear map such that $j^2 = -\text{id}_W$. (W is a model for the quaternions.) Then $\mathbb{S}_{\mathbb{C}}^* \cong S_0^* \otimes W^*$ and the complexified vector representation is $V_{\mathbb{C}} = \Lambda^2 S_0^*$. Then $V_{\mathbb{C}}$ carries a nondegenerate symmetric bilinear form $\Lambda^2 S_0^* \times \Lambda^2 S_0^* \rightarrow \mathbb{C}$ constructed using the inverse volume form ν^{-1} . A quaternionic structure on S_0 induces a real structure on both $\Lambda^2 S_0^*$ and on $S_0^* \otimes W^*$. One needs to tune these structures so that the signature of the induced symmetric bilinear form on V is $(1, 5)$ (as opposed to $(5, 1)$). The symmetric pairing $\Gamma: S^* \times S^* \rightarrow V$ is

$$(41.11) \quad \Gamma(s'_0 \otimes w', s''_0 \otimes w'') = \epsilon^{-1}(w' \wedge w'') s'_0 \wedge s''_0, \quad s'_0, s''_0 \in S_0^*, \quad w', w'' \in W^*.$$

The 4-dimensional supersymmetric gauge theory that leads to Donaldson invariants has supersymmetric symmetry type a dimensional reduction of the 6-dimensional relativistic symmetry type

$$(41.12) \quad G_6 = \text{Spin}_{1,5} \times \text{Sp}_1 \longrightarrow \text{O}_{1,5}^\dagger$$

with the extension to a super Lie group given by $\mathbb{S}^*$ and the pairing (41.11). That dimensional reduction has relativistic symmetry type

$$(41.13) \quad G_4 = \text{Spin}_{1,3} \times (\mathbb{R}^2 \times \text{U}_2) \longrightarrow \text{O}_{1,3}^\dagger.$$

The $\mathbb{R}^2$ subgroup comes from translations in the spacelike directions along which we dimensionally reduce. The double cover Spin_2 of rotations in that plane also appears in the internal symmetry subgroup of (41.13), but the central element equals that of Sp_1 , so they combine to the factor $\text{Sp}_1 \times_{\mu_2} \text{Spin}_2 \cong \text{U}_2$. It turns out that this is the relativistic symmetry type appropriate to field

theories that are superconformal; for the theory that includes Donaldson invariants, only the subgroup

$$(41.14) \quad G_4 = \text{Spin}_{1,3} \times (\mathbb{R}^2 \times \text{SU}_2) \longrightarrow \text{O}_{1,3}^\uparrow$$

acts as a symmetry.

We write explicit formulas for the complex super Lie algebra of this supersymmetry type as follows. First, for complex infinitesimal translations in $\mathbb{M}^4$ set

$$(41.15) \quad \begin{aligned} P_{1\dot{1}} &= \partial_0 + \partial_1 \\ P_{2\dot{2}} &= \partial_0 - \partial_1 \\ P_{1\dot{2}} &= \partial_2 - i\partial_3 \\ P_{2\dot{1}} &= \partial_2 + i\partial_3 \end{aligned}$$

Minimal supersymmetry in four dimensions has a 4-dimensional vector space of odd translations, for which we choose a basis of the complexification of the form $Q_1, Q_2, \bar{Q}_1, \bar{Q}_2$. The brackets are

$$(41.16) \quad \begin{aligned} [Q_a, Q_b] &= 0 \\ [\bar{Q}_{\dot{a}}, \bar{Q}_{\dot{b}}] &= 0 \\ [Q_a, \bar{Q}_{\dot{b}}] &= 2P_{a\dot{b}} \end{aligned}$$

where $a, b, \dot{a}, \dot{b} = 1, 2$.

For the dimensional reduction (41.13) with 8-dimensional space of odd translations, introduce a basis $S, \bar{S}$ for the complexification of the vector space of central charges, which is the complexified Lie algebra $\text{Lie}(\mathbb{R}^2) \otimes \mathbb{C}$ of the translation group $\mathbb{R}^2$ in (41.13). The complex basis of $\mathbb{S}^* \otimes \mathbb{C}$ is $Q_a^{(i)}, \bar{Q}_{b(j)}$ for $a, b, i, j = 1, 2$. The brackets are now

$$(41.17) \quad \begin{aligned} [Q_a^{(i)}, \bar{Q}_{b(j)}] &= -2\delta_j^i P_{ab} \\ [Q_a^{(i)}, Q_b^{(j)}] &= \epsilon^{ij} \epsilon_{ab} S \end{aligned}$$

as well as the complex conjugate of the second equation. Here $\epsilon^{12} = \epsilon_{12} = 1$ and ϵ is skew-symmetric.

41.4 Representations of the super Poincaré group

We now generalize the discussion of [Lecture 17](#) to super Poincaré groups. Fix a dimension n and a real spinor representation $\text{Spin}_{1,n-1} \curvearrowright \mathbb{S}^*$. The super Poincaré group $\mathcal{P}_{n|\mathbb{S}^*}$ of [Definition 41.1](#) fits into the group extension

$$(41.18) \quad 1 \longrightarrow \mathbb{R}^{1,n-1} \longrightarrow \mathcal{P}_{n|\mathbb{S}^*} \longrightarrow \text{Spin}_{1,n-1} \ltimes \Pi\mathbb{S}^* \longrightarrow 1$$

analogous to (17.1), where recall $\Pi\mathbb{S}^*$ is the odd vector space of odd translations. In Harish–Chandra pair notation (Definition 37.13) the quotient super Lie group in (41.18) is the pair

$$(41.19) \quad (\mathrm{Spin}_{1,n-1}, \mathfrak{spin}_{1,n-1} \oplus \mathbb{S}^*).$$

Set $V = \mathbb{R}^{1,n-1}$. The decomposition of the vector space V^* of characters of V into orbits of the quotient group $\mathrm{Spin}_{1,n-1} \ltimes \Pi\mathbb{S}^*$ is the same as in §16.4, but the stabilizer groups are now super Lie groups. The little groups defined in Definition 17.5 are now modified to groups isomorphic to (recall that m is the mass)

$$(41.20) \quad \begin{aligned} m > 0 : & \quad \mathrm{Spin}_{n-1} \ltimes \Pi\mathbb{S}^* \\ m = 0 : & \quad \mathrm{Spin}_{n-2} \ltimes \Pi\mathbb{S}^* \end{aligned}$$

which we could also write in Harish–Chandra pair notation, analogous to (41.19). At a positive energy character $\theta \in \overline{P}^* \subset V^*$, the translation subgroup $\mathbb{R}^{1,n-1}$ acts by

$$(41.21) \quad \xi \mapsto e^{-i\theta(\xi)/\hbar}.$$

Hence a representation of a little group (41.20) is constrained: the Lie bracket of two elements of $\mathbb{S}^*$ is a translation, hence is represented by (41.21). Consider, then, the bilinear form

$$(41.22) \quad b_\theta(Q_1, Q_2) := \theta([Q_1, Q_2]) = -2\theta(\Gamma(Q_1, Q_2)), \quad Q_1, Q_2 \in \mathbb{S}^*.$$

The positivity (40.35) guarantees that b_θ is nonpositive semidefinite. More precisely, it is negative definite if $m > 0$ and semidefinite if $m = 0$. Furthermore, in the latter case the kernel is half-dimensional (with the exception of some low dimensional cases). An irreducible representation of the little super groups is a *complex* Clifford module $W^0 \oplus W^1$ for the Clifford algebra of $(\mathbb{S}^*, b_\theta)$ equipped with an intertwining action of the Spin group.

Remark 41.23.

- (1) The spin-statistics Theorem 15.4 applies: the central element $\epsilon \in \mathrm{Spin}$ acts as the grading operator on $W^0 \oplus W^1$.
- (2) The CRT Theorem 15.6 also applies.
- (3) An irreducible unitary positive energy representation of the super Poincaré group $\mathcal{P}_{n|\mathbb{S}^*}$ restricts to a finite sum of irreducible unitary positive energy representations of the non-super Poincaré group $\mathcal{P}_n$. In other words, irreducible unitary positive energy representations of the super Poincaré group consist of a finite *multiplet* of relativistic particles.

To illustrate, consider $n = 3$. First, we enumerate the relativistic particle representations of $\mathcal{P}_3$. For positive mass $m > 0$ the little group is Spin_2 , and its irreducible complex representations are labeled by a *spin* $j \in \frac{1}{2}\mathbb{Z}$. There are names of the particles for small values of $|j|$:

$$(41.24) \quad \begin{array}{ll} j = 0 & \text{massive scalar particle} \\ j = \pm 1/2 & \text{massive spinor particle} \\ j = \pm 1 & \text{massive half-vector particle} \end{array}$$

In the massless $m = 0$ case, the little group is $\text{Spin}_1 \cong \mu_2$, and so there are two inequivalent irreducible complex representations:

$$(41.25) \quad \begin{array}{ll} \text{trivial rep} & \text{massless scalar particle} \\ \text{sign rep} & \text{massless spinor particle} \end{array}$$

Now we turn to the super Poincare group $\mathcal{P}_{3|\mathbb{S}^*}$. As in §41.2, let $\mathbb{S}^*$ be the 2-dimensional real representation of $\text{Spin}_{1,2} \cong \text{SL}_2\mathbb{R}$. The vector representation is $V \cong \text{Sym}^2 \mathbb{S}^*$. For massive $m > 0$ representations of $\mathcal{P}_{3|\mathbb{S}^*}$ the little group Spin_2 sits in $\text{Spin}_{1,2} \cong \text{SL}_2\mathbb{R}$ as $\text{SO}_2 \subset \text{SL}_2\mathbb{R}$. At $\theta \in P^*$ with positive mass, the bilinear form b_θ in (41.22) is nondegenerate, so the resulting Clifford algebra is isomorphic to $\text{Cliff}_2^{\mathbb{C}} \cong \text{End}(W^0 \oplus W^1)$, where $\dim W^0 = \dim W^1 = 1$. There is an intertwining action of Spin_2 on $W^0 \oplus W^1$, and if the spin on W^0 is $j \in \frac{1}{2}\mathbb{Z}$, then the spin on W^1 is $j + 1/2$. The spin-statistics theorem tells that in fact $j \in \mathbb{Z}$. Note, then, that an irreducible representation of $\mathcal{P}_{3|\mathbb{S}^*}$ consists of two relativistic particles of adjacent spins: one bosonic and one fermionic. The basic examples appear in the following table.

$$(41.26) \quad \begin{array}{c|ccccc} j & -1 & -1/2 & 0 & 1/2 & 1 \\ \hline \text{massive scalar} & & 1 & 1 & & \\ \text{massive scalar} & & & 1 & 1 & \\ \text{massive half vector} & & & & 1 & 1 \\ \text{massive half vector} & 1 & 1 & & & \\ \text{massive vector} & 1 & 1 & & 1 & 1 \end{array}$$

There is only one isomorphism class of massless ($m = 0$) irreducible representation of $\mathcal{P}_{3|\mathbb{S}^*}$. The bilinear form b_θ is degenerate; the relevant Clifford algebra is isomorphic to $\text{Cliff}_1^{\mathbb{C}}$. An irreducible complex module still has dimension 1|1. The intertwining action of $\text{Spin}_1 \cong \mu_2$ is constrained by spin-statistics: it acts trivially on the even vectors and by the sign representation on the odd vectors. Hence the massless super multiplet of $\mathcal{P}_{3|\mathbb{S}^*}$ consists of a massless scalar particle and a massless spinor particle.

Now consider the dimensional reduction to $n = 2$. We established notation in (41.8)–(41.9). Compute

$$(41.27) \quad \frac{1}{4}[Q_+ \pm Q_-, Q_+ \pm Q_-] = -\partial_0 \pm \zeta.$$

Argue as in §38.1 to deduce that if H is the Hamiltonian—the self-adjoint operator that quantizes ∂_0 —and $\hat{\zeta}$ is the self-adjoint operator that quantizes ζ , then

$$(41.28) \quad H \pm \hat{\zeta} \geq 0$$

In an irreducible representation $\hat{\zeta}$ acts as a constant, since ζ is central. Rewrite (41.28) as

$$(41.29) \quad H \geq |\hat{\zeta}|.$$

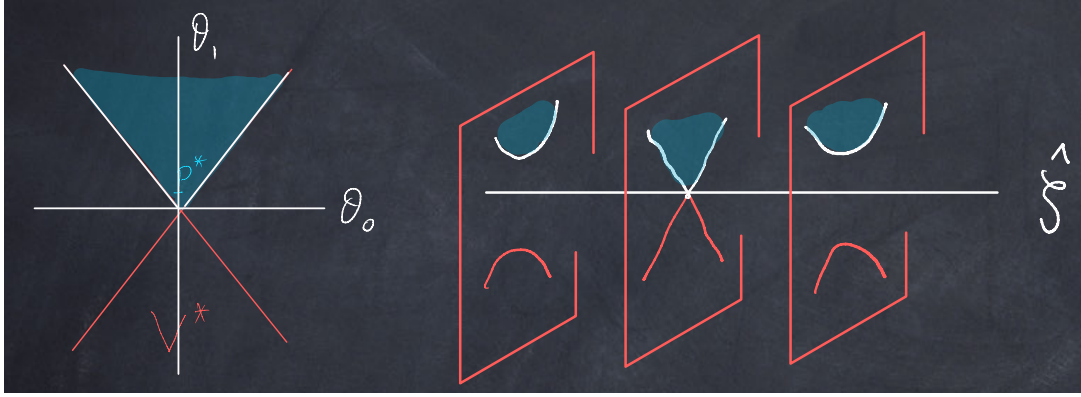


FIGURE 118. Energy-momentum space in 2 dimensions, without and with central charge

This is an example of a *BPS bound*. Observe that if $\hat{\zeta} \neq 0$, then the representation is massive. A vector ψ that satisfies $H\psi = |\hat{\zeta}|\psi$ is a *BPS state*. For a BPS state ψ , either $(\hat{Q}_+ + \hat{Q}_-)\psi = 0$ or $(\hat{Q}_+ - \hat{Q}_-)\psi = 0$. So in this case a BPS state annihilates half of the supersymmetry.

We illustrate the BPS bound in [Figure 118](#). On the left is the standard picture with no central charge. An irreducible representation with central charge has weights (under translation) that lie in a slice with constant $\hat{\zeta}$, and the BPS bound [\(41.29\)](#) forces them to be massive if $\hat{\zeta} \neq 0$. The BPS bound [\(41.29\)](#) is more invariantly written as

$$(41.30) \quad m \geq |\hat{\zeta}|,$$

where m is the mass of the representation in the sense of [Theorem 17.16](#).

Lecture 42: Twisting supersymmetric theories

The technique of “twisting” supersymmetric theories to define topological invariants was introduced by Witten [\[W7\]](#), first in the context of defining Donaldson’s invariants of 4-manifolds, a topic we take up in the last lecture of the course. There is a large literature on twisting after Witten’s paper. In this lecture we explain twistings. Our account here is inspired in part by [\[ES, ESW\]](#).

Twistings are of Wick-rotated theories, so we begin by telling extra features in the Wick rotation of a supersymmetric theory. We still obtain a representation of a bordism category, now with target *super* topological vector spaces to accommodate the fermionic states that are inevitably part of a supersymmetric theory. But the odd translations induce a split supermanifold structure on the objects and morphisms in the bordism category. We do not explain supermanifolds in this version of the notes, and so our account is not complete in some aspects.

The data that defines a twisting is a nonzero odd translation Q that squares to zero in the super Lie algebra of the theory. There are no such real vectors, so we must look in the complexification. That is fine since when Wick rotating the odd translations necessarily complexify. The structure

group of a twisting is the stabilizer subgroup of Q inside the full structure group of the Wick-rotated theory. We give some examples, including the 4-dimensional case relevant to the theory that produces Donaldson invariants.

Section 42.4 is a brief general discussion of symmetries in Wick-rotated field theory. This is a large topic of which we only scratch the surface. We apply it to illustrate special properties of correlation functions of observables annihilated by the infinitesimal symmetry induced by Q . We hope to return to these arguments in a future version of the notes.

42.1 Super Euclidean symmetry types and Wick-rotated supersymmetric theories

We first Wick rotate a super relativistic symmetry type (**Definition 40.31**), just as in §21.4 we Wick rotated non-super relativistic symmetry types.

Recall the data of an n -dimensional super relativistic symmetry type is a homomorphism $\lambda: G_n \rightarrow O_{1,n-1}^\uparrow$ and a super Lie group $(\mathcal{G}_n, \text{Lie } \mathcal{G}_n \oplus \mathbb{S}^*)$, where $\mathcal{G}_n$ is the Lie group that fits into the diagram

$$(42.1) \quad \begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ & & K & \xlongequal{\quad} & K & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{R}^{1,n-1} & \longrightarrow & \mathcal{G}_n & \longrightarrow & G_n \longrightarrow 1 \\ & & \parallel & & \downarrow \bar{\lambda} & & \downarrow \lambda \\ 1 & \longrightarrow & \mathbb{R}^{1,n-1} & \longrightarrow & \mathcal{I}_{1,n-1}^\uparrow & \longrightarrow & O_{1,n-1}^\uparrow \longrightarrow 1 \end{array}$$

and $\mathbb{S}^*$ is a real representation of G_n . Wick rotation of the homomorphism $\lambda: G_n \rightarrow O_{1,n-1}^\uparrow$ is (21.26), reproduced here:

$$(42.2) \quad \begin{array}{ccccccc} 1 & \longrightarrow & K & \longrightarrow & G_n & \xrightarrow{\lambda} & O_{1,n-1}^\uparrow \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & K(\mathbb{C}) & \longrightarrow & G_n(\mathbb{C}) & \longrightarrow & O_n(\mathbb{C}) \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & K & \longrightarrow & G_n^E & \xrightarrow{\lambda^E} & O_n \end{array}$$

The homomorphism $\lambda^E: G_n^E \rightarrow O_n$ is a Euclidean symmetry type, and there is an associated affine version $\tilde{\lambda}^E: \mathcal{G}_n^E \rightarrow \text{Euc}_n$ that fits into the diagram

$$(42.3) \quad \begin{array}{ccccccc} & & & 1 & & 1 & \\ & & & \downarrow & & \downarrow & \\ & & & K & \xlongequal{\quad} & K & \\ & & & \downarrow & & \downarrow & \\ 1 & \longrightarrow & V_E & \longrightarrow & \mathcal{G}_n^E & \longrightarrow & G_n^E \longrightarrow 1 \\ & & \parallel & & \downarrow \tilde{\lambda}^E & & \downarrow \lambda^E \\ 1 & \longrightarrow & V_E & \longrightarrow & \text{Euc}_n & \longrightarrow & O_n \longrightarrow 1 \end{array}$$

We would like to extend $\mathcal{G}_n^E$ to a super Lie group with odd translations $\mathbb{S}^*$, but the problem is that the Wick-rotated group G_n^E may not act on $\mathbb{S}^*$.

Example 42.4. Set $n = 3$, $G_3 = \text{Spin}_{1,2} \cong \text{SL}_2\mathbb{R}$, and $\mathbb{S}^* = \mathbb{R}^2$. Then $\mathbb{S}^* = \mathbb{R}^2$ is not a real representation of $G_3^E = \text{Spin}_3 \cong \text{SU}_2$.

The fix is simple: complexify $\mathbb{S}^*$. This is a general feature when Wick rotating spinors: they complexify. In this case, simply observe that $G_n(\mathbb{C}) \curvearrowright \mathbb{S}_{\mathbb{C}}^*$ is the complexification of the representation $G_n \curvearrowright \mathbb{S}^*$, and it restricts to a complex representation $G_n^E \curvearrowright \mathbb{S}_{\mathbb{C}}^*$. So the “super Lie group” we seek is the Harish–Chandra pair

$$(42.5) \quad (\mathcal{G}_n^E, \text{Lie } \mathcal{G}_n^E \oplus \mathbb{S}_{\mathbb{C}}^*).$$

Remark 42.6. The scare quotes indicate that (42.5) is not strictly a super Lie group, but rather is a variation in which the odd summand of the Lie algebra is complex. Following [DM, §4.8] we term this a *cs Lie group*. (‘cs’ = ‘complex super’) We encounter the general manifold version of this concept momentarily.

Recall that according to the Segal axioms a Wick-rotated field theory is a linear representation

$$(42.7) \quad F: \text{Bord}_n(G_n^E) \longrightarrow \text{stVect}.$$

of the bordism category of differential G_n^E -manifolds; see [Lecture 23](#), especially §23.8. Here we use *super* topological vector spaces to incorporate fermionic states (§10.2). This remains in the supersymmetric case, but there is additional structure. Namely, if X is a differential G_n^E -manifold, then associated to the representation $G_n^E \curvearrowright \Pi\mathbb{S}_{\mathbb{C}}^*$ on the parity-reversal of $\mathbb{S}_{\mathbb{C}}^*$ —an odd vector space—is an odd complex vector bundle

$$(42.8) \quad \Pi\mathbb{S}_{\mathbb{C}}^*(X) \longrightarrow X.$$

The pair consisting of X and this odd complex vector bundle is a split *cs manifold*; see [Remark 42.6](#) for the terminology and see [DM] for background on supermanifolds. Therefore, the supersymmetric theory gives rise to a representation of a bordism category of these split half-complex supermanifolds.

42.2 Square-zero odd translations and twisted field theories

Continuing with the super relativistic symmetry type, recall that the translation component of the Lie bracket

$$(42.9) \quad \Gamma: \mathbb{S}^* \times \mathbb{S}^* \longrightarrow \text{Lie } \mathbb{R}^{1,n-1}$$

satisfies a positivity condition (40.35). In particular, there are no nonzero solutions to the equation $\Gamma(Q, Q) = 0$. So to find *square-zero odd translations* we look in the complexification $\mathbb{S}_{\mathbb{C}}^*$.

Definition 42.10. Let $\mathcal{N} \subset \mathbb{PS}_{\mathbb{C}}^*$ denote the *null quadric*

$$(42.11) \quad \mathcal{N} = \{Q \in \mathbb{S}_{\mathbb{C}}^* : \Gamma(Q, Q) = 0\} / \mathbb{C}^\times.$$

Following [ES, ESW] and the references therein, we use points of $\mathcal{N}$ to define *twistings* of supersymmetric field theories. In fact, the group G_n^E acts on $\mathcal{N}$ and points in the same orbit exhibit similar features, so it is the orbit analysis that is relevant. Also, to define a twisted theory from a point in the null quadric, we choose a lift $Q \in \mathbb{S}_{\mathbb{C}}^*$.

There are three important pieces of data associated with a square-zero odd translation.

Definition 42.12. Let $Q \in \mathbb{S}_{\mathbb{C}}^*$ have square zero.

- (1) The *stabilizer* of Q under the linear action $G_n^E \curvearrowright \mathbb{S}_{\mathbb{C}}^*$ is $H_Q \subset G_n^E$.
- (2) The *image* of $\Gamma(Q, -): \mathbb{S}_{\mathbb{C}}^* \rightarrow \mathbb{C}^n$ is $D_Q \subset \mathbb{C}^n$.
- (3) The *fixed subspace* of the linear action $H_Q \curvearrowright \mathbb{S}_{\mathbb{C}}^*$ is $S_Q \subset \mathbb{S}_{\mathbb{C}}^*$.

If $Q' \in S_Q$, then $\Gamma(Q', Q') \in \mathbb{C}^n$ is fixed by H_Q , so if $H_Q \curvearrowright \mathbb{C}^n$ is irreducible, then $\Gamma(Q', Q') = 0$.

Definition 42.13. We say that a square-zero odd translation Q is *topological* if $D_Q = \mathbb{C}^n$.

We will see that a topological square-zero odd translation determines a *topological* field theory.

Definition 42.14. Let Q be a square-zero odd translation that is topological. A *descendant structure* is an H_Q -invariant subspace $T \subset \mathbb{S}_{\mathbb{C}}^*$ such that

- (i) $\Gamma(Q, -): T \rightarrow \mathbb{C}^n$ is an isomorphism; and
- (ii) T is isotropic: $\Gamma(G_1, G_2) = 0$ for all $G \in T$.

Observe that since Q is fixed by H_Q , the map $\Gamma(Q, -): \mathbb{S}_{\mathbb{C}}^* \rightarrow \mathbb{C}^n$ is H_Q -equivariant. The nomenclature ‘descendant structure’ will become clear at the end of the lecture. The equivalent of a descendant structure was used in [MW]; see their (2.8) and their references [30].

Remark 42.15.

- (1) If H_Q is compact, which is true if we assume the group K of internal symmetries is compact, then a descendant structure always exists. Furthermore, since the representation $G_n^E \curvearrowright \mathbb{C}^n$ factors through $G_n^E \rightarrow O_n$, so too does its restriction to H_Q factor through O_n , and then H_Q acts on T by the complexified vector representation.
- (2) If the representation $H_Q \curvearrowright \mathbb{C}^n$ is irreducible, so too is $H_Q \curvearrowright T$, and then Definition 42.14(ii) is automatic if $n \geq 2$: $\text{Sym}^2 T$ does not contain a copy of the vector representation.

We return to a general square-zero odd translation Q that is not necessarily topological.

Remark 42.16. Let $\overline{H}_Q \subset G_n^E$ be the stabilizer of the line $\mathbb{C} \cdot Q \in \mathcal{N}$ under the projective action $G_n^E \curvearrowright \mathcal{N}$. Then there is a split short exact sequence

$$(42.17) \quad 1 \longrightarrow H_Q \longrightarrow \overline{H}_Q \rightrightarrows \mathbb{T} \longrightarrow 1$$

The homomorphism $\overline{H}_Q \rightarrow \mathbb{T}$ tracks the image of Q , and in the other direction the homomorphism $\mathbb{T} \rightarrow \overline{H}_Q$ is the inclusion of scalar multiplication. (We assume K , hence G_n^E , is compact, so only scalar multiplication by $\mathbb{T} \subset \mathbb{C}^\times$ lies in G_n^E .) It follows that $\mathbb{T}$ is central and there is a direct product splitting $\overline{H}_Q \cong \mathbb{T} \times H_Q$. In the twisted field theory defined below, Q acts as a differential on the vector space attached to a closed $(n-1)$ -manifold, and the central $\mathbb{T}$ acts to induce a $\mathbb{Z}$ -grading with respect to which the differential has degree $+1$. (The last assertion follows from the fact that $\mathbb{T}$ acts on Q with weight one. Also, in a projective theory the $\mathbb{T}$ -action is projective, hence the grading may be by a $\mathbb{Z}$ -torsor.)

In general, from a square-zero odd translation we define a *twisted* variant of the Wick-rotated field theory (42.7). Namely, the homomorphism $H_Q \subset G_n^E \rightarrow \mathrm{O}_n$ gives rise to a notion of differential H_Q -manifolds, and then to a bordism category of such. Furthermore, via the inclusion $H_Q \hookrightarrow G_n^E$ there is a differential G_n^E -manifold associated to a differential H_Q -manifold, hence a map from the bordism category of H_Q -manifolds to the bordism category of G_n^E -manifolds.

Definition 42.18. Let $Q \in \mathcal{N}$ and suppose F is a Wick-rotated field theory as in (42.7). The associated Q -*twisted* theory is the composition

$$(42.19) \quad F_Q: \mathrm{Bord}_n(H_Q) \longrightarrow \mathrm{Bord}_n(G_n^E) \longrightarrow st\mathrm{Vect}.$$

Observe that the linear action $H_Q \curvearrowright \mathbb{S}_{\mathbb{C}}^*$ associates an odd vector bundle (42.8) to a differential H_Q -manifold, and the H_Q -fixed point $Q \in \mathbb{S}_{\mathbb{C}}^*$ determines a flat (covariant constant) section of that bundle, as does any $Q' \in S_Q$. Also, the subspace $D_Q \subset \mathbb{C}^n$ is an invariant subspace of the representation $H_Q \curvearrowright \mathbb{C}^n$, and so on a differential H_Q -manifold X it induces a complex distribution

$$(42.20) \quad \mathcal{D}_Q \longrightarrow X,$$

i.e., a subbundle of the complexified tangent bundle to X .

Lemma 42.21. *The distribution $\mathcal{D}_Q \rightarrow X$ is integrable.*

Proof. The differential H_Q -structure is a principal H_Q -bundle $P \rightarrow X$ with connection and an isomorphism of the induced principal O_n -bundle with the orthonormal frame bundle and its Levi-Civita connection. The distribution (42.20) is associated to the bundle $P \rightarrow X$, hence it inherits a covariant derivative. If ξ_1, ξ_2 are sections of $\mathcal{D}_Q \rightarrow X$, then since the Levi-Civita connection is torsionfree the Lie bracket is

$$(42.22) \quad [\xi_1, \xi_2] = \nabla_{\xi_1} \xi_2 - \nabla_{\xi_2} \xi_1.$$

But ∇ is a covariant derivative on $\mathcal{D}_Q \rightarrow X$, from which the Lie bracket also belongs to this distribution, i.e., is a section of $\mathcal{D}_Q \rightarrow X$. This proves integrability. $\square$

On a differential H_Q -manifold X , the subspace $S_Q \subset \mathbb{S}_{\mathbb{C}}^*$ of H_Q -fixed points induces a space of flat global sections of the odd complex vector bundle (42.8). These give rise to infinitesimal symmetries in the field theory, as we discuss below.

42.3 Examples

42.3.1 $n = 1$. Suppose $G_1^E = \text{Spin}_1 \times O(U)$ where U is a finite dimensional inner product space. (Little changes if we replace $O(U)$ by $\text{SO}(U)$.) Let S be the spinor (sign) representation of $\text{Spin}_1 \cong \mu_2$, and fix a basis element $\psi \in S$. Then $\mathbb{S}^* = S \otimes U$ and $\Gamma: S \otimes U \times S \otimes U \rightarrow \mathbb{R}$ is multiplication times the inner product:

$$(42.23) \quad \Gamma(x_1\psi \otimes u_1, x_2\psi \otimes u_2) = x_1x_2 \langle u_1, u_2 \rangle, \quad x_1, x_2 \in \mathbb{R}, \quad u_1, u_2 \in U.$$

If $\dim U$ is even, then $\mathbb{S}_{\mathbb{C}}^* = W \oplus \overline{W}$ is the sum of complex conjugate maximal isotropic subspaces. Then the null quadric $\mathcal{N}$ contains $\mathbb{P}W$ and $\mathbb{P}\overline{W}$ as components, but there are more components if $\dim U \geq 4$. The stabilizer H_Q of $Q \in W$ is isomorphic to the unitary group U_{m-1} . As in [Remark 42.16](#) scalar multiplication by unit norm complex numbers in $\mathbb{T} = U_1$ fix the image of Q in the null quadric.

If $\dim U = 2$, then the null quadric $\mathcal{N} = \mathbb{P}W \cup \mathbb{P}\overline{W}$ consists of two points. If u_1, u_2 is a real orthonormal basis of U , then the two points are scalar multiples of $Q = \psi \otimes (u_1 + iu_2)$ and $\overline{Q} = \psi \otimes (u_1 - iu_2)$. As for the associated data in [Definition 42.12](#), the stabilizer subgroup H_Q has order two, the nonzero element being $(\epsilon, -1) \in \text{Spin}_1 \times O(U)$, where $\epsilon \in \text{Spin}_1$ is the central element. The induced symmetry type is isomorphic to $\text{Spin}_1 \rightarrow O_1$, so the twisted theory is a theory of spin 1-manifolds. The subspace $D_Q = \mathbb{C}$, so according to [Definition 42.13](#) these square-zero odd translations are topological. The span of $\overline{Q}$ is a descendant structure. Finally, $S_Q = \mathbb{S}_{\mathbb{C}}^*$.

If $\dim U = 4$, then choose complex conjugate bases Q', Q'' for W and $\overline{Q}', \overline{Q}''$ for $\overline{W}$. Observe that $Q' + \overline{Q}''$ is a square-zero odd translation that does not project to a point of $\mathbb{P}W$ or $\mathbb{P}\overline{W}$. For a nonzero vector Q in W or in $\overline{W}$, which is necessarily square-zero, the situation is similar to that in the previous paragraph. The stabilizer group is $H_Q \cong \text{Spin}_1 \times U_1$; the subspace $D_Q = \mathbb{C}^2$, so that the twist is topological, and S_Q is again 2-dimensional, spanned by Q and $\overline{Q}$.

42.3.2 $n = 4$. We consider $N = 2$ supersymmetry in $n = 4$ dimensions; see [Remark 40.36\(5\)](#) for the nomenclature. There are two common possibilities for the Euclidean symmetry type:¹²¹

$$(42.24) \quad G_4^E = \text{Spin}_4 \times \text{SU}_2 / \langle (\epsilon, -1) \rangle \quad \text{or} \quad G_4^E = \text{Spin}_4 \times \text{U}_2 / \langle (\epsilon, -1) \rangle$$

The second corresponds to superconformal theories, whereas the first is for more general supersymmetric theories. In both cases we divide by a cyclic group of order two; $\epsilon \in \text{Spin}_4$ is the central element. The subgroup of internal symmetries is

$$(42.25) \quad K = \langle \epsilon \rangle \times \text{SU}_2 / \langle (\epsilon, -1) \rangle \quad \text{or} \quad \langle \epsilon \rangle \times \text{U}_2 / \langle (\epsilon, -1) \rangle$$

¹²¹Here we omit central charges, which are included in [§41.3](#).

In both cases we write the complexified space of odd translations as

$$(42.26) \quad \mathbb{S}_{\mathbb{C}}^* = S' \otimes W \oplus S'' \otimes \overline{W}$$

where each of S', S'', W is a two-dimensional hermitian vector spaces, and S', S'' are equipped with a compatible volume form. (In the first case in (42.24) the vector space W also carries a compatible volume form.) There is a decomposition $\text{Spin}_4 \cong \text{SU}_2' \times \text{SU}_2''$ as the product of two commuting SU_2 subgroups so that the action of G_4^E on $\mathbb{S}_{\mathbb{C}}^*$ has $\text{SU}_2' \curvearrowright S'$, $\text{SU}_2'' \curvearrowright S''$, and $\text{U}_2 \curvearrowright W$ by standard representations; the action $\text{U}_2 \curvearrowright \overline{W}$ is by the complex conjugate of the standard representation. Observe that the element $(\epsilon, -1)$ acts trivially on $\mathbb{S}_{\mathbb{C}}^*$. The complexified (infinitesimal) even translations are identified with $S' \otimes S''$. The symmetric pairing (42.9), after complexification, has each summand in (42.26) maximal isotropic and pairs them nondegenerately:

$$(42.27) \quad \Gamma(\psi' \otimes w', \psi'' \otimes \bar{w}'') = \langle \bar{w}'', w' \rangle \psi' \otimes \psi'', \quad \psi' \in S', \quad \psi'' \in S'', \quad w', w'' \in W.$$

The null vectors we focus on lie in $\mathbb{P}(S' \otimes W)$ or $\mathbb{P}(S'' \otimes \overline{W})$; we choose the former for convenience. (There are additional null vectors of the form $\psi' \otimes w' + \psi'' \otimes \bar{w}'$, where $\langle \bar{w}'', w' \rangle = 0$.) One model of the action of

$$(42.28) \quad G_4^E \cong \text{SU}_2' \times \text{SU}_2'' \times (\text{S})\text{U}_2 / \langle (-1, -1, -1) \rangle$$

on $S' \otimes W$ is to identify $S' \otimes W$ with 2×2 matrices Q and then $A' \times A'' \times A$ acts on Q to produce $A'QA^{-1}$. Observe that on nonzero Q there is an orbit of rank 1 matrices and an orbit of rank 2 matrices. Assume that Q has rank 2, so in this representation we can take Q to be the identity matrix. Then the stabilizer subgroup is the diagonal $A' = A$, which for the first case in (42.24) is

$$(42.29) \quad H_Q \cong \text{SO}_4 = \text{SU}_2 \times \text{SU}_2'' / \langle (-1, -1) \rangle.$$

Thus the twisted theory is a theory of oriented 4-manifolds. This is the sort of theory we study in the final lecture of the course.

Continuing now with the first case in (42.24), observe that $\overline{W} \cong W$ as representations of SU_2 . We claim that the square-zero odd translation Q of the previous paragraph is topological: $D_Q = \mathbb{C}^4$. To see this, choose unitary bases ψ'_1, ψ'_2 of S' , ψ''_1, ψ''_2 of S'' , and w_1, w_2 of W . Set

$$(42.30) \quad Q = \psi'_1 \otimes w_1 + \psi'_2 \otimes w_2.$$

Then $\Gamma(Q, \psi''_j \otimes \bar{w}_i) = \psi'_i \otimes \psi''_j$, and since the right hand side runs over a basis of $S' \otimes S''$ for $i, j \in \{1, 2\}$, we see that $\Gamma(Q, -)$ is surjective, as claimed. This argument also shows that the subspace $T = S'' \otimes W \subset \mathbb{S}_{\mathbb{C}}^*$ is a descendant structure.

Finally, S_Q is the line spanned by Q . This can be deduced by decomposing (42.26) under the action of $H_Q \cong \text{SO}_4$:

$$(42.31) \quad \begin{aligned} S' \otimes W &\cong \mathbb{C} \oplus \Lambda_+^2(\mathbb{C}) \\ S'' \otimes \overline{W} &\cong V_{\mathbb{C}} \end{aligned}$$

where V is the vector representation and Λ_+^2 the self-dual 2-form representation of SO_4 , both appearing complexified here. The only fixed vectors are multiples of Q .

42.4 Symmetries in Wick-rotated field theory

This is a preliminary to invariance properties of correlation functions in twisted supersymmetric theories (Definition 42.18). We hope to flesh these ideas out more in a future version of the notes.

Suppose $F: \text{Bord}_n(G_n^E) \rightarrow t\text{Vect}$ is a Wick-rotated field theory of differential G_n^E -manifolds. Then to a closed $(n-1)$ -dimensional¹²² differential G_n^E -manifold Y the theory F assigns a topological vector space $F(Y)$. A diffeomorphism $\psi: Y \rightarrow Y'$ of differential G_n^E -manifolds induces an isomorphism $\psi_*: F(Y) \rightarrow F(Y')$ of topological vector spaces.

Example 42.32. Suppose $G_n^E = \text{O}_n \times K$ is a direct product. Then a differential G_n^E -manifold is a Riemannian n -manifold¹²³ equipped with a principal K -bundle with connection. If the latter is the product K -connection—the product bundle equipped with the product connection—then the Lie group K acts as symmetries that equal the identity on the manifold and so are a global gauge transformation on the principal bundle. So a closed Riemannian $(n-1)$ -manifold Y has a state space $F(Y)$ that is equipped with a representation of K . This is the sense in which this symmetry type encodes “global” K -symmetry.

Example 42.33. A common example in $n = 2$ -dimensions has K a finite group. Then a principal K -bundle is a Galois covering space $\tilde{Y} \rightarrow Y$ over a 1-manifold Y , and the topological vector space attached to S^1 with a Galois covering is the state space of a *twisted sector*. The covering $\tilde{Y} \rightarrow S^1$ is characterized by an element of K , up to conjugacy, and the centralizer subgroup of that element acts on the state space of the twisted sector. More canonically, one obtains a vector bundle of super topological vector spaces over the groupoid of Galois covers of the circle, which is equivalent to the action groupoid $K//K$ of K acting on itself by conjugation.

Next, suppose $X: Y_0 \rightarrow Y_1$ and $X': Y'_0 \rightarrow Y'_1$ are bordisms, each equipped with a differential G_n^E -structure, and $\varphi: X \rightarrow X'$ is an isomorphism of these bordisms. Then the diagram

$$(42.34) \quad \begin{array}{ccc} F(Y_0) & \xrightarrow{F(X)} & F(Y_1) \\ (\partial_0 \varphi)_* \downarrow & & \downarrow (\partial_1 \varphi)_* \\ F(Y'_0) & \xrightarrow{F(X')} & F(Y'_1) \end{array}$$

commutes. When $Y_1 = Y'_1 = \emptyset$ this specializes to

$$(42.35) \quad \begin{array}{ccc} F(Y_0) & \xrightarrow{F(X)} & \mathbb{C} \\ (\partial_0 \varphi)_* \downarrow & & \nearrow F(X') \\ F(Y'_0) & \xrightarrow{F(X')} & \mathbb{C} \end{array}$$

The invariance of correlation functions of point observables (§23.6) is a corollary of (42.35). Namely, suppose $\varphi: X \rightarrow X'$ is a diffeomorphism of closed n -dimensional differential G_n^E -manifolds.

¹²²Recall from §23.2 that an object Y in the bordism category is a closed $(n-1)$ -manifold embedded in a germ of an n -manifold and cooriented inside that germ; see also §24.3.

¹²³which can be an n -dimensional germ of an $(n-1)$ -dimensional manifold

Let $x_1, \dots, x_k$ be distinct points of X , and write their images under φ as $x'_1, \dots, x'_k$. There is a topological vector space $\mathcal{O}_x$ of point observables at each $x \in X$, and φ induces an isomorphism $\varphi_* \mathcal{O}_x \rightarrow \mathcal{O}'_{\varphi(x)}$, where $\mathcal{O}'$ is the topological vector space of point observables on X' . Then

$$(42.36) \quad \langle \varphi_* \psi_1 \cdots \varphi_* \psi_k \rangle_{X'} = \langle \psi_1 \cdots \psi_k \rangle_X, \quad \psi_i \in \mathcal{O}_{x_i}.$$

Finally, suppose given a 1-parameter group (or a germ) of diffeomorphisms φ_t of a closed n -dimensional differential G_n^E -manifold X . Let $\dot{\varphi}$ denote the infinitesimal generator. Apply $d/dt|_{t=0}$ to (42.36):¹²⁴

$$(42.37) \quad \sum_i \langle \psi_1 \cdots \psi_{i-1} \dot{\varphi}_* \psi_i \psi_{i+1} \cdots \psi_n \rangle_X = 0.$$

42.5 Symmetries in Q -twisted supersymmetric field theories

We apply these general symmetry principles to the twisted theory (Definition 42.18) associated to a square-zero odd translation Q . This is a theory of differential H_Q -manifolds. Recall from the last remarks in §42.2 that Q induces a covariant constant global section of the vector bundle $\Pi\mathbb{S}_{\mathbb{C}}^*(X) \rightarrow X$ over a closed differential H_Q -manifold. This is an odd vector field on the split cs manifold, so it is an infinitesimal odd symmetry. For each $x \in X$ this odd vector field induces¹²⁵ an odd linear map $\hat{Q}: \mathcal{O}_x \rightarrow \mathcal{O}_x$ on the $\mathbb{Z}/2\mathbb{Z}$ -graded topological vector space of point observables at x , and since Q has square zero it follows that $\hat{Q}^2 = 0$, i.e., $\hat{Q}$ is a differential on $\mathcal{O}_x$. The cohomology of $\hat{Q}$ plays a large role in the theory.

Consider a correlation function $\langle \psi_1 \cdots \psi_k \rangle_X$ in which each ψ_i is $\hat{Q}$ -closed: $\hat{Q}\psi_i = 0$. Assume in addition that some ψ_j is $\hat{Q}$ -exact: $\psi_j = \hat{Q}\psi'_j$ for some ψ'_j . Apply (42.37) with ψ_j replaced by ψ'_j and with $\dot{\varphi}_* = \hat{Q}$ to deduce

$$(42.38) \quad \langle \psi_1 \cdots \psi_k \rangle_X = 0.$$

Now assume that Q is topological in the sense of Definition 42.13.¹²⁶ Suppose x_t is a path in X and ψ_t is a “constant” path of $\hat{Q}$ -closed point observables along x_t .¹²⁷ At each point x_t there exists $G \in \Pi\mathbb{S}_{\mathbb{C}}^*(X)_{x_t}$ such that $[Q, G]$ is the tangent vector to the path. (We can choose G in the subbundle of $\Pi\mathbb{S}_{\mathbb{C}}^*(X) \rightarrow X$ associated to the descendant structure T in Definition 42.14.) Then

$$(42.39) \quad \left. \frac{d}{dt} \right|_{t=0} \langle \psi_t \rangle = \langle [\hat{Q}, G] \psi_t \rangle = \langle \hat{Q} \hat{G} \psi_t \rangle - \langle \hat{G} \hat{Q} \psi_t \rangle = 0.$$

¹²⁴This requires more thought to see how this “Lie derivative” acts on the topological vector spaces of point observables.

¹²⁵but see footnote ¹²⁴. Also, the second half of this sentence requires more argument.

¹²⁶Some of the following applies to more general twistings. For example, in some cases the distribution $\mathcal{D}_Q \rightarrow X$ over a complex manifold X is the $(0,1)$ -summand of the complexified tangent bundle, and then one argues for holomorphicity of certain correlation functions rather than topological invariance.

¹²⁷The sense of this constancy needs more thought as well.

In the last step we use (42.38) and the fact that $\hat{Q}\psi_t = 0$. The argument extends to correlation functions of multiple observables. The conclusion is that correlation functions of $\hat{Q}$ -closed point observables are independent of the points at which the observables are supported, hence they are topological invariants. Furthermore, by (42.38) they only depend on the $\hat{Q}$ -cohomology class of the point observables.

42.6 Descendant defects

This section is a first go to expose common statements in the physics literature, but as in other parts of this lecture more thought is required.

As a preliminary, assume given a general linear symmetry type $\lambda_n: G_n \rightarrow \mathrm{GL}_n\mathbb{R}$ as in §22.2. Let

$$(42.40) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{R}^n & \dashrightarrow & \mathcal{G}_n & \dashrightarrow & G_n \longrightarrow 1 \\ & & \parallel & & \downarrow \tilde{\lambda}_n & & \downarrow \lambda_n \\ 1 & \longrightarrow & \mathbb{R}^n & \longrightarrow & \mathrm{Aff}_n & \longrightarrow & \mathrm{GL}_n\mathbb{R} \longrightarrow 1 \end{array}$$

be the associated affine symmetry type. On a differential (G_n, λ_n) -manifold (Definition 22.16) one has a principal G_n -bundle $P \rightarrow X$ with connection together with an isomorphism of the associated $\mathrm{GL}_n\mathbb{R}$ -bundle with the frame bundle. The linear group G_n in (42.40) is the structure group of the bundle and the translation group $\mathbb{R}^n$ is accounted for infinitesimally by the connection. More precisely, there are horizontal vector fields $\partial_1, \dots, \partial_n$ on P that are the horizontal lifts of the corresponding frame of the tangent space below. If $e^1, \dots, e^n$ is the standard basis of $(\mathbb{R}^n)^*$, and $\epsilon(-)$ denotes exterior multiplication, then in a well-defined sense

$$(42.41) \quad d = \epsilon(e^k) \partial_k$$

acts on differential forms as the Cartan differential, assuming that the connection is torsionfree. Note that the expression $e^k \partial_k$ can be interpreted as the tautological element of $\Omega_X^1(TX)$. Yet again, and without the torsionfree hypothesis, one can interpret $e^k \partial_k$ as the covariant derivative operator on any vector bundle associated to the principal G_n -bundle $P \rightarrow X$.

Returning to our twisted supersymmetric theory of differential H_Q -manifolds, and assuming a descendant structure as in Definition 42.14, let $G_1, \dots, G_n$ be the basis of T that corresponds to the standard basis $e_1, \dots, e_n$ of $\mathbb{C}^n$ under the isomorphism $\Gamma(Q, -): T \rightarrow \mathbb{C}^n$. Define

$$(42.42) \quad \hat{G} = \epsilon(e^k) \hat{G}_k$$

This is an operator on differential form-valued sections of the vector bundle $\mathcal{O} \rightarrow X$ of point observables (or presumably a dense subbundle $\mathcal{O}^\infty \rightarrow X$). From the definitions we deduce

$$(42.43) \quad [\hat{Q}, \hat{G}] = d.$$

This is the start of a “descent” construction from $\hat{Q}$ -closed point observables to higher dimensional defects. (See [Lecture 31](#) for defects in the context of topological field theory.) Again, these manipulations are formal in this version of the notes. Let ψ be a point observable. Set

$$(42.44) \quad \begin{aligned} \psi^{(1)} &:= \hat{G}\psi \\ \psi^{(2)} &:= \hat{G}\psi^{(1)} \\ \psi^{(3)} &:= \hat{G}\psi^{(2)} \\ &\dots \end{aligned}$$

so that $\psi^{(j)}$ is an operator-valued j -form. Then one can verify that

$$(42.45) \quad d\psi^j = \hat{Q}\psi^{(j+1)}.$$

We do not reproduce the computation here since we have not explicated the meaning of the symbols to our satisfaction. Nonetheless we proceed to illustrate how this “descent equation” combines with Stokes’ theorem to produce topological quantities.

Form extended observables by integrating $\psi^{(j)}$ over a j -dimensional cycle in X . For example, if $C \subset X$ is a curve from x to x' , then the formal computation

$$(42.46) \quad \hat{Q} \int_C \psi^{(1)} = \int_C \hat{Q}\psi^{(1)} = \int_C d\psi = \psi_{x'} - \psi_x$$

is meant to show that in correlation functions of $\hat{Q}$ -closed observables we can replace ψ_x with $\psi_{x'}$. This recovers the conclusion of [\(42.39\)](#). If now C is a closed curve, then [\(42.46\)](#) demonstrates that $\int_C \psi^{(1)}$ is a $\hat{Q}$ -closed observable, and then a similar argument to that above demonstrates that in correlation functions of $\hat{Q}$ -closed operators we can replace C with any homologous closed curve. These are the flavor of arguments that locate correlation functions with topological meaning inside the larger twisted theory.

Lecture 43: Introduction to mirror symmetry

The interaction of mathematics and quantum field theory has yielded many fruits for mathematics. In this lecture we explain the origins of one of them: mirror symmetry. This is an involution on certain supersymmetric quantum field theories in 2 dimensions, discovered in [\[LVW\]](#) but with antecedents in other works. For a detailed history and a much more detailed account of mirror symmetry we refer to the books [\[HKKPTVVZ, GY, ABCDGMSSW\]](#) and the references therein. There are many more recent works as well, both in the physics and mathematics literature.

We begin by elucidating the super relativistic symmetry type of these theories, which describes “(2, 2) supersymmetry in two dimensions”. This is a double of the type described in [§41.2](#). There

are several possible interpretations of the code “(2, 2) supersymmetry in two dimensions”, and we single out one that carries an involution. In [Remark 43.18](#) we describe a variation with nonzero central charges that is also important.

The involution gives rise to an involution on quantum field theories of this type, both theories in Minkowski spacetime and Wick-rotated theories. The mechanism of the involution in the Wick-rotated case is the same as the one that gives rise to the renormalization flow, as we point out in [§43.3](#).

We then examine twistings for this super relativistic type, of which there are two distinguished topological twists: the *A-twist* and the *B-twist*. They are exchanged by mirror symmetry. We conclude by briefly telling some of the main characters in the case of supersymmetric σ -models, which is what gives rise to many of the ensuing developments in algebraic geometry.

43.1 The super relativistic symmetry type in two dimensions and its involution

In [§41.2](#) we discussed supersymmetry in $n = 2$ dimensions with odd translations of dimension two. Recall that in two dimensions the vector representation

$$(43.1) \quad V = \mathbb{R}^{1,1} = V_+ \oplus V_-$$

is reducible; see [Figure 117](#). In that figure ∂_0 is timelike, ∂_1 is spacelike, and ∂_+, ∂_- , defined in [\(41.8\)](#), form a basis of null vectors. The group $\mathrm{SO}_{1,1}^\uparrow \cong \mathbb{R}^{>0}$ acts on V_+ with weight $+1$ and on V_- with weight -1 . The double cover $\mathrm{Spin}_{1,1}$ is isomorphic to $\mu_2 \times \mathbb{R}^{>0}$. The central element $\epsilon \in \mu_2$ acts trivially on V .

Our interest now is the double of the supersymmetry in [§41.2](#). Let S^+, S^- be real lines equipped with isomorphisms

$$(43.2) \quad \begin{aligned} (S_+)^{\otimes 2} &\xrightarrow{\cong} V_+ \\ (S_-)^{\otimes 2} &\xrightarrow{\cong} V_- \end{aligned}$$

and set

$$(43.3) \quad \mathbb{S}^* = S_+ \otimes \mathbb{R}^2 \oplus S_- \otimes \mathbb{R}^2.$$

The relativistic symmetry type is

$$(43.4) \quad G_2 = \mathrm{Spin}_{1,1} \times (\mathrm{SO}_2)_+ \times (\mathrm{SO}_2)_- / \langle (\epsilon, -1, -1) \rangle \longrightarrow \mathrm{O}_{1,1}^\uparrow$$

where the homomorphism is projection onto the first factor followed by the inclusion $\mathrm{SO}_{1,1}^\uparrow \hookrightarrow \mathrm{O}_{1,1}^\uparrow$. Let $\mathcal{G}_2$ be the affine group that corresponds to the linear group G_2 .

In the super relativistic symmetry type defined by the super Lie group $(\mathcal{G}_2, \mathrm{Lie} \mathcal{G}_2 \oplus \mathbb{S}^*)$, the central element in the group $\mathrm{Spin}_{1,1} \cong \mu_2 \times \mathbb{R}^{>0}$ acts on $S_\pm$ as -1 and $\mathbb{R}^{>0}$ acts on $S_\pm$ with

weight ± 1 ; the group $(\mathrm{SO}_2)_\pm$ acts on the corresponding factor of $\mathbb{R}^2$ by the vector representation. The bracketing relations are similar to (41.9), which in the obvious notation has nonzero brackets

$$(43.5) \quad \begin{aligned} [s_+ \otimes e_i, s_+ \otimes e_j] &= -2\delta_{ij}\partial_+ \\ [s_- \otimes e_i, s_- \otimes e_j] &= -2\delta_{ij}\partial_- \end{aligned}$$

We have set the central charges to zero for convenience, but one can include them in the discussion as well; see Remark 43.18 below. This super relativistic symmetry type is one possible incarnation of “(2, 2) supersymmetry in two dimensions”.

Now Wick rotate. The Euclidean group

$$(43.6) \quad G_2^E = \mathrm{Spin}_2 \times \mathbb{T}_+ \times \mathbb{T}_- / \langle (\epsilon, -1, -1) \rangle$$

acts on the complexified space of odd translations

$$(43.7) \quad \mathbb{S}_\mathbb{C}^* = S_\mathbb{C}^+ \otimes \mathbb{C}^2 \oplus S_\mathbb{C}^- \otimes \mathbb{C}^2.$$

Write $Q_+, \bar{Q}_+$ for the basis of $S_\mathbb{C}^+ \otimes \mathbb{C}^2$ with

$$(43.8) \quad \begin{aligned} Q_+ &= s_+ \otimes \frac{e_1 + ie_2}{\sqrt{2}} \\ \bar{Q}_+ &= s_+ \otimes \frac{e_1 - ie_2}{\sqrt{2}} \end{aligned}$$

There is a similar basis $Q_-, \bar{Q}_-$ of $S_\mathbb{C}^- \otimes \mathbb{C}^2$. The nonzero brackets are

$$(43.9) \quad \begin{aligned} [Q_+, \bar{Q}_+] &= -2\partial_+ \\ [Q_-, \bar{Q}_-] &= -2\partial_- \end{aligned}$$

Identifying $\mathrm{Spin}_2 \cong \mathbb{T}$, the action of $(\mu, \lambda_+, \lambda_-) \in \mathrm{Spin}_2 \times \mathbb{T}_+ \times \mathbb{T}_-$ on $\mathbb{S}_\mathbb{C}^*$ is

$$(43.10) \quad \begin{aligned} (\mu, \lambda_+, \lambda_-) \cdot Q_+ &= \mu\lambda_+Q_+ \\ (\mu, \lambda_+, \lambda_-) \cdot \bar{Q}_+ &= \mu\bar{\lambda}_+\bar{Q}_+ \\ (\mu, \lambda_+, \lambda_-) \cdot Q_- &= \bar{\mu}\lambda_-Q_- \\ (\mu, \lambda_+, \lambda_-) \cdot \bar{Q}_- &= \bar{\mu}\bar{\lambda}_-\bar{Q}_- \end{aligned}$$

The root of mirror symmetry is an involution α of the super Lie group $(\mathcal{G}_2, \mathrm{Lie} \mathcal{G}_2 \oplus \mathbb{S}^*)$. This automorphism α acts as $A \mapsto A^{-1}$ on $(\mathrm{SO}_2)_-$ while simultaneously acting as a reflection on the $\mathbb{R}^2$ factor in the second summand $S_- \otimes \mathbb{R}^2$ of (43.3), say the reflection $e_1 \mapsto e_1, e_2 \mapsto -e_2$; it is the identity otherwise. This induces an involution α^E of the Wick-rotated cs Lie group $(\mathcal{G}_2^E, \mathrm{Lie} \mathcal{G}_2^E \oplus \mathbb{S}_\mathbb{C}^*)$ that acts as

$$(43.11) \quad \begin{aligned} \lambda &\longmapsto \lambda^{-1} && \text{on } \mathbb{T}_-, \\ Q_- &\longmapsto \bar{Q}_- && \text{on } S_\mathbb{C}^- \otimes \mathbb{C}^2. \end{aligned}$$

43.2 Mirror symmetry

Mirror symmetry acts as an involution on supersymmetric quantum field theories with the super relativistic symmetry type of §43.1. This is true both for Wightman theories in Minkowski spacetime and for Wick-rotated theories.

In the former case, a relativistic quantum mechanical theory is a positive energy unitary representation of the super Lie group $(\mathcal{G}_2, \text{Lie } \mathcal{G}_2 \oplus \mathbb{S}^*)$. The mirror theory simply precomposes this representation with the involution α . The other data that defines a Wightman quantum field theory (§21.3) is unchanged.

As a preliminary to constructing an involution on Wick-rotated theories, observe that if $\pi: P \rightarrow M$ is a principal G -bundle for a Lie group G , and if $\alpha^E: G \rightarrow G$ is an automorphism of G , then there is a transformed principal G -bundle which is the same map $\pi: P \rightarrow M$ but with G -action on P precomposed with α^E . Applied to the group $G = G_2^E$, this induces an automorphism of the bordism category in (42.7) and hence to Wick-rotated theories. Recall that we view Wick-rotated supersymmetric theories as representations of a bordism category of cs manifolds. The involution on the bordism category of cs manifolds also acts on the odd vector bundle (42.8) via the exchange $Q_- \leftrightarrow \overline{Q}_-$ that is part of the automorphism of the cs Lie group $(\mathcal{G}^E, \text{Lie } \mathcal{G}_2^E \oplus \mathbb{S}_{\mathbb{C}}^*)$.

43.3 Digression: renormalization group flow

This is a formal remark to say that precomposition with an automorphism of the bordism category is ubiquitous in Wick-rotated field theory. As an important case, in the general context of **Axiom System 23.42** in a physical theory we should assume that the sheaf $\mathcal{F}$ of fields comes equipped with a fibering

$$(43.12) \quad \lambda: \mathcal{F} \longrightarrow \text{Riem},$$

where Riem is the sheaf of Riemannian metrics. For background fields from symmetry, (43.12) is induced from the Euclidean symmetry type. Now the sheaf Riem carries an action of $\mathbb{R}^{>0}$: namely, $u \in \mathbb{R}^{>0}$ scales a Riemannian metric g by $g \mapsto u^2 g$. Assume that this 1-parameter group of automorphisms lifts to a 1-parameter group of automorphisms of $\mathcal{F}$:

$$(43.13) \quad \begin{array}{ccc} \mathcal{F} & \xrightarrow{\varphi_u} & \mathcal{F} \\ \downarrow & & \downarrow \\ \text{Riem} & \xrightarrow{g \mapsto u^2 g} & \text{Riem} \end{array}$$

The *renormalization group flow* is the 1-parameter group of theories obtained by precomposition with the flow induced on the bordism category $\text{Bord}_n(\mathcal{F})$ by (43.13).

43.4 Twisted theories and mirror symmetry

There are many square-zero odd translations in (43.7). For example, each of Q_+ , Q_- is square-zero. However, neither is topological: the image of bracketing with either gives a 1-dimensional

subspace of the complexified space of infinitesimal even translations. Rather, among the topological square-zero odd translations we isolate two:

$$(43.14) \quad \begin{aligned} Q_A &:= Q_+ + Q_- \\ Q_B &:= Q_+ + \overline{Q}_- \end{aligned}$$

These give rise to the *A-twist* and *B-twist*, respectively.

We determine the data in [Definition 42.12](#) for these two topological square-free odd translations. From [\(43.10\)](#) we see that the stabilizer subgroup of Q_A inside G_2^E is

$$(43.15) \quad \begin{aligned} \mathrm{SO}_2 \cong \mathrm{Spin}_2 / \langle (\epsilon) \rangle &\longrightarrow \mathrm{Spin}_2 \times \mathbb{T}_+ \otimes \mathbb{T}_- / \langle (\epsilon, -1, -1) \rangle \\ \mu &\longmapsto (\mu, \quad \bar{\mu}, \quad \mu) \end{aligned}$$

where we identify $\mathrm{Spin}_2 \cong \mathbb{T}$. Similarly, the stabilizer subgroup of Q_B is

$$(43.16) \quad \begin{aligned} \mathrm{SO}_2 \cong \mathrm{Spin}_2 / \langle (\epsilon) \rangle &\longrightarrow \mathrm{Spin}_2 \times \mathbb{T}_+ \otimes \mathbb{T}_- / \langle (\epsilon, -1, -1) \rangle \\ \mu &\longmapsto (\mu, \quad \bar{\mu}, \quad \bar{\mu}) \end{aligned}$$

So the twisted theories are theories of oriented 2-manifolds. To verify that Q_A and Q_B are topological compute

$$(43.17) \quad \begin{aligned} [Q_A, \overline{Q}_+] &= [Q_B, \overline{Q}_+] = -2\partial_+ \\ [Q_A, \overline{Q}_-] &= [Q_B, Q_-] = -2\partial_- \end{aligned}$$

Hence the maps $[Q_A, -]$ and $[Q_B, -]$ are surjective. Finally, the fixed subspace for each of Q_A, Q_B is two-dimensional. For Q_A it is spanned by Q_+ and Q_- ; for Q_B it is spanned by Q_+ and $\overline{Q}_-$.

The involution α^E in [\(43.11\)](#) exchanges Q_A and Q_B .

Now consider the twisted Wick-rotated theories, as in [Definition 42.18](#). Let $Q = Q_A$ or $Q = Q_B$. The super topological vector space attached to S^1 carries odd differentials $\hat{Q}_+$ and either $\hat{Q}_-$ ($Q = Q_A$) or $\hat{\overline{Q}}_-$ ($Q = Q_B$) induced from the basis elements of the fixed subspace. If these operators are Fredholm, then they have well-behaved cohomologies. The cohomology of Q is called the *chiral rings* associated to the twisted theory. One argues that this cohomology is independent of the Riemannian metric, so it is a topological invariant of the theory. Observe that the $(\mathbb{T}_+ \times \mathbb{T}_-)$ -action induces a compatible $(\mathbb{Z} \times \mathbb{Z})$ -grading, or at least a grading by a $(\mathbb{Z} \times \mathbb{Z})$ -torsor.

Remark 43.18. There are theories of this type with nonzero central charges, so as in [Definition 40.31](#) with a modified relativistic symmetry type in which there is a nonzero central Lie algebra. We already gave an example with less supersymmetry in [\(41.7\)](#). There is a possible two dimensional space of central charges, though a given theory may only exhibit a subspace of these. If the central charges are nonzero, then the internal symmetry group is *not* [\(43.4\)](#) since the central charges must be central (see [Remark 40.36\(4\)](#)). As an example consider the relativistic symmetry type

$$(43.19) \quad G'_2 = \mathrm{Spin}_{1,1} \times \mathbb{R}^2 \times \mathrm{SO}_2 / \langle (\epsilon, 0, -1) \rangle \longrightarrow \mathrm{O}_{1,1}^\uparrow$$

where $\mathrm{SO}_2 \subset (\mathrm{SO}_2)_+ \times (\mathrm{SO}_2)_-$ is the diagonal subgroup. The $\mathbb{R}^2$ factor is central in the associated super Lie group $(\mathcal{G}'_2, \mathrm{Lie} \mathcal{G}'_2 \oplus \mathbb{S}^*)$ and there are new nonzero brackets among odd translations, in addition to (43.5).¹²⁸ Let ζ_1, ζ_2 be a basis of $\mathrm{Lie}(\mathbb{R}^2)$, and let $\langle -, - \rangle$ and $\omega(-, -)$ be the standard inner product and symplectic form on $\mathbb{R}^2$, respectively. Then the additional nonzero bracket is

$$(43.20) \quad [s_+ \otimes v_1, s_- \otimes v_2] = 2\langle v_1, v_2 \rangle \zeta_1 + 2\omega(v_1, v_2) \zeta_2, \quad v_1, v_2 \in \mathbb{R}^2.$$

For the modified Wick-rotated cs Lie group $((\mathcal{G}'_2)^E, \mathrm{Lie}(\mathcal{G}'_2)^E \oplus \mathbb{S}^*_\mathbb{C})$ this implies

$$(43.21) \quad \begin{aligned} [Q_+, Q_-] &= 0 \\ [Q_+, \overline{Q}_-] &= 2(\zeta_1 - i\zeta_2) \end{aligned}$$

Two consequences: there is no mirror symmetry involution; and whereas Q_A is square-zero, Q_B is not. (A better statement is that mirror symmetry maps the modified super relativistic symmetry type (43.19) to a close cousin in which Q_B is square-zero and Q_A is not.)

It follows from (43.10) that the stabilizer subgroup of Q_A in G'_2 is the trivial group $1 \subset G'_2$, hence the Q_A -twisted theory with nonzero central charge (43.20) is a theory of *framed manifolds*.

43.5 Supersymmetric σ -models

This is a large topic and we only give a few paragraphs of introduction. For more see [W11, HKKPTVVZ, GY], the references therein, and more recent works. related to the basic σ -model: gauge theories, gauged σ -models, Landau–Ginzburg models, ...

In a σ -model the fluctuating field is a map into a fixed manifold M . In a supersymmetric σ -model there is also an odd fluctuating field. For the super relativistic symmetry type of §43.1, the manifold M must be Kähler. In the A-twist of this model, the chiral ring is isomorphic to the complex cohomology $H^\bullet(M; \mathbb{C})$. It arises as the cohomology of $d = \partial + \overline{\partial}$ on differential forms; the operators $\partial, \overline{\partial}$ are $\hat{Q}_+, \hat{Q}_-$. The bigrading pointed out before Remark 43.18 is the usual bigrading on complex cohomology induced from the Kähler structure. The main interest here is the ring structure, which we have not discussed in this version of the notes: it is the *quantum cohomology* product.

The B-twist is only defined if the manifold M is *Calabi–Yau*, which here means that it is Kähler with vanishing Ricci curvature. Then the chiral ring is

$$(43.22) \quad \bigoplus_{q,p} H^q(M; \Omega^p).$$

Mirror symmetry predicts that the A-twist of the supersymmetric σ -model into a Calabi–Yau manifold M is isomorphic to the B-twist of another supersymmetric model. In many cases the latter is a supersymmetric σ -model into a Calabi–Yau manifold $\widetilde{M}$, but in many cases it is a different theory. Many mathematical developments have ensued from this prediction.

¹²⁸The representation of the diagonal SO_2 on the tensor product $(S_+ \otimes \mathbb{R}^2) \otimes (S_- \otimes \mathbb{R}^2) \subset \mathrm{Sym}^2 \mathbb{S}^*$ has a 2-dimensional subspace invariant under G'_2 , and it is that subspace that can be identified with the space of central charges.

Lecture 44: Introduction to Seiberg–Witten theory

The early 1990s saw great advances in understanding the *dynamics* of supersymmetric quantum field theories. By ‘dynamics’ we mean the following. A full quantum field theory is defined at all distance scales, in particular at small distances, which in a relativistic theory is equivalent to high energies. One can consider it to be the “question” to which an “answer” is an effective description of the low energy theory. That description is what is meant by ‘dynamics’; see the discussion in [W11, §1.0]. Seiberg in particular made great strides in analyzing theories with $N = 1$ supersymmetry, as exposed in the lectures [Sei1]. Seiberg–Witten [SeWi] built on these techniques to analyze $N = 2$ supersymmetric gauge theories. In [W7] Witten had already identified Donaldson invariants as correlation functions in a twisted Wick-rotated version of this theory. (His paper introduced twistings for this purpose.) In [W12] he applies the Seiberg–Witten work to re-express Donaldson invariant in terms of new nonlinear partial differential equations in four-dimensional Riemannian geometry: the *Seiberg–Witten equations*. We have no pretense that in one lecture we can do justice to these developments. The lecture course [W11] tells the story in much more detail, and there are many other expositions available as well, such as [Mo].

We begin in §44.1 with a lightning summary of Donaldson’s construction [Do] of invariants of smooth 4-manifolds; the book [DK] has a detailed account. On a 4-manifold X these invariants are intersection numbers on the moduli space $\mathcal{M}_X$ of anti-self-dual connections as embedded in the moduli space $\mathcal{B}_X$ of all connections.

For the rest of the lecture we turn to the physics story. We begin by recalling the relevant super relativistic symmetry type, and we list the relevant particle multiplet representations. Then we introduce the classical Yang–Mills theory in Minkowski spacetime. It has a superspacetime formulation that only uses a single function, the *holomorphic prepotential*, and this highly constrains the deformations that occur in the effective low energy dynamics [Sei2]. The geometry of the holomorphic prepotential, called *special Kähler geometry*, is the subject of §44.4. A global expression is as the base of an algebraic integrable system [DoWi, F15].

So the problem of solving for the dynamics is one of finding the appropriate algebraic integrable system. That is precisely what Seiberg–Witten do in [SeWi]. Their solution is Conclusion 44.25. We defer to the original reference and their lecture series for the detailed arguments.

In the final Section 44.6 we apply their work to the twisted Wick-rotated theory. The basic principles were explained in Lecture 42. We first outline Witten’s identification in [W7] of Donaldson invariants as correlation functions. The Seiberg–Witten solution re-expresses them as correlation functions in the low-energy theory. On a 4-manifold X with $b_2^+(X) > 1$ the situation is simplest, as is well-known, and in this case we indicate the emergence of the Seiberg–Witten equations and of the formula that uses them to express Donaldson invariants. This was the catalyst for many mathematical developments.

We report here on many physical arguments that do not rest on an infrastructure of mathematical definitions, theorems, and proofs.

44.1 Donaldson invariants of 4-manifolds

The kernels of linear partial differential operators, such as the Laplace operator and the Dirac operator, produce topological invariants of smooth manifolds. Nonlinear analogs of this idea have had profound applications in geometry. The 4-dimensional instanton equations that Donaldson exploits are a prime example. They are based on self-duality in 4-dimensional Riemannian geometry [AHS].

The starting point is an exceptional isomorphism of Lie groups

$$(44.1) \quad \text{Spin}_4 \cong \text{SU}_2 \times \text{SU}_2$$

which has the immediate consequence

$$(44.2) \quad \text{SO}_4 \cong (\text{SU}_2 \times \text{SU}_2) / \langle (-1, -1) \rangle.$$

The spin representation $S = S^+ \oplus S^-$ is the sum of the vector representations of the two SU_2 factors in (44.1). The splitting (44.2) is reflected in the splitting of the representation of SO_4 on the second exterior power of the vector representation:

$$(44.3) \quad \Lambda^2 = \Lambda^{2,+} \oplus \Lambda^{2,-}$$

This is the eigenspace decomposition of the Hodge star operator. Observe that $\Lambda^{2,\pm} \cong \text{Sym}^2 S^\pm$.

On an oriented 4-manifold X , the decomposition (44.3) induces a decomposition of 2-forms

$$(44.4) \quad \Omega_X^2 = \Omega_X^{2+} \oplus \Omega_X^{2-}$$

and it extends to a decomposition of vector bundle-valued 2-forms as well. Thus if A is a connection on X , then the curvature $F_A = F_A^+ + F_A^-$ decomposes as the sum of self-dual and anti-self-dual components. The anti-self-dual equation

$$(44.5) \quad F_A^+ = 0$$

imposes that the curvature be purely anti-self-dual. This is an elliptic equation for the connection A , once one mod out by gauge transformations. (Recall from §18.1 that connections are only local when considered as objects in a groupoid, that is, when taking into account isomorphisms of connections.) Ellipticity implies that the space $\mathcal{M}_X \subset \mathcal{B}_X$ of isomorphism classes of anti-self-dual connections is finite dimensional inside the space of isomorphism classes of all connections. One proves that, after a perturbation, $\mathcal{M}_X$ is a smooth manifold or at least has controlled singularities. It is orientable as well. Crucially, one knows its ends and therefore can control its noncompactness. Each of these three properties of $\mathcal{M}_X$ requires significant work to prove. See [DK] for a detailed account of the theory.

Donaldson [Do] uses the moduli space $\mathcal{M}_X$ to define invariants of X as follows. First, there is a universal bundle over the Cartesian product $\mathcal{B}_X \times X$, and it has a characteristic class of degree 4 on

this product. Slanting with a homology class in X yields cohomology classes on $\mathcal{B}_X$. In principle the invariants are cup products of these evaluated on the fundamental homology class of $\mathcal{M}_X \subset \mathcal{B}_X$, but since $\mathcal{M}_X$ is not compact there is no such fundamental class. Donaldson controls this noncompactness using representative dual finite codimension cycles for these cohomology classes, and then intersecting them with each other and with $\mathcal{M}_X$ to define the invariants.

The story of this lecture begins with a realization of these intersection numbers as correlation functions in a twisted Wick-rotated supersymmetric quantum field theory. Eventually they appear (formally) as integrals over $\mathcal{B}_X$ that can be identified with the intersection numbers Donaldson defines. We now turn to the physics that leads to these correlation functions.

44.2 Representations of $N = 2$ supersymmetry in 4 dimensions

The super relativistic symmetry type $G_4 \rightarrow O_{1,3}^\uparrow$ we use was described in §41.3, specifically in (41.14). Notice that it includes a 2-dimensional space of central charges, as in (41.17). In the complex basis used there the central charge S is a complex number, and as in (??) there is a BPS bound

$$(44.6) \quad m \geq |S|$$

for representations of the corresponding super Lie group $(\mathcal{G}_4, \text{Lie } \mathcal{G}_4 \oplus \mathbb{S}^*)$, where m is the mass of the representation.

We list the relevant irreducible representations of this super Lie group in a table similar to (41.26). For massless representations the little group is Spin_2 , though there are internal symmetries as well, and we organize representations according to the spin j of the representation of Spin_2 :

j	-1	$-1/2$	0	$1/2$	1
massless half hypermultiplet		1	2	1	
massless hypermultiplet		2	4	2	
massless vector multiplet	1	2	2	2	1

The half hypermultiplet representation does not have a realization in field theory. The pure Yang–Mills theory we consider does not have any massless hypermultiplets; we only encounter massless vector multiplets. When restricted to the non-super Lie subgroup, the massless vector multiplet splits as a sum of an ordinary massless vector representation, two spinor representations, and two scalar representations. The linear version of the Yang–Mills theory below (which is for gauge group $\mathbb{T}$) realizes this representation in field theory. Note that the two scalar representations appear as a single complex scalar field.

The classification of BPS representations is the same as in (44.7).

There is a single relevant massive representation of this super Lie group:

j	-1	$-1/2$	0	$1/2$	1
massive vector multiplet	1	4	6	4	1

Here the little group is Spin_3 and we are showing the decomposition under $\text{Spin}_2 \subset \text{Spin}_3$ according to the spin j .

44.3 Classical supersymmetric Yang–Mills theory

Let the gauge group be SU_2 , though the theory exists for any Lie group. Fix also an SU_2 -invariant inner product $\langle -, - \rangle$ on the Lie algebra $\mathfrak{su}_2$. We indicate a family of theories parametrized by

$$(44.9) \quad \tau_0 = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}, \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}, \quad g \in \mathbb{R}^{>0}.$$

The theory has a lagrangian description. The fluctuating fields on Minkowski spacetime $\mathbb{M}^4$ are nonlinear version of linear fields that correspond (as in [Lecture 19](#)) to the fields in the massless vector multiplet [\(44.7\)](#). Namely, there is an SU_2 -connection on $\mathbb{M}^4$, a scalar field $\sigma: \mathbb{M}^4 \rightarrow \mathfrak{su}_2 \otimes \mathbb{C}$ with values in the complexified Lie algebra (or, more properly, complexified adjoint bundle), and there are spinor fields that we do not detail. Set the spinor fields to zero to obtain the bosonic lagrangian:^{[129](#)}

$$(44.10) \quad \left\{ -\frac{1}{2}|F_A|^2 + \langle d_A \sigma, d_A \bar{\sigma} \rangle - \frac{1}{2} |[\bar{\sigma}, \sigma]|^2 \right\} |d^4 x| + \frac{\theta}{16\pi^2} \langle F_A \wedge F_A \rangle.$$

The full lagrangian can be found in [\[DF3, \(10.45\)\]](#).

Remark 44.11. This theory has a formulation in *Minkowski superspacetime*, which is an affine supermanifold $\mathbb{M}^{4|8}$. The superspacetime formulation makes the entire super Lie group symmetry manifest, and it is much simpler: there is a single complex Lie-algebra valued scalar field Σ , which is the curvature of a constrained connection, and the action is determined by a single holomorphic function on the complex Lie algebra $\mathfrak{su}_2 \otimes \mathbb{C}$:

$$(44.12) \quad \mathcal{F}(\Sigma) = \frac{\tau}{2} \langle \Sigma, \Sigma \rangle$$

The function $\mathcal{F}$ is called the *holomorphic prepotential*. See [\[DF3, §10.2\]](#) for details.

To determine the classical vacua, choose a splitting $\mathbb{M}^4 = \mathbb{R} \times \mathbb{E}^3$ of spacetime into time $\times$ space, restrict to static fields (fields invariant under time translation), and minimize the integral of the *energy density* over a slice $\{t\} \times \mathbb{E}^3$. The energy density works out to be the lagrangian density with the sign of the potential energy term reversed; see [\[DF2, §2.10\]](#) for an exposition of energy in lagrangian field theory. That integral—the *energy* of the classical static field—is independent of t . The minima are the classical vacua, static fields with zero total energy.^{[130](#)} From the first term we deduce that A is flat, hence is a product connection. From the second we deduce that σ is a constant. Taking into account automorphisms of connections, then, the moduli space of classical vacua^{[131](#)} is

$$(44.13) \quad \left\{ \text{minima of } V = \frac{1}{2} |[\bar{\sigma}, \sigma]|^2 \right\} / SU_2$$

¹²⁹The last term is a 4-form, so an orientation is required to turn it into a density.

¹³⁰Supersymmetry gives zero as a lower bound of energy; see [Proposition 38.4](#).

¹³¹One speaks of a moduli space of vacua when in fact it is a moduli space of theories; see the discussion in [\[W11, §1.0\]](#).

where $\sigma \in \mathfrak{su}_2 \otimes \mathbb{C}$ and SU_2 acts by conjugation. Write $\sigma = \gamma + i\delta$ for $\gamma, \delta \in \mathfrak{su}_2$. Then since the minima occur where $V = 0$, we must have $[\gamma, \delta] = 0$, so γ, δ are simultaneously diagonalizable. Hence up to conjugacy we can diagonalize σ as

$$(44.14) \quad \sigma = \frac{1}{\sqrt{2}} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad a \in \mathbb{C}.$$

There is a residual action of the Weyl group μ_2 on $\mathbb{C}$: it exchanges a and $-a$. In other words,

$$(44.15) \quad u = \text{tr } \sigma^2 = a^2 \in \mathbb{C}$$

parametrizes the classical vacua. In the business the space $\mathbb{C}_u$ of classical vacua is called the *u-plane*.

Remark 44.16.

- (1) One should regard this as a stack rather than a space. At each point there is a low-energy theory that approximates the original theory at low energies. The fields are a connection with structure group the automorphism group at that point, and a scalar field with values in the tangent space at that point. The stabilizer subgroup is isomorphic to $\mathbb{T}$ at every nonzero vacuum $u \neq 0$. At these points it is the *Higgs mechanism* that gives the other components of the SU_2 -connection a mass; see [DF2, Example 2.204] for an illustration. (The Higgs mechanism dates from the 1960s and is an important feature of the Standard Model.) At $u = 0$ there are additional massless fields, including a full SU_2 -connection.
- (2) A point on a positive dimensional component of a moduli stack of theories represents a *gapless* quantum field theory: there are massless scalar fields.

44.4 Special Kähler manifolds and integrable systems

Supersymmetry constrains the geometry of the moduli space of vacua of a theory. In the case at hand that moduli space is a *special Kähler manifold*. Special Kähler geometry was initiated in this context in [SiTo, G]; see also [Sei2]. The following global intrinsic definition was introduced in [F15].

Definition 44.17. Let M be a Kähler manifold with complex structure $I: TM \rightarrow TM$. A *special Kähler structure* on M is a flat torsionfree symplectic covariant derivative ∇ on the tangent bundle $TM \rightarrow M$ such that

$$(44.18) \quad d_{\nabla} I = 0,$$

where $d_{\nabla}: \Omega_M^1(TM) \rightarrow \Omega_M^2(TM)$ is induced from the covariant derivative.

The metric and the special Kähler structure on an n -dimensional complex manifold are encoded locally, but not uniquely, in a single $\mathbb{C}$ -valued holomorphic function $\mathcal{F}$ of n variables $z^1, \dots, z^n$

called the *holomorphic prepotential*; it is what appears in the physics literature. The Kähler form is

$$(44.19) \quad \omega = \frac{\sqrt{-1}}{2} \operatorname{Im} \left(\frac{\partial^2 \mathcal{F}}{\partial z^i \partial \bar{z}^j} \right) dz^i \wedge \overline{dz}^j.$$

This imposes the positivity condition

$$(44.20) \quad \operatorname{Im}(\tau_{ij}) := \operatorname{Im} \left(\frac{\partial^2 \mathcal{F}}{\partial z^i \partial \bar{z}^j} \right) > 0.$$

Remark 44.21.

- (1) The holomorphic prepotential $\mathcal{F}$ is essentially the lagrangian in the superspace formulation of the vector multiplet. The basic example is (44.12).
- (2) In a family of these quantum field theories parametrized by a special Kähler manifold, the lagrangian description is local on the parameter space. One covers the parameter space by charts with a local lagrangian, and then overlaps are related by electromagnetic duality; see [F15, §5] for an elaboration.
- (3) It is a feature of supersymmetric quantum field theories that special geometries are attached to moduli spaces. The pertinent geometric structure depends on the dimension and the details of the super relativistic symmetry type. It is a powerful constraint on dynamics.

In the presence of integrality, the matrix (τ_{ij}) that appears in (44.20) is the period matrix of a family of abelian varieties. This locates special Kähler manifolds as bases of holomorphic integrable systems, as we now explain.

We begin with a reminder of the real case, in which the base geometry is a *flat affine structure*. A *real integrable system* is a smooth fiber bundle¹³² $\pi: X \rightarrow M$ together with a symplectic structure on X such that the fibers of π are lagrangian tori in X . The symplectic structure induces an isomorphism $T(X/M) \xrightarrow{\cong} \pi^* T^* M$, where $T(X/M) = \ker \pi_* \rightarrow X$ is the vertical tangent bundle. Thus we can identify the tangent space $T_m M$, $m \in M$, as the cohomology space $H^1(X_m; \mathbb{R})$ of the torus fiber $X_m = \pi^{-1}(m)$. Integral cohomology $H^1(X_m; \mathbb{Z}) \subset H^1(X_m; \mathbb{R})$ induces a full lattice inside the tangent bundle, which determines a flat torsionfree connection on M , the flat affine structure.

The holomorphic analog induces a special Kähler structure rather than a flat affine structure.

Definition 44.22. An *algebraic integrable system* is a holomorphic fiber bundle $\pi: X \rightarrow M$, a holomorphic symplectic form $\eta \in \Omega_X^{2,0}$, and a smoothly varying class $[\rho_m] \in H^{1,1}(X_m) \cap H^2(X_m; \mathbb{Z})$, $m \in M$, such that the fibers of π are Lagrangian tori and $[\rho_m]$ is a positive polarization, i.e., gives the fiber X_m the structure of a torsor over an abelian variety.

In this lecture we only need the special case in which $\dim M = 1$ and the fibers of π are elliptic curves, though we do allow singular fibers. In that case the period map is a holomorphic function

$$(44.23) \quad \tau: M \longrightarrow \mathbb{H}/\operatorname{PSL}_2 \mathbb{Z}$$

¹³²One should allow singular fibers.

Analogous to the real case, one constructs the special Kähler covariant derivative from the integrable system [F15, §3]. The role of integrable systems in special Kähler geometry was introduced in [DoWi].

44.5 The Seiberg–Witten solution

The conclusion of the preceding discussion is that a supersymmetric field theory of the type we are studying is specified by an algebraic integrable system. For the “classical” theory in §44.3 the base of the integrable system is the u -plane $\mathbb{C}$ with a singularity at $u = 0$. On the complement $\mathbb{C} \setminus \{0\}$, the integrable system is constant in the sense that the fiber elliptic curve has period given by the constant τ_0 in (44.9). Here the function (44.23) into the moduli stack of elliptic curves has been lifted, but there is an indeterminacy of $\{\pm 1\} = \mu_2 \subset \mathrm{SL}_2\mathbb{Z}$. By an argument [W11, §17.2.7] that we do not reproduce here, there is a holonomy of -1 around the origin. In other words, the family is globally constant when lifted to the “ a -plane” for the parameter a in (44.14), but the descent to the u -plane has monodromy of order two. The Kähler metric is

$$(44.24) \quad \mathrm{Im}\tau \, da \, d\bar{a} = \frac{\mathrm{Im}\tau \, du \, d\bar{u}}{|u|^2}.$$

In [SeWi] Seiberg and Witten determine the algebraic integrable system that describes the low energy behavior of the supersymmetric Yang–Mills theory.

Conclusion 44.25. *The low energy theory is described by the universal elliptic curve $\pi: X \rightarrow M$ for the modular group $\Gamma(2) \subset \mathrm{SL}_2\mathbb{Z}$.*

Here $\Gamma(2)$ is the subgroup of matrices congruent to the identity modulo two. The base $M = \mathbb{C}_u$ has singularities at $u = \pm 1$. The Kähler metric on $\mathbb{C} \setminus \{-1, 1\}$ is not complete: the singularities at ± 1 are at finite distance. The moduli space M is the quantum deformation of the classical moduli space, and it describes the low energy physics as a family of supersymmetric Yang–Mills theories with gauge group the circle $\mathbb{T}$. There is no global lagrangian formulation, only local lagrangians that are transformed by locally constant maps to $\mathrm{SL}_2\mathbb{Z}$ on the overlaps. There are electric and magnetic charges that form a 2-dimensional lattice Λ_m at each theory $m \in M$, and this lattice is part of the geometry of the integrable system, as explained above. At the singularities $u = \pm 1$, a BPS hypermultiplet that is otherwise massive becomes massless.

Remark 44.26. I feel that more remains to be said about the structure of M as a stack—not as the stack that is the global quotient of the upper half plane by $\Gamma(2)$, but as the stack that describes the low energy theory. In particular, the automorphism group at each point is isomorphic to $\mathbb{T}$, but can we distinguish the automorphism groups at $u = \pm 1$?

There is much physics reasoning behind Conclusion 44.25; see [SeWi] and [W11, Lecture 18] for detailed expositions. A few of the principles are:

- (1) Quantum effects are bounded. Hence the quantum moduli space M has the same ends as the classical moduli space $\mathbb{C}_u$.
- (2) For large u the quantum and classical vacua are isomorphic, since the SU_2 theory is asymptotically free and the $\mathbb{T}$ theory is infrared free.

- (3) The monodromy of the family of elliptic curves at ∞ can be deduced from asymptotic freedom and a 1-loop computation.
- (4) Arguments about electromagnetism, in particular the physics of magnetic monopoles, play an important role.
- (5) Seiberg–Witten use perturbations to $N = 1$ supersymmetric Yang–Mills theory and the fact that the latter has two gapped vacua.

The Seiberg–Witten work has many physical ramifications, but our focus is on the topological consequences, to which we now turn.

44.6 Wick rotation, twisting, localization, and Donaldson $\rightarrow$ Seiberg–Witten

More details for most of what we say may be found in [W11, Lecture 19].

First, a general remark. On Minkowski spacetime $\mathbb{M}^4$ we have a family of quantum field theories parametrized by the moduli space M of Conclusion 44.25. However, when we work on a closed 4-manifold X we need to sum over the vacua, which means that correlation functions become integrals over M . This is because on the noncompact $\mathbb{M}^4$, at least near ∞ in M , the vacua are the values at ∞ in $\mathbb{M}^4$ of the scalar field σ , and it takes infinite action to change that value. However, on a compact oriented Riemannian manifold X there is no such action barrier and one must sum over all values of σ . It turns out that on a manifold with Betti number $b_2^+(X) > 1$, for the topological correlation functions that define Donaldson’s invariants the integral reduces to a sum of contributions at the singularities $u = \pm 1$. For the case $b_2^+ = 1$, there is a nontrivial integral over the u -plane; see [MW].

Formulating the Wick-rotated theory on an arbitrary oriented Riemannian 4-manifold X is achieved by the topological twisting discussed in §42.3.2. There is a distinguished complex square-free odd translation Q , and its stabilizer group (42.29) is SO_4 . On an oriented Riemannian 4-manifold, there is an odd operator $\hat{Q}$ that acts on the topological vector space of point observables as well as on defects with positive dimensional support. We argued in §42.5 that correlation functions of $\hat{Q}$ -closed observables are topological invariants, and furthermore that they only depend on the $\hat{Q}$ -cohomology class of the observables.

The correlation functions have a path integral expression as an integral over the supermanifold of fluctuating fields, the Wick-rotated and twisted counterparts to the connection A , scalar field σ , and spinor fields introduced in §44.2. On that infinite dimensional supermanifold, Q manifests as an odd vector field ξ_Q . A key assertion is that the integral localizes to the points where ξ_Q vanishes, that is, to the Q -fixed points on the space of fluctuating fields. This is a kind of Stokes’ theorem, and it can also be argued by scaling up the metric that the integral localizes. We refer to [Petal] and the references therein.

On the space of bosonic fields, the vanishing $\xi_Q = 0$ is expressed as first order differential equations called the *BPS equations*; they are a classical analog of equality in the BPS bound (44.6). In the SU_2 -theory described in §44.3, the BPS equations are

$$(44.27) \quad \begin{aligned} F_A^+ &= 0 \\ d_A \sigma &= 0 \end{aligned}$$

At an irreducible SU_2 -connection A we have $\sigma = 0$, and so (44.27) reduces to the anti-self-dual equation (44.5). This means that in the SU_2 -theory, the correlation functions we are interested in localize to integrals over the moduli space $\mathcal{M}_X$ discussed in §44.1. A more careful argument would use localization to (formally) identify the correlation functions with the intersection numbers that define the Donaldson invariant.

The relevant Q -closed observables were already identified in [W7]. They appear in a semi-classical description as functions on the infinite dimensional supermanifold of fields. The point observables are invariant polynomials on $\mathfrak{su}_2$ applied to σ . The basic one is $\text{tr } \sigma^2$. Then, as in §42.6, one constructs descendant defects supported on submanifolds of X . These too already appear in [W7]. They correspond exactly to the cohomology classes on $\mathcal{B}_X$ constructed by slant product in §44.1. The integral of their product, once localized to the vanishing set of ξ_Q , matches the definition of Donaldson's invariants.

Remark 44.28. As explained in §44.1, careful analysis of the nonlinear PDE (44.5) is required to construct these invariants rigorously. The physics approach assumes that the correlation functions are well-defined, and then uses localization to argue that these correlation functions agree with the intersection numbers defined by Donaldson.

Up to this point we have only used the high energy SU_2 -theory. The Seiberg–Witten solution gives a new computation of the same correlation functions in the low energy theory. Namely, since the correlation functions in question are topological invariants, they should be computable at any scale. The SU_2 -theory is the description at high energy, or short distance. The same correlation functions should be computable in the low energy theory, i.e., the theory in the Seiberg–Witten solution Conclusion 44.25. The task then is to locate the appropriate correlation functions in that theory, and then apply localization to derive a semi-classical description as intersection numbers. This is precisely what is done in [W12]. We give only a brief summary here.

Remark 44.29. The shape of this argument is familiar from the heat equation approach to index theory for linear elliptic differential operators [ABP]. Namely, for a Dirac-type operator D on a closed manifold one considers the 1-parameter family of heat operators $C_t = e^{-tD^2}$, $t \in \mathbb{R}^{>0}$. As $t \rightarrow 0$ the heat operator approaches projection onto the kernel of D , which shows that the (super) trace of C_t approaches the index of D . One proves that this trace is independent of t , so it can be computed in the $t \rightarrow \infty$ limit. This limit localizes on the manifold and leads to the formula for the index as an integral of characteristic differential forms.

The low energy theory at each of $u = \pm 1$ is a supersymmetric Yang–Mills theory with gauge group $\mathbb{T}$, so has a massless vector multiplet, and there is a massless hypermultiplet as well; see (44.7). (The latter is the BPS hypermultiplet that becomes massless; see the text before Remark 44.26.) The theory has a description in terms of fluctuating fields and an lagrangian analogous to that in §44.3. An argument with electromagnetic duality gives the detailed description, in which the fluctuating field is a differential spin^c structure rather than a $\mathbb{T}$ -connection; see [W13, §4]. Then one must identify the correlation functions of the (twisted Wick-rotated) high energy theory as correlation functions in the (twisted Wick-rotated) low energy theory. Finally, as in the high energy theory, the path integral that expresses the correlation function localizes to the vanishing locus of an odd vector field ξ_Q , which is cut out by BPS equations analogous to (44.27). For the

low energy theory at $u = \pm 1$ —an $N = 2$ supersymmetric $\mathbb{T}$ gauge theory with hypermultiplet—the relevant BPS equations are the *Seiberg–Witten equations*. Behind each sentence in this paragraph is a detailed justification that is beyond the scope of these lectures; these justifications are not at the level of mathematical proof.

We conclude by stating the Seiberg–Witten equations. As above, X is an oriented Riemannian 4-manifold. The fluctuating bosonic fields are a spin^c -connection α and a positive spinor field ψ . The Seiberg–Witten equations take the form

$$(44.30) \quad \begin{aligned} D_\alpha \psi &= 0 \\ F_\alpha^+ &= q(\psi) \end{aligned}$$

Here D_α is the Dirac operator of the spin^c -connection, $F^+ \alpha$ is the self-dual component of the spin^c -curvature, and $q(-)$ is a quadratic function from positive spinor fields to self-dual 2-forms. There is a well-developed mathematical theory of these equations using techniques from global analysis. The physics argument leads to conjectural relationships between Donaldson invariants and Seiberg–Witten invariants—the latter defined by counting solutions to (44.30)—but sadly we must stop here.

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Problem Set # 1

Math262a: Quantum theory from a geometric viewpoint I

These problems are supplementary to the lectures and the lecture notes. I hope you tackle many of them, even if you don't complete them. Many are open-ended. They provide a gateway to engage with the material in the class. I suggest you form small groups to discuss the class material, including these problems. Those getting a grade in the course should regularly hand in some solutions. Please come to office hours and the discussion section to discuss these problems, and also take advantage of Discord as a platform for discussion.

Problems

1. In this problem you will derive the symplectic form on the space of classical trajectories of a particle, for simplicity a particle of mass m moving on the standard Euclidean line $\mathbb{E}^1$ with coordinate x . Let $V: \mathbb{E}^1 \rightarrow \mathbb{R}$ be a smooth function, which is the potential energy of the particle. Let $\mathcal{F} = \text{Map}(\mathbb{R}, \mathbb{E}^1)$ be the space of smooth possible particle trajectories.
 - (a) What is the space $N \subset \mathcal{F}$ of classical trajectories, i.e., trajectories that satisfy Newton's law? Consider the special cases $V = 0$ and $V(x) = kx^2/2$ for some $k > 0$.
 - (b) Review the derivation of Newton's law from the calculus of variations (Euler-Lagrange equations) applied to the *action* function

$$S(\gamma) = \int \left[\frac{1}{2} m \dot{x}(t)^2 - V(x(t)) \right] dt$$

where $x: \mathbb{R} \rightarrow \mathbb{E}^1$ is a smooth motion.

- (c) Rephrase this computation using calculus on $\mathcal{F} \times \mathbb{R}$. Observe that the Euler-Lagrange equations only depend integrand in the action—the *lagrangian*—not its integral. Write δ for the de Rham differential on $\mathcal{F}$ and d for the de Rham differential $d = dt \cdot \partial/\partial t$ on $\mathbb{R}$. Then the total de Rham differential on $\mathcal{F} \times \mathbb{R}$ is $\delta + d$. In particular, $(\delta + d)^2 = 0$. A differential form on $\mathcal{F} \times \mathbb{R}$ has *type* (p, q) , $p \in \mathbb{Z}^{\geq 0}$, $q \in \{0, 1\}$, if it is nonzero only when evaluated on p vectors tangent to $\mathcal{F}$ and q vectors tangent to $\mathbb{R}$. Calculus on the Cartesian product $\mathcal{F} \times \mathbb{R}$ is depicted in the diagram

$$\begin{array}{c|ccc}
 & 0 & 1 & \cdots \\
 \hline
 1 & & & \\
 & d \uparrow & & \\
 0 & & \xrightarrow{\delta} &
 \end{array}
 \begin{array}{c}
 \mathcal{F} \\
 \\
 \mathbb{R}
 \end{array}$$

(If one writes the lagrangian and action as a *density* rather than a differential form—use $|dt|$ in place of dt —which we should do since we should not use an orientation on $\mathbb{R}$ to define this

mechanical system, then the formatting of the square makes more sense: starting from the top, the vertical degrees are then written as $|0\rangle$ and $| - 1\rangle$.) Let

$$e: \mathcal{F} \times \mathbb{R} \longrightarrow \mathbb{E}^1$$

$$x, t \longmapsto x(t)$$

be the evaluation function. Show that the integrand above can be written as

$$L = \left[\frac{1}{2} m \dot{e}^2 - V \circ e \right] dt$$

and that it is an element of $\Omega^{0,1}(\mathcal{F} \times \mathbb{R})$. (Be sure to define $\dot{e}$, which is a directional derivative of e along a certain vector field on $\mathcal{F} \times \mathbb{R}$.)

- (d) Compute $\delta L \in \Omega^{1,1}(\mathcal{F} \times \mathbb{R})$. Carry out the integration by parts in the following form: find $\theta \in \Omega^{1,0}(\mathcal{F} \times \mathbb{R})$ so that $\delta L + d\theta$ is a function times $\delta e \wedge dt$, i.e., it has no $\delta \dot{e} \wedge dt$ term.
- (e) Show that $\delta L + d\theta$ vanishes on N , and in fact that vanishing defines N .
- (f) Show that the restriction of $\delta\theta \in \Omega^{2,0}(\mathcal{F} \times \mathbb{R})$ to $N \times \{t_0\}$ is independent of t_0 . Show that this restriction is a symplectic form. Identify it with the symplectic form you may have run into before in this context.

2. In the previous problem specialize to $V(x) = kx^2/2$ for fixed $k > 0$.

- (a) Fix $t \in \mathbb{R}$ and consider the observables (functions on N) given by $\mathcal{O}_1(x) = x(0)$ and $\mathcal{O}_2(x) = x(t)$. Choose a pure state σ , which is a point of N . Compute the expectation values $\langle \mathcal{O}_1 \rangle_\sigma$ and $\langle \mathcal{O}_2 \rangle_\sigma$. Compute the correlation function $\langle \mathcal{O}_2 \mathcal{O}_1 \rangle_\sigma$. Investigate the dependence on t and on σ .
- (b) Now consider the mixed *Gibbs* state, which is a probability measure on the space N of classical solutions. Its probability density is a constant times $e^{-\beta E}$, where $\beta > 0$ is a constant and $E: N \rightarrow \mathbb{R}$ is the energy function $E = m\dot{x}^2/2 + kx^2/2$. Compute the Gibbs state (a function times the standard measure on the $(x, \dot{x})$ -plane) and evaluate the expectation values in part (a) in the Gibbs state.

3. Verify the axioms in §1.2 of the lecture notes for the case of a classical mechanical system.

4. In the context of a general mechanical system, let σ_1, σ_2 be states and let A, A' be observables. Fix $t \in [0, 1]$. Prove that for the convex combination $\sigma = t\sigma_1 + (1-t)\sigma_2$ the standard deviations $\Delta_\rho B$ of the probability distributions ρ_B obtained from the various states ρ and observables B satisfy

$$\Delta_\sigma A \Delta_\sigma A' \geq t(\Delta_{\sigma_1} A)(\Delta_{\sigma_1} A') + (1-t)(\Delta_{\sigma_2} A)(\Delta_{\sigma_2} A')$$

In particular, conclude that

$$\Delta_\sigma A \geq \min(\Delta_{\sigma_1} A, \Delta_{\sigma_2} A).$$

(Recall that the standard deviation of a probability measure on $\mathbb{R}$ is the square root of the integral of $(\lambda - \mu)^2$ times the probability measure on the real line $\mathbb{R}_\lambda$ with coordinate λ , where μ is the *mean* or *expected value*, i.e., μ is the integral of λ times the probability measure.)

Problem Set # 1

Math 262a: Quantum theory from a geometric viewpoint I

Due: September 20

Problems

1. Let N be a smooth manifold of dimension $n = 2m$, and suppose $\omega \in \Omega_N^2$.
 - (a) Prove that ω is nondegenerate (at each point of N) iff $\omega^{\wedge m} \in \Omega_N^{2m}$ is everywhere nonzero.
 - (b) Now assume that ω is a symplectic form. Prove that the subspace $\mathcal{X}_N^\omega \subset \mathcal{X}_N$ of symplectic vector fields is closed under Lie bracket of vector fields.
 - (c) Define a Lie algebra structure on Ω_N^0 by transport from the Lie bracket on $\mathcal{X}_N^\omega$ via the symplectic gradient map $f \mapsto \xi_f$ on functions. Prove the formula

$$\{f, g\} = \xi_f \cdot g = \omega(\xi_f, \xi_g), \quad f, g \in \Omega_N^0.$$

- (d) A *Hamiltonian vector field* is a symplectic gradient of a function. Is the subspace of Hamiltonian vector fields invariant under Lie bracket?
2. Let V be a real symplectic vector space, and suppose A is an affine space over V . Feel free to work with the model spaces.
 - (a) Define the space of affine polynomial functions on A and prove that it is closed under the Poisson bracket on smooth functions.
 - (b) Prove that the space of affine linear functions is also invariant. Do you recognize the resulting Lie algebra?
 - (c) Prove that the space of affine quadratic functions is also invariant. Do you recognize the resulting Lie algebra?
 - (d) Does the pattern continue?
3. (a) Prove that an almost symplectic manifold admits an almost complex structure. (An almost complex structure on a smooth manifold N is a section I of $\text{End}(TN) \rightarrow TN$ such that $I^2 = -\text{id}$ holds at every point.)
 - (b) Prove that S^4 does not admit an almost symplectic structure. (Hint: Use characteristic classes)

4. Let (M, g) be a Riemannian manifold, and suppose $V: M \rightarrow \mathbb{R}$ is a smooth (potential energy) function. Fix $m > 0$, which we take to be the mass of a particle moving on M . Let $\mathcal{F}$ be the space of smooth maps $x: \mathbb{R} \rightarrow M$. Treat $\mathcal{F}$ as an infinite dimensional manifold, at least formally, assuming differential calculus carries over. Consider the Lagrangian

$$L = \left\{ \frac{m}{2} \langle \dot{x}, \dot{x} \rangle - V(x) \right\} |dt|,$$

where $x: \mathcal{F} \times \mathbb{R} \rightarrow M$ is the evaluation map. (You may replace ‘ $|dt|$ ’ with dt .) Compute the Euler-Lagrange equations, the symplectic form, and the Hamiltonian. (You will need some familiarity with Riemannian geometry.)

5. Define the notion of a symmetry of a general mechanical system. Does a symmetry have to preserve the direction (arrow) of time, or can a symmetry reverse time flow?

Problem Set # 3

Math 262a: Quantum theory from a geometric viewpoint I

Due: September 29

Problems

1. Consider a classical particle moving on the Euclidean line $\mathbb{E}^1$ with standard metric and coordinate x . Fix $a > 0$ and set $V(x) = \frac{1}{2}(x^2 - a^2)^2$. What are the classical trajectories of least energy? Answer the same question for a classical particle moving in the *inverted* potential $-V$. For $T > 0$ is there a classical trajectory $\gamma: [-T/2, T/2]$ with initial position $\gamma(-T/2) = -a$ and final position $\gamma(T/2) = a$, and if so give a formula or characterize it? What happens if $T \rightarrow \infty$?
2. (a) Prove the formula (5.14) in the notes.
(b) Generalize the measurement depicted in Figure 9 of the notes to arbitrary states.
3. This concerns the particle on a ring $M = \mathbb{E}^1/2\pi L\mathbb{Z}$ of length $2\pi L$, both classical and quantum.
(a) Work out the details of Example 3.57 in the notes.
(b) The quantum theory has Hilbert space $\mathcal{H} = L^2(M; \mathbb{C})$ and the Hamiltonian is the unbounded self-adjoint operator

$$H = \frac{1}{2} \left(i \frac{\partial}{\partial x} + \frac{1}{2\pi} \theta \right)^2.$$

Diagonalize H , and in particular compute its spectrum. Is there periodicity in θ ? How does that compare to the classical system?

4. Let $\mathcal{H}$ be a separable complex Hilbert space. A continuous linear operator $T: \mathcal{H} \rightarrow \mathcal{H}$ is *compact* if the closure of the image $T(D) \subset \mathcal{H}$ of the closed unit disk $D \subset \mathcal{H}$ is compact. Prove the following.

Theorem. *Let $T: \mathcal{H} \rightarrow \mathcal{H}$ be a positive self-adjoint compact operator. Prove that there exists a complete orthonormal basis $\{e_n\}_{n=1}^\infty$ and positive numbers $\mu_1 \geq \mu_2 \geq \dots$ such that*

- (i) $T\psi_n = \mu_n\psi_n$;
- (ii) For any $c > 0$ there is a finite number of $\mu_n > c$;
- (iii) $\lim_{n \rightarrow \infty} \mu_n = 0$.

The completeness means that the closure of the span of the set $\{e_n\}$ equals $\mathcal{H}$. One approach is to consider the quadratic form $\psi \mapsto \langle \psi, T\psi \rangle$ on the unit ball D . Since T is compact, this form achieves its maximum, say at a unit norm vector e_1 . Show that T preserves the orthogonal complement to $\mathbb{C} \cdot e_1$, and iterate the argument.

5. Consider the toric code, as defined in §5.5 of the notes, formulated on a 2-dimensional manifold Y equipped with a triangulation $\Gamma \subset Y$ (or, better, a CW structure).
- (a) Prove that any two elements in the set of operators $\{H_v, H_f\}_{v,f}$ commute. These operators are defined in (5.24).
 - (b) Prove that the kernel is as stated in the notes: see (5.27).
 - (c) Describe the eigenspace for the minimum nonzero eigenvalue.
 - (d) For any edge e in Γ consider the permutation of $\mathcal{D}(\Gamma, \Gamma^0)$ that modifies a double cover at a single interior point of Γ : cut open the edge and reglue the double cover after applying the deck transformation (swap sheets) at the point. This induces an observable $\mathcal{O}_e$. Fix a vacuum state σ , a positive time $T > 0$, times $t_1, t_2 \in (0, T)$, and edges e_1, e_2 . You may suppose σ is induced from the delta function at a fixed double cover of Y . What can you say about the correlation function of $\mathcal{O}_{e_1}(t_1)$ and $\mathcal{O}_{e_2}(t_2)$ over the interval $[0, T]$ with initial and final state σ ? Investigate how your answer depends on T and on whether the edges share a vertex and/or a face. (I have not tried this, so it may very well be the start of a project on the toric code model. Incidentally, it is fine if you want to collaborate with one other person on a class project.)

Problem Set # 4

Math 262a: Quantum theory from a geometric viewpoint I

Due: October 6

Problems

1. The construction of the principal $\mathbb{R}$ -bundle with connection $T \rightarrow N$ in §3.4 of the notes is stated poorly; here is another attempt.

Quite generally, if $\pi: M \rightarrow N$ is a map and if $\mathcal{O}$ is a contravariant local mathematical object on M —a cocycle for a cohomology class, a differential form, fiber bundle over M , a connection on a principal bundle over M , etc.—then there are two general ways we can think of obtaining a mathematical object $\bar{\mathcal{O}}$ on N . First, there might be a *pushforward* or *umker map* or *integration* that produces $\pi_*\mathcal{O}$. In general, the geometric nature of $\bar{\mathcal{O}}$ is different from that of $\mathcal{O}$. For example, there might be a degree shift (down by the relative dimension of the map π). The second general construction is *descent*: an object $\bar{\mathcal{O}}$ on N and an isomorphism of $\mathcal{O}$ with the pullback $\pi^*\bar{\mathcal{O}}$. The descended object $\bar{\mathcal{O}}$ has the same geometric nature as $\mathcal{O}$. It is this second scenario that applies here. We put ourselves in the situation of §3.4.

- (a) Consider the product principal $\mathbb{R}$ -bundle

$$p: P = (N \times \mathbb{R}_t) \times \mathbb{R}_s \longrightarrow N \times \mathbb{R}_t$$

Prove that the real-valued 1-form

$$\Theta = ds + p^*\gamma \in \Omega_P^1$$

is a connection. Compute its curvature $\Omega \in \Omega_{N \times \mathbb{R}}^2$.

- (b) Prove that the curvature descends under the projection

$$\pi: N \times \mathbb{R} \longrightarrow N$$

In other words, construct a 2-form $\omega \in \Omega_{N \times \mathbb{R}}^2$ such that $\Omega = \pi^*\omega$.

- (c) Now descend $p: P \rightarrow N \times \mathbb{R}$ and its connection Θ to a principal $\mathbb{R}$ -bundle $T \rightarrow N$ with connection. Be sure to produce the requisite isomorphism that proves you have a descent. (Hint: the fiber T_n over $n \in N$ is the $\mathbb{R}$ -torsor of parallel sections of the restriction of $p: P \rightarrow N \times \mathbb{R}$ to $\pi^{-1}(n) = \{n\} \times \mathbb{R}$.) How does this relate to the construction in the notes?
- (d) Consider the translation action of $\mathbb{R}_t$ on $N \times \mathbb{R}_t$. Define a lift to P such that (i) the connection 1-form Θ is invariant, and (ii) if $\xi = \partial/\partial t$ denotes the vector field that generates this lifted translation action, then the contraction $\iota_\xi \Theta$ vanishes. Conclude that the 1-form Θ descends to the quotient of P by the translation action.
- (e) More generally, suppose $\pi: Q \rightarrow N$ is a principal G -bundle for some Lie group G . What are the descent conditions on a differential form $\Omega \in \Omega_Q^\bullet$? In other words, characterize the image of the injective linear map $\pi^*: \Omega_N^\bullet \rightarrow \Omega_Q^\bullet$.

2. Consider the simple harmonic oscillator: a mass one particle on $\mathbb{E}_x^1$ with potential $V(x) = x^2/2$. The Hilbert space is $\mathcal{H} = L^2(\mathbb{E}^1; \mathbb{C})$ and the Hamiltonian is

$$H = \frac{1}{2} \left(-\frac{d^2}{dx^2} + x^2 \right)$$

- (a) Compute eigenfunctions for the four smallest eigenvalues of H .
 - (b) State and prove the spectral decomposition for H , namely, $\mathcal{H}$ is the Hilbert space completion of the direct sum of the eigenspaces.
 - (c) Recall that the classical phase space is $N = \mathbb{E}_x^1 \times \mathbb{R}_y^1$ with symplectic form $\omega = dy \wedge dx$. Verify that the 3-dimensional real vector space of affine functions on N is closed under Poisson bracket. Verify the same for the 6-dimensional subspace of affine quadratic functions. What about the affine cubic functions?
 - (d) The affine functions under Poisson bracket form the Heisenberg Lie algebra. Construct a representation by unbounded operators on $\mathcal{H}$ (which are skew-adjoint.) Can you extend the representation to the affine quadratic functions?
 - (e) Construct an antiunitary operator on $\mathcal{H}$ which implements the time-reversal symmetry that preserves the operator x and sends $\partial/\partial x \mapsto -\partial/\partial x$. Show that the induced operator on $\mathbb{P}\mathcal{H}$ has order 2. What about the operator on $\mathcal{H}$?
3. Let $V: \mathbb{E}^1 \rightarrow \mathbb{R}$ be a smooth function and consider a quantum mass $m > 0$ particle moving on $\mathbb{E}^1$ with potential energy V . An eigenfunction with eigenvalue $E \in \mathbb{R}$ is an L^2 function $\psi: \mathbb{E}^1 \rightarrow \mathbb{C}$ that satisfies the *Schrödinger equation*

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + (V(x) - E)\psi(x) = 0.$$

- (a) Prove that if ψ_1, ψ_2 are nonzero eigenfunctions with the same eigenvalue E , then ψ_2 is proportional to ψ_1 . In other words, eigenspaces of the Hamiltonian are 1-dimensional. Observe that, depending on V , there may be no eigenspaces, the eigenspaces may span the space of L^2 functions, or anything in between. (Hint: Consider the *Wronskian* $\psi_1\psi_2' - \psi_1'\psi_2$. You may need to use some facts about elliptic second-order ordinary differential equations. For example, solutions are smooth functions and their zero set contains no accumulation points.)
- (b) Prove that each eigenspace contains a real-valued function.
- (c) Prove that if the Hamiltonian is gapped, then the unique up to a constant eigenfunction for the minimal eigenvalue is a nonvanishing function.

4. Let W be a finite dimensional real vector space. Recall that in lecture we discussed the bosonic harmonic oscillator as a representation of an algebraic operator algebra generated by $W \oplus W^*$ on the complex vector space $\text{Sym}^\bullet W_{\mathbb{C}}^*$. The goal now is to introduce inner products and complete to a Hilbert space. For this, suppose W is endowed with a real inner product.
- (a) Induce a hermitian inner product on $W_{\mathbb{C}}^*$.
 - (b) Induce a hermitian inner product on $\text{Sym}^\bullet W_{\mathbb{C}}^*$.
 - (c) What is the adjoint of the operator $A(w)$, $w \in W$, defined in lecture?
 - (d) Is the Hamiltonian (formally) self-adjoint?
5. The abelian group $\mathbb{T} \subset \mathbb{C}$ of unit norm complex numbers has an automorphism $\lambda \mapsto \lambda^{-1}$. Let $\mu_2 \subset \mathbb{T}$ be the group $\{\pm 1\}$ of square roots of unity.
- (a) Classify group extensions (up to isomorphism)

$$1 \longrightarrow \mathbb{T} \longrightarrow G \longrightarrow \mu_2 \longrightarrow 1$$

in which μ_2 acts on $\mathbb{T}$ by the nontrivial automorphism.

- (b) Classify group extensions

$$1 \longrightarrow \mathbb{T} \longrightarrow G \longrightarrow \text{O}_2 \longrightarrow 1$$

in which O_2 acts nontrivially on $\mathbb{T}$ via the determinant $\text{O}_2 \longrightarrow \mu_2$.

Problem Set # 5

Math 262a: Quantum theory from a geometric viewpoint I

Due: October 13

Problems

1. Recall the 2-component Lie group PQ constructed in lecture (and in the lecture notes). I write ‘PQ’, but you are welcome to work this problem with PQ_n and a finite dimensional quantum mechanical system. Given a right PQ-torsor and a Hamiltonian H (where does it live?), construct the data of a quantum mechanical system, that is, the data in Axiom System 1.1 of the notes. There are many parts to this, so first make a list of the data you must produce and then knock them out one-by-one. Can you generalize to a parametrized family of data? To a piece of data with symmetry? Can you combine these?
2. For $n \in \mathbb{Z}^{\geq 0}$ the standard Clifford algebra Cl_{+n} is the unital algebra over $\mathbb{R}$ generated by $e_1, \dots, e_n$ subject to the relation

$$e_i e_j + e_j e_i = 2\delta_{ij}, \quad 1 \leq i, j \leq n.$$

Cl_{-n} is similar, but the relation is $e_i e_j + e_j e_i = -2\delta_{ij}$. The complex version, for which the sign is irrelevant but we take the plus sign for definiteness, is denoted $Cl_n^{\mathbb{C}}$. The abstract version for a vector space U (complex or real) with a symmetric bilinear form B is the free unital algebra $Cl(U, B)$ generated by U subject to

$$u_1 u_2 + u_2 u_1 = B(u_1, u_2), \quad u_1, u_2 \in U.$$

These Clifford algebras are superalgebras.

- (a) Construct an isomorphism of superalgebras $Cl_2^{\mathbb{C}} \xrightarrow{\cong} \text{End}(\mathbb{C}^{1|1})$.
- (b) Define the tensor product $A_1 \otimes A_2$ of superalgebras A_1, A_2 . Mind your Koszul sign rule!
- (c) Let (U_i, B_i) be real vector spaces and symmetric bilinear forms. Construct an isomorphism

$$Cl(U_1, B_1) \otimes Cl(U_2, B_2) \xrightarrow{\cong} Cl(U_1 \oplus U_2, B_1 \oplus B_2).$$

- (d) Construct an isomorphism of superalgebras $Cl_{+8} \xrightarrow{\cong} \text{End}(\mathbb{R}^{8|8})$. (You may want to apply (c).)
- (e) Observe that $\mathbb{R}^n \subset Cl_{\pm n}$. For $\xi \in \mathbb{R}^n$ a vector of unit norm, check that the map $\zeta \mapsto -\xi \zeta \xi^{-1}$ reflects $\zeta \in \mathbb{R}^n$ in the hyperplane perpendicular to ξ .
- (f) For $\theta \in \mathbb{R}$ define $g_\theta = \cos(\theta/2) + \sin(\theta/2)e_1 e_2$. What transformation does g_θ induce on $\mathbb{R}^n$?
- (g) Show that “quadratic” elements $\frac{1}{2}A^{ij}e_i e_j$ in the Clifford algebra are closed under commutation. Can you identify this Lie algebra? (Here the A^{ij} are scalars and the expression is summed over the repeated indices i, j .)

3. For computations in superalgebra, say over $\mathbb{R}$, it is wise to extend scalars from $\mathbb{R}$ to the commutative superalgebra $\mathbb{R}[\eta_1, \dots, \eta_N]$ for arbitrary N . Multiply odd elements by a variable η_i to obtain an even element, and then use the even elements to compute, thereby eliminating signs until the end of the computation. For example, if Q_1, Q_2 are odd elements of a super Lie algebra, then compute the sign rule for the Lie bracket as follows:

$$[\eta_1 Q_1, \eta_2 Q_2] = -[\eta_2 Q_2, \eta_1 Q_1] = +\eta_2 \eta_1 [Q_2, Q_1] = -\eta_1 \eta_2 [Q_2, Q_1].$$

On the other hand,

$$[\eta_1 Q_1, \eta_2 Q_2] = -\eta_1 \eta_2 [Q_1, Q_2],$$

and comparing we deduce the correct rule for the Lie bracket:

$$[Q_1, Q_2] = [Q_2, Q_1].$$

- (a) Revisit part (b) of the previous problem and apply these “even rules”.
 - (b) Deduce the proper signs for commutativity in the symmetric algebra $\text{Sym}^\bullet(V)$ of a super vector space V .
 - (c) Deduce the Jacobi identity for a super Lie algebra $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$. Verify that (i) $\mathfrak{g}^0$ is an (ungraded) Lie algebra, (ii) $\mathfrak{g}^1$ is a $\mathfrak{g}^0$ -module, (iii) Lie bracket induces a symmetric pairing $\mathfrak{g}^1 \times \mathfrak{g}^1 \rightarrow \mathfrak{g}^0$ of $\mathfrak{g}^0$ -modules, and (iv) for $x \in \mathfrak{g}^1$ we have $[x, [x, x]] = 0$. Conversely, show that the data (i)–(iv) determine a super Lie algebra.
 - (d) Define the opposite superalgebra to a superalgebra. What is the opposite superalgebra to $\text{Cl}(U, B)$?
4. This problem identifies the double cover of low dimensional orthogonal groups. Many parts do not require background in Lie group theory.
- Let V be a 4-dimensional complex vector space and fix a nonzero volume form $\mu \in \bigwedge^4 V^*$. Define a $\mathbb{C}$ -valued bilinear form b on $\bigwedge^2 V$ by $b(\alpha, \beta) = \langle \mu, \alpha \wedge \beta \rangle$ for $\alpha, \beta \in \bigwedge^2 V$.
- (a) Choose a basis $\{e_1, e_2, e_3, e_4\}$ of V and the induced basis $\{e_i \wedge e_j : i < j\}$ of $\bigwedge^2 V$. Then choose $\mu = e^1 \wedge e^2 \wedge e^3 \wedge e^4$, where $\{e^1, e^2, e^3, e^4\}$ is the dual basis. Show that b is nondegenerate.
 - (b) Define the homomorphism

$$\pi: \text{Aut}(V, \mu) \longrightarrow \text{Aut}(\bigwedge^2 V, b)$$

which maps a volume-preserving automorphism of V to a bilinear form-preserving automorphism of $\bigwedge^2 V$. Write the corresponding map of Lie algebras. Prove that the latter is an isomorphism.

- (c) Deduce that the image of π is open and a subgroup, whence π maps onto the identity component of $\text{Aut}(\bigwedge^2 V, b)$. (Show that the latter group has two components.)

- (d) Prove that the kernel of π is $\{\pm \text{id}\}$ so that π is a 2:1 covering. Deduce that $\text{SL}_4(\mathbb{C})$ is a double covering of $\text{SO}_6(\mathbb{C})$.
- (e) Now choose $\omega \in \wedge^2 V^*$ with $(\omega \wedge \omega)/2 = \mu$. For example, take $\omega = e^1 \wedge e^2 + e^3 \wedge e^4$. Let $W \subset \wedge^2 V$ be the annihilator of ω . Then use the restriction of the map π above to define

$$\pi: \text{Aut}(V, \omega) \longrightarrow \text{Aut}(W, b).$$

As before, prove that π is a 2:1 covering map. Deduce that $\text{Sp}_4(\mathbb{C})$ is a double covering of $\text{SO}_5(\mathbb{C})$.

- (f) Write $V = U_1 \oplus U_2$ as the direct sum of 2-dimensional subspaces. Choose nonzero $\omega_1 \in \wedge^2 U_1^*$ and $\omega_2 \in \wedge^2 U_2^*$. Construct a decomposition $\wedge^2 V^* \cong \wedge^2 U_1^* \oplus \wedge^2 U_2^* \oplus U_1^* \otimes U_2^*$. Then, following the ideas in previous parts, construct a 2:1 covering

$$\pi: \text{Aut}(U_1, \omega_1) \times \text{Aut}(U_2, \omega_2) \longrightarrow \text{Aut}(U_1^* \otimes U_2^*, b).$$

(In the definition of b take $\mu = \omega_1 \wedge \omega_2$.) Deduce that $\text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$ is a double covering of $\text{SO}_4(\mathbb{C})$.

- (g) Construct a double covering $\pi: \text{SL}_2(\mathbb{C}) \rightarrow \text{SO}_3(\mathbb{C})$.
- (h) From the beginning introduce a hermitian inner product on V and restrict to the subgroups that preserve this inner product. What do you learn?

Problem Set # 6

Math 262a: Quantum theory from a geometric viewpoint I

Due: October 27

Problems

1. In this problem you can work over any field, for example the complex numbers.

(a) Let

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

be a short exact sequence of vector spaces. Show that the space of splittings has a natural structure as an affine space over $\text{Hom}(V'', V')$. Generalize to short exact sequences of groups, Lie algebras, ...

(b) Let V be a finite dimensional vector space and $W \subset V$ a subspace. Let A_W denote the set (or, better, space) of subspaces $W' \subset V$ such that $V = W \oplus W'$. Show that A_W has a natural structure of an affine space over $\text{Hom}(V/W, W)$. Give a geometric construction for the sum of an element of A_W and a linear map $V/W \rightarrow W$.

(c) Use (b) to construct an atlas of charts on the Grassmannian $Gr_k(V)$ of k -dimensional subspaces of V . (Index the charts by subspaces of dimension $n - k$, where $\dim V = n$. What are the transition functions?)

(d) Let M be a Galilean spacetime with translation group V equipped with a codimension one subspace W . Let $C, C' \subset M$ be affine lines that are transverse to the simultaneity foliation, i.e., C, C' are *affine worldlines*. Produce a linear map $V/W \rightarrow W$ and interpret it as the relative velocity of the affine worldlines (in order). Define the acceleration of single affine worldline $C \subset M$ as a linear map $(V/W)^{\otimes 2} \rightarrow W$.

2. (a) Let $\mathcal{O}$ be a C^* -algebra. Prove that the space of states is convex. Let S be a compact Hausdorff space and $\mathcal{O} = C^0(S; \mathbb{C})$ the algebra of continuous functions on S . What are the states on $\mathcal{O}$? What are the pure states?

(b) Let $(\mathcal{H}, U_t)$ be the (linear) data of a QM system. Identify a C^* -algebra of “observables”. (I use scare quotes since ‘observables’ are the real points.) What are its states in the sense of C^* -algebras? Does this agree with the QM states as defined in §1.4 of the notes?

3. Investigate the existence of time-reversal symmetries in the toric code (§5.5). In the particle on a ring (§7.3). Consider both a single system and families of systems.

4. In lecture we (will) construct the symplectic manifold that is the phase space of a classical relativistic particle. Do the same for a classical Galilean particle.

5. Take up Problem 4 on Problem Set #5. Deduce the chart of low dimensional spin groups (dimensions ≤ 6) that appeared in lecture and (will appear) in the notes. You previously did the Euclidean case; now do the Lorentz case.
6. Consider Maxwell's equations in whatever form is familiar to you. Define an action of the group of isometries of Minkowski spacetime on the variables (fields) in those equations. Are the equations invariant? If not, is there a subgroup under which they are invariant?

Problem Set # 7

Math 262a: Quantum theory from a geometric viewpoint I

Due: November 3

Problems

1. (a) Let M be an affine space over a real vector space V , and suppose $C \subset M$ is a submanifold of dimension k . Let $\text{Gr}_k(V)$ denote the Grassmannian of k -dimensional subspaces of V . The *Gauss map*

$$\Gamma: C \longrightarrow \text{Gr}_k(V)$$

assigns to each $p \in C$ its tangent space $T_p C \subset V$. Prove that for $p \in C$ the differential

$$d\Gamma_p: T_p C \longrightarrow \text{Hom}(T_p C, V/T_p C)$$

can be interpreted as a *symmetric* bilinear form

$$T_p C \times T_p C \longrightarrow V/T_p C$$

- (b) Suppose $\dim M = 2$ and $k = 1$. How is the differential of the Gauss map related to the classical curvature of a (cooriented) plane curve.
2. (a) Recall that the phase space $\mathcal{M}$ of the classical relativistic particle in a Minkowski spacetime M is the manifold of all timelike affine lines $C \subset M$. The isometry group $\text{O}(M)$ acts on $\mathcal{M}$. Prove that the subgroup $\text{O}^\uparrow(M)$ acts preserving the symplectic form and its complement in $\text{O}(M)$ acts by reversing the symplectic form (to its negative).
- (b) Repeat the analysis of the classical relativistic particle for the classical Galilean particle.
3. (a) Let M be a smooth manifold equipped with a transitive action of a Lie group G . Fix $p \in M$ and let $G_p \subset G$ be the stabilizer subgroup at p . Define the notion of a G -invariant smooth measure on M . Prove that G -invariant smooth measures form either an $\mathbb{R}^{>0}$ -torsor or form an empty set. Show that the former possibility prevails if G_p is compact.
- (b) Let G be a Lie group. Prove that there exist left-invariant smooth measures on G .
- (c) Investigate the existence of bi-invariant smooth measures on G . What condition(s) on G guarantee their existence? Give an example of G and a proof of non-existence for that G .
4. Consider standard Minkowski spacetime $\mathbb{M}^n$ with coordinates $t, x^1, \dots, x^{n-1}$ and Lorentz metric $c^2 dt^2 - (dx^1)^2 - \dots - (dx^{n-1})^2$. For $m > 0$ consider the mass shell $\mathcal{O}_m$ in $(\mathbb{R}^n)^*$ defined as the manifold of vectors $E dt + p_i dx^i$ that satisfy $E^2/c^2 - \sum (p_i)^2 = m^2 c^2$ and $E > 0$. Write a formula for a smooth density on $\mathcal{O}_m$ that is invariant under the group $\text{O}^\uparrow(\mathbb{M}^n)$. Is it invariant under $\text{O}(\mathbb{M}^n)$?

Problem Set # 8

Math 262a: Quantum theory from a geometric viewpoint I

Due: November 10

I'm giving fewer problems this week in the hope that you will be able to catch up if you've fallen behind. Please come see me if you'd like to discuss the class.

Problems

1. (a) Construct a finite dimensional representation of the Lie group $\mathrm{SL}_2\mathbb{R}$.
(b) Prove that a *unitary* irreducible representation of $\mathrm{SL}_2\mathbb{R}$ is either the trivial representation or is infinite dimensional.
(c) Conclude the same for a representation of the Lorentz group $\mathrm{Spin}_{1,n-1}$ for $n \geq 3$.
2. A spin group $\mathrm{Spin}_{p,q}$ ($p, q \in \mathbb{Z}^{\geq 0}$) has a *vector representation* on $V = \mathbb{R}^{p,q}$ that acts via the quotient $\mathrm{SO}_{p,q}$. (The vector space $\mathbb{R}^{p,q}$ is $\mathbb{R}^{p+q}$ with a nondegenerate symmetric bilinear form of the indicated signature.) Recall the chart (12.24) of low dimensional spin groups for $p = n, q = 0$ (positive definite) and $p = 1, q = n - 1$ (Lorentz). Let S be a *real* spin representation, which in dimensions 3–6 are the real representations underlying the defining representations. That is a bit vague, but enough for you to find a few examples in the chart to play with, and you can think about dimensions 1,2 as well. The question: Is there a $\mathrm{Spin}_{p,q}$ -invariant nonzero symmetric bilinear map

$$\Gamma: S \times S \longrightarrow V?$$

Investigate in examples. Can you make a conjecture based on your work?

Problem Set # 9

Math 262a: Quantum theory from a geometric viewpoint I

Due: November 17

The first exercise makes precise why when one considers the presheaf of double covers on manifolds, one must use groupoids (and not equivalence classes of double covers) to achieve locality. The same principle applies to connections for a fixed gauge group, also known as “gauge fields”. We use the notation $A = (A_0, A_1)$ for (small) groupoids, where A_0 is the set of objects and A_1 is the set of morphisms.

Problems

1. Let A, A', B be groupoids.

- (a) Suppose

$$\begin{array}{ccc} & A' & \\ & \downarrow \alpha' & \\ A & \xrightarrow{\alpha} & B \end{array}$$

is a diagram of functors. Define the pullback

$$\begin{array}{ccc} C & \xrightarrow{\gamma'} & A' \\ \downarrow \gamma & & \downarrow \alpha' \\ A & \xrightarrow{\alpha} & B \end{array}$$

as follows. Objects of C are triples $(a, a', \theta) \in A_0 \times A'_0 \times B_1$ where $\theta: \alpha(a) \rightarrow \alpha'(a')$. A morphism $(a, a', \theta) \rightarrow (b, b', \phi)$ in C is a pair $(f, f') \in A_1 \times A'_1$ such that $f: a \rightarrow b$, $f': a' \rightarrow b'$, and the diagram

$$\begin{array}{ccc} \alpha(a) & \xrightarrow{\theta} & \alpha'(a') \\ \alpha(f) \downarrow & & \downarrow \alpha(f') \\ \alpha(b) & \xrightarrow{\phi} & \alpha'(b') \end{array}$$

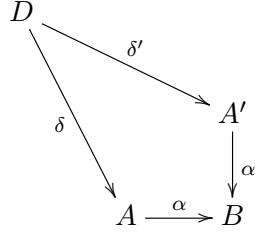
commutes.

- (b) Suppose $\beta: A \rightarrow B$ is a map of groupoids and $b \in B_0$ an object. Compute the pullback of

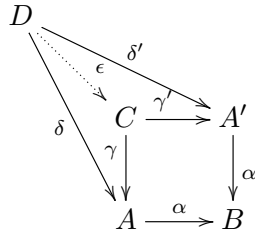
$$\begin{array}{ccc} & A & \\ & \downarrow \beta & \\ * & \xrightarrow{b} & B \end{array}$$

This is the fiber of β over b .

(c) Suppose that



is a commutative diagram in the sense that we are given a natural *isomorphism* $\varphi: \alpha\delta \rightarrow \alpha'\delta'$. Prove that the morphisms δ, δ' factor through the pullback:



What precisely do we mean by the commutativity of the diagram? What kind of uniqueness do we have?

- (d) Formulate the universal property satisfied by the pullback in this context. What uniqueness is satisfied?
- (e) Carefully define the groupoid double covers as a presheaf on the category of smooth manifolds and smooth maps. Formulate the local $\rightarrow$ global condition for this to be a sheaf and verify it. If you don't want to tackle the general case, then formulate the gluing for the cover of S^1 consisting of two overlapping open intervals; the intersection is the disjoint union of two open intervals.
2. Let $\mathbb{M}^1$ be standard 1-dimensional Minkowski spacetime: the affine line $\mathbb{A}_t^1$ with metric $c^2 dt^2$, where $c > 0$ is the speed of light. Consider the Klein-Gordon equation for mass $m > 0$:

$$\left(\frac{1}{c^2} \partial_t^2 + \frac{c^2}{\hbar^2} m^2 \right) \phi = 0,$$

where $\phi: \mathbb{M}^1 \rightarrow \mathbb{R}$ is a real-valued function. What is the space $\mathcal{M}$ of global solutions? Compute the symplectic form

$$\omega(\dot{\phi}_1, \dot{\phi}_2) = *d\dot{\phi}_1 \cdot \dot{\phi}_2 - *d\dot{\phi}_2 \cdot \dot{\phi}_1$$

evaluated at some $t \in \mathbb{M}^1$. Here $\dot{\phi}_1, \dot{\phi}_2$ are tangents to $\mathcal{M}$. Prove that the form is independent of t . What happens under Fourier transform? Formulate a dictionary to the simple harmonic oscillator.

Problem Set # 10

Math 262a: Quantum theory from a geometric viewpoint I

Due: December 1

The first exercise makes precise why when one considers the presheaf of double covers on manifolds, one must use groupoids (and not equivalence classes of double covers) to achieve locality. The same principle applies to connections for a fixed gauge group, also known as “gauge fields”. We use the notation $A = (A_0, A_1)$ for (small) groupoids, where A_0 is the set of objects and A_1 is the set of morphisms.

Problems

1. Let A, A', B be groupoids.

- (a) Suppose

$$\begin{array}{ccc} & A' & \\ & \downarrow \alpha' & \\ A & \xrightarrow{\alpha} & B \end{array}$$

is a diagram of functors. Define the pullback

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as follows. Objects of C are triples $(a, a', \theta) \in A_0 \times A'_0 \times B_1$ where $\theta: \alpha(a) \rightarrow \alpha'(a')$. A morphism $(a, a', \theta) \rightarrow (b, b', \phi)$ in C is a pair $(f, f') \in A_1 \times A'_1$ such that $f: a \rightarrow b$, $f': a' \rightarrow b'$, and the diagram

$$\begin{array}{ccc} \alpha(a) & \xrightarrow{\theta} & \alpha'(a') \\ \alpha(f) \downarrow & & \downarrow \alpha(f') \\ \alpha(b) & \xrightarrow{\phi} & \alpha'(b') \end{array}$$

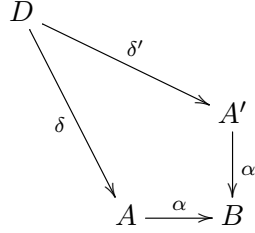
commutes.

- (b) Suppose $\beta: A \rightarrow B$ is a map of groupoids and $b \in B_0$ an object. Compute the pullback of

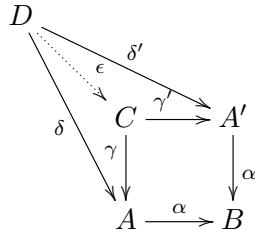
$$\begin{array}{ccc} & A & \\ & \downarrow \beta & \\ * & \xrightarrow{b} & B \end{array}$$

This is the fiber of β over b .

(c) Suppose that



is a commutative diagram in the sense that we are given a natural *isomorphism* $\varphi: \alpha\delta \rightarrow \alpha'\delta'$. Prove that the morphisms δ, δ' factor through the pullback:



What precisely do we mean by the commutativity of the diagram? What kind of uniqueness do we have?

- (d) Formulate the universal property satisfied by the pullback in this context. What uniqueness is satisfied?
- (e) Carefully define the groupoid double covers as a presheaf on the category of smooth manifolds and smooth maps. Formulate the local $\rightarrow$ global condition for this to be a sheaf and verify it. If you don't want to tackle the general case, then formulate the gluing for the cover of S^1 consisting of two overlapping open intervals; the intersection is the disjoint union of two open intervals.

2. Let $\mathbb{M}^1$ be standard 1-dimensional Minkowski spacetime: the affine line $\mathbb{A}_t^1$ with metric $c^2 dt^2$, where $c > 0$ is the speed of light. Consider the Klein-Gordon equation for mass $m > 0$:

$$\left(\frac{1}{c^2} \partial_t^2 + \frac{c^2}{\hbar^2} m^2 \right) \phi = 0,$$

where $\phi: \mathbb{M}^1 \rightarrow \mathbb{R}$ is a real-valued function. What is the space $\mathcal{M}$ of global solutions? Compute the symplectic form

$$\omega(\dot{\phi}_1, \dot{\phi}_2) = *d\dot{\phi}_1 \cdot \dot{\phi}_2 - *d\dot{\phi}_2 \cdot \dot{\phi}_1$$

evaluated at some $t \in \mathbb{M}^1$. Here $\dot{\phi}_1, \dot{\phi}_2$ are tangents to $\mathcal{M}$. Prove that the form is independent of t . What happens under Fourier transform? Formulate a dictionary to the simple harmonic oscillator.

Problem Set # 11

Math 262b: Quantum theory from a geometric viewpoint II

Due: February 9

The numbering on the problem sets (and the lecture notes) is a continuation from the first semester.

Problems

1. (a) Let S be a set with composition laws $\circ_1, \circ_2: S \times S \rightarrow S$ and distinguished element $1 \in S$. Assume (i) 1 is an identity for both $\circ_1$ and $\circ_2$; and (ii) for all $s_1, s_2, s_3, s_4 \in S$ we have

$$(s_1 \circ_1 s_2) \circ_2 (s_3 \circ_1 s_4) = (s_1 \circ_2 s_3) \circ_1 (s_2 \circ_2 s_4).$$

Prove that $\circ_1 = \circ_2$ and that this common operation is commutative and associative.

- (b) Let C be a symmetric monoidal category. Apply (a) to $C(1, 1) = \text{Hom}_C(1, 1)$, where $1 \in C$ is the tensor unit.
2. Let $y \in C$ be a dualizable object in a symmetric monoidal category, and suppose $(y^\vee, c, e)$ and $(\widetilde{y}^\vee, \tilde{c}, \tilde{e})$ are two sets of duality data. Prove there is a unique map $(y^\vee, c, e) \rightarrow (\widetilde{y}^\vee, \tilde{c}, \tilde{e})$.
3. For each of the following symmetric monoidal categories determine all of the dualizable objects.
 - (a) $(\text{Top}, \amalg)$, the category of topological spaces and continuous maps under disjoint union.
 - (b) $(\text{Ab}, \oplus)$, the category of abelian groups and homomorphisms under direct sum.
 - (c) $(\text{Mod}_R, \otimes)$, the category of R -modules and homomorphisms under tensor product, where R is a commutative ring.
 - (d) $(\text{Vect}_{\mathbb{F}}, \oplus)$, the category of vector spaces over a field $\mathbb{F}$ under direct sum.
 - (e) $(\text{Vect}_{\mathbb{F}}, \otimes)$, the category of vector spaces over a field $\mathbb{F}$ under tensor product.
 - (f) $(\text{Set}, \times)$, the category of sets and functions under Cartesian product.
4. Explain why each of the following fails to be a natural map $\eta: F \rightarrow G$ of symmetric monoidal functors $F, G: C \rightarrow D$.
 - (a) F, G are the identity functor on Vect_k for some field k , and $\eta(V): V \rightarrow V$ is multiplication by 2 for each vector space V .
 - (b) C, D are the category of algebras over a field k , the functor F maps $A \mapsto A \otimes A$, the functor G is the identity, and $\eta(A): A \otimes A \rightarrow A$ is multiplication.

5. Let B, C be symmetric monoidal categories, let $F, G: B \rightarrow C$ be symmetric monoidal functors, let $\eta: F \rightarrow G$ be a symmetric monoidal natural transformation, and suppose $y \in B$ is a dualizable object.
- (a) Prove that $\eta(y): F(y) \rightarrow G(y)$ is invertible. (Hint: $\eta(y)^{-1} = \eta(y^\vee)^\vee$.)
 - (b) Prove that a morphism between topological field theories is invertible.
6. In this problem you construct a simple TQFT $F: \text{Bord}_{\langle 0,1 \rangle} \rightarrow \text{Vect}_{\mathbb{Q}}$. For any manifold M let $\text{Bun}_G(M)$ denote the groupoid of principal G -bundles over M .
- (a) For a compact 0-manifold Y , define $F(Y)$ as the vector space of functions $\text{Bun}_G(Y) \rightarrow \mathbb{Q}$. Say what you mean by such functors on a groupoid.
 - (b) For a *closed* 1-manifold X define

$$F(X) = \sum_{[P] \in \pi_0 \text{Bun}_G(X)} \frac{1}{\# \text{Aut}(P)},$$

where the sum is over equivalence classes of principal G -bundles. Extend this to all bordisms $X: Y_0 \rightarrow Y_1$.

- (c) Check that F is a symmetric monoidal functor. (Do you need to specify more data?)
 - (d) Calculate F on a set of duality data for the point $\text{pt} \in \text{Bord}_{\langle 0,1 \rangle}$. Use it to compute $F(S^1)$. Compare to the formula above for $X = S^1$.
7. Let G be a finite group and let $F_G: \text{Bord}_{\langle 1,2 \rangle} \rightarrow \text{Vect}_{\mathbb{C}}$ be finite gauge theory. Let $X: S^1 \rightarrow \emptyset^1$ be the Möbius band. Compute $F_G(X)$.

Problem Set # 12

Math 262b: Quantum theory from a geometric viewpoint II

Due: March 8

Problem #3 is a repeat, just in case...

Problems

1. Write a careful proof of Lemma 27.36 in the notes.
2. Write the details of the computations sketched in the text following that lemma.
3. Let G be a finite group and let $F_G: \text{Bord}_{\langle 1,2 \rangle} \rightarrow \text{Vect}_{\mathbb{C}}$ be finite gauge theory. Let $X: S^1 \rightarrow \emptyset^1$ be the Möbius band. Compute $F_G(X)$.
4. You need to know about spin structures in dimension 1 for this problem. Let F be a 1-dimensional topological field theory of spin manifolds with values in the category of super vector spaces. Recall that the symmetry in the symmetric monoidal structure involves the *Koszul sign*.
 - (a) Suppose F assigns the finite dimensional super vector space $V = V^0 \oplus V^1$ to a point. Compute the value of F on each of the two spin circles.
 - (b) Is there a point defect that changes from one spin structure to the other?
5. In this problem you encounter a simple instance of electromagnetic duality. Let A be a finite abelian group, and let $F_A: \text{Bord}_2 \rightarrow \text{Alg}$ be the associated finite gauge theory.
 - (a) Compute the partition function $F_A(X)$ for a closed surface X .
 - (b) Let $A^\vee$ be the Pontrjagin dual group of characters of A , and let $f_{A^\vee}$ be the theory that counts maps into $A^\vee$. Compute $f_{A^\vee}(X)$.
 - (c) For $\lambda \in \mathbb{C}^\times$ let E_λ be the theory whose partition function is $\lambda^{\text{Euler}(X)}$. Conjecture relation amongst the theories $F_A, f_{A^\vee}, E_\lambda$ for a specific value of λ .

Problem Set # 13

Math 262b: Quantum theory from a geometric viewpoint II

Due: March 22

Problems

1. Recall the cochain model for differential cohomology discussed in lecture. You can find some text about it in M. J. Hopkins and I. M. Singer, *Quadratic Functions in Geometry, Topology, and M-Theory*, J. Diff. Geom. 70 (2005), 329–452, [arXiv:math/0211216](#) as well as in D. S. Freed and A. Neitzke, *3d spectral networks and classical Chern-Simons theory*, [arXiv:2208.07420](#). Use this model to produce an isomorphism

$$\check{H}^1(M) \xrightarrow{\cong} \text{Map}(M, \mathbb{R}/\mathbb{Z})$$

for any smooth manifold M . In what sense is your isomorphism *natural* in M ?

2. Continuing with the cochain model, for any smooth manifold M and any nonnegative integer n , produce an isomorphism of $\check{H}^n(M)$ with the abelian group of *differential characters* introduced by Cheeger and Simons. (You should look up their paper.) A differential character is a pair (f, ω) in which $f: Z_{n-1}(M) \rightarrow \mathbb{R}/\mathbb{Z}$ is a homomorphism and $\omega \in \Omega_M^n$ is a differential form such that for $c \in C_n(M)$ we have $f(\partial c) = \int_c \omega$. (For this to make sense, c is a *smooth* singular chain; see Warner's book on smooth manifolds for example.) As a start, you might prove that ω is closed. You might also see how the group of differential characters fits into the same exact sequences that the differential cohomology group does.
3. Construct an invertible 1-dimensional topological quantum field theory of spin manifolds

$$\alpha: \text{Bord}_{\langle 0,1 \rangle}(B\text{Spin}_1) \longrightarrow s\text{Vect}$$

that maps a spin point to the odd line Π . What is the partition function on each spin circle?

4. Let $\mathcal{G} = s\text{Line}$ denote the Picard groupoid whose objects are super lines and whose morphisms are invertible even maps of superlines. Compute $\pi_0\mathcal{G}$, $\pi_1\mathcal{G}$, and the k -invariant $\pi_0\mathcal{G} \otimes \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_1\mathcal{G}$ that connects them.
5. Repeat for the geometric realization of the bordism category $\text{Bord}_{\langle 0,1 \rangle}(BSO_1)$ of oriented 0- and 1-manifolds. This was sketched in lecture. Compute $\pi_0\mathcal{G}$, $\pi_1\mathcal{G}$, and the k -invariant $\pi_0\mathcal{G} \otimes \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_1\mathcal{G}$ that connects them.

Problem Set # 14

Math 262b: Quantum theory from a geometric viewpoint II

Due: April 5

Problems

1. Deduce the Jacobi identity for a super Lie algebra $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$. Verify that (i) $\mathfrak{g}^0$ is an (ungraded) Lie algebra, (ii) $\mathfrak{g}^1$ is a $\mathfrak{g}^0$ -module, (iii) Lie bracket induces a symmetric pairing $\mathfrak{g}^1 \times \mathfrak{g}^1 \rightarrow \mathfrak{g}^0$ of $\mathfrak{g}^0$ -modules, and (iv) for $x \in \mathfrak{g}^1$ we have $[x, [x, x]] = 0$. Conversely, show that the data (i)–(iv) determine a super Lie algebra.
2. Let M be a smooth manifold, let $\mathcal{X}_M$ denote the Lie algebra of smooth vector fields on M with Lie bracket, and consider the $\mathbb{Z}$ -graded Lie algebra

$$\begin{aligned}\mathfrak{g} &= \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \\ &= \mathcal{X}_M \oplus \mathcal{X}_M \oplus \mathbb{R}\langle d \rangle\end{aligned}$$

The brackets can be defined by the action on differential forms $\Omega_M^\bullet$, where $\xi \in \mathfrak{g}^{-1}$ acts by contraction, $\xi \in \mathfrak{g}^0$ acts by Lie derivative, and d is the differential. Compute all Lie brackets in this graded Lie algebra.

3. Let C be a symmetric monoidal category, and suppose $f: x \rightarrow x$ is a morphism on a dualizable object. The *categorical trace* of f is defined as the morphism

$$1 \longrightarrow x \otimes x^* \xrightarrow{f \otimes \text{id}} x \otimes x^* \longrightarrow x^* \otimes x \longrightarrow 1$$

where the first map is coevaluation, the third is the symmetry, and the last is evaluation. The *categorical dimension* of x is the categorical trace of id_x .

- (a) Verify that in the category of vector spaces, the categorical trace is the usual trace. Where does dualizability enter?
- (b) Verify that in the category of super vector spaces, if $T: V \rightarrow V$ is an endomorphism of $V = V^0 \oplus V^1$, written as $T = \begin{pmatrix} T_0^0 & T_1^0 \\ T_0^1 & T_1^1 \end{pmatrix}$, then the categorical trace is the supertrace $\text{Tr } T_0^0 - \text{Tr } T_1^1$.
- (c) Compute the categorical dimension of a point in $\text{Bord}_{\langle 0,1 \rangle}$. Do the same in the bordism category of oriented manifolds. Of spin manifolds. Of 2-framed manifolds.
- (d) Apply the preceding to deduce the partition functions of various circles in 1-dimensional topological field theories with values in super vector spaces.

4. Consider the super Lie algebra

$$\mathfrak{g} = \mathbb{R}\langle P \rangle \oplus \mathbb{S}$$

in which $\mathbb{S}$ is a 2-dimensional real vector space with inner product $\Gamma: \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{R}$ and the nontrivial bracket is

$$[Q, Q'] = -2\Gamma(Q, Q')P, \quad Q, Q' \in \mathbb{S}.$$

Determine all elements of $\mathbb{S} \otimes \mathbb{C}$ that square to zero. Repeat for $\dim \mathbb{S} = 4$.