Simultaneous Uniformization of Blaschke Products

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I Statement of Results

In 1960 Bers proved (a more general form of) the following result:

Theorem 1 Let Γ_0 and Γ_1 be Fuchsian groups uniformizing two compact Riemann surfaces of genus g. Then there exists a quasifuchsian group Γ whose limit set Λ is a quasicircle, such that Γ is conjugate to Γ_0 and Γ_1 respectively on the uniformizations of the two components of $\mathbb{P}^1 - \Lambda$.

Here we establish an analogous result in the realm of rational functions.

Theorem 2 Let $B_0(z)$, $B_1(z)$ be two expanding Blaschke products of degree $n > 1$. Then there exists a rational function $R(z)$ whose Julia set J is a quasicircle, such that R is conjugate to B_0 and B_1 respectively on the uniformizations of the two components of $\mathbb{P}^1 - J$.

The idea of the proof is simple. Two copies of the closed unit disk can be pasted along their boundaries to form $\mathbb{P}^1$, in such a way that the complex analytic structures are compatible and B_0 matches up with B_1 on the overlap. A regularity theorem allows us to conclude that B_0 and B_1 define an analytic map R on the whole of $\mathbb{P}^1$.

The 'conformal surgery' described above is justified by a simple proposition which may be interesting in its own right. Let J be a Jordan curve in $\mathbb{P}^1$, and let U and V denote the two components of the complement of J. Then there exist conformal maps f and g carrying the unit disk Δ onto U and V; these maps extend by continuity to homeomorphisms of the closed disk, and the map $\phi = g^{-1} f$, a homeomorphism of $\partial\Delta = S^1$ to itself, is the **boundary correspondence** associated to J. The map ϕ is **quasisymmetric** if there exists a constant k such that

$$\frac{1}{k} \leq \frac{|\phi(x)-\phi(x+y)|}{|\phi(x)-\phi(x-y)|} \leq k$$

for all $x,y \in S^1$, where x+y denotes additon of angles on the circle. (More simply, a quasisymmetric function is a one-dimensional quasiconformal mapping in the sense of bounded circular dilatation.)

A Jordan curve $J \subset \mathbb{P}^1$ is a **quasicircle** if it is the image of a Euclidean circle under a quasiconformal map.

Theorem 3 Given any quasisymmetric function $\phi: S^1 \to S^1$, there exists a quasicircle J inducing the boundary correspondence ϕ.

Remark It is known the the boundary correspondence is quasisymmetric for any quasicircle, so the above characterizes homeomorphisms of S^1 which arise in this way. In fact, there is a bijection

[quasisymmetric ϕ] / [$\phi \sim A\phi B$, A,B conformal automorpisms of Δ] $\leftrightarrow$
[quasicircles J] / [$J \sim AJ$, A = conformal automorphism of $\mathbb{P}^1$]

We are unaware of a similar characterization of ϕ for arbitrary Jordan curves J.

II Sketch of the Proofs

For properties of quasiconformal maps used below, we refer to [Ahlfors].

A **Blaschke product** B(z) of degree n is a map of the form

$$B(z) = e^{i\theta} \prod_{i=1}^{n} \left[\frac{z + a_i}{1 + \bar{a}_i z}\right]$$

where $|a_i| < 1$. B carries the disk onto itself and defines a degree n map of the unit circle to itself.

Given B_0 and B_1, a pair of Blaschke products of degree $n > 1$, there exists a homeomorphsim ϕ of the unit circle to itself such that $\phi B_0 = B_1 \phi$. In fact, there exist exactly (n-1) such ϕ, corresponding to the (n-1) fixed points on S^1 of an arbitrary Blaschke product of degree n.

Lemma 4 The map ϕ is quasisymmetric.

Proof Let f_0, f_1 denote lifts of B_0 and B_1 to the universal cover $\mathbb{R}$ of S^1. Normalize these maps so that $f_0(0) = f_1(0) = 0$. (Different choices of fixed points to lift to 0 give rise to the n-1 possible ϕ.) The lift of the conjugating map, which we continue to denote by ϕ, is given by

$$\phi(x) = \lim \phi_n(x) \qquad \text{where} \qquad \phi_n(x) = f_1^{(-n)} f_0^{(n)}(x)$$

and $f^{(n)}$ denotes f composed with itself n times.

Now observe that f_0', f_1' are greater than some constant strictly greater than one (B_0 and B_1 are expanding on the unit circle), while the nonlinearity of f_0 and f_1, measured by $|f_0''/f_0'|$ and $|f_1''/f_1'|$, is bounded above (by compactness of S^1). A well-known distortion theorem [Sullivan, Lemma 1] then implies that

$$|\phi_n''(x)| \, / \, |\phi_n'(x)| = |(\log \phi_n')'|$$

is bounded above by a constant independent of x and n. By the mean value theorem,

$$\frac{|\phi_n(x+y) - \phi_n(x)|}{|\phi_n(x-y) - \phi_n(x)|} = \frac{|\phi_n'(a)|}{|\phi_n'(b)|} \qquad \text{for some } a,b \in [x-y, x+y]$$

so our bound on the logarithmic derivative of $\phi_n{}'$ implies the sequence of functions ϕ_n are uniformly quasisymmetric. Therefore the limit ϕ is itself quasisymmetric.

□

Assuming Theorem 3, Theorem 2 follows easily from the above Lemma.

Proof of Theorem 2 Given B_0 and B_1, choose a homeomorphism ϕ conjugating their actions on the unit circle. By Theorem 3 and the preceding Lemma, there exists a quasicircle J whose induced boundary correspondence is ϕ. Gluing in B_0 and B_1 via the uniformizations of the components of the complement of J, we obtain a continuous endomorphsim R(z) which maps $\mathbb{P}^1$ to itself and is analytic outside of J. Furthermore R(z) is locally quasiconformal on J, so it has square-integrable distributional derivatives; since J has measure zero, R is weakly analytic, and hence (by Weyl's lemma) analytic. Therefore R is a rational function.

Since iterates of the original Blaschke products are normal on the interior of the disk, the Julia set of R is contained in J. But we easily verify that the fixed points of R are dense in J, so J is exactly the Julia set.

□

Proof of Theorem 3 Let $\phi: S^1 \to S^1$ be a given quasisymmetric function. Then ϕ extends to a quasiconfomal map Φ of the closed unit disk to itself. Let μ denote the complex dilatation of Φ^{-1}, and extend μ to all of $\mathbb{P}^1$ by setting $\mu = 0$ on $\Sigma = \mathbb{P}^1 - \Delta$. By the measurable Riemann mapping theorem, there exists a quasiconformal map $\Psi: \mathbb{P}^1 \to \mathbb{P}^1$ such that $\Psi_{\bar{z}} / \Psi_z = \mu$ almost everywhere.

Let $J = \Psi(S^1)$. We claim J induces the boundary correspondence ϕ. Indeed, Ψ carries Σ conformally onto one component of $\mathbb{P}^1 - J$ and $\Psi\Phi$ carries Δ conformally onto the other -- this is a consequence of our choice of μ. The boundary correspondence is simply the map on S^1 defined by $(\Psi)^{-1}(\Psi\Phi) = \phi$, as claimed.

□

Example Consider the pair of degree 2 Blaschke products B_0, B_1, given by

$$B_i(z) = \frac{z(z + a_i)}{(1 + \bar{a}_i z)} \qquad (i=0,1)$$

We claim these maps are simultaneously uniformized by the rational function

$$R(z) = \frac{(z^2 + a_0 z)}{(1 + a_1 z)}$$

Indeed, Theorem 2 guarantees such a rational function exists, and R(z) is essentially the only such rational function of degree two with the correct derivatives at its attracting fixed points.

For n > 2, there does not seem to be any explicit description of R(z) for general B_0 and B_1.

$$\frac{4}{\log|f'(x)|}$$

and consequently it is never sharp.

References

L. Ahlfors, Lectures on Quasiconformal Mappings, Van Nostrand, 1966.

L. Bers, 'Simultaneous Uniformization', Bulletin of the American Mathematical Society, 66 (1960), pp.95-97

____, 'On boundaries of Teichmuller Spaces and on Kleinian Groups: I', Annals of Mathematics, 91 (1970), pp.570-600.

D. Sullivan, 'Conformal Dynamical Systems', in Geometric Dynamics, Springer Lecture Notes 1007 (1983), pp.725-752.