Geometrically finite maps and Kleinian groups

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Geometrically finite rational maps

Let $f(z)$ be a rational map, C its set of critical points, $P = \cup_1^\infty f^n(C)$ its post-critical set and J its Julia set. The map f is *expanding* if $\overline{P} \cap J = \emptyset$. It is well-known that f is expanding iff some fixed iterate of f uniformly expands the spherical metric on the Julia set; these maps are also called *hyperbolic* or *Axiom A*.

Let the space Rat_d (respectively Poly_d) of rational (polynomial) maps of degree d be equipped with the topology of uniform convergence. A well-known and fundamental problem is to resolve the following:

Conjecture. The expanding maps form a dense subset of Rat_d and Poly_d.

Cf. [MSS] where this is related to the problem of invariant measurable line fields supported on the Julia set. (It is known that the set of expanding maps is open).

In many ways an expanding rational map is well-behaved (cf. [Sul]); it is like a Kleinian group with compact convex core in $\mathbf{H}^3$.

More generally, let us say a rational map is *geometrically finite* if $\overline{P} \cap J$ is a finite set. Equivalently, every critical point in the Julia set is pre-periodic. These maps should be compared to geometrically finite Kleinian groups.

For a geometrically finite rational map f:

Problem 1. Show that either the Julia set J is the whole sphere and the action of f on J is ergodic, or the Hausdorff dimension δ of J is less than 2. In the latter case, what can be said about the δ-dimensional measure of J and the dynamics with respect to this measure class?

Problem 2. Show every component of J is locally connected.

Problem 3. Develop for f an analogue of the Haken decomposition for 3-manifolds. For example, if J is disconnected, can f be constructed by surgery from rational maps with connected Julia sets?

Problem 4. Extend Thurston's combinatorial theory of critical finite rational maps (those for which $|P| < \infty$) to all geometrically finite maps. That is, describe f up to combinatorial equivalence rel $\overline{P}$ by a finite amount of topological data, and characterize those combinatorial types which arise as rational maps.

Convex hyperbolic 3-manifolds

Let N be a complete hyperbolic 3-manifold presented as the quotient of $\mathbf{H}^3$ by the action of a Kleinian groups Γ. The *convex core* of N is the quotient of the convex hull of the limit set. All closed geodesics in N are contained in the convex core.

Question. Suppose $\pi_1(N)$ is generated by n elements. Is there an upper bound R_n to the radius of an embedded ball entirely contained in the convex core of N? (Here R_n should depend only on n).

The question has a positive answer when N is a quasifuchsian group. By results of Thurston [Th, Ch. 13] there is a pleated surface near every point in the convex core, and this provides an upper bound on the injectivity radius.

The question also has an (easy) positive solution for hyperbolic 2-manifolds, and we know of no counterexample for hyperbolic manifolds of any dimension.

Critically finite rational maps on $\mathbf{P}^n$

A basic tool aiding the study of critically finite rational maps on the Riemann sphere is the Poincaré metric on the complement of the post-critical set P (assuming $|P| > 2$). This metric is expanded by f. One should be able to apply the same sort of arguments to critically finite rational maps $f : \mathbf{P}^n \to \mathbf{P}^n$, $n > 1$, such that the complement of the post-critical set is Kobayashi hyperbolic.

More precisely, say f is *critical finite* if there exist (possibly reducible) hypersurfaces $V \subset W \subset \mathbf{P}^n$ such that

$$f : (\mathbf{P}^n - W) \to (\mathbf{P}^n - V)$$

is a covering map.

Problem. Are there nontrivial examples of critically finite maps with $\mathbf{P}^n - V$ Kobayashi hyperbolic? How do they behave dynamically?

References

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[Sul] D. Sullivan. Conformal dynamical systems. In *Geometric Dynamics*, pages 725–752, Springer-Verlag Lecture Notes No. 1007, 1983.

[Th] W. P. Thurston. *Geometry and Topology of Three-Manifolds.* Princeton lecture notes, 1979.