Hölder Continuity of the Boundary Correspondence for Isomorphic Fuchsian Groups

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Let $g: R \to R'$ be a diffeomorphism between compact Riemann surfaces represented as quotients of the unit disc Δ by Fuchsian groups Γ and Γ' respectively. Then g lifts to a map F of the disk to itself and extends to a continuous map on the boundary of the disk. Let $f: S^1 \to S^1$ denote the restriction of the lifted map F to the boundary; we call f the **boundary correspondence** induced by g. The map g also induces an isomorphism between Γ and Γ'; for any $\gamma \in \Gamma$, we denote by γ' the corresponding element in Γ'. The map f is uniquely determined by this isomorphism, because it must induce a correspondence between the fixed point of elements in Γ and Γ', and these are dense in S^1.

It is known that analytic properties of f are related to geometric properties of R and R'. For example, if f is absolutely continuous, then in fact f is a Möbius transformation (restricted to the boundary) and R and R' are conformally equivalent Riemann surfaces [Kuusalo, Mostow]. Here we obtain a geometric interpretation of the modulus of Hölder continuity of f.

The map f is in Hölder class C^α if there is a constant M such that

$$d(fx,fy) \leq M\, d(x,y)^\alpha$$

where d is the Euclidean metric on S^1. The **Hölder modulus** of f is α (briefly, $H.mod(f) = \alpha$) if $f \in C^{\alpha+\varepsilon}$ and $f \notin C^{\alpha-\varepsilon}$ for all $\varepsilon > 0$.

Theorem 1 The Hölder modulus of the boundary correspondence f is given by

$$\alpha = \inf_{\gamma \in \Gamma} \frac{\text{length}(\gamma')}{\text{length}(\gamma)}$$

Here **length** refers to the length in the Poincare' metric of the corresponding geodesic on R or R'.

It is easy to see that α is an upper bound for H.mod(f). The opposite inequality follows from a study of the 'coarse hyperbolic geometry' of the map F between the universal covers of R and R'.

We say F is **K-Lipschitz** if

$$\rho(Fx,Fy) \leq M + K\,\rho(x,y)$$

for some constant M, where ρ denotes the Poincare' metric on the disk. Notice F is a **quasi-isometry**, i.e. F and F^{-1} are each K-Lipschitz for some finite K. (Compactness of R and R' implies that the derivatives of g and g^{-1} are bounded; apply the mean value theorem.) K is the **Lipschitz modulus** of F (briefly, L.mod(F) = K) if F is $(K+\varepsilon)$ (but not $(K-\varepsilon)$) - Lipschitz for all $\varepsilon > 0$.

Theorem 2 The Lipschitz modulus of the lifted map F is given by

$$K = \sup_{\gamma \in \Gamma} \frac{\text{length}(\gamma')}{\text{length}(\gamma)}.$$

Theorem 1 follows easily from Theorem 2 and the following observation.

Proposition 3 If K is the Lipschitz modulus of F^{-1}, then the Hölder modulus of f is $\geq$ 1/K.

Proof of Theorem 1

Suppose h: U $\to$ V is a homeomorphism of open neighborhoods of x=0 on the real line, and satisfies h(ax) = a'h(x), where a, a' > 0. Then H.mod(h) $\leq \beta$ = log a'/log a, since h(x) must look roughly like x^{β}.

If we replace the disk by the upper half-plane, and normalize so that two corresponding Möbius transformations γ and γ' have their repelling fixed points at the origin, then $\gamma(z) = az$, $\gamma'(z) = a'z$ where log a = length(γ) and log a' = length(γ'); since f conjugates γ to γ', the inequality H.mod(f) $\leq \alpha$ follows from the preceding remark.

On the other hand, Theorem 2 and Proposition 3 imply H.mod(f) $\geq \alpha$, so H.mod(f) = α as claimed.

□

Proof of Proposition 3

It is enough to show that H.mod(f) $\geq$ 1/K when F^{-1} is K-Lipschitz. By preliminary composition with a Möbius transformation, we may assume that F fixes 0, the center of the disk. Let x and y be points on the boundary of the disk mapped to x' and y' by F. Let δ (δ') be the unique geodesic joining x to y (x' to y') and let p' be the point on δ' closest to 0. Since F is a quasi-isometry, there is a constant r, independent of δ and δ', such that $F^{-1}(\delta')$ lies entirely within a tube of radius r about δ [Thurston, 5.40]. In particular, $p = F^{-1}(p')$ lies within r of δ. Hence

$$\rho(0,\delta) \leq r + \rho(0,p) \leq (r+M) + K\,\rho(0,p') = (r+M) + K\,\rho(0,\delta')$$

where we have used the assumption that F^{-1} is K-Lipschitz. Now it is elementary to check that the Euclidean distance between x and y is nearly proportional to $\exp(-\rho(0,\delta))$ for x and y close together; exponentiating the above expression, we obtain

$$|x-y| \geq C\,|x'-y'|^{K}$$

for some constant C, and hence f (the restriction of F to the boundary) satisfies a Hölder condition with exponent 1/K, as claimed.

□

To prove Theorem 2, consider the subset $T(\varepsilon)$ of the orbit of 0 defined as follows:

$$T(\varepsilon) = \{\gamma(0) : \gamma \in \Gamma \text{ and } \rho(0,\gamma'0) \leq (1+\varepsilon)\,\text{length}(\gamma')\}$$

Say a set E in Δ is **coarsely dense** if there exists a constant M such that $\rho(x,E) \leq M$ for all $x \in \Delta$. Theorem 2 follows easily from the following:

Proposition 4 $T(\varepsilon)$ is coarsely dense for all $\varepsilon > 0$.

Proof of Theorem 2

We again assume F is normalized so that $F(0) = 0$. Since $R = \Delta/\Gamma$ is compact, any geodesic on R has a lift which passes within a fixed distance of 0. If γ denotes the corresponding group element, then $\rho(0,\gamma 0) \leq \text{length}(\gamma) + M$ for a fixed constant M. On the other hand, $\rho(F0,F\gamma 0) = \rho(0,\gamma' 0) \geq \text{length}(\gamma')$; by taking high powers of γ, we can ignore the constant M and deduce $\text{L.mod}(F) \geq K$.

To prove $\text{L.mod}(F) \leq K$, it suffices to show that for all $\varepsilon > 0$, $\rho(0,Fx) \leq M + (1+\varepsilon)K\rho(0,x)$, where M may depend upon ε. Since F is already a quasi-isometry, we need only verify this condition for all x in the coarsely dense subset $T(\varepsilon)$. But if $x = \gamma 0$ is an element of $T(\varepsilon)$, then

$$\rho(0,x) = \rho(0,\gamma 0) \geq \text{length}(\gamma) \geq \text{length}(\gamma)\,\frac{\rho(0,\gamma' 0)}{(1+\varepsilon)\text{length}(\gamma')}$$

so

$$\rho(0,Fx) = \rho(0,\gamma' 0) \leq (1+\varepsilon)\left[\sup \frac{\text{length}(\gamma')}{\text{length}(\gamma)}\right]\rho(0,x)$$

as desired.

$\square$

Here is a geometric argument for Proposition 4. Construct the geodesic δ joining 0 to $\gamma 0$ and measure the angle between the tangent to δ at $\gamma 0$ and the image of the tangent to δ at 0 under the derivative map $(\gamma)'$. If the angle is zero, then γ stabilizes δ and hence $\rho(0,\gamma 0) = \text{length}(\gamma)$. More generally, if the angle is bounded away 180° from 90°, then for $\text{length}(\gamma)$ large the ratio of $\rho(0,\gamma 0)$ to $\text{length}(\gamma)$ is nearly 1. The angle cannot be nearly 90° for <u>all</u> points in the orbit of 0 which are near $\gamma 0$, because of mixing properties for the geodesic flow on a hyperbolic surface. Hence any point in the orbit of 0 is uniformly close to a point in $T(\varepsilon)$, and therefore $T(\varepsilon)$ is coarsely dense.

We will prove two lemmas to make this argument precise.

Lemma 5 Let $M(z)$ be a hyperbolic Möbius transformation stabilizing the geodesic μ. Then

$$\frac{\text{length}(\mu)}{\rho(0,M0)} \sim 1 + \frac{\log|\cos\frac{1}{2}\arg M'(0)|}{\rho(0,M0)}$$

as $\text{length}(\mu) \to \infty$.

Proof Write $M(z) = (az + b)/(\bar{b}z + \bar{a})$ with $|a|^2 - |b|^2 = 1$ and $\text{Re}\, a \geq 0$. A calculation shows $\rho(0,M0) \simeq 2\log|a|$ and $\text{length}(\mu) \simeq 2\log|\text{Re}(a)|$. If $a = r\exp(i\theta)$, then $\log|\text{Re}\, a| = \log|a| + \log|\cos\theta|$ and consequently

$$\frac{\text{length}(\mu)}{\rho(0,M0)} \simeq 1 + \frac{\log|\cos\theta|}{\rho(0,M0)}$$

On the other hand, $M'(0) = \bar{a}^{-2}$, so $\theta = \frac{1}{2} \arg M'(0)$.

□

Lemma 6 There exists a finite set $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that for all $\gamma \in \Gamma$,

$$\sup_i \left| \cos \tfrac{1}{2} \arg (\gamma\gamma_i)'(0) \right| \geq k > 0.$$

for some fixed constant k.

Proof Write

$$\gamma(z) = \frac{az + b}{\bar{b}z + \bar{a}}, \quad \gamma_i(z) = \frac{c_i z + d_i}{\bar{d}_i z + \bar{c}_i}$$

Then $\arg (\gamma\gamma_i)'(0) = 2 \arg (ac_i + b\bar{d}_i)$. Now transitivity of the geodesic flow on the unit tangent bundle to R implies that the set of pairs ($\arg c_i$, $\arg d_i$) are dense mod π, as γ_i ranges over all elements of Γ. It easily follows that we can extract a finite subset of such γ_i such that the Lemma holds.

□

Proof of Proposition 4

Consider $T(\varepsilon)$ for some fixed $\varepsilon > 0$. By Lemmas 5 and 6, there is a finite set of elements γ_i in Γ such that for all $\gamma \in \Gamma$ with length(γ) sufficiently large, $(\gamma\gamma_i)(0)$ is in $T(\varepsilon)$ for at least one choice of i. Since γ is an isometry, this implies $\rho(\gamma 0, T(\varepsilon)) \leq M$ where $M = \sup \rho(0, \gamma_i(0))$. But $R = \Delta/\Gamma$ is compact, so the orbit of 0 is coarsely dense (and remains so even if we delete the finite set of $\gamma 0$ with length(γ) insufficiently large); therefore $T(\varepsilon)$ itself is a coarsely dense subset of the disk.

□

Questions and Remarks

1. By Mostow rigidity, if we replace R and R' by compact hyperbolic manifolds of dimension $n \geq 2$ in the above discussion, then the lifted map F is a coarse isometry, the boundary map f is conformal and the lengths of corresponding geodesics on R and R' are equal. Hence Theorems 1 and 2 remain true.

2. There is an interesting similarity between the quantities

$$K(\Gamma,\Gamma') = \sup_{\gamma\in\Gamma} \frac{\text{length}(\gamma')}{\text{length}(\gamma)}$$

and

$$L(\Gamma,\Gamma') = \sup_{\gamma\in\Gamma} \frac{\text{module}(\gamma')}{\text{module}(\gamma)}$$

where module(γ) denotes the modulus of the largest annulus imbedded in R with core curve γ. Kerchoff has shown that $\frac{1}{2} \log L(\Gamma,\Gamma')$ is exactly the Teichmüller distance between R and R'. It is easy to see $K \leq L$ and one would expect them both to be good measures of the distance between R and R'; on the other hand, Kerchoff's result implies that L is symmetric in Γ and Γ', while this is certainly not true of K. What is the relationship between these quanities asymptotically and in the small?

References

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