

Riemann surfaces, dynamics and hyperbolic geometry

Course Notes — Harvard University — Math 275

C. McMullen

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1 Introduction

This course concerns the interplay between topology, geometry and dynamics in dimensions 2 and 3. It centers around the notion of a hyperbolic manifold, $M^n = \mathbb{H}^n/\Gamma$, $n = 2$ or 3 .

There are remarkable connections with:

1. Topology — for example the mapping–class group of a surface, knot theory, the classification of 3–manifolds;
2. Algebraic geometry — the complex and symplectic structures on moduli space \mathcal{M}_g and its universal cover, Teichmüller space \mathcal{T}_g ; the construction of Bers’ embedding; an analytic proof of the Shafarevich conjecture;
3. Number theory — the structure of the group Γ in $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$, and its relation to arithmetic groups and quaternion algebras; reflection groups and Coxeter diagrams; and
4. Dynamics — on several levels, including (i) analysis of surface diffeomorphisms (pseudo–Anosov theory); (ii) ergodic theory at infinity (Mostow rigidity); (iii) behavior of horocycles and geodesic planes (Ratner rigidity); and (iv) the topological characterization of critically finite rational maps;

and many other topics.

We begin with some examples of hyperbolic manifolds and orbifolds.

Surfaces.

1. The *geometrization theorem* says that every compact surface S can be modeled on S^2 , \mathbb{E}^2 or \mathbb{H}^2 . That is, S admits a metric of constant curvature.

For surfaces with boundary, one can choose the boundary to be *geodesic* or, in the hyperbolic case, to be *cusps*.

2. Wallpaper groups. The study of \mathbb{E}^2 is surprisingly rich if one includes orbifolds; this leads to the 17 types of wallpaper patterns. A key here is the existence of glide reflections: how is the Klein bottle covered by \mathbb{R}^2 ?

3. *The regular right-angled pentagon.* For higher genus one can start with the fact that a surface of genus two can be tiled by 8 right angled hyperbolic pentagons. Since every surface of higher genus covers one of genus two, this settles all the remaining surfaces.
4. Once a Riemann surface is presented as $X = \mathbb{H}/\Gamma$, it has plenty of meromorphic functions, as observed by Poincaré: one can form $\sum_{\Gamma} g^*(q)$, where $q = q(z) dz^2$ is a meromorphic quadratic differential on the unit disk, say with a pole of order k at the origin and otherwise holomorphic and $\int_{\Delta_K} |q| < \infty$ for some compact set K .

By using poles of orders 1 and 2, one can get local coordinates near the image of the origin. By compactness, one can show that X embeds in \mathbb{P}^n (and then \mathbb{P}^3).

3-manifolds. The 8 geometries. Most of these uniformize manifolds of dimension $2 + 1/2$. They can be organized as follows:

| | | |
|-------------------------|----------------|---------------------------------------|
| S^3 | \mathbb{E}^3 | \mathbb{H}^3 |
| $S^2 \times \mathbb{R}$ | * | $\mathbb{H}^2 \times \mathbb{R}$ |
| Sol | Nil | $\widetilde{\text{SL}_2(\mathbb{R})}$ |

Spaces built from the (regular) dodecahedron. Recall that the dodecahedron's vertices, edges and faces satisfy

$$V - E + F = 20 - 30 + 12 = 2 = \chi(S^2).$$

The Poincaré homology sphere. The first space is obtained by identifying opposite faces of a dodecahedron with a small twist (the smallest possible — $2\pi/10$.) One can check that the edges come together in *groups of 3*, and that the vertices come together in *groups of 4*. The inside of the dodecahedron provides a single 3-cell. After checking that the links of vertices are spheres, we conclude that the quotient space is a 3-manifold with

$$V - E + F - C = 20/4 - 30/3 + 12/2 - 1 = 5 - 10 + 6 - 1 = 0.$$

If we choose the dodecahedron D to be spherical with dihedral angles of 120° , then we get a smooth metric structure along the edges. Also, the links

of vertices are spherical tetrahedra, so we get a spherical structure there as well. This shows that

$$M^3 = D / \sim \cong S^3 / \Gamma$$

for some group Γ . In fact Γ has order 120; it is the $\mathbb{Z}/2$ central extension of A_5 . Thus $H_1(M^3, \mathbb{Z}) = 0$, and M^3 is an (integral) homology sphere. This space is a counterexample to Poincaré's first formulation of the Poincaré conjecture.

The Seifert-Weber manifold. Now twist by 3/10ths of a revolution. In this case we find the edges come together in groups of 5, and all the vertices are identified. We get a 3-manifold with

$$V - E + F - C = 20/20 - 30/5 + 6 - 1 = 1 - 6 + 6 - 1 = 0.$$

In this case we can choose D to be a hyperbolic dodecahedron with dihedral angles of 72° , and we find the link of the unique vertex in the quotient is S^2 tiled by 20 triangles in an icosahedral pattern. Thus

$$M^3 = D / \sim \cong \mathbb{H}^3 / \Gamma$$

for some *infinite* group Γ .

The figure eight knot complement. Let $M^3 = S^3 - K$ where K is the figure 8 knot. The Wirtinger presentation for the figure 8 turns out to have the relations:

$$ac = da, ca = bc, dc = bd, \quad \text{and} \quad ba = db,$$

which can be simplified to:

$$ac\bar{a}ca = ca\bar{c}ac.$$

One can now try to find a representation of this group into $SL_2(\mathbb{C})$ that will give M^3 a hyperbolic structure. Since all generators are peripheral, we expect a and c to be parabolic, so we can set

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix};$$

then solve for t . (!)

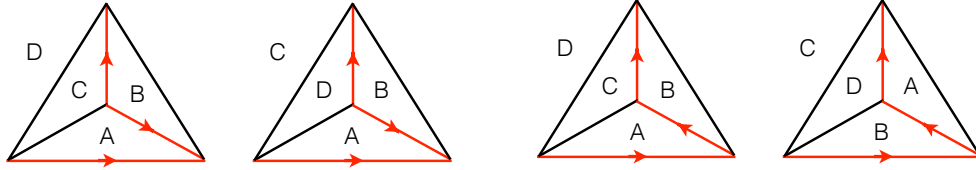


Figure 1. Gluing pattern for the figure eight knot complement, and its twisted cousin.

In detail: one gets two equations:

$$t^2 - t + 1 = 0 \quad \text{and} \quad t(t^2 - t + 1) = 0;$$

their simultaneous solutions are given by the primitive 6th roots of one.

Remark on relations. The Wirtinger presentation is always redundant: one relation can always be deleted. To see this, observe that if we label the generators g_1, \dots, g_n cyclically around the knot, then the relations describe how to conjugate g_i to obtain g_{i+1} . The last crossing transforms g_n to g_1 , but this relation can also be obtained by transform g_1 to g_2 and ultimately to g_n .

Construction with tetrahedra. In fact one can build the figure eight knot complement using a pair of regular ideal tetrahedra. These should be glued along their faces so the resulting complex X has 2 edges and a single vertex (and 4 faces and 2 cells); Thus

$$\chi(X) = -2 + 4 - 2 + 1 = 1.$$

Removing the vertex gives Euler characteristic zero, which shows the boundary is a torus.

Fibering. It is a challenging exercise to show this is really the figure eight knot complement. One method is to show that the figure eight knot complement fibers over the circle, with fiber a punctured torus and monodromy $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. This can be done using the language of Murasugi sums: the spanning surface comes from a sum of two Hopf bands with twists in opposite directions. (The same direction would yield the trefoil knot).

One then should try to see geometrically that the union of two tetrahedra also fibers.

One can also build the *twisted* figure eight–knot complement, with monodromy $-\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, using two ideal tetrahedra. Their gluing patterns are shown in Figure 1.

These two manifolds have isomorphic double covers.

Arithmeticity. It is known that the figure eight knot is the only knot whose complement is an arithmetic 3–manifold.

3-manifolds as moduli spaces. It is well-known that $\mathcal{M}_1 \cong \mathbb{H}^2/\mathrm{SL}_2(\mathbb{Z})$ is both a hyperbolic orbifold and the moduli space of Riemann surfaces of genus 1.

Is there an interpretation of a space such as

$$M^3 = \mathbb{H}^3/\mathrm{SL}_2\mathbb{Z}[i]$$

as a suitable moduli space? The answer is *yes*, but of course the moduli space must have some real aspect, since the dimension is odd.

In fact, there is a natural map $M^3 \rightarrow \mathcal{A}_2$ into the moduli space of principally polarized Abelian varieties A such that the image consists of those A admitting an *anti-holomorphic* endomorphism $J : A \rightarrow A$ such that $J^2 = -I$ and J is self-adjoint for the symplectic form.

The underlying geometry is that every point $p \in \mathbb{H}^3$ determines an *antipodal map* $J_p : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, well-defined up to sign, such that

$$J_p(iv) = -iJ_p(v) \quad \text{and} \quad J_p^2 = -I.$$

This map acts on $\widehat{\mathbb{C}} = \partial\mathbb{H}^3$ by an anti-holomorphic involution, and any such involution arises for some point in \mathbb{H}^3 .

We can now use J_p to give $A = \mathbb{C}^2/\mathbb{Z}[i]^2$ a new complex structure, compatible with its symplectic form, in such a way that multiplication by i becomes an anti-holomorphic endomorphism of A . For details see [GT].

Constructions with octahedra. Let $W \subset S^3$ be the *Whitehead link* — the simplest two–component link consisting of a pair of unknots with linking number zero, but still intertwined.

Theorem 1.1 *The Whitehead link complement $S^3 - W \cong \mathbb{H}^3/\Gamma$ admits a complete hyperbolic structure with $\Gamma \subset \mathrm{SL}_2\mathbb{Z}[i]$. It can be constructed from a regular ideal octahedron via suitable face pairings.*

Note that the dihedral angles of such an octahedron are 90° .

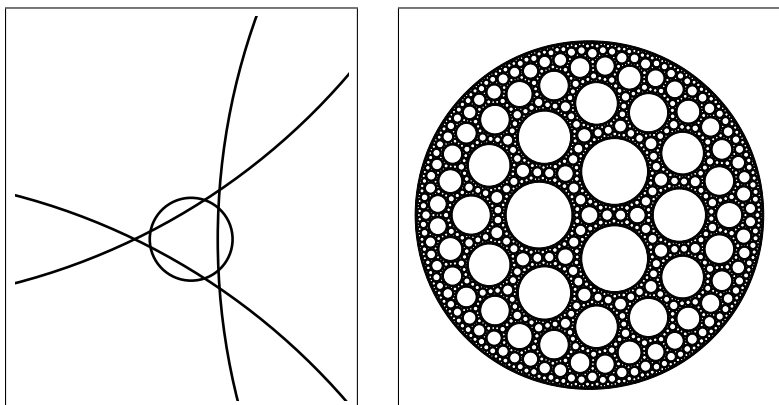


Figure 2. A regular tetrahedron with dihedral angles of $\pi/7$, and the limit set of its reflection group.

It is easy to see that the Borromean rings B have the property that $S^3 - B$ is a double cover of $S^3 - W$. Thus this link complement also admits a hyperbolic structure.

Jacks and tinker-toys. One way to produce a lot of compact hyperbolic 3-manifolds from simple pieces is to use orbifold building block. Here is one such construction.

The basic block is a regular tetrahedron T_n with dihedral angles of π/n . For $n = 6$ we obtain an ideal tetrahedron. For $n \geq 7$ or more, we obtain a tetrahedron of infinite volume, with ‘vertices outside \mathbb{H}^3 ’. See Figure 2 for the case $n = 7$. When $n = \infty$ the circles become tangent, and the corresponding reflection group’s limit set is the Apollonian gasket.

For $n \geq 7$ we can truncate the infinite volume ends of T_n with orthogonal planes to obtain a compact polyhedral P_n , with 4 new faces that are hyperbolic equilateral triangles with angles π/n . Since all dihedral angles are $\pi/2$ or π/n , we can regard P_n as a compact *orbifold*, namely the quotient of \mathbb{H}^3 by the reflection group it defines.

Now suppose we have a graph G with degree 4 at every vertex. Fix $n \geq 7$. Then replace each vertex by a copy of P_n , and then glue corresponding triangles together along an edges. There are some additional choices involved — twisting along edges and ordering the edges around a vertex. In any case, the end result is a more complicated orbifold, with a collapsing map to G . Passing to a manifold cover, we obtain a compact hyperbolic manifold

with a map from $\pi_1(M)$ to $\pi_1(G)$, itself a free group. This provides rather elementary proof that such ‘large’ cocompact groups exists.

2 Mostow rigidity

One of the most important results in mathematics is:

Theorem 2.1 (Mostow rigidity) *Let $f : M_1 \rightarrow M_2$ be a homotopy equivalence between a pair of finite-volume hyperbolic manifolds of dimension $n \geq 3$. Then f can be deformed to an isometry.*

In this section we will examine one or more proofs of this result.

Models.

1. The ball model: $\mathbb{H}^n = \{x \in \mathbb{R}^n : |x| < 1\}$, with the metric

$$\rho = \frac{2|dx|}{1 - |x|^2}.$$

2. The upper half-space model, $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ with the metric

$$\rho = \frac{|dx|}{x_n}.$$

These two models are conformally equivalent: the geodesics are arcs of circles orthogonal to the boundary.

3. The Minkowski model in $\mathbb{R}^{n,1}$: here the inner product comes from the form

$$\langle x, x \rangle = x_1^2 + \cdots + x_n^2 - t^2,$$

and $\mathbb{H}^n = \{x : \langle x, x \rangle = -1 \text{ and } t > 0\}$.

4. The Klein model in $\mathbb{R}\mathbb{P}^n$: this is obtained by taking the lines through \mathbb{H} in the Minkowski model. In an affine chart, it is given by the unit ball again, but now the geodesics have become straight lines.

Two geodesics in \mathbb{H} . The hyperbolic metric (of constant curvature -1) in \mathbb{H} is given by $\rho = |dz|/y$. The simplest geodesic is the imaginary axis I , parameterized at unit speed by

$$\gamma(t) = i \exp(t).$$

The second geodesic is $|z| = 1$, parameterized by

$$\gamma(t) = i \operatorname{sech}(t) + \tanh(t).$$

Here we have used the identity $\sinh^2(t) + 1 = \cosh^2(t)$ and divided through by $\cosh^2(t)$.

Note that

$$\gamma'(t) = -i \operatorname{sech}(t) \tanh(t) + \operatorname{sech}^2(t) = \operatorname{sech}(t)(-i \tanh(t) + \operatorname{sech}(t))$$

satisfies

$$|\gamma'(t)| = \operatorname{sech}(t) = \operatorname{Im} \gamma(t),$$

which shows $\gamma(t)$ has unit speed. From this example one can easily prove:

Theorem 2.2 *Let $\gamma \subset \mathbb{H}^n$ be a geodesic, and ρ a parallel path at distance t . Then the natural projection $\rho \rightarrow \gamma$ is contracting by a factor of $\cosh(t)$.*

Corollary 2.3 *On a hyperbolic surface, a collar of radius t about a simple closed geodesic γ of length L has boundary components of length $L \cosh(t)$.*

A geodesic in the Minkowski model. In $\mathbb{R}^{n,1}$ a natural unit-speed geodesic is given by $\gamma(s) = (x(s), t(s)) = (\sinh(s), \cosh(s))$. From this we see:

Theorem 2.4 *A sphere of radius s in \mathbb{H}^n is isometric to a sphere of radius $\sinh(s)$ in Euclidean space \mathbb{R}^n .*

Proof. The sphere of radius s about the point $(x, t) = (0, 1)$ in Minkowski space is the intersection of the plane P defined by $t = \cosh(s)$ with \mathbb{H}^n . The restriction of the Minkowski metric to P is the same as the Euclidean metric on \mathbb{R}^n , and (x, t) lies in the intersection if and only if $|x| = \sinh(s)$. ■

From this one can easily derive formulas for areas and volumes of spheres and balls; but the fundamental principle they rest on is the easily formulate statement above.

Corollary 2.5 *In \mathbb{H}^2 we have:*

$$\begin{aligned} \text{length}(S^1(p, s)) &= 2\pi \sinh(s) \quad \text{and} \\ \text{area}(B^2(p, s)) &= 2\pi(\cosh(s) - 1) = 4\pi \sinh^2(s/2). \end{aligned}$$

(Note the double angle formula $\cosh(2s) = 1 + 2\sinh^2(s)$.)

Corollary 2.6 *In \mathbb{H}^3 we have:*

$$\text{area}(S^2(p, s)) = 4\pi \sinh^2(s) \quad \text{and} \quad \text{vol}(B^3(p, s)) = \pi(\sinh[2t] - 2t).$$

Quasi-isometry. A map $f : X \rightarrow Y$ between metric spaces is a *quasi-isometry* if

$$d(x, y) \asymp d(f(x), f(y))$$

for all $x, y \in X$. More precisely, there is a constant $K \geq 1$ such that

$$(1/K)d(x, y) \leq d(f(x), f(y)) \leq Kd(x, y).$$

It is a *coarse* quasi-isometry if the equation above holds whenever $d(x, y)$ is sufficiently large; equivalently, one replaces the lower bound by

$$(1/K)d(x, y) - K.$$

The map is a *coarse surjection* if every point in Y is within a uniformly bounded distance of $f(X)$.

For example, the inclusion $\mathbb{Z}^n \rightarrow \mathbb{R}^n$ is a coarse surjection. More generally one can show:

Theorem 2.7 *Let $\widetilde{M} \rightarrow M$ be the universal cover of a compact manifold with a Riemannian metric. Choose generators for $\pi_1(M)$, and metrize it using the word metric. Then the inclusion*

$$\pi_1(M) \rightarrow \widetilde{M}$$

coming from the orbit of a point is a coarse surjective quasi-isometry.

The idea that the universal cover can be reconstructed, coarsely from the fundamental group alone is one of the central ideas underlying Mostow rigidity.

Proof of Mostow rigidity. Next we outline the proof of Mostow rigidity. For simplicity, we assume $f : M^n \rightarrow N^n$ is a homotopy equivalence between *compact* hyperbolic manifolds, with homotopy inverse $g : N^n \rightarrow M^n$. We also may assume that f and g are *smooth* (this is not essential).

1. We first lift f and g to maps F and G on the universal cover \mathbb{H}^n of domain and range, and show that each is a coarse quasi-isometry.

Indeed, there is a $K > 1$ such that F and G expand distances by at most a factor of K , and also $GF(x)$ is within distance K of x for all x . Thus, writing $|x - y|$ for $d(x, y)$, we have:

$$|F(x) - F(y)| \leq K \cdot |x - y|$$

and

$$|x - y| - 2K \leq |GF(x) - GF(y)| \leq K|F(x) - F(y)|,$$

which gives

$$\frac{|x - y|}{K} - 2 \leq |F(x) - F(y)|.$$

Hence F is a (coarse) quasi-isometry

2. As a consequence, F sends geodesics to quasigeodesics.
3. We now use negative curvature to prove an essential fact:

Any quasi-geodesic lies within a uniformly bounded neighborhood of a geodesic.

For the proof, let $\gamma : \mathbb{R} \rightarrow \mathbb{H}^n$ be a quasi-geodesic. Let δ_T be the unique geodesic joining $\gamma(-T)$ to $\gamma(T)$, for $T \gg 0$. Suppose γ makes an excursion outside the ball $B(\delta_T, R)$, leaving and re-entering this ball at the endpoints of the interval $[s, t] \subset [-T, T]$. Then the length of the excursion is at most $K|s - t|$.

We can now bound the distance from $\gamma(s)$ to $\gamma(t)$ below by projecting this excursion to δ_T , and adding segments of length R at the beginning and end. Since the projection contracts by a factor of $\cosh(R)$, we find:

$$\frac{|s - t|}{K} - K \leq |\gamma(s) - \gamma(t)| \leq 2R + \frac{K|s - t|}{\cosh R}.$$

It follows that

$$|s - t| \cdot \left(\frac{1}{K} - \frac{K}{\cosh R} \right) \leq 2R + K.$$

Now choose R large enough that the coefficient of $|s - t|$ is positive. We then obtain an upper bound $|s - t| \leq S(K)$. Thus the excursion has length at most $KS(K)$, and hence γ stays within distance $R + KS(K)$ of δ_T (for all T).

We can now let $T \rightarrow \infty$ and pass to a subsequence to obtain a geodesic δ with $\gamma \subset B(\delta, R(K))$.

4. It follows that F extends continuously to a homeomorphism on $S^{n-1} = \partial\mathbb{H}^n$, with inverse the extension of G .

To prove this it is useful to work in the ball model, and adjust F and G by homotopy so that $F(0) = G(0) = 0$. Fix a small number δ , and suppose $x, y \in S^{n-1}$ and $|F(x) - F(y)| > \delta$. Then the geodesic from $F(x)$ to $F(y)$ comes within distance R of 0. Here $R = R(\delta)$ is approximately $\log(1/\delta)$. Using the fact that G is a quasi-isometry, and the geodesic tracking property, we conclude that the geodesic from x to y comes within distance $KR + K$ of 0. It follows that $|x - y| > \epsilon$, where ϵ is approximately $\exp(-KR - K)$, and only depends on δ . Thus $|x - y| < \epsilon$ implies $|F(x) - F(y)| < \delta$, which gives continuity.

Checking the bounds throughout, one finds that the boundary values of F are Hölder continuous of exponent $1/K$, whenever F is a K -quasi-isometry.

5. Next we show that F is quasiconformal.
6. In dimensions $n \geq 3$, it follows that DF is defined everywhere, and that its conformal distortion provides measurable, a $\Gamma = \pi_1(M^n)$ -invariant unit ellipsoid in each tangent space to S^{n-1} .
7. We then use ergodic theory for Γ acting on S^{n-1} to show almost every ellipsoid is round. It follows that $F|_{S^{n-1}}$ is a conformal map.
8. Let F_0 denote the unique extension of $F|_{S^{n-1}}$ to an isometry of \mathbb{H}^n . It is then easy to show that F_0 descends to an isometry $f_0 : M^n \rightarrow N^n$ homotopic to f .

■

Ergodic theory. Let us discuss the ergodic theory of a discrete group Γ acting on S_∞^{n-1} in more detail. Let $M = M^n = \mathbb{H}^n/\Gamma$.

As a warmup we observe:

Theorem 2.8 *If M is compact, then the action of Γ on S_∞^{n-1} is ergodic.*

For the proof the following general fact is useful. Given an integrable measure function f on S_∞^{n-1} , let $F = \text{ex}(f)$ denote its *visual extension* to \mathbb{H}^n .

The value of F at the center of the ball model for \mathbb{H}^n is simply the average of f with respect to the round measure on the sphere. Its value at $p \in \mathbb{H}^n$ is the average of f with respect to the visual measure v_p . This extension gives the unique *natural* map

$$\text{ex} : C(S_\infty^{n-1}) \rightarrow C(\overline{\mathbb{H}^n}),$$

meaning $\text{ex}(g \cdot f) = g \cdot \text{ex}(f)$ for all $g \in \text{Isom}(\mathbb{H}^n)$, and $\text{ex}(f)|_{S_\infty^{n-1}} = f$.

Theorem 2.9 *The visual extension $F = \text{ex}(f)$ is harmonic (in the hyperbolic metric) for any integrable function f on the sphere.*

Proof. Consider the map $f \mapsto \Delta \text{ex}(f)$. This natural map annihilates the constants. It follows that it is identically zero. ■

Proof of Theorem 2.8. Note there is no invariant measure! What this means is that any measure Γ -invariant subset $A \subset S_\infty^{n-1}$, either A or its complement has measure zero.

To see this, let A be such a set and let $F = \text{ex}(\chi_A)$. By naturality, F descends to a harmonic function on M . By compactness and the maximum principle, F is constant. In fact, by the Lebesgue density theorem, this constant is 0 or 1. In the first case A has measure zero, and in the second case A has full measure. ■

Aside: the ergodic theorem. The ergodic theorem concerns a *measure preserving* map $T : X \rightarrow X$ on a *probability space* (X, m) . Assuming T is invertible, its study translates into the theory of *unitary* operators U on the Hilbert space $H = L^2(X, m)$. This operator is given by:

$$U(f) = f \circ T;$$

it satisfies $U^* = U^{-1}$.

The von Neumann ergodic theorem can be expressed in terms of an arbitrary unitary operator U in terms of the averaging function,

$$S_n(f) = \frac{1}{n} \sum_{i=0}^{n-1} U^i f.$$

Note that $\|S_n(f)\| \leq 1$. Let

$$H^U = \text{Ker}(U - I)$$

denote orthogonal projection to the set of U -invariant vectors. Of course $S_n|_{H^U}$ is the identity. The main result is the following:

Theorem 2.10 *For any unitary operator $U \in \mathcal{B}(H)$, we have $S_n(f) \rightarrow g = \pi(f)$, where π is orthogonal projection to the closed subspace $H^U = \text{Ker}(U - I)$.*

Lemma 2.11 *For any operator $A \in \mathcal{B}(H)$, we have $H = \overline{A^*(H)} \oplus \text{Ker}(A)$.*

Proof. Let $S = A^*(H)$. Then for all $y \in S^\perp$ and all $x \in H$, we have

$$\langle A^*x, y \rangle = \langle x, Ay \rangle = 0,$$

and thus $Ay = 0$. Conversely, if $Ay = 0$, then $y \in S^\perp$. Thus $\text{Ker}(A) = S^\perp$. Since $(S^\perp)^\perp = \overline{S}$, the Lemma follows. ■

Proof of Theorem 2.10. By the Lemma above, we have $H = \overline{(U - I)(H)} \oplus H^U$. Thus any $f \in H$ can be expressed in the form

$$f = \pi(f) + g - Ug + \delta,$$

where $\|\delta\|$ is small. For the coboundary $g - Ug$ we have $S_n(g - Ug) \rightarrow 0$, since we get a telescoping series. Of course $S_n(\pi(f)) = \pi(f)$. Thus for large n , $S_n(f)$ is close to $\pi(f) + S_n(\delta)$. Since $\|\delta\|$ was small, this shows that $S_n(f)$ converges to $\pi(f)$. ■

The significantly stronger and more difficult Birkhoff–Khinchin ergodic theorem says what happens pointwise.

Theorem 2.12 *If $f \in L^1(X, m)$, then $S_n(f)(x) \rightarrow F(x)$ for almost every $x \in X$. Here F is a T -invariant function, namely the conditioning of f on the algebra of T -invariant measurable sets.*

For $f \in L^2(X, m)$, $F = \pi(f)$.

Ergodicity on $S_\infty^{n-1} \times S_\infty^{n-1}$. Next we show:

Theorem 2.13 *If M has finite volume, then the geodesic flow on the unit tangent bundle $T_1(M)$ is ergodic.*

Proof. Let f be a continuous function on $T_1(M)$. By the ergodic theorem, the averages of f along geodesics in forward or backwards time converge in $H = L^2(T_1(M))$, and almost surely, to the projection f_0 of f to the closed subspace H^A of functions invariant under the geodesic flow.

Now consider the lifts of f and f_0 to F and F_0 on $T_1(\mathbb{H}^n)$. We apply the Hopf argument. Since F_0 is invariant the geodesic flow, its value only depends on the endpoints (x, y) of a given geodesic γ in S_∞^{n-1} . Since F is continuous, and two geodesics with the same forward endpoints on S_∞^{n-1} are asymptotic, the forward average of F along γ does not depend on x . The backward average does not depend on y . Thus F_0 is constant, and hence H^A consists only of the constant functions. This is equivalence to ergodicity of the geodesic flow. ■

Corollary 2.14 *The action of Γ on $S_\infty^{n-1} \times S_\infty^{n-1}$ is ergodic.*

Proof. An invariant function on the product of spheres gives an invariant function for the geodesic flow on $T_1(\mathbb{H}^n)$ which descends to an invariant function on $T_1(M)$, and the latter must be constant by ergodicity. ■

Remark. Using the horocycle flow, it is easy to show that the geodesic flow on $T_1(M)$ is actually *mixing*. In fact general theorems in homogeneous dynamics show that for any irreducible lattice Γ in a semisimple Lie group G , the action of any noncompact Lie subgroup $H \subset G$ on $L^2(\Gamma \backslash G)$ is mixing.

Corollary 2.15 *In dimension 3, if M is compact then there is no Γ -invariant measurable line field on S_∞^2 .*

Proof. Given a pair of point x, y in the sphere, we can use parallel transport along the geodesic γ joining them to compute the angle $\theta(x, y)$ between the line at x and the line at y . This gives a nonconstant Γ -invariant function of (x, y) , a contradiction. ■

Blowup argument. Using the geometric topology and a blow-up argument, the last step in the proof of Mostow rigidity can be enhanced as follows. (We focus on dimension 3 for concreteness.)

Theorem 2.16 *Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic manifold with injectivity radius bounded above. Then there is no measurable Γ -invariant line field on S_∞^2 .*

See [Mc, §2].

The Hausdorff topology. In any compact metric space X , there is a natural *Hausdorff topology* on the set $\text{Cl}(X)$ of closed subsets $F \subset X$, defined by the metric

$$d(F_1, F_2) = \inf\{r > 0 : B(F_1, r) \supset F_2 \text{ and } B(F_2, r) \supset F_1\}.$$

It is easy to show:

Proposition 2.17 *If X is a compact metric space, so is $\text{Cl}(X)$.*

Compactness of X is not essential: a similar construction works when the one-point compactification of X is metrizable, e.g. when X is a separable, locally compact metric space, and the same result holds.

The geometric topology. Now let G be a Lie group with finitely many components. It is easy to show that if $H_n \subset G$ is a sequence of closed subgroups, and $H_n \rightarrow L \subset G$ in the Hausdorff topology, then L is also a closed subgroup of G .

Discreteness is also preserved, provided a definite neighborhood of the identity is avoided. Thus we have:

Proposition 2.18 *Let U be a neighborhood of the identity in G . Then the set of discrete subgroups $\Gamma \subset G$ meeting U only in $\{e\}$ is compact.*

Example: Mahler's compactness criterion; lattices. A lattice $\Lambda \subset \mathbb{R}^n$ is *unimodular* if $\text{vol}(\mathbb{R}^n/\Lambda) = 1$. The space of unimodular lattices with a chosen orientation is given by:

$$\mathcal{L}_n = \text{SL}_n(\mathbb{Z}) \backslash \text{SL}_n(\mathbb{R}).$$

In view of the Proposition above, it is easy to prove:

Theorem 2.19 (Mahler) *For each $r > 0$, the set $\mathcal{L}_n(r)$ of unimodular lattices with shortest (nonzero) vector of length $\geq r$ is compact.*

Many results in number theory, such as finiteness of the ideal class group and the rank of the unit group, follow from this important observation.

Hyperbolic manifolds. Now let $G = \text{Isom}(\mathbb{H}^n)$. Then a *torsion-free* discrete group $\Gamma \subset G$ gives rise to a manifold $M^n = \Gamma \backslash \mathbb{H}^n$ with a natural *baseframe* $p \in FM_n$. This baseframe is defined by fixing once and for all a standard basepoint p_0 in $F\mathbb{H}^n$ (say the origin in the ball model) and a frame at this point (say the standard basis for \mathbb{R}^n).

Conversely, the pair (M^n, p) uniquely determines Γ , since there is a unique way to identify its universal cover with \mathbb{H}^n so that p_0 lies over p . In brief:

$$\{\text{Baseframed hyperbolic manifolds } (M^n, p)\} = \{\text{Torsion-free groups } \Gamma \subset \text{Isom } \mathbb{H}^n\}.$$

The case of \mathbb{H}^2 .

Theorem 2.20 *Let $f : X \rightarrow Y$ be a diffeomorphism between compact hyperbolic Riemann surfaces, and let $F : S^1 \rightarrow S^1$ be the boundary values of its lift to the universal cover. Then either f is homotopic to an isometry, and F is a Möbius transformation, or*

$$F'(x) = 0$$

almost everywhere.

Other proofs of Mostow rigidity. Gromov famously proved Mostow rigidity using the fact that there is a unique tetrahedron of maximal volume in 3 dimensions (later generalized to $n \geq 3$), and at the same time gave a direct proof that hyperbolic volume is a topological invariant, namely it is the L^1 norm of the fundamental class up to a constant factor [Gr], [Rc].

Thurston gave a very geometric proof of a weakening of Mostow rigidity, showing that the set of hyperbolic structures on an acylindrical 3-manifold is *compact* [Th]. This proof also works for open manifolds, and is a key step in the original proof of the geometrization theorem for Haken 3-manifolds.

The thick–thin decomposition. We aim to deduce some consequence from Mostow rigidity. First we describe a basic part of our picture of hyperbolic manifolds.

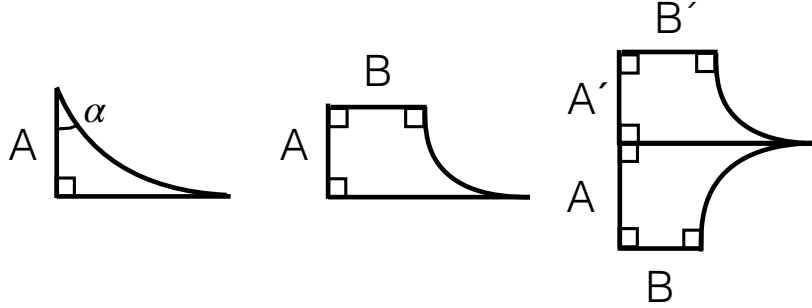


Figure 3. Trigonometry in \mathbb{H}^2 .

Theorem 2.21 *For each dimension n there exists an ϵ_n such that for any hyperbolic manifold M^n , the ϵ_n -thin part of M^n consists of cylinders around short simple geodesics, and standard horoball neighborhoods of cusps.*

Examples: $n = 2$. In this case we have standard collar neighborhoods around short simple geodesics and cusps. The basic fact is that for a short geodesic of length L , the standard collar around L has boundary components of length $\asymp 1$, separated by distance $\asymp \log(1/L)$, forming an annulus of modulus $\asymp 1/L$.

A more precise statement is that the collar of width $S(L)$ about a simple geodesic of length L is always embedded, where $S(L)$ is chosen so that

$$\sinh S(L) \sinh(L/2) = 1$$

This frequently-occurring expression has the following source:

Theorem 2.22 *Let $Q \subset \mathbb{H}^2$ be a quadrilateral with one ideal vertex, and all other angles 90° . Let A and B be the lengths of its two finite sides. Then:*

$$\sinh(A) \sinh(B) = 1.$$

Figure 3 (middle) illustrates this quadrilateral. The triangle at the left satisfies:

Lemma 2.23 *Let $T \subset \mathbb{H}^2$ be a triangle with one right angle, one ideal vertex and the remaining angle α . Let A be the length of its finite side. Then:*

$$\sin \alpha \cosh A = 1.$$

Proof. Put the ideal vertex at infinity in the upper half plane model, and be the segment on $|z| = 1$ starting at i and running to $(x, y) = (\tanh A, \operatorname{sech} A)$. Then we find the vertical line through (x, y) meets the unit circle in angle α , and hence the line from the origin to (x, y) meets the real axis in angle α . It follows that $y = \sin \alpha = \operatorname{sech} A$. ■

Proof of Theorem 2.22. Cut Q into two triangles along the geodesic from its ideal vertex to the opposite vertex v . The result is a pair of triangles as in the Lemma, with angles $\alpha + \beta = \pi/2$, because we had an ideal vertex at p . Thus $\sin(\alpha) = \cos(\beta)$, and hence

$$\begin{aligned}\sin^2(\alpha) \cosh^2(A) &= \sin^2(\alpha)(1 + \sinh^2(A)) = 1 \quad \text{and} \\ \sin^2(\beta) \cosh^2(B) &= (1 - \sin^2(\alpha))(1 + \sinh^2(B)) = 1.\end{aligned}$$

Eliminating $\sin^2(\alpha)$ gives the relation $\sinh^2(A) \sinh^2(B) = 1$. ■

Corollary 2.24 *Let P be a pair of pants with one cusp and two boundary components of lengths $2A, 2A'$. Choose B such that $\sinh(A) \sinh(B) = 1$, and similarly for B' . Then the cuffs of width B and B' about the respective boundary components are disjoint.*

Proof. Such a pair of pants can be constructed by gluing together two quadrilaterals of the type described above, and then doubling. (See Figure 3 (right).) By construction, the cuffs are disjoint. ■

Corollary 2.25 *On a hyperbolic surface, the collars of width $S(L_i)$ about disjoint simple closed curves of lengths L_1 and L_2 are disjoint.*

Proof. Use coverings to reduce to the case of a pair of pants, and then use the Schwarz lemma to reduce to the case of where one cuff is a cusp. ■

Theorem 2.26 *Any sufficiently short closed geodesic on a hyperbolic surface is simple.*

Proof. Passing to a covering space, we reduce to the case of a closed geodesic on a pair of pants, and then by the Schwarz Lemma to the case of a triply-punctured sphere. There, the shortest geodesic comes from the matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ in $\Gamma(2)$ of trace $6 = 2 \cosh(L/2)$, which gives $L = 2 \cosh^{-1}(3) = 3.5254\dots$ ■

Similar ideas prove the general collar lemma with specific bounds.

Corollary 2.27 *Each boundary component of the standard collar around a simple geodesic of length L has itself total length*

$$L \cosh(S(L)) \sim 2$$

as $L \rightarrow 0$.

Proof. Note that as $L \rightarrow 0$, we have $S(L) \rightarrow \infty$ and hence

$$\cosh(S(L)) \sim \sinh(S(L)) = 1/\sinh(L/2) \sim 2/L.$$

■

Corollary 2.28 *We can choose the standard collar neighborhood of a cusp to be bounded by a horocycle of length 2.*

Note that this bound is sharp for the triply-punctured sphere.

Examples: $n \geq 4$. For $n = 2$, a collar neighborhood of a short geodesic can disconnect the surface. But for $n \geq 3$, the thick part is *connected*, and for $n \geq 4$, with finite volume, its boundary components are either of the form $S^{n-2} \times S^1$ (for a short geodesic) or $(S^1)^n$ (for a cusp).

Thus for $n \geq 4$, the fundamental group of the thick part is *isomorphic* to the fundamental group of M^n itself. This plus Mostow rigidity shows:

Theorem 2.29 *For $n \geq 4$, and a given V , there are only finitely many hyperbolic manifolds with $\text{vol}(M^n) \leq V$.*

Proof. The thick part can be covered by $O(V)$ embedded balls, which can meet in only a finite number of patterns, providing only finitely many possibilities for $\pi_1(M^n)$. By Mostow rigidity, the fundamental group itself determines M^n . ■

Examples: $n = 3$. In the special case of dimension 3, a rank two cusp and a short geodesic both contribute a copy of $S^1 \times S^1$ to the boundary of the thin part. In this case $\pi_1(M_{\text{thick}})$ surjects onto $\pi_1(M)$, but there may be some collapsing.

In fact, an analysis of *Dehn filling* shows that whenever M^3 has cusps, one can find an infinite sequence of *closed* manifolds K_n with $\text{vol}(K_n) \rightarrow \text{vol}(M)$ from below. On the level of volumes, this analysis shows:

Theorem 2.30 *The set of volumes of hyperbolic 3-manifolds is a closed, well-ordered subset of \mathbb{R} , isomorphic to ω^ω .*

3 Hyperbolic surfaces

In this section we discuss the *flexibility* of hyperbolic surfaces, both compact and of finite volume. We will use pairs of pants to construct Fenchel–Nielsen coordinates on Teichmüller space, as well as a smooth model for the Deligne–Mumford compactification of moduli space, $\mathcal{M}_{g,n}$.

Teichmüller space: definitions. We begin by defining the *Teichmüller space* \mathcal{T}_g of marked compact Riemann surfaces of genus g .

Let Σ_g denote an oriented topological surface of genus g . A point in \mathcal{T}_g is specified by a Riemann surface X of genus g together with a *marking* homeomorphism,

$$f : \Sigma_g \rightarrow X.$$

Every point in \mathcal{T}_g arises in this way.

Two markings in the same homotopy class are considered *equivalent*, and they determine the same point in \mathcal{T}_g . Two different marked Riemann surfaces (X, f) and (Y, g) determine the same point in \mathcal{T}_g if there is a holomorphic isomorphism $\alpha : X \rightarrow Y$ compatible with markings; this means that (Y, g) and $(Y, \alpha \circ f)$ are equivalent.

Teichmüller space for flat surfaces. Let us start by constructing \mathcal{T}_1 , the Teichmüller space of complex tori.

Let $E \cong \mathbb{C}/L$ be a compact complex torus, and let $f : \Sigma_1 \rightarrow E$ be an orientation–preserving homeomorphism. Every Riemann surface of genus 1 carries a unique flat conformal metric, up to scale. Equivalently, E can be presented as a quotient \mathbb{C}/L , and the lattice L is determined up to $L \mapsto \lambda L$, $\lambda \in \mathbb{C}^*$.

Now suppose E has a marking $f : \Sigma_1 \rightarrow E$. Choose an oriented basis (a, b) for $H_1(\Sigma_1, \mathbb{Z})$. Then $f(a)$ can be represented by a unique simple closed geodesic α on E . Scaling, we can assume the length of α is one.

If we cut E open along α , we obtain a Euclidean cylinder $C \cong S^1 \times [0, y]$ of height $y > 0$ with geodesic boundary of length 1. The original surface E is obtained by gluing $S^1 \times 0$ to $S^1 \times 1$. There is a unique way to glue with no twist; this results in a Euclidean torus $E(y)$, isomorphic to $\mathbb{C}/(\mathbb{Z} \oplus iy\mathbb{Z})$. For this torus, we can choose the marking so that $f(b)$ is represented by the vertical geodesic $[0, iy]$.

Twists. We can also identify the ends of C by rotating through distance x , then gluing. In this case the result is a torus $E(\tau = x + iy)$ isomorphic to $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. There is a unique such gluing so that $E(\tau)$ is our original

marked surface (E, f) . The twist is chosen so that $f(b)$ is sent to the geodesic $[0, \tau]$.

It follows that a point in \mathcal{T}_1 is uniquely determined by the *length* and *twist* parameters (x, y) , which together correspond to a point $\tau \in \mathbb{H}$. Summing up, we have:

Theorem 3.1 *The choice of an oriented basis (a, b) for $H_1(\Sigma_1, \mathbb{Z})$ determines a natural isomorphism $\mathcal{T}_1 \rightarrow \mathbb{H}$.*

Representations and the mapping–class group. Identifying $\pi_1(\Sigma_1)$ with \mathbb{Z}^2 , we can also identify \mathcal{T}_1 with a space of *representations*:

$$\mathcal{T}_1 = \text{Hom}^+(\mathbb{Z}^2, \mathbb{C})/\mathbb{C}^*,$$

where the $+$ indicates that the pullback of the symplectic form on \mathbb{C} should be positive on any positively oriented basis for \mathbb{Z}^2 . This insures, in particular, that $\rho(\mathbb{Z}^2) = L$ is discrete.

Any two markings of E differ by an element of the *mapping–class group*

$$\text{Mod}_1 = \text{Mod}(\Sigma_1) = \text{Diff}^+(\Sigma_1)/\text{Diff}_0(\Sigma_1) \cong \text{SL}_2(\mathbb{Z}).$$

By changing the marking, we obtain an action of Mod_1 on \mathcal{T}_1 with quotient the *moduli space* of Riemann surfaces of genus 1,

$$\mathcal{M}_1 = \mathcal{T}_1/\text{Mod}_1 \cong \mathbb{H}/\text{SL}_2(\mathbb{Z}).$$

Note: the element $-I$ in $\text{SL}_2(\mathbb{Z})$ acts trivially, because $z \mapsto -z$ is always an automorphism of E .

Symplectic form. The intersection pairing on $H_1(\Sigma_1, \mathbb{Z})$ gives rise to a natural symplectic form on \mathcal{T}_1 . Given a symplectic basis (a, b) for homology, this form can be expressed as:

$$\omega = d\ell \wedge d\tau = C \cdot \frac{dx \wedge dy}{y^2},$$

i.e. it is given by the hyperbolic area form. Here the area of the torus is given by $A(x, y) = y$, and ℓ and τ are the length and twist along a *after* normalizing by the area; that is,

$$\ell = \frac{1}{\sqrt{A}} = y^{-1/2} \quad \text{and} \quad \tau = \frac{x}{\sqrt{A}} = xy^{-1/2}.$$

This formula has a nice generalization to the hyperbolic case, as we will see below.

Aside: Siegel space. The case of genus 1 is special because a complex torus $E = \mathbb{C}/L$ is isomorphic to its Jacobian variety. Thus the moduli space of Riemann surfaces and the moduli space of Abelian varieties coincide for this genus: we have $\mathcal{M}_1 \cong \mathcal{A}_1$. The space \mathcal{A}_g , in general, carries a natural Kähler metric and hence a symplectic structure, which coincides with the one just described in the case of genus $g = 1$. However for higher genus, the geometric structures on \mathcal{M}_g and \mathcal{A}_g are very different.

Teichmüller space for hyperbolic surfaces. We now proceed to construct \mathcal{T}_g , $g \geq 2$. The definitions are the same: we fix an oriented smooth surface Σ_g , and a point in \mathcal{T}_g is an equivalence class of marked surfaces $[X, f]$, where $f : \Sigma_g \rightarrow X$ is an orientation-preserving homeomorphism.

Building blocks.

Theorem 3.2 *Given any three real numbers with $0 \leq \theta_1, \theta_2, \theta_3$ and $\sum \theta_i < \pi$, there exists a unique hyperbolic triangle with these internal angles, up to isometry.*

Theorem 3.3 *Given any three real numbers with $0 \leq L_1, L_2, L_3$, there exists a unique hyperbolic pair of pants with these boundary lengths.*

The proof in both cases hinges on an analysis of the quadratic form

$$Q(a) = \begin{pmatrix} 1 & -a_1 & -a_2 \\ -a_1 & 1 & -a_3 \\ -a_2 & -a_3 & 1 \end{pmatrix}.$$

If we set $a_i = (z_i + 1/z_i)/2$, then we find:

$$\det Q(a) = -\frac{(z_1 z_2 + z_3)(z_1 z_3 + z_2)(z_1 + z_2 z_3)(z_1 z_2 z_3 + 1)}{4z_1^2 z_2^2 z_3^2}.$$

It follows that $\det Q(a)$ vanishes exactly when

$$z_1^{\pm 1} z_2^{\pm 1} z_3^{\pm 1} = -1$$

for some choice of signs. Note that replacing z_i with z_i^{-1} does not change the value of a_i .

Triangles. Setting $z_i = \exp(i\theta_i)$, we find this condition singles out the Euclidean triangles. Thus, provided $\sum \theta_i < \pi$, $Q(a)$ has constant signature, which can be easily shown to be $(2, 1)$. Changing coordinates to put $Q(a)$ in standard form, we obtain 3 geodesics in the Minkowski model for hyperbolic space that determine the desired triangle.

Pairs of pants. Setting $z_i = \exp L_i$, we find $z_i \geq 1$ so $Q(a)$ has constant signature in this case as well, and again the signature is $(2, 1)$. Passing to Minkowski space, we obtain the desired hyperbolic pair of pants, i.e. a triple of oriented geodesics with $d(g_i, g_j) = L_k$ for all permutations (i, j, k) of $(1, 2, 3)$.

Examples in the unit disk. Let us illustrate this result and at the same time describe the connection between geodesics in the Minkowski and disk model for \mathbb{H}^2 .

An oriented geodesic in Minkowski space corresponds to a point $p(x, y, t) \in \mathbb{R}^{2,1}$ with $p^2 = 1$, or equivalently

$$x^2 + y^2 = t^2 + 1.$$

Projectivizing, we obtain the point

$$z = [x : y : t] = [x/t : y/t : 1] = (x + iy)/t$$

outside the unit disk Δ . The endpoints of the corresponding geodesic $\gamma \subset \Delta$ are simply the two points $a, b \in S^1$ where the tangent line to S^1 passes through z . Since γ is orthogonal to S^1 at these points, the lines az and bz are radii of the circle C containing γ as a subarc. Since $|a| = |b| = 1$, the radius r of C satisfies, by the Pythagorean rule,

$$r^2 = |z|^2 - 1 = (x^2 + y^2 - t^2)/t^2 = (1/t)^2,$$

and thus $r = 1/t$.

Let us apply this reasoning to explicitly construct a regular hyperbolic quadrilateral $Q \subset \Delta$ with angles of 60° . Two adjacent sides of Q correspond to geodesics specified by p_1 and p_2 with $\langle p_1, p_2 \rangle = -\cos(\pi/3) = -1/2$. If we center Q at the origin and arrange that two of its sides are orthogonal to the real axis, then we find

$$p_1 = (x, 0, t), p_2 = (0, x, t), x^2 = 1 + t^2 \quad \text{and} \quad \langle p_1, p_2 \rangle = -t^2 = -1/2.$$

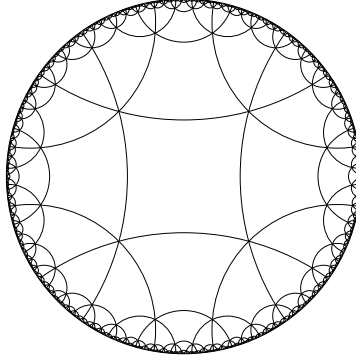


Figure 4. The reflection group for a regular quadrilateral with 60° angles.

Thus $t^2 = 1/2$, $x^2 = 3/2$, $z = x/t = \sqrt{3}$ and $r = 1/t = \sqrt{2}$. Thus Q is bounded by the arcs of the 4 symmetrically placed Euclidean circles in \mathbb{C} , all of radius $\sqrt{2}$, and one of them centered at $z = \sqrt{3}$.

The corresponding reflection group is shown in Figure 4. This image was generated by the `lim` code shown in Figure 3. This code is a shell script; `lim` is run from the command line, the options specify a clipping window, and the program generates the orbits of the first 4 circles — described by `c x y r` — under the group generated by reflections through the next 4 circles.

Fenchel–Nielsen coordinates.

Theorem 3.4 *Each pair of pants decomposition S of Σ_g determines a system of length and twist coordinates (ℓ_i, τ_i) , $i = 1, \dots, 3g - 3$ on Σ_g , providing a bijection*

$$\mathcal{T}_g \cong \mathbb{R}_+^S \times \mathbb{R}^S \cong \mathbb{R}^{6g-6}.$$

Changes of coordinates are real–algebraic.

Corollary 3.5 *Teichmüller space is homeomorphic to an open ball.*

Corollary 3.6 *The moduli space \mathcal{M}_g is an orbifold $K(\pi, 1)$, with $\pi = \text{Mod}_g / Z(\text{Mod}_g)$.*

Here $Z(\text{Mod}_g)$ is the center of the mapping–class group, equal to $\mathbb{Z}/2$ for $g = 1$ and 2 and otherwise trivial.

Compactness theorems.

```

lim -w -1.1 -1.1 1.1 1.1 -c 0 0 1 <<eof | ps2pdf - > quad.pdf
c 1.73205080757 0 1.41421356237
c 0 1.73205080757 1.41421356237
c -1.73205080757 0 1.41421356237
c 0 -1.73205080757 1.41421356237
r 1.73205080757 0 1.41421356237
r 0 1.73205080757 1.41421356237
r -1.73205080757 0 1.41421356237
r 0 -1.73205080757 1.41421356237
eof
open quad.pdf

```

Figure 5: Code for Figure 4.

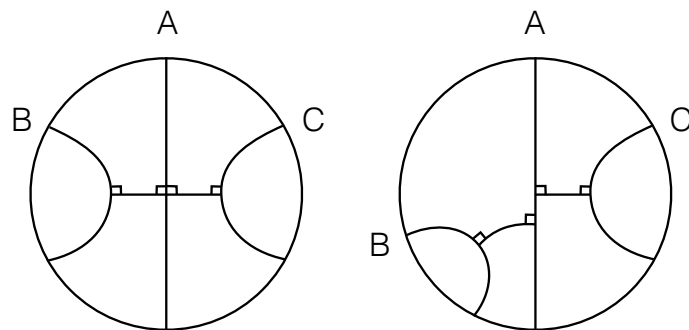


Figure 6. Twist parameters as seen in the universal cover.

Theorem 3.7 *There are only finitely many pair of pants decompositions, up to the action of Mod_g . In fact they correspond to trivalent graphs of rank g .*

Theorem 3.8 *Every $X \in \mathcal{M}_g$ admits a pair of pants decomposition $S = (\gamma_i)$ with $L(\gamma_i) = O(g)$.*

Proof. Choose a pair of pants decomposition of $X \in \mathcal{M}_g$ by a greedy algorithm: let γ_1 be the shortest simple geodesic, let γ_{n+1} be the shortest simple geodesic disjoint from those chosen so far, and let $S = (\gamma_1, \dots, \gamma_{3g-3})$.

Using the fact that the area of X is $O(g)$, we find that $L(\gamma_1) = O(\log g)$. (Consider a maximal embedded ball about any point.) Then, cutting along the geodesics chosen so far and taking a maximal embedded collar of width w about its boundary, we obtain an incompressible pair of pants inside X . Its area is at least $w(\sum \ell_i) = O(1)$. Provided this sum is not too small, we find $w = O(1)$. By considering the geodesics bounding this pair of pants, we conclude that $\ell_{n+1} \leq (\max \ell_i) + O(1)$. A similar argument, using the thick-thin decomposition, works when $\sum \ell_i$ is small to achieve the same conclusion. It follows by induction that $\ell_k = O(k + \log g)$. Since the maximum value of k is $3g - 3$, and $\log g = O(g)$, we conclude that $L(\gamma_i) = O(g)$ for all i . ■

Counting graphs. For work on counting the number of trivalent graphs on $2n$ vertices, see for example (Robinson and Wormald, 1983).

The end of moduli space. Let $L(X) > 0$ denote the length of the shortest closed geodesic on $X \in \mathcal{M}_g$. This geodesic is necessarily simple and $L(X)$ is sometimes called the *systole* of X . Let $\mathcal{M}_g(r)$ denote the locus where $L(X) \geq r > 0$.

Theorem 3.9 *For each $r > 0$, the set $\mathcal{M}_g(r) \subset \mathcal{M}_g$ is compact. Equivalently, $L : \mathcal{M}_g \rightarrow \mathbb{R}_+$ is proper.*

Proof 1. Since every $X \in \mathcal{M}_g$ has the same volume, every X in $\mathcal{M}_g(r)$ can be covered with $N(r)$ embedded geodesic balls. It follows that $\mathcal{M}_g(r)$ is compact in the geometric topology, and this is easily seen to coincide with the usual topology on \mathcal{M}_g . ■

Proof 2. Consider a sequence $X_n \in \mathcal{M}_g(r)$. Let S_n be a greedy collection of simple geodesics on X_n forming a pants decomposition with all $\gamma \in S_n$ of length $O(g)$. Pass to a subsequence so that S_n is constant up to the action of the mapping-class group.

Then can then choose markings $f_n : \Sigma_g \rightarrow X_n$ for all n such that $S_n = f(S)$ for a fixed pants decomposition S of Σ_g . Then in Fenchel–Nielsen coordinates subordinate to S , we have $X_n = (\ell_i(n), \tau_i(n))$ with $r \leq \ell_i(n) \leq Cg$ for all i and n . Composing with Dehn twists, we can also assume that $0 \leq \tau_i(n) \leq Cg$ for all i and n . Passing to a subsequence, we can assume that these Fenchel–Nielsen coordinates converge, and $\lim \ell_i(n) > 0$ for all i . Then $X_n \rightarrow X \in \mathcal{T}_g$, where X is the marked hyperbolic surface described by the limiting coordinates. ■

The moduli space $\mathcal{M}_{g,n}$. A (connected) finite volume hyperbolic surface X has type (g, n) if it has genus g and n cusps. One can canonically attach to X a compact Riemann surface with marked points, (\bar{X}, P) , such that $X = \bar{X} - P$.

Such a surface X can be assembled from pairs of pants where some boundary geodesics have become cusps; formally, we regard these as boundaries of length $\ell = 0$. Then the description of $\mathcal{T}_{g,n}$ and $\mathcal{M}_{g,n}$ goes through as in the compact case ($n = 0$). To mark X , we use the interior of a compact surface $\Sigma_{g,n}$ with genus g and n boundary components.

Stable curves. A *stable curve*, from the perspective of hyperbolic geometry, is the data (Y, τ) of a possibly disconnected but finite area hyperbolic surface Y , together with a pairing τ between some or all of the cusps of Y , such that (Y, τ) is ‘combinatorially connected’: one can pass between any two components of Y by jumping across paired cusps.

One can attach to Y a complex analytic space \bar{Y} with normal crossings (modelled on $xy = 0$ in \mathbb{C}^2) at points P corresponding to the cusps pairings, such that Y is isomorphic to $\bar{Y} - P$. Then \bar{Y} is connected.

We say (Y, τ) has type (g, n) if there exists a system D of simple closed curves on $\Sigma_{g,n}$ such that Y is homeomorphic to the interior of $\Sigma_{g,n} - D$, with its paired cusps corresponding to the components of D .

The Deligne–Mumford compactification. We let $\bar{\mathcal{M}}_{g,n}$ denote the set of all stable curves of type (g, n) . With a suitable topology, $\bar{\mathcal{M}}_{g,n}$ is compact and it contains $\mathcal{M}_{g,n}$ as a dense open subset.

For simplicity, let us consider the construction of $\bar{\mathcal{M}}_g$. One can start by

forming the *augmented* Teichmüller space \mathcal{T}_g^* , and then defining

$$\overline{\mathcal{M}}_g = \mathcal{T}_g^* / \text{Mod}_g.$$

A point of \mathcal{T}_g^* is specified by a system of simple curves $D \subset \Sigma_g$, and a stable curve (Y, eta) , together with a marking

$$f : (\Sigma_g - D) \rightarrow Y$$

compatible with τ . Given a pants decomposition $S \supset D$, we can regard Y Fenchel–Nielsen coordinates as a point with $\ell_i = 0$ for the curves $S_i \in D$. The twist parameters τ_i are then irrelevant. In this way we obtain a neighborhood of Y in \mathcal{T}_g^* , and hence a topology on $\overline{\mathcal{M}}_g$.

Put differently, let \mathbb{Z}^D denote the subgroup of Mod_g generated by Dehn twists on the curves in D . We can then form the quotient

$$\mathcal{M}_g^D = \mathcal{T}_g / \mathbb{Z}^D \cong (\mathbb{R}_+ \times S^1)^D \times (\mathbb{R}_+ \times \mathbb{R})^{S-D}$$

using the ‘polar coordinates’ $(\ell_i, \tau_i/\ell_i)$ on the first $|D|$ factors. Each of these factors can be compactified by adding a point to account for the case $\ell_i = 0$. Adding these in, we obtain a manifold together with a natural map,

$$\pi_D : \overline{\mathcal{M}}_g^D \rightarrow \overline{\mathcal{M}}_g.$$

These charts give $\overline{\mathcal{M}}_g$ the natural structure of an orbifold.

Stratification. The boundary of moduli space falls in to strata of codimension $2d$, corresponding to systems of simple closed curves $D \subset \Sigma_g$ (up to Mod_g) with $|D| = d$. In the stratum of codimension $2d$, exactly d simple curves have been pinched to create stable curves with $2d$ cusps. It is therefore clear that the boundary consists of the closure of its codimension two strata.

Now given any two simple closed curves D_1 and D_2 on Σ_g , one can find an element $f \in \text{Mod}_g$ such that D_1 and $f(D_2)$ are disjoint (and non-isotopic). It follows that any two codimension two strata have closures which meet. Consequently, we have:

Theorem 3.10 *The moduli space \mathcal{M}_g has only one end. Equivalently, $\partial \mathcal{M}_g$ is connected.*

Example: $g = 2$. In general the strata are parameterized by products of lower-dimensional moduli spaces. For $g = 2$, the strata of complex dimension d are given by:

$$\begin{array}{l|l} d = 2 & A \cong \mathcal{M}_{1,1} \times \mathcal{M}_{1,1} \quad B \cong \mathcal{M}_{1,2} \\ d = 1 & C \cong \mathcal{M}_{1,1} \times \mathcal{M}_{0,3} \quad D \cong \mathcal{M}_{0,4} \\ d = 0 & E \cong \mathcal{M}_{0,3} \times \mathcal{M}_{0,3} \quad F \cong \mathcal{M}_{0,3} \times \mathcal{M}_{0,3}. \end{array}$$

Here $C \subset \bar{A} \cap \bar{B}$, while $D \subset \bar{B} - \bar{A}$. The strata of dimension zero are naturally in bijection with the trivalent graphs on $2g - 2$ vertices.

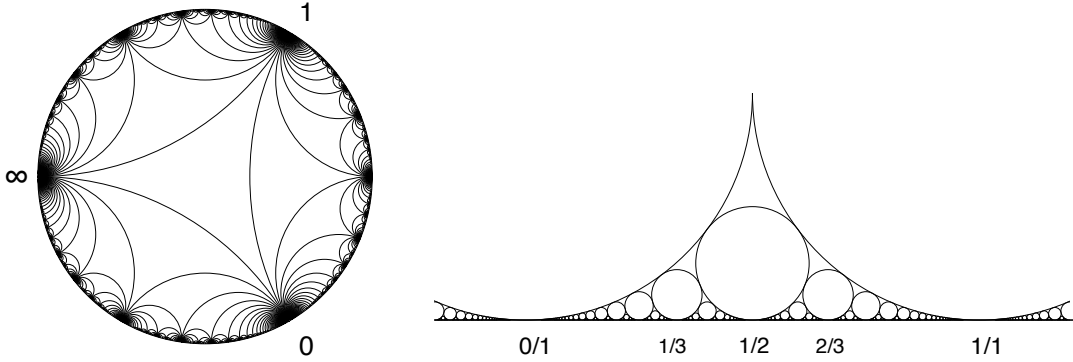


Figure 7. Farey geodesics and the Farey circle packing.

Example: $(g, n) = (1, 1)$. The augmented Teichmüller space in this case is given by

$$\mathcal{T}_{1,1}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}.$$

A neighborhood basis for $x = p/q \in \partial\mathcal{T}_{1,1}^*$ is given by the horoballs resting on x . Each boundary point corresponds to an isotopy class of simple closed curve on $\Sigma_{1,1}$.

The *Farey tessellation* of \mathbb{H} is given by the orbit under $\text{Mod}_{1,1} \cong \text{SL}_2(\mathbb{Z})$ of the ideal triangle with vertices $0, 1$ and ∞ . The geodesics in this tessellation

join a/b to c/d if and only if

$$\left| \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = 1.$$

This condition means that the corresponding simple closed curves meet exactly once.

The *Farey circle packing* of \mathbb{H} consists of the region B_∞ above the line $y = 1$ – a horoball resting on $x = \infty$ – together with the ball $B_{a/c}$ resting on a/c with diameter $1/c^2$. Two circles in this packing touch if and only if the determinant condition above is satisfied. In this case we say a/b and c/d are *adjacent*. It follows that if a/b and c/d are adjacent, then each is adjacent to their Farey sum,

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+b}{c+d}.$$

(This sum is obtained by ‘freshman addition’.) One can thus easily verify:

Theorem 3.11 *The set of all rationals in $[0, 1]$, expressed in lowest terms, can be generated by starting with $0/1$ and $1/1$ and then repeatedly performing Farey sum on adjacent numbers.*

The prime number theorem. As an aside, we remark that the prime number theorem is equivalent to the statement that:

$$\sum_{a/b \in [0,1] : b \leq N} \exp(2\pi a/b) = o(N).$$

(There are about N^2 terms in the sum.) The same statement with $o(N)$ replaced by $O(N^{1/2+\epsilon})$ is equivalent to the Riemann hypothesis.

The curve complex. The graph formed by the Farey geodesics is a version of the *curve complex* $\mathcal{C}_{g,n}$. The vertices of $\mathcal{C}_{g,n}$ are given by (nontrivial) simple closed curves α on $\Sigma_{g,n}$; a collection of vertices $(\alpha_0, \dots, \alpha_n)$ spans an n -simplex if the intersection number $i(\alpha_i, \alpha_j) = 0$ for all i, j . (For the case of $(1, 1)$ we have used intersection number one.)

See [Iv] and [Ha] for more details. The curve complex \mathcal{C}_g is homotopy equivalent to a wedge of spheres of dimension $2g - 2$.

4 Complex surfaces

In this section we describe the perspective on Riemann surfaces and their Teichmüller spaces afforded by complex analysis and quasiconformal mappings.

Teichmüller's theorem. One of our main goals will be to understand:

Theorem 4.1 (Teichmüller) *Given $X, Y \in \mathcal{T}_g$, there exists a unique extremal quasiconformal map*

$$f : X \rightarrow Y$$

compatible with markings. This map is an affine stretch in local coordinates governed by a pair of holomorphic quadratic differentials on X and Y .

Here *extremal* means that f has the minimum possible dilatation $K(f)$ among all quasiconformal maps in its isotopy class. We remark that f is typically *not smooth* at the zeros of the associated quadratic differential q . Provided f is not conformal, its Beltrami coefficient has the form

$$\mu(f) = k \frac{\bar{q}}{|q|}$$

for a unique $q \in Q(X)$ of norm 1, and a unique real constant $k = (K - 1)/(K + 1) \in (0, 1)$, where $K = K(f) > 1$.

Tangent and cotangent spaces. For more context, we remark that the holomorphic tangent and cotangent spaces at $X \in \mathcal{T}_g$ have the form, by general principles,

$$T_X \mathcal{T}_g \cong H^1(X, \Theta) = H^1(X, K_X) \quad \text{and} \quad T_X^* \mathcal{T}_g \cong H^0(X, K_X^2) = Q(X).$$

Here K_X denotes the canonical bundle to X at the same time as its sheaf of holomorphic sections, while Θ is the sheaf of holomorphic vector fields on X , and $Q(X) \cong 3g - 3$ is the vector space of holomorphic quadratic differentials on X . We also have, using the exact sequence of sheaves,

$$0 \rightarrow \Theta \rightarrow (\text{qc vector fields}) \xrightarrow{\bar{\partial}} (L^\infty \text{ Beltrami differentials}) \rightarrow 0,$$

an isomorphism

$$T_X \mathcal{T}_g \cong M(X)/\bar{\partial}V(X) \cong M(X)/Q(X)^\perp,$$

where $M(X)$ is the space of global L^∞ Beltrami differentials on X . This isomorphism is proved using Weyl's lemma and duality between L^∞ and L^1 . Indeed, the tangent and cotangent spaces are dual Banach spaces with respect to the L^1 -norm

$$\|q\|_T = \int_X |q|$$

on $Q(X)$, and the L^∞ quotient norm:

$$\|\mu\|_T = \inf_v \|\mu + \bar{\partial}v\|_\infty.$$

We remark that from the perspective of hyperbolic geometry, one can endow $X = \mathbb{H}/\Gamma$ with the sheaf $\mathfrak{sl}_2(\mathbb{R})$ of locally isometric vector fields, and then

$$\Gamma_X \mathcal{T}_g \cong H^1(X, \mathfrak{sl}_2(\mathbb{R})).$$

Moreover, using the hyperbolic metric ρ on X the Weil–Petersson metric on the cotangent space is simply the L^2 -norm:

$$\|q\|_{\text{WP}}^2 = \int_X \rho^{-2} |q|^2.$$

In this section we will concentrate on the Teichmüller metric.

Extremal length. To motivate the statement and proof of Teichmüller's theorem we will first focus on a particular *conformal invariant* of a Riemann surface, one that can be defined using *all* conformal metrics at once. This powerful invariant is the *extremal length* of a family of paths Γ on X . Given a rectifiable path $\gamma : [0, 1] \rightarrow X$ and a Borel metric $\rho = \rho(z) dz$, $\rho(z) \geq 0$, the length of γ in this metric is defined by

$$L(\gamma, \rho) = \int_0^1 \rho(\gamma(t)) \gamma'(t) dt.$$

If γ is not rectifiable then the length is infinite. We let

$$L(\Gamma, \rho) = \inf_{\gamma \in \Gamma} L(\gamma, \rho)$$

denote the length of the shortest path in the family, and let

$$A(X, \rho) = \int_X \rho^2$$

denote the area of X . Then the extremal length is defined by:

$$\lambda(\Gamma) = \sup_{\rho} \lambda(\Gamma, \rho) = \frac{L(\Gamma, \rho)^2}{A(X, \rho)}.$$

Note that to make $L(\Gamma, \rho)$ large, ρ must assign significant length to *all* the paths γ in Γ . Provided these paths sweep out a significant part of X , the area $A(X, \rho)$ is then also forced to be large, and these two quantities compete to define the extremal length.

Moduli of quadrilaterals. A *quadrilateral* is a Jordan domain $Q \subset \mathbb{C}$ with 4 marked points on its boundary, producing two pairs of opposite edges, the a -pairs and the b -pairs. The prototypical example is $Q = [0, a] \times [0, b]$. We let Γ_b denote all paths joining the opposite a -edges, and similarly for Γ_a . In the Euclidean metric, we have

$$L(\Gamma_a, |dz|) = a, \quad L(\Gamma_b, |dz|) = b \quad \text{and} \quad A(Q, |dz|) = ab.$$

Theorem 4.2 *We have $\lambda(\Gamma_a) = a/b$ and $\lambda(\Gamma_b) = b/a$.*

Proof. Using the Euclidean metric we see that $\lambda(\Gamma_a) \leq b/a$. The reverse inequality follows from the Cauchy-Schwarz theorem: on the one hand, we have

$$\left(\int_Q \rho \cdot 1 \right)^2 \leq \int_Q 1 \int_Q \rho^2 = ab \cdot A(Q, \rho);$$

on the other hand, we have

$$\int_Q \rho = \int_0^b dy \int_0^a \rho(x, y) dx \geq b \cdot L(\Gamma_b, \rho),$$

which gives

$$\lambda(\Gamma_a, \rho) \leq (ab)/b^2 = a/b.$$

The same reasoning applies to $\lambda(\Gamma_b)$. ■

Marked quadrilaterals. In general, we define the modulus of a quadrilateral by $\text{mod}(Q) = \lambda(\Gamma_a)$. By the Riemann mapping theorem (and the theory of elliptic functions), every quadrilateral is isomorphic to a rectangular model of height one, $Q_a = [0, a] \times [0, 1]$, uniquely determined by the condition $\text{mod}(Q_a) = \text{mod}(Q) = a$. Thus $\text{mod}(Q)$ is a *complete invariant* of Q ; the moduli space of quadrilaterals is isomorphic to \mathbb{R}_+ .

Flipped quadrilaterals. Let Q' denote the quadrilateral Q with its a -sides replaced by its b -sides. Since it holds in the rectangular case, we have:

Corollary 4.3 *Any quadrilateral Q satisfies $\text{mod}(Q) \text{mod}(Q') = 1$.*

In view of this remarkable result, any given conformal metric ρ on Q provides upper and lower bounds for its modulus: we have:

$$\lambda(\Gamma_a, \rho) \leq \text{mod}(Q) \leq 1/\lambda(\Gamma_b, \rho).$$

Using this idea with the Euclidean metric, we obtain:

Theorem 4.4 (Besikovitch) *Let $Q \subset \mathbb{C}$ be a quadrilateral, and let A and B be the minimum Euclidean distances between its a -sides and its b -sides respectively. Then $\text{area}(Q) \geq AB$.*

Distortion of extremal length.

Stretch theorem for a quadrilateral.

5 Dynamics of elements of the mapping–class group

Let us say $f \in \text{Mod}_g$ is:

1. Of *finite order* if $f^n = \text{id}$ for some n ;
2. *Reducible* if f has infinite order, and there exists a nontrivial system of disjoint simple closed curves $S \subset \Sigma_g$ such that $f(S) = S$ up to isotopy; and
3. *Pseudo–Anosov* if there exists a $\lambda > 1$ such that for any pair of simple closed curves α and β , we have

$$i(\alpha, f^n(\beta)) \sim C\lambda^n$$

as $n \rightarrow \infty$, where $C > 1$ only depends on α and β .

In all three cases there are good geometric representatives of f , to be elaborated below.

Theorem 5.1 (Thurston) *Every $f \in \text{Mod}_g$ is either finite order, reducible or pseudo–Anosov.*

The proof we will give, following Bers, uses Teichmüller theory (both for the proof and to define what a pseudo–Anosov mapping is). Thurston’s original proof is based on the action of f on a natural compactification of Teichmüller space,

$$\overline{\mathcal{T}}_g = \mathcal{T}_g \cup \mathbb{P}\mathcal{ML}_g \cong B^{6g-6},$$

on which the mapping–class group acts. The geometry of f is revealed by studying its fixed points in this compactification.

From a modern perspective, these two approaches are linked by the natural isomorphism

$$Q\mathcal{T}_g \cong \mathcal{ML}_g \times \mathcal{ML}_g - D.$$

Here (λ_1, λ_2) lie in D if there exists a measured lamination μ that has zero intersection number with both.

Fibered 3–manifolds. The *mapping torus* of $f \in \text{Mod}_g$ is the 3–manifold defined by

$$T_f = [0, 1] \times \Sigma_g / (1, x) \sim (0, f(x)).$$

There is a fibration $T_f \rightarrow S^1$ with fiber Σ_g and monodromy f .

The following much deeper result is a complement to the classification:

Theorem 5.2 (Thurston) *The mapping–torus T_f is a hyperbolic manifold if and only if f is pseudo–Anosov.*

Theorem 5.3 *Let q be the quadratic differential associated to a pseudo–Anosov map f . Then $\mathcal{F}(q)$ is uniquely ergodic.*

Proof. Let $f : X \rightarrow X$ be the geometric realization of the given mapping–class, as a real–linear map for (X, q) with Df diagonal. Passing to a branched cover, we can assume that $q = \omega^2$ for some holomorphic 1–form ω . Writing

$$\omega = \alpha + i\beta,$$

we find, remarkably, that the harmonic forms α and β are eigenforms for the action of f ; that is,

$$f^*(\alpha) = \pm\lambda^{-1}\alpha \quad \text{and} \quad f^*(\beta) = \pm\lambda\beta,$$

where $\lambda > 1$ is the stretch factor of f . Moreover, $\lambda = \|f\|$ for the Hodge norm on $H^1(X, \mathbb{R})$. It is geometrically clear that for any harmonic form

$\gamma \neq 0$ orthogonal to these two, $f^*(\gamma)$ is *not* harmonic, and thus the Hodge norm of its class satisfies

$$\|f^*(\gamma)\| < \lambda \cdot \|\gamma\|.$$

Thus the iterates of any homology class converge, projectively, to $[\beta]$. It follows that there is only one transverse invariant measure, up to scale, supported on $\mathcal{F}(\omega)$. ■

What is a pseudo-Anosov mapping?

Let $f \in \text{Mod}_g$ be a mapping-class on Σ_g . The following are equivalent.

1. There exists a pair of transverse measured foliations \mathcal{F}_\pm on Σ_g preserved by f , except that their transverse measures are scaled by $\lambda^{\pm 1}$ for some $\lambda > 1$.
2. There exists a Riemann surface $X \in \mathcal{T}_g$ such that the Teichmüller mapping $F : X \rightarrow X$ in the isotopy class $[f]$ has the same initial and terminal quadratic differential.
3. There exists a Teichmüller geodesic $\gamma \subset \mathcal{T}_g$ invariant under f , with $f|_\gamma$ acting by translation by $L = \log \lambda > 0$.
4. There exists a quadratic differential $(X, q) \in Q\mathcal{T}_g$ and a diagonal matrix $a \in \text{SL}_2(\mathbb{R})$ such that $a \cdot (X, q) = f \cdot (X, q)$, and $L = d(X, f \cdot X) > 0$.
5. There exists a quadratic differential $(X, q) \in \mathcal{T}_g$ and a real-affine map $F : (X, q) \rightarrow (X, q)$ such that (i) F is in the isotopy class of f and (ii) the spectral radius of its derivative satisfies $\rho(DF) = \lambda > 1$.
6. Same as above, but with DF a diagonal matrix.

6 Problems

Problems marked with (*) are especially challenging.

1. (i) Describe a pattern for constructing a closed hyperbolic surface X of genus 2 out of right-angled pentagons. Is more than one pattern possible?

- (ii) Determine $G = \text{Isom}(X)$ for your chosen pattern. Include orientation-reversing automorphisms.
 - (iii) Describe X/G as an orbifold.
 - (iv) Show explicitly that X can be presented as a degree two holomorphic covering space of the Riemann sphere, branched over six points.
 - (v) *Identify these six points. (For example, find a polynomial of degree 5 or 6 such that X is isomorphic to the surface $y^2 = p(x)$).
2. Let $\overline{\Delta}(p, q, r) \subset \text{Isom } \mathbb{H}$ denote the group generated by reflections in the sides of a triangle with internal angles π/p , π/q and π/r .
Find (p, q, r) such that $\overline{\Delta}(p, q, r)$ contains the group Γ generated by reflections in the sides of a right-angled regular pentagon.
 3. Let Γ be the group generated by reflections in the sides of a right-angled regular pentagon P in \mathbb{H} .
 - (i) Show there is a group $\Gamma' \subset \text{Isom } \mathbb{H}$ such that \mathbb{H}/Γ' is a compact surface of genus 2 and $\Gamma' \cap \Gamma$ has finite index in both groups.
 - (ii) Show that Γ does not *contain* a subgroup Γ' such that \mathbb{H}/Γ' has genus two.
 4. Let $G = \langle A, B \rangle$ be the free group on two generators, identified with the fundamental group of a punctured torus $X = (\mathbb{R}^2/\mathbb{Z}^2) - \{0\}$.
 - (i) Find an element of $\text{Aut}(G)$ that represents a right Dehn twist on A .
 - (ii) Find an element of $\text{Aut}(G)$ that acts by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on $H_1(X, \mathbb{Z})$.
 5. Can a Möbius band be embedded in \mathbb{R}^3 in such a way that its boundary is a round circle?
 6. Let $S = \{z \in \mathbb{H} : \text{Re}(z) \in [0, 1]\}$, and let $X = S/g$, where $g(z) = 2z + 1$ glues one edge of S to the other.
Prove that X , with the induced hyperbolic metric, is not complete (as a metric space). What is its metric completion?
 7. Show that the edges of an ideal triangle in \mathbb{H} have natural midpoints (even though they are infinitely long).

8. Let A, B be two isometric, Euclidean equilateral triangles. Describe, up to homeomorphism, all the 2-manifolds that can be obtained by gluing the edges of A and B together in pairs.
9. Let $Q_t \subset \mathbb{H}$ be the ideal quadrilateral with (distinct) vertices $0, 1, \infty$ and t .
- (i) Show that opposite edges of Q_t can be identified so that $X = Q_t / \sim$ is a finite-volume hyperbolic surface of genus one with one cusp.
- (ii) Show that, for $t = -1$, this identification can be chosen such that X is conformally isomorphic to the square torus, $\mathbb{C}/\mathbb{Z}[i]$, with a point removed.
- (iii) By varying t , show that every rectangular torus, $X_y = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}iy - \{0\}$, can be given a complete hyperbolic structure of finite volume.
10. Let S be the (non-orientable) connect sum of a torus and real projective space; it has Euler characteristic -1 . Construct a hyperbolic metric on S (for example, by building S using hyperbolic polygons, or by constructing a discrete group such that $S \cong \mathbb{H}/\Gamma$).
11. Let $K \subset S^3$ be the figure eight-knot.
- (i) Find a presentation for the fundamental group G_1 of a punctured-torus bundle over S^1 with monodromy $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ as a semidirect product.
- (ii) Find a presentation for $G_2 = \pi_1(S^3 - K)$ using Wirtinger's method. (One answer:
- $$G_2 = \langle a, b : ab\bar{a}ba = ba\bar{a}b \rangle.$$
- (iii) Prove that $G_1 \cong G_2$.
- (iv) Find an explicit faithful representation of either group in to $\mathrm{PSL}_2(\mathbb{Z}[\omega])$, where $\omega = \exp(2\pi i/3)$. (Just sketch how you would prove the representation is faithful.)
- Hint: the generators of the Wirtinger presentation must be parabolics.
12. (i) Let $T_A \rightarrow S^1$ be a 3-manifold that fibers over the circle, with fiber a torus, and monodromy given by $A \in \mathrm{SL}_2(\mathbb{Z})$. Show that

$$H_1(T_A, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}^2 / (A - I)\mathbb{Z}^2.$$

- (ii) Suppose $H_1(T_A, \mathbb{Z}) \cong \mathbb{Z}$. What are the possibilities for the trace of A ?
 - (iii) Suppose we remove from T_A the zero section. Show that this does not change $H_1(T_A, \mathbb{Z})$.
 - (iv) Give a matrix A corresponding to the trefoil knot complement.
13. Let M^3 be the twisted version of the figure eight knot complement (with monodromy $-\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$). Compute $H_1(M^3, \mathbb{Z})$ and that conclude this manifold is not a knot complement.
14. Let $K \subset S^3$ be the figure-eight knot, and let L be the ‘minimally twisted 5-chain’. (Each component of L is an unknot, and they are connected like a necklace.)
Show that there degree 5 covering map $f : S^3 - L \rightarrow S^3 - K$.
15. Let $A(z) = 1/z$, $B(z) = z + i$ and $C(z) = z + 1$.
- (i) Show that A , B and C can be represented by matrices in $G = \text{PSL}_2 \mathbb{Z}[i] \subset \text{PSL}_2 \mathbb{C}$.
 - (ii) Does $G = \langle A, B, C \rangle$?
 - (iii) What is a fundamental domain for the action of G on \mathbb{H}^3 ?
 - (iv) Describe explicitly the orbifold $\mathbb{H}^3 / \text{SL}_2(\mathbb{Z}[i])$.
16. What is the torus \mathbb{C}/Λ associated to the cusp of the figure eight knot complement $M^3 = S^3 - K$? In other words, if we write $M = \mathbb{H}/\Gamma$, where $\Gamma \subset \text{SL}_2 \mathbb{Z}[i]$, what is the stabilizer Λ of ∞ in Γ ?
17. Find the Wirtinger presentation for the Whitehead link. Simplify it so it just has two generators and one relation.
- (i) Prove the relation abelianizes to zero.
 - (ii) Find a pair of parabolic matrices in $\text{PSL}_2(\mathbb{Z}[i])$ that satisfy the required relation.
18. Let M^3 be a compact, orientable 3-manifold with boundary. Prove that $\chi(M) = (1/2)\chi(\partial M)$.
19. Suppose we glue together some 3D polyhedral to obtain a compact space X with one vertex p , and the link of p is a torus. What is $\chi(X)$?

20. Show, more generally, that if construct a complex K by gluing together 3D polyhedra in an orientable way, leaving no open faces, then $\chi(K) = \sum_V g_i$, where (v_i) is the set of vertices in K and g_i is genus of the link of v_i .

Conclude that K is a 3-manifold if and only if $\chi(K) = 0$.

21. Let $M^3 \rightarrow S^1$ be the 3-manifold that fibers over S^1 with fiber a torus and monodromy a Dehn twist, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(i) Prove that $\pi_1(M^3)$ is a nilpotent group, in fact it is a central extension of \mathbb{Z}^2 by \mathbb{Z} .

(ii) Prove that M^3 is homeomorphic to the quotient $H(\mathbb{R})/H(\mathbb{Z})$ of the Heisenberg group by its integral points, where

$$H(A) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in A \right\}.$$

22. What 3-manifold do you get from a regular dodecahedron if you glue opposite faces with a 180° twist?
23. Draw a picture of the 1-skeleton of the dodecahedron, and then (i) work out explicitly how the edges are identified to make the Seifert-Weber space, and (ii) show there is only one vertex in the quotient.
24. Let M^3 be the Seifert-Weber manifold. Calculate $H_1(M^3, \mathbb{Z})$.
25. Show that the fundamental group of the Seifert-Weber manifold is commensurable to the reflection group for a tetrahedron with the Coxeter diagram:

$$O \equiv O - O \equiv O.$$

26. Let $M^3 = T_1\Sigma_2$ be the unit tangent bundle of a surface of genus two.
- (i) Find a presentation for $\pi_1(M^3)$.
- (ii) Compute $H_1(M^3, \mathbb{Z})$.

- (iii) Let $Z \subset \widetilde{\mathrm{SL}_2(\mathbb{R})}$ denote the center of the universal cover of $\mathrm{SL}_2(\mathbb{R})$. Using a hyperbolic structure on Σ_2 , we can write $M^3 = \widetilde{\mathrm{SL}_2(\mathbb{R})}/\Gamma$. What is the index $[Z : Z \cap \Gamma]$?
27. Let $d > 1$ be a square-free integer and let \mathcal{O} be the ring of integers in the field $K = \mathbb{Q}(\sqrt{-d})$ of \mathbb{C} . Show that the number of cusps of $\mathbb{H}/\mathrm{SL}_2(\mathcal{O})$ is equal to the class number $h(d)$ of K .
28. Let L_n denote the unlink with n components: this means L_n consists of n unknotted loops in n disjoint balls in S^3 . Let $M^3 = S^3 - L_n$.
- (i) Show that M^3 admits a complete hyperbolic metric.
- (ii) Show that M^3 has no complete hyperbolic metric of finite volume.
29. Let $M^3 \subset S^3$ be the complement of the (p, q) torus knot, where $\gcd(p, q) = 1$ and $\min(p, q) \geq 2$.
- (i) Show that $\pi_1(M^3)$ is a central extension of $(\mathbb{Z}/p) * (\mathbb{Z}/q)$.
- (ii) Show that M^3 admits no hyperbolic structure.
- (iii) What kind of geometry is appropriate for M^3 ?
30. Show that the unit tangent bundle to the orbifold $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ is homeomorphic to the complement of the trefoil knot in S^3 .
31. Give an explicit example of a quasimetric homeomorphism $f : S^1 \rightarrow S^1$ such that $f'(x)$ is 0 or ∞ at every point where the derivative exists.
32. Let A denote the Banach space of L^1 quadratic differentials on the Riemann sphere, locally of the form $q = q(z) dz^2$ where $q(z)$ is in $L^1(\mathbb{C})$.
- (i) Prove that $\|q\|_1 = \int_{\widehat{\mathbb{C}}} |q|$ is invariant under the action of $\mathrm{SL}_2(\mathbb{C})$ on A .
- (ii) Prove that the dual A^* of A can be naturally identified with the space of L^∞ Beltrami differentials, $\mu = \mu(z) d\bar{z}/dz$.
- (iii) Prove that in the norm topology, the natural action of $\mathrm{SL}_2(\mathbb{C})$ on A is continuous.
- (iv) Prove that $\mathrm{SL}_2(\mathbb{C})$ acts continuously on A^* in the weak* topology, but not in the norm topology.

33. Prove that the Beltrami differential $\mu = d\bar{z}/dz$ on $\widehat{\mathbb{C}}$ has a compact orbit under the action of $\mathrm{SL}_2(\mathbb{C})$, using the weak* topology as in the preceding exercise.

What is the formula for a typical element in its orbit?

34. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be the homeomorphism given in polar coordinates by $f(r, \theta) = (r^\alpha, \theta)$.
- (i) Assuming $0 < \alpha < 1$, find the Beltrami coefficient $\mu = f_{\bar{z}}/f_z$.
 - (ii) Show that dilatation satisfies $K(f) = 1/\alpha$. (Here $K = (1+k)/(1-k)$, where $k = \|\mu\|_\infty$.)
 - (iii) Show that for $\alpha \geq 1$, we have $K(f) = \alpha$.
 - (iv) Using $\alpha > 1$, given an example of a quasiconformal map $f : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$K'(f) = \sup_{z \in \mathbb{C}, r > 0} \frac{\sup_{|z-w|=r} |f(z) - f(w)|}{\inf_{|z-w|=r} |f(z) - f(w)|} > K(f).$$

(v) Describe how $f(z)$ behaves in the cylindrical metric $|dz|/|z|$ on \mathbb{C}^* . Does something like (v) occur in this metric?

35. Prove that the area of a sphere of radius $r < \pi$ in S^n is the same as the area of a sphere of radius $\sin(r)$ in \mathbb{R}^n .
36. *Boundary of the free group.* Let $G = \langle a, b \rangle$ be the free group on two generators.
- (i) Let Γ be the Cayley graph of G with these generators. Show that G is an infinite tree with degree 4 at every vertex.
 - (ii) Make Γ into a metric space by declaring every edge to have length one. Given $g \in G$ and $e = \mathrm{id}$, describe $d(g, e)$ in terms of group theory.
 - (iii) How many points are there at distance n from e in G ?
37. (Continuation). Rescale the metric on Γ so that an edge at distance n from e (in the original metric) now has length $1/n^2$ (in the new metric). Let $\bar{\Gamma}$ be the metric completion of Γ in this new metric, and let

$$\partial G = \bar{\Gamma} - \Gamma.$$

- (i) Show that ∂G is a Cantor set, and that the points of ∂G can be identified with infinite reduced words in the generators a, b and their inverses.
 - (ii) The group G acts on itself by right and left multiplication. Show that both of these actions extend continuously to homeomorphisms of $\bar{\Gamma}$.
 - (iii) Show that one of these actions is trivial on $\partial\Gamma$.
38. Show that the Hausdorff dimension of ∂K is infinite.
39. (Continuation). (i) Show that every automorphism of G extends continuously to ∂G .
- (ii) Consider the unique automorphism α of G sending (a, b) to (a, ab) . Find all of its fixed points in ∂G .
 - (iii) Describe the dynamics of α on ∂G . That is, describe the long-term behavior of $\alpha^n(x)$ for every point $x \in \partial G$.
40. (i) Prove that the hyperbolic plane \mathbb{H}^2 has the thin triangles property: there exists a constant $r > 0$ such that, for any triangle T with sides A, B and C , T is contained in the r -neighborhood of $A \cup pB$.
- (ii) What is the best possible value of r when T is an ideal triangle?
 - (iii) Prove that this value works for any triangle.
41. Let \mathfrak{H}_g denote the Siegel upper-halfplane, consisting of $g \times g$ symmetric matrices $Z = X + iY$ such that Y is positive definite. The natural metric on \mathfrak{H}_g is given by

$$ds^2 = \operatorname{tr}(Y^{-1} dZ Y^{-1} d\bar{Z}).$$

Let $S \subset \mathfrak{H}_2$ be the subspace consisting of matrices of the form $Z = \begin{pmatrix} \tau & x \\ x & -\bar{\tau} \end{pmatrix}$, where $\operatorname{Im}(\tau) > 0$. Show that S is isometric to \mathbb{H}^3 (up to a constant factor).

42. (i) Define the geodesic curvature of a path in a Riemannian manifold in terms of its deviation from a tangent geodesic.
- (ii) Show that the geodesic curvature of a circle of radius s in hyperbolic space is given by $1/\tanh(s)$.
 - (iii) Conclude that a horosphere has geodesic curvature 1.

43. Show that in the Poincaré ball model for \mathbb{H}^n , the hyperbolic metric is comparable to $|dx|/d(x, \partial B)$. Here $|dx|$ is the Euclidean metric.
44. Find a good approximation to the hyperbolic metric in the Klein ball model B for \mathbb{H}^n . (Working near a boundary point p , the tangent space can be decomposed into a subspace parallel to $T_p(\partial B)$ and an orthogonal line. Express the approximation as a multiple of the Euclidean metric on each of these subspaces.)
45. Let $G = \mathbb{Z} * \mathbb{Z}$ be the free group on two generators.
- (i) Show there is a hyperbolic surface X of finite volume with $\pi(X) \cong G$.
 - (ii) Show that G , with its usual word metric, is not quasi-isometric to \mathbb{H}^2 .
 - (iii) Can you reconcile (i) and (ii)?
46. Let $\gamma : \mathbb{R} \rightarrow \mathbb{H}^n$ be a unit speed path with geodesic curvature at every point less than a constant $k < 1$. Prove that γ is a quasi-geodesic.
47. Let $\gamma : \mathbb{R} \rightarrow \mathbb{H}^n$ be a unit speed piecewise geodesic path, consisting of geodesic segments of length $\leq L$, with consecutive segments making an angle of at least $\alpha > 0$ at their common endpoints.
- (i) Show that for fixed α , if L is sufficiently large then γ is a quasi-geodesic.
 - (ii) Give a more precise relationship between L and α that guarantees γ is a quasigeodesic.
48. Recall that a homeomorphism $f : \mathbb{R} \rightarrow \mathbb{R}$ is K -quasi-symmetric if for any two adjacent intervals of the same length, $I, J \subset \mathbb{R}$, we have $|f(I)| \leq K|f(J)|$.
- (i) Prove that a quasi-symmetric map is Hölder continuous.
 - (ii) Give an example of a Hölder continuous map that is not quasi-symmetric.
49. Let $f : [0, 1] \rightarrow [0, 1]$ be the usual Cantor step function (defined by $f(0.x_1x_2x_3\dots) = 0.y_1y_2y_3\dots$, where x_i are digits in base 2, y_i are digits in base 3, and $y_i = 2x_i$.)

- (i) Prove that $f'(x) = 0$ almost everywhere.
- (ii) Let $g(x) = f(x) + x$. Prove that $g : [0, 1] \rightarrow [0, 2]$ is a homeomorphism.
- (iii) Is $g(x)$ quasi-symmetric?
50. Let $f : S^1 \rightarrow S^1$ be a C^∞ expanding covering map, meaning $\inf_x |f'(x)| > 1$. Prove that f is ergodic: that is, for any measurable set $A \subset S^1$ satisfying $f^{-1}(A) = A$, either A or $S^1 - A$ has measure zero.
- Hint: suppose $m(A) > 0$; choose an interval $I \subset S^1$ where $m(A \cap I)/m(I) > 1 - \epsilon$, and then iterate f to conclude that A has density close to 1 on the whole circle. To carry out this argument, one needs to control how the nonlinear mapping f distorts the density of A .
51. Let $G = \text{Isom}(\mathbb{R}^n)$. Show there exists a sequence of discrete, cyclic subgroups $H_n \subset G$ that converge geometrically to a subgroup $L \cong \mathbb{Z}^2$ generated by a pair of translations.
- (Hint: take the generator h_n of H_n to be a screw motion stabilizing a geodesic parallel to the z -axis at distance $n/2\pi$ from the origin. One can arrange that $h_n(x)$ converges to $x + (1, 0, 0)$, while $h_n^n(x)$ converges to $x + (0, 1, 0)$.)
52. Let \mathcal{C} denote the closure of the set of discrete, infinite cyclic subgroups of $\text{SL}_2(\mathbb{R})$.
- (i) Show every group in \mathcal{C} is either discrete or connected.
- (ii) What connected subgroups lie in \mathcal{C} ? In particular, does $\text{SO}(2)$ lie in \mathcal{C} ?
53. Find the centers and radii (c_i, r_i) of $n \geq 5$ circles in \mathbb{C} that meet Δ in n arcs bounding a regular n -sided polygon with internal angles of 90° .
54. (Continuation.) Use `lim` to draw the corresponding tiling of Δ , for $n = 5, 6$ and 7 .
55. Calculate the length of the shortest geodesic on the triply-punctured sphere $X = \widehat{\mathbb{C}} - \{0, 1, \infty\}$. (Note that $X \cong \mathbb{H}/\Gamma(2)$.)

56. Given $g \in \text{Isom } \mathbb{H}^3$, let $T(g, r)$ be the set of points $x \in \mathbb{H}^3$ with $d(x, gx) < r$. (i) Show that (g, r) is a convex. (ii) Show that if g is hyperbolic, then $T(g, r)$ is a neighborhood of the unique geodesic γ that it stabilizes. (iii) Given an example where $T(g^{100}, r)$ is not contained in $T(g, r)$. (Hint: consider an isometry that acts on $\widehat{\mathbb{C}}$ by $g(z) = \exp(2\pi i/100 + \epsilon)z$.)
57. Let $g(z) = \lambda z$, $|\lambda| > 1$, be an isometry of \mathbb{H}^3 fixing 0 and ∞ . Let γ be the geodesic joining 0 to ∞ , let $C(\gamma, r)$ be the set of points at distance r from γ , and let $T(r) = C(\gamma, r)/\langle g \rangle$.
- (i) Show that the hyperbolic metric, restricted to $C(\gamma, r)$, is intrinsically flat.
- (ii) Compute a lattice $L(r) = \mathbb{Z} \oplus \tau(r)\mathbb{Z}$ such that $T(r)$ is isomorphic to $\mathbb{C}/L(r)$.
- (iii) Show that as $r \rightarrow \infty$, $T(r)$ converges to the torus $\mathbb{C}^*/\langle \lambda \rangle$.
- (iv) Does $\tau(r)$ move along a geodesic in \mathbb{H} ?
58. Find all the trivalent graphs of rank $g = 3$ and $g = 4$. (These describe pants decompositions in the corresponding genera).
59. For each genus g we have a directed graph Γ_g whose vertices correspond to systems S of disjoint simple closed curves on Σ_g , up to the action of Mod_g , and whose edges connect S to S' if S can be obtained from S' by erasing one curve.
- (i) Draw this graph for genus $g = 2$ and $g = 3$.
- (ii) Prove that Γ_g is connected for all g .
- (iii) Explain how (ii) shows that the moduli space \mathcal{M}_g has only one end.
60. Let X denote a finite volume hyperbolic surface of genus one, with exactly one cusp. (Topologically, X is a torus with a point removed.)
- (i) Show there exists a immersed smooth loop $\gamma \subset X$ that crosses itself exactly once, such that $\pi_1(\gamma)$ maps onto $\pi_1(X)$.
- (ii) Show there exists a *no* closed geodesic γ , crossing itself exactly once, such that $\pi_1(\gamma)$ maps onto $\pi_1(X)$.
- (iii) In case (ii), what is the index of $\pi_1(\gamma)$ in $\pi_1(X)$?

61. (*Volume of $\mathcal{M}_{1,1}$.*) (Cf.[Mir].) Let $F(x) = 1/(1 + e^x)$. McShane's identity states that for any $X \in \mathcal{M}_{1,1}$, we have

$$\sum F(L(\gamma)) = 1/2,$$

where the sum is over all (unoriented) simple closed geodesics $\gamma \subset X$. Use this identity to show that the Weil–Petersson volume of $\mathcal{M}_{1,1}$ is $\pi^2/6$. (This value is given by another method in [Pen]).

You may use the fact that $\int_0^\infty x/(1 + e^x) dx = \pi^2/12$.

(Hint: Use Fenchel–Nielsen coordinates to show that the volume of $\mathcal{M}_{1,1}$ is equal to $\int_U F(\ell) d\ell \wedge d\tau$, where U is the region in \mathbb{R}_+^2 where $0 < \tau < \ell$.)

62. (Quantitative version of Mahler's compactness theorem.) Give an explicit constant $C(n, r)$ such that, for every lattice $L \subset \mathbb{R}^n$ with $\text{vol}(\mathbb{R}^n/L) = 1$ and

$$r = \inf\{|x| : x \in L, x \neq 0\},$$

has a basis (v_1, \dots, v_n) with $|v_i| \leq C(n, r)$.

63. Let $f : X \rightarrow Y$ be an orientation–preserving homeomorphism between compact Riemann surfaces. It is sometimes said:

The map f is a Teichmüller mapping if, away from finitely many points in X , we can choose local analytic coordinates on domain and range such that f is an affine stretch; that is, $f(x + iy) = x/A + iAy$ for some $A > 1$.

In this case the dilatation $K(f) = A^2$.

(i) Show that, provided $K(f) > 1$, a map f as above locally determines a holomorphic quadratic differential q on X , such that $\mu(f)$ is proportional to $\bar{q}/|q|$.

(ii) Show that the statement above is false. For a counterexample, let $X = \mathbb{C}^*/2\mathbb{Z}$ be a Riemann surface of genus 1, choose $A > 1$, and let $f : X \rightarrow X$ be the map that has the form of an affine stretch, as above, when lifted to \mathbb{C}^* .

64. Let E_y denote the rectangular torus $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}iy)$, $y > 0$, and let E_y^* be the hyperbolic Riemann surface obtained by removing $z = 0$ from E_y .

Show that as $y \rightarrow \infty$, the length $L(E_y^*)$ of the shortest closed geodesic is asymptotic to $\ell = 1/y$.

65. (Continuation.) (i) Explain why one expects the corresponding twist parameter on $\mathbb{H} \cong \mathcal{T}_{1,1}$ to be asymptotic to $\tau = x/y$.
- (ii) Let $\rho = \rho(z) dz^2$ be the unique metric on \mathbb{H} whose area form is given by $d\tau \wedge dl$. What is the corresponding metric σ on the punctured disk $\Delta^* \cong \mathbb{H}/\mathbb{Z}$ in the coordinate $q = \exp(2\pi iz)$?
- (iii) Show that (Δ^*, σ) is incomplete. (The metric σ near $q = 0$ is an approximate model for $\mathcal{M}_{1,1}$ in the Weil–Petersson metric.)
66. An inclusion $A \subset B$ between annuli is *essential* if $\pi_1(A)$ maps isomorphically to $\pi_1(B)$. Prove that for such an inclusion, $\text{mod}(A) \leq \text{mod}(B)$, and equality holds only if $A = B$.
67. (i) Prove that if $A \subset \mathbb{C}$ is an annulus, and $\text{mod}(A)$ is sufficiently large, then A contains a round circle separating its boundary components.
- (ii) Give a specific constant $m > 0$ such that there exists an annulus A with $\text{mod}(A) > m$ that does not contain such a circle.
68. Let $q = z^2 dz^2$ on \mathbb{C} . Give an explicit example of a quasiconformal map $f(z)$, defined near $z = 0$, that is an affine stretch in the flat coordinates locally determined by q . Prove that $f(z)$ is not differentiable at $z = 0$. (Hint: the fact that $q(z) = (z dz)^2$ makes the solution easier.)
69. Let $A \subset \mathbb{C}$ be an annulus, and let $u : A \rightarrow \mathbb{R}$ be the unique harmonic function equal to 0 on the inner boundary and 1 on the outer boundary of A . Prove that $\int_A |\nabla u|^2 = 1/\text{mod}(A)$.
70. Let $A \subset \mathbb{C}$ be the annulus obtained from a square with sides of length 3 by removing the central unit subsquare.
- (i) Compute upper and lower bounds for $\text{mod}(A)$, using the Euclidean metric.
- (ii) Compute an approximation to $\text{mod}(A)$ numerically, by solving the Dirichlet problem as above.
71. Let $\Delta \subset \mathbb{C}$ denote the unit disk, and let A_n denote the annulus $\Delta - [-1/n, 1/n]$. Show that $\text{mod}(A_n) \sim C \cdot \log n$ as $n \rightarrow \infty$, and compute

the constant C . (Hint: this is easy if you replace $[-1/n, 1/n]$ by the closed disk of radius $1/n$.)

72. Prove that for $g \geq 2$, the mapping-class group Mod_g does not embed as a subgroup of $\text{Isom } \mathbb{H}^n$ (for any n). In particular, M_g is not homeomorphic to a hyperbolic orbifold when $g \geq 2$ (although it is when $g = 1$).
73. (i) Prove that $G = KAK$, where $G = \text{SL}_2(\mathbb{R})$, $K = \text{SO}(2)$ and A is the subgroup of diagonal matrices of determinant one.

(ii) Find the polar decomposition of $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; that is, write $B = UAV$ with $U, V \in \text{SO}(2)$ and A diagonal.

74. Let (X, q) be a nonzero quadratic differential in \mathcal{QM}_g , $g \geq 2$, and let $L \subset X$ be a leaf of the foliation $\mathcal{F}(q)$, disjoint from the zeros $Z(q)$. Prove that either L is a closed leaf (isomorphic to S^1), or \bar{L} is a subsurface of X bounded by leaves of $\mathcal{F}(q)$. In particular, $\mathcal{F}(q)$ has no leaf with endpoints in $Z(q)$, then every leaf of $\mathcal{F}(q)$ is dense.

75. Let $H^\infty(\Delta)$ denote the space of all bounded harmonic functions on the unit disk, equipped with the sup-norm. Show that any $u \in H^\infty(\Delta)$ has boundary values

$$U(z) = \lim_{r \rightarrow 1^-} u(rz)$$

that are well-defined for almost every $z \in S^1$. Show that $u \mapsto U$ gives an isomorphism:

$$H^\infty(\Delta) \cong L^\infty(S^1).$$

In particular, u is uniquely determined by its boundary values (via the Poisson kernel).

76. Let $A^\infty(\Delta)$ denote the space of all bounded *analytic* functions on the unit disk. Since analytic functions are harmonic, every $f \in A^\infty(\Delta)$ has boundary values $F(z)$ in $L^\infty(S^1)$.

Prove that $F(z)$ cannot vanish on a set of positive measure in S^1 . (Hint: We may assume $f(0) \neq 0$. Use the fact that $v(z) = \log |f(z)|$ is harmonic away from its poles to relate $v(0)$ to its average over S^1 .)

The latter would be $-\infty$ if $F(z)$ were to vanish on a set of positive measure.)

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