Barycentric subdivision, martingales and hyperbolic geometry

June 23, 2011

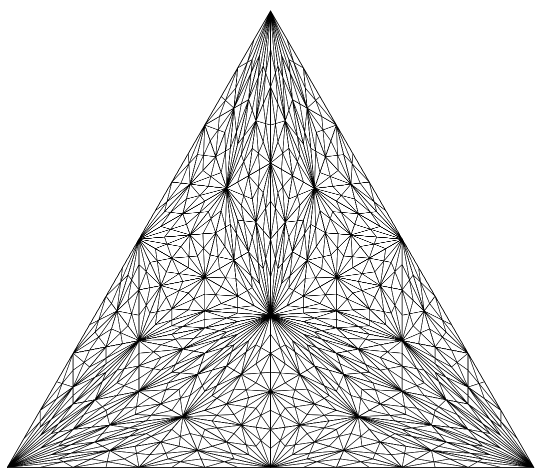

Figure 1. The 4th generation of barycentric subdivision.

1 Introduction

> "Finally, each BCS stage also makes the simplices not only smaller but "skinnier", i.e. it tends to increase their aspect ratio (the ratio between the longest and shortest edge)."
> —Wikipedia, *Barycentric subdivision*, 2008.

A better definition of the aspect ratio of a triangle T is $\alpha(T) = \text{area}(T)/L(T)^2$, where $L(T)$ is the length of the long side. It is easy and common for $\alpha(T)$

to go to zero even though the ratio between the longest and shortest edge does not.

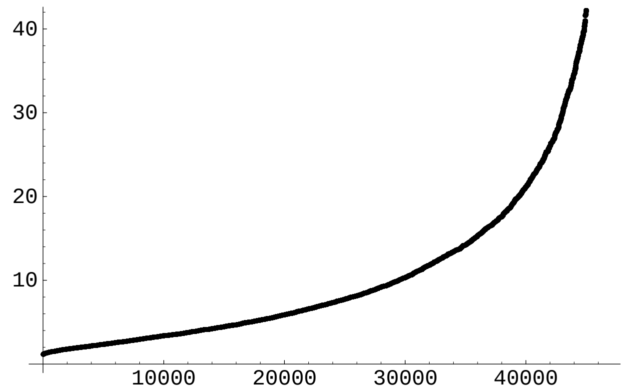

Figure 2. Sorted list of the aspect ratios of the 46,656 triangles obtained at the 6th generation of subdivision for an equilateral triangle.

Theorem 1.1 *Iterated barycentric subdivision results in almost all triangles being long and thin.*

(However, the ratio between the longest and shortest edge oscillates and obeys a definite distribution.) (What we prove is actually the stronger 'pointwise' statement: $\alpha(T_n) \to 0$ almost surely for iterated random subdivision of a given triangle.)

Moduli of marked triangles. A marked triangle is a triangle with a chosen ordering of its vertices, up to similarity. The space of marked triangles is the upper halfplane, with z corresponding to the ordered vertices $(0, 1, z)$. (Note that the triangle defined by $\overline{z}$ is marked isometric to the one defined by z.) The group S_3 acts naturally, changing the marking, by reflections through the circles $|z| = 1$ and $|z - 1| = 1$, and through the line $\operatorname{Re} z = 1/2$. (These three circles pass through the sixth root of unity).

The common fixed point of S_3 is the equilateral triangle, $z = (1 + \sqrt{-3})/2$. The three axes of the reflections in S_3 correspond to the isoceles triangles. The aspect ratio is $\alpha(T(z)) = \min_{S_3} 2 \operatorname{Im}(gz)$, which is a proper function on $\mathbb{H}$; the locus $\alpha(z) \geq A$ is the intersection of 3 horoballs resting on $\{0, 1, \infty\}$.

Action of barycentric subdivision. Now one of the triangles obtained by subdividing (a, b, c) has vertices $(a, (a + b)/2, (a + b + c)/3)$. This means

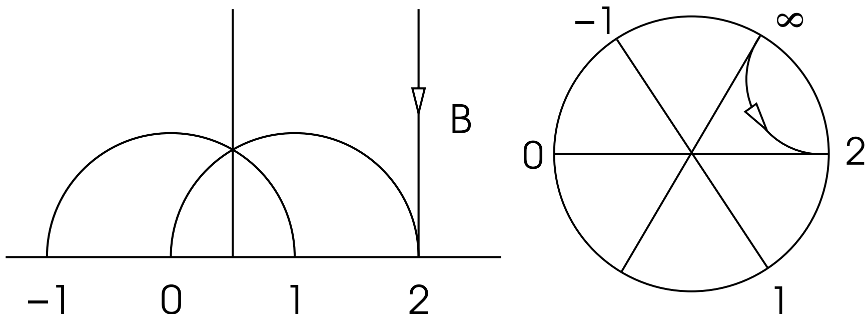

Figure 3. Generators of the barycentric subdivision group acting on $\mathbb{H}$: 3 reflections and $B(z) = 2(z+1)/3$.. The triple point corresponds to the equilateral triangle. At right: the same configuration in the disk model.

$B(z) = 2(z+1)/3$ gives the new triangle up to similarity. To obtain the iterated barycentric subtriangles, one alternates random elements of S_3 with B.

Let a 'random generator' mean the composition $g(z) = B(A(z))$ where A is chosen at random from S_3. The distribution of a random generator is the measure μ on G with an equal mass at each of the points $BA : A \in S_3$). (Note that BS_3 generates the same group as $\langle B, S_3\rangle$, since it contains $BI = B$.)

If $t_0 \in \mathbb{H}$ represents the initial triangle T_0, and $T_0 \subset T_1 \subset T_2 \subset \cdots$ is a random chain of subtriangles obtained by barycentric subdivision, then

$$t_n = g_n \cdots g_1(t_0)$$

for a random sequence of elements $g_i \in BS_3$ (chosen independently with respect to the measure μ).

Proposition 1.2 *The group* $\Gamma \subset \mathrm{Isom}(\mathbb{H}) = \mathrm{PGL}_2(\mathbb{R})$ *generated by* S_3 *and* B *has no invariant finite measure on* S^1.

Proof. The only invariant measures for $\langle B\rangle$ are supported on its fixed points, 2 and ∞. But this set is not invariant under S_3. ∎

Random walks Let (g_i) be a sequence of group elements chosen independently in G with distribution μ. The associated random walk based at z_0 is given by

$$z_n = g_1 g_2 \cdots g_n(z_0).$$

Note that

$$d(z_n, z_{n-1}) = d(z_0, g_n(z_0)) = O(1).$$

Note also that the 'walk' is actually taking place on the frame bundle of $\mathbb{H}$, not on $\mathbb{H}$ itself; we need direction information to take the next step.

A theorem of Furstenberg [Fur, Thm. 1.13] guarantees that a random orbit goes off to infinity so long as the support of μ is not too small.

Theorem 1.3 *Let μ wbe a measure on $G = \mathrm{PGL}_2(\mathbb{R})$ whose support generates a group Γ with no finite invariant measure on S^1. Then the random walk based on μ converges to S^1 with probability one.*

Note that this random walk is the reverse of the process on triangles; however if we show $d(z_0, z_n) \to \infty$ then we also have $d(t_0, t_n) \to \infty$. So the theorem on aspect ratios follows from the result above.

Proof.

1. There exists a stationary measure ν on S^1 for the action of random generators. I.e. $E(g_* \nu) = \nu$. This is immediate by iterated convolution and weak limits.

2. The measures $\mu_n = (g_1 \cdots g_n)_* \nu$ converge almost surely in the weak topology on the space of measure on S^1. This is by Doob's theorem — the measures μ_n form a martingale, since ν is stationary. That is,

   $$E(\nu_n | g_1, \ldots, g_k) = \mu_k.$$

3. There is almost surely a *subsequence* such that $w_n = g_1 \cdots g_n \to \infty$ in G. Indeed, since Γ has no invariant measure, its closure is noncompact; thus there exists finite words in the generating set which are as large as we want; and such words occur infinitely often in any random word (with probability one), so the product is unbounded, and thus a subsequence diverges.

   Now along a further subsequence, $w_n / \|w_n\|$ converges to a rank one matrix; geometrically, this means it pushes all of $S^1 - p_1$ towards a second point p_2.

   Thus $\mu = \lim \mu_n$ has support consisting of at most the two points $\{p_1, p_2\}$. (Indeed, μ is supported at a single point unless v has an atom at p_1.)

4. Now Γ is contained in the stabilizer of the support S of ν. Since Γ has no invariant measures on S^1, S has infinitely many points (otherwise uniform measure on S would be Γ-invariant). But the only way μ can have smaller support than ν is that $g_1 \cdots g_n \to \infty$ in G. (Otherwise, a subsequence of $g_1 \cdots g_n$ would accumulate on some $h \in G$, and then we would have $\mu = h_*(\nu)$.

5. It now follows directly z_n can accumulate only on p_1 or p_2. Since z_n is taking bounded steps, only one is possible (otherwise z_n would accumulate on a whole arc from p_1 to p_2 in S^1.

■

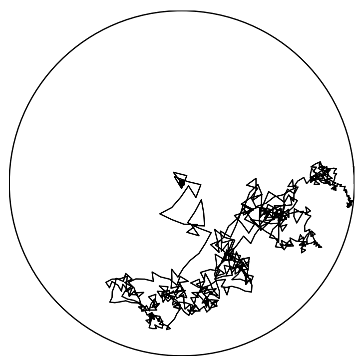

Figure 4. A random walk in the hyperbolic metric.

Remarks.

1. All triangles at level n have the same area. Thus we can also conclude: a point chosen at random is very likely to lie in a needle-like triangle.

2. The points $z_n \in \mathbb{H}$ have the property that the hyperbolic distance satisfies $\liminf d(p, z_n)/n > 0$ for any $p \in \mathbb{H}$. This follows from [Fur, Thm. 8.6]. Put differently, we have $\limsup(1/n)\log\alpha(T(z_n)) < 0$. To see this, check that $\log\alpha(g(z)) \asymp \log\|g(z)\|$ for $g \in \mathrm{SL}_2(\mathbb{R})$. (For example we have $\mathrm{Im}(gi) = 1/(c^2+d^2)$ if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.)

3. In fact, the group $\Gamma \subset \mathrm{Isom}(\mathbb{H}) = \mathrm{PGL}_2(\mathbb{R})$ generated by S_3 and B is dense.

   Proof: Γ contains reflections through vertical lines that get arbitrarily close together, e.g. the images of $\mathrm{Re}\, z = 1/2$ under B. Since the product of two such reflections is a small translation, $\overline{\Gamma}$ contains the group N of all translations $z \mapsto z+t$, $t \in \mathbb{R}$. It also contains the group $N^t = RNR$ where R is reflection through $|z| = 1$. In terms of matrices, $N = \{\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)\} \ : \ t \in \mathbb{R}\}$ and N^t is its transpose. It is elementary to show these groups generate $\mathrm{SL}_2(\mathbb{R})$; in fact, $NN^tN = \mathrm{SL}_2(\mathbb{R})$

4. The measure ν is just the hitting measure for the random walk, which (in the way we have formulated it) is independent of the basepoint z_0.

5. The measure ν has no atoms. Indeed, suppose $\nu(p) = m > 0$ is an atom of maximal mass. Then the preimages of p under elements g in the support of μ must also be atoms of mass m, since $\nu(p) = (\mu * \nu)(p)$ is a convex combination of their measures. It follows that the atoms of maximal mass are invariant under Γ, which is impossible (since Γ has no finite orbits).

6. Q. Does the largest angle go to π almost surely?

   If $z_n \to \mathbb{R} \cup \{\infty\}$ but the largest angle of $T(z_n)$ does not go to π, then a subsequence of z_n must converge radially to 0, 1, or ∞. Since ν has no atoms, it is cleary that the probability that z_n is close to $0, 1$ or ∞ is small for large n.

   So we *can* assert that after n subidivisions, most of the triangles have largest angle close to π.

   Still, the stronger *pointwise* assertion, that the largest angle of $T(z_n)$ tends to π, is also very likely true. This says z_n spends only a finite amount of time in any cone with vertex at $z = 0$.

7. Q. Does the ratio of longest to shortest side tend to infinity?

   A. No. The distribution of this random variable, call it $\beta(z_n)$, converges to the distribution of z with respect to ν restricted to $[1, \infty)$. Knowing this distribution is the same as knowing ν.

8. Furstenberg's theorem just requires that Γ has no invariant measure on S^1. For this it is sufficient that Γ has no abelian subgroup of finite index. (The case of $\mathrm{SL}_2(\mathbb{R})$ is special here, since its maximal compact is abelian).

9. Note: the hypothesis of [Fur, Thm. 1.13] is slightly too restrictive to literally include our case, although the proof given in Furstenberg works, as seen above. This lacuna is rectified in Thm. 8.6 in which G is defined as the *closure* of the group generated by the support of

μ (see p.423). Note that μ is *not* required to be absolutely continuous in [Fur, §8.3])

Tetrahedra, etc. Exactly the same argument can be used to prove:

Theorem 1.4 *Iterated barycentric subdivision in $\mathbb{R}^n$ results in almost all simplices having small aspect ratios.*

To set this up, the space of marked simplices in $\mathbb{R}^n$ is $O_n(\mathbb{R})\backslash \mathrm{PGL}_n(\mathbb{R})$. There is a natural action of S_{n+1}, and barycentric subdivision is given by the right action of

$$B = \begin{pmatrix} 1/2 & 0 & 0 & \ldots & 0 \\ 1/3 & 1/3 & 0 & \ldots & 0 \\ & \ldots & \ldots & & 0 \\ 1/(n+1) & 1/(n+1) & \ldots & 1/(n+1) & \end{pmatrix}.$$

Since B has distinct eigenvalues, the only invariant measures for B acting on $\mathbb{RP}^{n-1}$ are atomic measures concentrated on its n fixed points. But this set is not invariant under S_{n+1}.

Theorem 1.5 *In $\mathbb{R}^n$ the aspect ratio also goes to zero.*

Experimentally, the ratio of the longest to shortest edge does *not* tend to zero in $\mathbb{R}^n$; however the angles between edges all tend to 0 or π. That is:

After many steps, the tetrahedron has become a needle.

Martingale convergence. The martingale theorem we have used above is also not hard to prove (cf. [Doob]). Suppose $X(i)$ is a martingale with values in $[-1,1]$ and $E(X(0)) = 0$. Fix any interval $[a,b]$, and let K be the number of times that $X(i)$ crosses $[a,b]$ from above to below. We will show that

$$E(K) = O(1/(b-a)).$$

This immediately implies $X(i)$ converges almost surely.

Proof. Fix a large index N. Define optional times $(B_1 \leq A_1 \leq B_2 \leq A_2 \ldots)$ so that $X(B_i) > b$, $X(A_i) < a$ and these indices are otherwise as small as possible. (If there is no such index, set it equal to N.)

Since $X(i)$ is a martingale, we have $E(X(A_i)) = E(X(B_i)) = 0$. On the other hand, we have:

$$\sum X(B_i) - X(A_i) \geq (b-a)\min(K,N) + O(1).$$

Indeed, each term with $1 \leq i \leq K$ contributes at least $(b-a)$ to the sum; the terms with $i = K+1$ contribute $O(1)$; and the remaining terms have $A_i = B_i = N$, so they contribute 0.

This shows $E(\min(K,N)) = O(1/(b-a))$ with a constant independent of N, and thus $E(K) = O(1/(b-a))$. ∎

Note that this argument gives an alternate proof that $f'(x)$ exists a.e for any monotone function $f : \mathbb{R} \to \mathbb{R}$.

This martingale result applies just as well to functions whose values are probability measures on a compact space X (such that the circle). This is by duality: using the fact that $M(X) = C(X)^*$, we obtain from a martingale $\mu_n \in M(X)$ a traditional martingale $\int f \, d\mu_n$ for every $f \in C(X)$, and convergence of these traditional martingales for all f implies (weak) convergence of the μ_n.

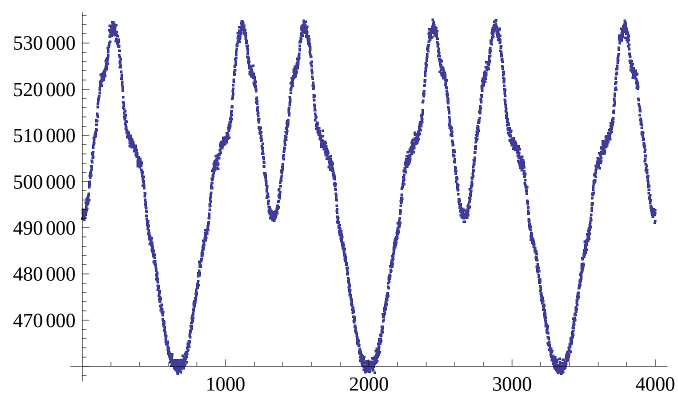

Figure 5. An approximation to the stationary density.

Experiments. Figure 5 shows an experimental approximation to the stationary (or harmonic) measure ν on S^1. Remarkably, the measure appears to be absolutely continuous. If true, it would provide a counterexample to the conjecture stated in the introduction to [Kai]. The absolutely continuity seems related to that of Bernoulli convolutions, see e.g. [PS]:

Theorem 1.6 *The random sums* $\sum \pm\lambda^{-n}$ *have an absolutely continuous density for almost all* $\lambda \in (1,2]$. *(In fact the density function is in* L^2*.)*

However Erdø"s showed the distribution is singular when λ is a Pisot number (and there are infinitely many Pisot numbers < 2).

Conjecture 1.7 *Let z_n be a random walk on $\mathbb{H}$ coming from equal weights on k independent motions. Suppose that $d(z_0, z_n) \sim nR$ and $k^n \gg \mathrm{vol}(B(z_0, nR))$ as $n \to \infty$. Then the hitting measure on S^1 is absolutely continuous.*

Markov partition. There is a good Markov partition for the boundary dynamics. Namely, we have $B(z) = 2(z+1)/3$ fixing $z = 2$, and also a reflection $R(z) = 1/2 - z$. So given $z < 1/2$, we can apply B until we get $z > 1/2$; then apply R to get it $< 1/2$, and continue. This is like the usual continued fraction algorithm, with the parabolic $T(z) = z + 1$ replaced by the hyperbolic $B(z)$.

This partition may be useful to prove that the stationary measure on S^1 is absolutely continuous.

Remarks. We have learned from a paper of Hough (2009) that our use of Furstenberg's theorem was also noticed earlier in [BBC].

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