Math 219 HW 7 Solutions

Problem. (74) (i) A simple path in G, starting at x and of length n, is determined by its vertices $x, v_1, \ldots, v_n$. Since v_1 is a neighbor of x, there are d choices for v_1. For $i \geq 2$, since v_i is a neighbor of v_{i-1}, and $v_i \neq v_{i-2}$ and $v_i \neq x$, there are at most $d-1$ choices for v_i. Thus, there are at most $d(d-1)^{n-1}$ such paths (and a single path when $n = 0$).

(ii) Write $E(|S_x|) = \sum_{y \in V(G)} P(y \in S_x)$, and note that $y \in S_x$ if and only if there is a simple path from x to y contained in S_x. Any simple path starting at x of length n contains $n+1$ vertices, so is contained in S_x with probability p^{n+1}. Letting $SP(x, y, n)$ be the set of simple paths from x to y of length n, we have

$$P(y \in S_x) \leq \sum_{n=0}^{\infty} |SP(x, y, n)| \cdot p^{n+1}$$

and $\mu_n = \sum_{y \in V(G)} |SP(x, y, n)|$, therefore

$$E(|S_x|) \leq \sum_{y \in V(G)} \sum_{n=0}^{\infty} |SP(x, y, n)| \cdot p^{n+1} = \sum_{n=0}^{\infty} \mu_n p^{n+1}$$

(iii) By (i)-(ii), $E(|S_x|) \leq p + \sum_{n=1}^{\infty} d(d-1)^{n-1} p^{n+1}$. If $p < \frac{1}{d-1}$, this sum is finite, so the probability that $|S_x| = \infty$ is 0. Taking a countable union over $V(G)$, the probability that S contains an infinite cluster is 0, so $p_c(G) \geq p$. Since $0 \leq p < \frac{1}{d-1}$ is arbitrary, $p_c(G) \geq \frac{1}{d-1}$.

(iv) The arguments in (i)-(iii) apply when the vertices in G have degree at most d. In this case, $d = 3$ and $p_c(G) \geq \frac{1}{2}$. Fix $p > \frac{1}{2}$, and for $n \geq 0$ let q_n be the probability that S_x contains a path from x to a vertex at distance n from x. Clearly, $q_0 = p$, the probability that $x \in S_x$. Now, $G \setminus x$ consists of two copies G_1, G_2 of G. Letting $x_1 \in G_1$ and $x_2 \in G_2$ be the 2 neighbors of x, any path from x to a distance $n+1$ vertex in G consists of an edge incident to x, followed by a path from one of the x_i to a distance n vertex in G_i. Thus,

$$q_{n+1} = p(1 - (1 - q_n)^2) = p q_n (2 - q_n)$$

The events defining q_n are nested, so $q_n \geq q_{n+1}$ for $n \geq 0$. Since $p > \frac{1}{2}$, we have $q_0 = p > 2 - \frac{1}{p} > 0$. If $q_n > 2 - \frac{1}{p}$, then

$$q_{n+1} = p q_n (2 - q_n) > p \left(2 - \frac{1}{p}\right) \frac{1}{p} = 2 - \frac{1}{p}$$

as well. Thus, $\lim_{n \to \infty} q_n \geq 2 - \frac{1}{p} > 0$. This means S_x is infinite with positive probability. Then S containing an infinite cluster is a tail event with positive probability, so by Kolmogorov's zero-one law, S contains an infinite cluster with probability 1. Then $p_c(G) \leq p$, and since $\frac{1}{2} < p \leq 1$ was arbitrary and $p_c(G) \geq \frac{1}{2}$, it must be that $p_c(G) = \frac{1}{2}$.