Math 213b HW9 Solutions

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Throughout, X denotes a compact Riemann surface.

Problem 63

Let

$$\alpha = z\left(y^2 - x^2\right) dxdy - x^2 y dxdz - xy^2 dydz$$

be a 2-form on $\mathbb{R}^3$. Check that $d\alpha = 0$, and then find a 1-form β such that $d\beta = \alpha$.

Solution

We compute

$$\begin{aligned} d\alpha &= \frac{\partial}{\partial z}\left(z\left(y^2 - x^2\right)\right) dzdxdy - \frac{\partial}{\partial y}\left(x^2 y\right) dydxdz - \frac{\partial}{\partial x}\left(xy^2\right) dxdydz \\ &= \left(\left(y^2 - x^2\right) + x^2 - y^2\right) dxdydz \\ &= 0 \end{aligned}$$

The 1-form

$$\beta = x^2 yz dx + xy^2 z dy$$

satisfies

$$\begin{aligned} d\beta &= \left(-\frac{\partial}{\partial y}\left(x^2 yz\right) + \frac{\partial}{\partial x}\left(xy^2 z\right)\right) dxdy - \frac{\partial}{\partial z}\left(x^2 yz\right) dxdz - \frac{\partial}{\partial z}\left(xy^2 z\right) dydz \\ &= \left(-x^2 z + y^2 z\right) dxdy - x^2 y dxdz - xy^2 dydz \\ &= \alpha \end{aligned}$$

Problem 64

Critique the following argument: since X is compact, it admits a finite covering by open sets $U_i \cong \Delta$; and since $H^p\left(\Delta, \mathcal{O}\right) = 0$ for all $p > 0$, by Leray's theorem $H^1\left(X, \mathcal{O}\right)$ is finite-dimensional.

Solution

We first note that by Theorem 6.5 in the course notes, since we are only dealing with the first cohomology group, the fact that $H^1\left(U_i, \mathcal{O}\right) = 0$ for every open set U_i in our cover is enough to use the open cover $\{U_i\}$ to compute $H^1\left(X, \mathcal{O}\right)$. However, the isomorphism

$$H^1\left(X, \mathcal{O}\right) \cong \check{H}^1\left(X, \{U_i\}, \mathcal{O}\right)$$

with $\{U_i\}$ a finite cover does not clearly imply that $H^1\left(X, \mathcal{O}\right)$ is finite-dimensional, since the spaces of Cech 1-cocycles and 1-coboundaries with respect to this cover are both infinite-dimensional.

Problem 65

Prove that $H^2(X,\mathcal{O})=0$.

Solution

We use the long exact sequence of cohomology groups associated to the short exact sequence

$$0\to\mathcal{O}\to\mathcal{E}^0\xrightarrow{\bar\partial}\mathcal{E}^{0,1}\to 0$$

Since $\mathcal{E}^0$ and $\mathcal{E}^{0,1}$ are sheaves of C^∞-modules, their higher sheaf cohomology groups vanish. Within the long exact sequence we have

$$H^1\left(X,\mathcal{E}^{0,1}\right)\to H^2(X,\mathcal{O})\to H^2\left(X,\mathcal{E}^0\right),$$

so $H^1\left(X,\mathcal{E}^{0,1}\right)=0$ and $H^2\left(X,\mathcal{E}^0\right)=0$ imply $H^2(X,\mathcal{O})=0$.

Problem 66

Let $\mathcal{Q}$ denote the sheaf of meromorphic 1-forms on X with zero residue at every pole. Show that

$$0\to\mathbb{C}\to\mathcal{M}\xrightarrow{d}\mathcal{Q}\to 0$$

is an exact sequence of sheaves. Deduce that the period map $\mathcal{Q}(X)\to H^1(X,\mathbb{C})$ is surjective. (Note that for $\omega\in\mathcal{Q}(X)$, $\int_{C_1}\omega=\int_{C_2}\omega$ for any pair of homologous curves avoiding the poles of ω.) (Here $\mathcal{M}$ is the sheaf of meromorphic functions, and you may use the fact that $H^1(X,\mathcal{M})=0$.)

Solution

Exactness is obvious except at $\mathcal{Q}$. A section on $\mathcal{Q}$ is given locally by a Laurent series $\sum_{n\geq -N}a_nz^ndz$ with $a_{-1}=0$, so a local antiderivative is given by $\sum_{n\geq -N}\frac{a_n}{n+1}z^{n+1}$. This shows $\mathcal{M}\xrightarrow{d}\mathcal{Q}$ is surjective on stalks. In the associated long exact sequence, we have

$$\mathcal{Q}(X)\xrightarrow{\delta}H^1(X,\mathbb{C})\xrightarrow{i}H^1(X,\mathcal{M})=0$$

so the connecting homomorphism $\mathcal{Q}(X)\xrightarrow{\delta}H^1(X,\mathbb{C})$ is surjective. We need to relate this map to the period map. We have a natural map

$$j:H^1(X,\mathbb{C})\to\operatorname{Hom}(H_1(X,\mathbb{Z}),\mathbb{C})\cong\operatorname{Hom}(\pi_1(X),\mathbb{C})$$

which is described as follows. Let (f_{ij}) be a 1-cocycle in $H^1(X,\mathcal{U};\mathbb{C})$ with respect to a Leray covering $\mathcal{U}$, and let C be a curve in X. Subdivide C into closed segments C_n, each of which is contained in a single open set $U_{i(n)}$, and let $j(f_{ij})(C)=\sum_n f_{i(n+1)i(n)}$, where the indices n are interpreted cyclically. We will show that the composition

$$\mathcal{Q}(X)\xrightarrow{\delta}H^1(X,\mathbb{C})\xrightarrow{j}\operatorname{Hom}(\pi_1(X),\mathbb{C})$$

is equal to the period map

$$\mathcal{Q}(X)\xrightarrow{\int_C\omega}\operatorname{Hom}(\pi_1(X),\mathbb{C})$$

Given $\omega\in\mathcal{Q}(X)$, $\delta\omega$ is defined by choosing an antiderivative f_i on each U_i and setting $(\delta\omega)_{ij}=f_i-f_j$. So if C is a curve in X and C_n a subdivision into segments compatible with the cover $\mathcal{U}$, and x_n is the starting point of C_n, then

$$j\circ\delta\omega(C)=\sum_n\delta\omega_{i(n+1)i(n)}=\sum_n\left(f_{i(n+1)}(x_{n+1})-f_{i(n)}(x_n)\right)=\sum_n\int_{C_n}\omega=\int_C\omega$$

Now, the equivalence of Cech and singular cohomology on manifolds and the fact that X is path-connected implies that j is an isomorphism, so since δ is a surjection, the period map $j\circ\delta$ is also a surjection.

Problem 67

Let $X = \mathbb{C}/\Lambda$ be a Riemann surface of genus one. Find an explicit pair of forms $\omega_1, \omega_2 \in \mathcal{Q}(X)$ whose periods form a basis for $H^1(X, \mathbb{C}) \cong \mathbb{C}^2$.

Solution

Consider the forms $dz, \wp(z)\,dz \in \mathcal{Q}(X)$ (we have $dz \in \mathcal{Q}(X)$ since dz is holomorphic, and $\wp(z)\,dz \in \mathcal{Q}(X)$ since $\wp(z)\,dz$ has a single pole of order 2). Suppose $adz + b\wp(z)\,dz$ is exact for some $a, b \in \mathbb{C}$. If $b \neq 0$, then this form has a single pole of order 2, so it must be of the form df for some meromorphic function $f : X \to \widehat{\mathbb{C}}$ with a single pole of order 1, which cannot exist since X has positive genus. So $b = 0$, and then adz has periods $a\Lambda$, which must all be zero if adz is exact, so $a = 0$. This shows that $dz, \wp(z)\,dz$ are linearly independent in $H^1(X, \mathbb{C}) \cong \mathbb{C}^2$, and thus a basis.

Problem 68

(i) Let u be a smooth function on $\mathbb{C}$ with compact support, and let $f = u * \log|z|$ (convolution). Prove that $\Delta f = Cu$ for some constant C, and determine the value of C.

(ii) Prove that if g is a smooth function on $\mathbb{C}$, $\Delta g = Cu$ and $g(z) \to 0$ as $z \to \infty$, then $g = f$.

(iii) Now suppose u is just a continuous function with compact support. Prove that $f \in C^1(\mathbb{C})$, i.e., f has a continous first derivative.

Solution

(i) Since u is smooth with compact support, we can calculate with distributions. If δ denotes evaluation at 0, then

$$\Delta \log|z| = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}\log|z| = 2\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}\log|z\overline{z}| = 2\frac{\partial}{\partial \overline{z}}\frac{1}{z} = 2\pi\delta$$

where the last equality comes from the proof of the Dolbeault lemma. Then

$$\Delta(u * \log|z|)(z) = \Delta\int_{\mathbb{C}} u(w)\log|z-w|\,|dw|^2 = \int_{\mathbb{C}} u(w)\,\Delta\log|z-w|\,|dw|^2 = 2\pi u(z)$$

(ii) $f-g$ is a harmonic function on $\mathbb{C}$. Since u is bounded (say by C) with compact support (say contained in $B_R(0)$), we have

$$\left|f - i\left(\int_{\mathbb{C}} u\right)\log|z|\right| = \left|\int_{\mathbb{C}} u(w)\left(\log|z-w| - \log|z|\right)|dw|^2\right| \leq C\int_{B_R(0)} \left|\log|z-w| - \log|z|\right|\,|dw|^2 = o(1)$$

as z goes to infinity. Therefore, $f(z) = \left(\int u\right)\log|z| + o(1)$, and by assumption $g = o(1)$. Since $f-g$ is harmonic with sublinear growth, the mean value property implies that it is constant. Therefore, $f = O(1)$, so $\int u = 0$ and in fact $f - g$ tends to 0 at infinity, hence $f - g = 0$.

(iii) We will show that f is differentiable and its derivative is

$$\frac{d}{dz}(u * \log|z|) = u * \frac{d}{dz}\log|z| = u * \frac{1}{2z}$$
$$\frac{d}{d\overline{z}}(u * \log|z|) = u * \frac{d}{d\overline{z}}\log|z| = u * \frac{1}{2\overline{z}}$$

It suffices to show that f is differentiable at $z = 0$. To do this we need to show

$$I(h) = \left|f(h) - f(0) - u * \frac{1}{2z}(0)\,h - u * \frac{1}{2\overline{z}}(0)\,\overline{h}\right| = o(|h|)$$

as $h \to 0$. Unpackaging the definition this is the same as showing

$$\lim_{h\to 0}\left|\frac{1}{h}\int_{\mathbb{C}} u(w)\left(\log\left|1 - \frac{h}{w}\right| + \operatorname{Re}\left(\frac{h}{w}\right)\right)|dw|^2\right| = 0$$

Fix $\varepsilon > 0$. Because $\log|1-t|$ is smooth in a neighorbhood of $t = 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that for $|t| < \delta$,

$$|\log|1-t| + \mathrm{Re}(t)| < \varepsilon |t|$$

We estimate the above integral by breaking it into two parts, where $\left|\frac{h}{w}\right| \leq \delta$ and $\left|\frac{h}{w}\right| \geq \delta$. For the first part,

$$\left|\frac{1}{h}\int_{\left|\frac{h}{w}\right|\leq\delta} u(w)\left(\log\left|1-\frac{h}{w}\right| + \mathrm{Re}\left(\frac{h}{w}\right)\right)|dw|^2\right| \leq \left|\frac{1}{h}\right|\int_{\left|\frac{h}{w}\right|\leq\delta}|u(w)|\,\varepsilon\left|\frac{h}{w}\right||dw|^2 \leq \varepsilon\int_{\mathbb{C}}\left|u(w)\frac{1}{w}\right||dw|^2 = C\varepsilon$$

for some constant C depending only on u. For the second part,

$$\begin{aligned}
&\left|\frac{1}{h}\int_{\left|\frac{h}{w}\right|\geq\delta} u(w)\left(\log\left|1-\frac{h}{w}\right| + \mathrm{Re}\left(\frac{h}{w}\right)\right)|dw|^2\right| \\
&= \left|\frac{1}{h}\right|\left|\int_{\left|\frac{1}{t}\right|\geq\delta} u(w)\left(\log\left|1-\frac{1}{t}\right| + \mathrm{Re}\left(\frac{1}{t}\right)\right)|h|^2|dt|^2\right| \\
&\leq |h|\int_{|t|\leq\frac{1}{\delta}}|u(th)|\left(|\log|t-1|| + |\log|t|| + \left|\frac{1}{t}\right||dt|^2\right) \\
&\leq C_\delta\|u\|_\infty|h|
\end{aligned}$$

where the constant C_δ depends only on δ, using the fact that the functions $\log|t|$, $\log|t-1|$, $\left|\frac{1}{t}\right|$ are locally integrable. Combining the two estimates, we see that for a fixed ε, we obtain

$$\left|\frac{I(h)}{h}\right| \leq C\varepsilon + O(|h|)$$

as $h \to 0$. Since ε can be chosen arbitrarily, this shows $I(h) = o(h)$ as $h \to 0$, which is what we want. Finally, the expression of $\frac{d}{dz}f$ as a convlution $u * \frac{1}{2z}$ shows that it is continuous, being the convolution of a compactly supported continuous function and a locally integrable function. The same argument for $\frac{d}{d\bar{z}}f$ shows that it is also continuous.

Problem 69

Prove that

$$0 \to \mathcal{H} \to \mathcal{E}^0 \to \mathcal{E}^2 \to 0$$

is an exact sequence of sheaves, where the middle maps are inclusion $\mathcal{H} \to \mathcal{E}^0$ and $\partial\overline{\partial} : \mathcal{E}^0 \to \mathcal{E}^2$.

Solution

Since harmonic functions are by definition annihilated by $\partial\overline{\partial}$, exactness is obvious except at $\mathcal{E}^2$. To check that $\partial\overline{\partial} : \mathcal{E}^0 \to \mathcal{E}^2$ is a surjective morphism, pick an open set U biholomorphic to the disc $\Delta \subset \mathbb{C}$ and a coordinate z on U, and let $f dz \wedge d\overline{z}$ be a 2-form on U. By the Dolbeault lemma, this 2-form is $\overline{\partial}$-exact, so $f dz \wedge d\overline{z} = \overline{\partial}(g dz)$ for some smooth function g on U. Taking complex conjugates, by the Dolbeault lemma again, the ∂-closed form $g dz$ is ∂-exact, so $f dz \wedge d\overline{z}$ is $\partial\overline{\partial}$-exact. This shows that every point admits a neighborhood on which every 2-form is $\partial\overline{\partial}$-exact, so $\partial\overline{\partial} : \mathcal{E}^0 \to \mathcal{E}^2$ is surjective on stalks and thus a surjective morphism of sheaves.