Advanced Complex Analysis
Homework 3
Due Tuesday, 18 February 2014

1. Show that for each $d \geq 1$, there exists a complex torus $X \cong \mathbb{C}/\Lambda$ and an analytic map $f : X \to X$ of degree d.

2. Show there is no proper holomorphic map $f : \Delta \to \mathbb{C}$.

3. Give an explicit example of a polynomial $p(x)$ of degree 6 such that the compact Riemann surface Y of genus 2 defined by the equation $y^2 = p(x)$ admits a nonconstant holomorphic map to a Riemann surface of genus one.

4. Let $f : X \to Y$ be a nonconstant map of degree 3 between compact Riemann surfaces. Show that if $\mathrm{Deck}(X/Y) \cong \mathbb{Z}/3$, then there is no $x \in X$ such that $\mathrm{mult}(f, x) = 2$. Does the converse hold?

5. Find a Belyi polynomial $p(z)$ of degree 5 such that $p^{-1}([0, 1])$ is homeomorphic to the letter Y, with the fork at $z = 0$.

   (The Belyi condition means $p(0) = 0$, $p(1) = 1$ and the critical values of p are contained in $\{0, 1\}$.)

6. Find a Belyi map $f : X \to \widehat{\mathbb{C}}$ where $X = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}i$ is the square torus. That is, find a holomorphic map from X to the Riemann sphere branched over just $\{0, 1, \infty\}$.

7. Show that for any finite group G, there exists a Galois covering map $f : X \to Y$ between compact Riemann surfaces such that $\mathrm{Deck}(X/Y) \cong G$.

   If $G = (\mathbb{Z}/2)^n$, what is the smallest possible genus for Y?

8. Let L be the union of two distinct lines through the origin in $\mathbb{C}^2$. What is $\pi_1(\mathbb{C}^2 - L)$? Give a suitable topological definition of the 'two-sheeted covering space Y of $\mathbb{C}^2$ branched over L.' Show that Y is not a manifold, and that Y is homeomorphic to the (singular quadric) surface in $\mathbb{C}^3$ given by $z^2 = xy$. Sketch the real points of this quadric.