

Advanced Complex Analysis Homework 11

Due Tuesday, 23 April 2013

Throughout X is a compact Riemann surface.

1. Let \mathcal{F} be a sheaf of locally free \mathcal{O} -modules of rank 1 on X . (This means $\mathcal{L}(U) \cong \mathcal{O}(U)$ on small enough open sets.) Show there is a line bundle $L \rightarrow X$ such that \mathcal{F} is isomorphic to the sheaf of sections of L .
2. Let $f : \mathbb{P}^n \rightarrow \tau^{-d}$ be a holomorphic section of a negative tensor power of the tautological bundle. Prove that f is given by a homogeneous polynomial of degree d on \mathbb{C}^{n+1} . (Hint: use Hartog's extension theorem.)

Then prove that any smooth complex hypersurface $H \subset \mathbb{P}^n$ representing the generator of $H^2(\mathbb{P}^n, \mathbb{Z})$ is a hyperplane.
3. Let $X \subset \mathbb{P}^3$ be a canonical curve of genus 4. Suppose 3 distinct points $P_1, P_2, P_3 \in X$ lie on a line L . Show that for any hyperplane H containing L , the other 3 points of $H \cap X$ also lie on a line.
4. Let $C \subset \mathbb{P}^n$ be a smooth, nondegenerate curve of degree d . Prove that the points of a generic hyperplane section $H \cap C$ are in general position: no 3 lie in a line, no 4 lie in a plane, ..., no n of them lie in a copy of \mathbb{P}^{n-2} .
5. Prove that any $2n + 1$ points E in general position in \mathbb{P}^n impose independent linear conditions on the space of quadrics. I.e. the space of quadrics passing through E has codimension $2n + 1$.
6. Suppose X is not hyperelliptic, and let $Q(X) \cong \mathbb{C}^{3g-3}$ denote the vector space of holomorphic quadratic differentials $q(z) dz^2$ on X . Prove that the natural map $\Omega(X) \otimes \Omega(X) \rightarrow Q(X)$ is surjective. (Hint: use the preceding results and the canonical embedding.)
7. What happens when X is hyperelliptic?