Advanced Complex Analysis: Homework 7

1. Let $\omega \in \Omega(X)$ be a holomorphic 1-form on a compact Riemann surface X, and let
$$\Lambda = \left\{ \int_\alpha \omega \ : \ \alpha \in \pi_1(X) \right\} \subset \mathbb{C}$$
be the group of periods of ω.

   (i) Show that Λ cannot be isomorphic to $\mathbb{Z}$.

   (ii) Show that if Λ is a lattice in $\mathbb{C}$, then there exists a nonconstant holomorphic map from X to a Riemann surface of genus one.

2. (i) Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves on a space X. Show that the sequence of groups
$$\mathcal{A}(X) \xrightarrow{\alpha} \mathcal{B}(X) \xrightarrow{\beta} \mathcal{C}(X)$$
is then also exact; that is, $\operatorname{Im}\alpha = \operatorname{Ker}\beta$.

   (ii) 'If $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is an exact sequence of sheaves, then $\mathcal{A}(X) \to \mathcal{B}(X) \to \mathcal{C}(X)$ is an exact sequence of groups.' Explain why this statment does not follow from (i). Show in fact that (ii) is false.

3. Let $X = \mathbb{R}/\mathbb{Z}$ and let $\mathcal{F}$ be the sheaf of linear functions $f : U \to \mathbb{R}$, meaning $f(x) = ax + b$ when lifted to $\mathbb{R}$. Let $D^2 : C^\infty \to C^\infty$ be the operator on the sheaf of smooth functions given by $D^2(f) = f''(x)$.

   (i) Show that $0 \to \mathcal{F} \to C^\infty \xrightarrow{D^2} C^\infty \to 0$ is an exact sequence of sheaves.

   (ii) Compute $\dim H^1(X, \mathcal{F})$.

   (iii) Using the covering $X = U_1 \cup U_2 = (0,1) \cup (-1/2, 1/2) \bmod \mathbb{Z}$, give an explicit representative for $\delta^*(f) \in H^1(X, \mathcal{F})$, where $f \in H^0(X, C^\infty)$ is the constant function $f(x) = 1$ and
$$\delta^* : H^0(X, C^\infty) \to H^1(X, \mathcal{F})$$
is the connecting homomorphism.

4. Let $\mathcal{Q}$ denote the sheaf of meromorphic 1-forms on a Riemann surface X with zero residue at every pole. Show that $0 \to \mathbb{C} \to \mathcal{M} \xrightarrow{d} \mathcal{Q} \to 0$ is an exact sequence of sheaves. Interpret the connecting homomorphism $\delta^* : \mathcal{Q}(X) \to H^1(X, \mathbb{C})$ in terms of periods.

5. Suppose $\Delta f = u$ where u is a smooth function on $\mathbb{C}$ with compact support, and $f(z) \to 0$ as $|z| \to \infty$.

   (i) Prove that f is proportional to $u * \log|z|$ (Here $f * g(z) = \int_{\mathbb{C}} f(w) g(z - w)\, |dw|^2$.)

   (ii) Now suppose u is just a continuous function with compact support, and $f = u * \log|z|$. Prove that $f \in C^1(\mathbb{C})$, i.e. f has a continuous first derivative.