Advanced Complex Analysis: Homework 5

1. Let $P = \{2, 3, 5, \ldots\}$ be the prime numbers, endowed with the cofinite topology (a set is closed iff it is finite or the whole space). Given an open set $U = P - \{p_1, \ldots, p_n\}$, let $\mathcal{F}(U)$ be the ring $\mathbb{Z}[1/p_1, \ldots, 1/p_n] \subset \mathbb{R}$, and let $\mathcal{F}(\emptyset) = (0)$.

   Show that with the natural restriction maps, $\mathcal{F}$ is a sheaf of rings over P. What is the stalk $\mathcal{F}_p$?

2. Let $p(x, y) = \sum_{0 \le i+j \le d} a_{ij} x^i y^j \in \mathbb{C}[x, y]$ be a polynomial defining a smooth Riemann surface $X^* \subset \mathbb{C}^2$ of degree d. Let $\pi : X^* \to \mathbb{C}$ be projection to the x-coordinate.

   (i) Show that for 'typical' p (i.e. for generic coefficients a_{ij}), the map π is proper of degree d.

   (ii) Determine the 'typical' number of critical points $|C(\pi)|$.

   (iii) Let $\overline{\pi} : X \to \widehat{\mathbb{C}}$ denote the compact branched covering obtained by completing $\pi : X^* \to \mathbb{C}$. Show that $\pi^{-1}(\infty)$ 'typically' consists of d points.

   (iv) Derive a formula for the genus of X in terms of d.

3. (i) Prove that for any sheaf of abelian groups $\mathcal{F}$ on a space X and any open set $U \subset X$, the group $\mathcal{F}(U)$ is naturally isomorphic to the group of continuous sections of the éspace étalé $|\mathcal{F}|$ over U.

   (ii) Give an example of a presheaf on a space X such that $\mathcal{F}(X) \cong \mathbb{Z}$ but $\mathcal{F}_x = (0)$ for every $x \in X$.

   (iii) Let $\mathcal{F}$ be the sheaf of locally constant functions on X with values in a fixed abelian group G. Give an explicit description of $|\mathcal{F}|$ (including its topology).

4. Let $U \subset \mathbb{C}$ be a connect open set containing $z = 0$. Show there is a power series $f(z) = \sum a_n z^n$, convergent near $z = 0$, which can be analytically continued to U but to no larger Riemann surface.

5. Given the first 3 nonzero terms in the solution to $P(T) = T^5 + zT + z = 0$ near $z = 0$ by a Puiseux series.

6. Let $U \subset \mathbb{C}$ be an open set. Prove that every complex-linear ring homomorphism $\chi : \mathcal{O}(U) \to \mathbb{C}$ is given by $\chi(f) = f(p)$ for some $p \in U$.