Advanced Complex Analysis: Homework 2

You may collaborate with others on this and future homework, but you should cite your collaborators as well as any references you use.

1. Let $A \subset X$ be a finite subset of a compact Riemann surface. Suppose $f : (X - A) \to Y$ is a 1-1 holomorphic map to another compact Riemann surface. Show that f extends to an isomorphism $F : X \to Y$.

2. Let Γ be a discrete subgroup of $\mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}_2(\mathbb{R})$ acting freely on $\mathbb{H}$. Show that the action of Γ is properly discontinuous.

3. Let $H = \{z \in \mathbb{C}^* : \mathrm{Im}(z) \geq 0\}$. Let X be the Riemann surface obtained from H by gluing each positive boundary point $x > 0$ to the point $-x^2 < 0$. Show that X can be realized explicitly as a domain $X \subset \mathbb{C}$. What is the image of ∂H in X?

4. Let P be a regular octagon in the plane, and let X be the closed surface of genus two obtained by gluing opposite sides of P together by isometries. Draw a (topological) picture of X showing the image of ∂P.

5. Correct problem 4.4(c) of Forster, Ch. I, then do problems 4.1 and 4.4.

6. Let $E \subset \mathbb{C}$ be a finite set of algebraic numbers. Show that there exists a polynomial $P : \mathbb{C} \to \mathbb{C}$, branched only over 0 and 1, such that $P(E) \subset \{0, 1\}$. (Hint: write P as a composition of many polynomials, first reducing to the case where $E \subset \mathbb{Q}$.)

7. Given $z \in S^1$ and $a \in \Delta$, show that the tangent to the hyperbolic geodesic $\overline{az}$ at a has slope
$$\alpha(z, a) = 2\arg(z - a) - \arg(z).$$
Letting $\arg z = \theta$ and $\dot{\alpha} = d\alpha/d\theta$, show that for $r > 0$, $\dot{\alpha}(z, r)$ agrees with the Poisson kernel $P_r(\theta)$. Using this calculation, describes the harmonic extension of $u|S^1$ to $a \in \Delta$ in terms of random hyperbolic geodesics from a to S^1.