Advanced Complex Analysis: Homework 10

Throughout X is a compact Riemann surface.

1. Critique the following argument: "We can cover X with a finite number of open sets $\mathfrak{U} = (U_i)$ with each $U_i \cong \Delta$. Since $H^1(\Delta, \mathcal{O}) = 0$, we have $H^1(X, \mathcal{O}) \cong H^1(\mathfrak{U}, \mathcal{O})$ by Leray's theorem, and thus $H^1(X, \mathcal{O})$ is finite-dimensional."

2. Prove that for any pair of effective divisors D, E on X, we have
$$h^0(D) + h^0(E) \le h^0(D + E) + 1. \tag{1}$$
(Hint: do not use Riemann-Roch.)

3. Let $D \ge 0$ be an effective divisor. We say D is *special* if there exists a holomorphic 1-form $\omega \neq 0$ with $(\omega) \ge D$; equivalently, if there exists a canonical divisor K with $D \le K$.

   Prove that any $D \ge 0$ satisfies $h^0(D) \ge 1 - g + \deg D$, and equality holds unless D is special.

4. Prove that any effective divisor with $\deg D < g$ is special, and any special divisor satisfies $\deg D \le 2g - 2$.

5. Prove that any special divisor $D \ge 0$ satisfies
$$\deg D - g < h^0(D) - 1 \le \frac{1}{2} \deg D.$$
(Hint: for the upper bound, use equation (1) above with with $E = K - D$, K is a canonical divisor.)

6. Let $\Lambda \subset \mathbb{C}$ be a lattice, let $p_1, \ldots, p_n \in X = \mathbb{C}/\Lambda$ be distinct points, and suppose $a_1 + \cdots + a_n = 0$. Construct explicitly a meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$, doubly-periodic with respect to Λ, with principal parts $a_i/(z - p)$ at the points $p = p_i \bmod \Lambda$, and with no other poles.

   Can this be done if $\sum a_i \neq 0$?