Math 212a: Advanced Real Analysis

Suggested solutions to Homework 8

Problem 49. Let $f(z)$ be a holomorphic function on the unit disk $\Delta \subset \mathbb{C}$. Show there exists a sequence of polynomials $p_n(z)$ such that $p_n|_\Delta \to f$ in $C(\Delta)$.

Solution. Consider the partial sums of the Taylor series expansion at 0,

$$p_n(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k.$$

Given a compact set K of Δ, choose a disk D_r containing K with $r < 1$, and choose $r < r' < 1$. By Cauchy integral formula, for any $z \in K \subset D_r$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial D_{r'}} \frac{f(w)}{w - z} dw,$$

and

$$p_n(z) = \frac{1}{2\pi i} \sum_{k=0}^{n} \int_{\partial D_{r'}} \frac{z^k f(w)}{w^{k+1}} dw = \frac{1}{2\pi i} \int_{\partial D_{r'}} \frac{f(w)(1 - (z/w)^{n+1})}{w - z} dw.$$

Hence

$$|f(z) - p_n(z)| \le \frac{1}{2\pi} \int_{\partial D_{r'}} \frac{|f(w)||z/w|^{n+1}}{|w| - |z|} |dw| \le \frac{r'}{r' - r} \left(\frac{r}{r'}\right)^{n+1} \sup_{|w|=r'} |f(w)|.$$

Taking $n \to \infty$, we conclude that $p_n \to f$ uniformly on compact set $K \subset \Delta$. □

Problem 50. (Continuation.) Show that for any $g \in \mathcal{D}'(\Delta)$, there exists an $f \in \mathcal{D}'(\Delta)$ such that $\overline{\partial} f = g$.

(Hint: write $g = \sum g_n$ as a sum of compactly supported distributions whose supports tend to $\partial\Delta$; solve $\overline{\partial} f_n = g_n$; and use the preceding problem to find polynomials such that $\sum (f_n - p_n)$ converges on Δ.)

Solution. For $n \ge 2$, let ϕ_n be a smooth function compactly supported in $D_{1-1/(n+1)}$ and identically 1 on $D_{1-1/n}$, and set ϕ_0 to be identically 0. Define $g_n = (\phi_{n+1} - \phi_n)g$. Then g_n is supported on $A_n = A_{1-1/n, 1-1/(n+1)} := \{z : 1 - 1/n \le |z| \le 1 - 1/(n+1)\}$, when $n \ge 2$, and on $A_1 = D_{1/2}$ when $n = 1$. Clearly $g = \sum g_n$.

Now since g_n is a compactly supported distribution, $g_n \in H^s(\mathbb{R}^2)$ for some $s \in \mathbb{R}$. By taking Fourier transform, we then find a tampered distribution $f_n \in H^{s+1}(\mathbb{R}^2)$ with $\overline{\partial} f_n = g_n$. Note that for $n \geq 3$, as g_n is supported on A_n, we have $g_n|_{D_{1-1/n}} = 0$. Thus $\overline{\partial} f_n|_{D_{1-1/n}} = 0$ and hence $f_n|_{D_{1-1/n}}$ is holomorphic. Choose a polynomial p_n so that the sup norm of $f_n - p_n$ over $D_{1-2/n}$ is less than $1/2^n$ (note that we shrink slightly the domain to apply the previous problem). We will also set $p_1 = p_2 = 0$.

Define $\Lambda_N := \sum_{n=1}^N (f_n - p_n)$. Given a compactly supported function ψ on Δ, choose n_0 so that $\operatorname{supp}(\psi) \subset D_{1-2/n_0}$. Then for $N \geq n_0$,

$$|\Lambda_{N+1}(\psi) - \Lambda_N(\psi)| = |(f_{N+1} - p_{N+1})(\psi)| \leq C 2^{-N-1} \|\psi\|_\infty$$

where C is a constant independent of N. Hence the sequence $\{\Lambda_N(\psi)\}$ forms a Cauchy sequence, and a unique limit exists. Thus $f = \lim_{N\to\infty} \Lambda_N$ is a well defined distribution on Δ. Since $\overline{\partial}\Lambda_N = \sum_{n=1}^N g_n$, we have $\overline{\partial} f = g$. □

Problem 54. (Harmonic polynomials.) Let P_d denote the space of real homogeneous polynomials p of degree d on $\mathbb{R}^n$, and let $H_d \subset P_d$ be the subspace of harmonic polynomials, where $\Delta p = 0$.

(a) Show that on $\mathbb{R}^2 \cong \mathbb{C}$, for $d > 0$ the space H_d is spanned by $\operatorname{Re} z^d$ and $\operatorname{Im} z^d$.

(b) Show that on $\mathbb{R}^2$, we have $P_d = H_d \oplus |x|^2 P_{d-2}$ for all d.

(c) Show the same is true on $\mathbb{R}^3$, and use it to compute $\dim H_d(\mathbb{R}^3)$.

(d) Let f be a tempered distribution on $\mathbb{R}^n$ such that $\Delta f = 0$. Show that $\widehat{f}$ is supported at the origin, and hence f is a harmonic polynomial.

Solution. (a, b) Given a real homogeneous polynomial $p(x,y)$ of degree d on $\mathbb{R}^2$, using $x = (z+\bar{z})/2$ and $y = (z-\bar{z})/(2i)$, we get a complex homogenous polynomial $q(z,w)$ such that $p(x,y) = q(z,\bar{z})$. Moreover, as p is real, if we write $q(z,w) = \sum_{k=0}^d a_k z^k w^{d-k}$, we must have $a_k = \overline{a}_{d-k}$. Conversely, any such polynomial gives a real homogeneous one. Thus the map $p \mapsto q$ is a linear isomorphism. Since on $\mathbb{R}^2$, the Laplace operator is, up to a constant, the operator $\partial\overline{\partial}$, we calculate

$$\partial\overline{\partial} q(z,\bar{z}) = \sum_{k=1}^{d-1} k(d-k) a_k z^{k-1} \bar{z}^{d-k-1}$$

This is a homogeneous polynomial of degree $d-2$ and is obviously one coming from a real homogenous polynomial of degree d. Hence if q is harmonic, then $a_k = 0$ for $k = 1, \ldots, d-1$. This gives (a). Note that for any $p(x,y) = q(z,\bar{z})$, $q(z,\bar{z}) = a_0\bar{z}^d + a_d z^d + |z|^2 \sum_{k=1}^{d-1} a_k z^{k-1}\bar{z}^{d-k-1}$. This gives the decomposition in (b), and noting that the dimensions match, this decomposition has to be a direct sum.

(c) A basis of P_d is given by monomials of the form $x^{d_1}y^{d_2}z^{d_3}$ with $d_1 + d_2 + d_3 = d$. Define an inner product $(\cdot,\cdot)_d$ on P_d by declaring this basis to be orthogonal and the norm of

$x^{d_1}y^{d_2}z^{d_3}$ to be $(d_1!d_2!d_3!)^{1/2}$. We have the Laplace operator $\Delta : P_d \to P_{d-2}$ and the operator $T : P_{d-2} \to P_d$ given by multiplying $x^2 + y^2 + z^2$. By checking on the basis set, we have $(Tp, q)_d = (p, \Delta q)_{d-2}$, i.e. $T = \Delta^*$. Then $P_d = \ker \Delta \oplus \operatorname{im}(T)$, as desired.

Since $\dim(P_d) = \binom{d+2}{2}$, we have $\dim(H_d) = \binom{d+2}{2} - \binom{d}{2} = 2d + 1$.

(d) Taking Fourier transform, we have $-|t|^2\widehat{f} = 0$. Thus $\widehat{f}$ is supported at the origin, and hence f is a polynomial (and harmonic by assumption). □

Problem 55. Show there exist compact supported smooth functions f_i on $\mathbb{R}^n$ such that $\|\Delta f_i\|_{L^2} = 1$ but $\|f_i\|_{L^2} \to \infty$. Why does this not contradict Theorem 7.10?

Solution. Let f be a (nontrivial) compactly supported smooth function on $\mathbb{R}^n$. Set $f_n(x) = n^{3/2}f(x/n)$, whose Laplacian cannot be trivial, so we may normalize f so that $\|\Delta f\|_2 = 1$. Then

$$\int f_n^2(x) = \int n^3 f^2(x/n)dx = n^4 \int f^2(y)dy \to \infty,$$

while $\Delta f_n(x) = n^{-2}n^{3/2}\Delta f(x/n) = n^{-1/2}\Delta f(x/n)$, and hence

$$\int |\Delta f_n(x)|^2 = n^{-1}\int |\Delta f(x/n)|^2 dx = \int |\Delta f(y)|^2 dy = 1.$$

In order to apply Theorem 7.10, we need to fix a compact set *a priori*. □

Problem 56. Let $P(x, D)$ be an elliptic differential operator of order N on $\mathbb{R}^n$. Suppose N is *odd*. (i) Prove that $n = 1$ or $n = 2$. (ii) Prove that if $n = 2$, then the symbol $P(x, t)$ cannot be a real–valued function.

Remark 0.1. We can prove that an odd degree polynomial in one variable with real coefficients has a root in $\mathbb{R}$, using either the intermediate value theorem, or the fundamental theorem of algebra (plus the action of complex conjugation). Here we similarly provide two different solutions to this problem, mirroring the two methods in one variable.

Solution. (Algebraic) The symbol $P_N(x, t)$ of $P(x, D)$ is a homogeneous polynomial of degree N in t. Write it as $P_1(x, t) + iP_2(x, t)$, where P_1, P_2 are real valued. First assume $n \geq 3$. Choose $x_0 \in \mathbb{R}^n$ so that at least one of $P_1(x_0, t)$ and $P_2(x_0, t)$ is nontrivial. The vanishing locus of P_1 is a subvariety of $\mathbb{P}^{n-1}$ of dimension $\geq n - 2$ and degree N. Note that the conjugation acts as an involution on this subvariety. So there exists an irreducible component of odd degree, fixed by conjugation. This irreducible component intersects the vanishing locus of P_2 again in a subvariety of odd degree and dimension $\geq n - 3$, and again we can choose an irreducible component of odd degree fixed by conjugation. Intersecting this component by a generic projective subspace of complementary dimension, we get a subvariety of dimension 0 of odd degree, fixed by conjugation; and there exists a projective subspace defined over $\mathbb{R}$ that is generic. Since the degree is odd, the conjugation action must have a fixed point. This gives a nontrivial real solution to $P_1(x_0, t) = P_2(x_0, t) = 0$.

Similarly in $n = 2$, if the symbol is real, then $P_N(x,t) = P_1(x,t)$. Following the argument above, we can then produce a nontrivial real solution to $P_1(x,t) = 0$ for some $x \in \mathbb{R}^2$. $\square$

Solution. (Analytic) As above, $P_N(x_0, t)$ is a homogeneous polynomial of degree N in t. We consider the restriction f of $P_N(x_0,t)$ to the unit sphere S^{n-1}. Then as P_N has odd degree, $f(-y) = -f(y)$ for every $y \in S^{n-1}$. Let γ be any smooth path from y to its antipode $-y$. Then $f(\gamma)$ gives a path in $\mathbb{C}$ from $\lambda := f(y)$ to $-\lambda$, avoiding zero by ellipticity (note that this is impossible in dimension $n = 2$ when $P_N(x,t)$ is real, proving (ii)). The concatenation of the path γ with the path $-\gamma$ gives a closed loop based at y, and its image is the concatenation of the path $f(\gamma)$ with $-f(\gamma)$, a closed loop γ_1 based at λ. When $n \geq 3$, S^{n-1} is simply connected, and hence γ_1 is homotopically trivial. But this is impossible: the line integral

$$\int_{f(\gamma)} \frac{1}{z} dz = \pi i + 2k\pi i \neq 0$$

for some integer k, and the integral over $-f(\gamma)$ is the same. Hence the integral of $1/z$ over the closed loop γ_1 is nonzero, and therefore γ_1 cannot be homotopically trivial. $\square$

Problem 57. Given a sequence of weights $w_n > 0$, let $\ell^2_w(\mathbb{Z})$ be the Banach space of sequences such that

$$\|a\|_w^2 = \sum_{\mathbb{Z}} w_n |a_n|^2 < \infty.$$

(i) Show that if $x_n/y_n \to \infty$ as $|n| \to \infty$, then the natural inclusion $T : \ell_x(\mathbb{Z}) \to \ell_y(\mathbb{Z})$ is a compact operator.

(ii) Make up a reasonable definition of Sobolev spaces $H^s(S^1)$ for functions on the unit circle in $\mathbb{C}$ (with Fourier series $f(z) = \sum a_n z^n$), and prove that the inclusion $H^r(S^1) \to H^s(S^1)$ is a compact operator whenever $r > s$.

Solution. (i) Let $a(i)$ be a sequence in $\ell_x(\mathbb{Z})$ with $\|a(i)\|_x \leq 1$. Define a_n and a subsequence $a(\sigma(i))$ inductively as follows. Since $x_1|a(i)_1|^2 \leq 1$, the sequence $\{a(i)_1\}_i$ is bounded. Up to passing to a subsequence, we may assume $a(i)_1 \to a_1$. Choose $\sigma(1)$ so that $y_1|a(\sigma(1))_1 - a_1|^2 < 1/2$. At the N-th step, we have a sequence $\{a(i)\}$ whose first $(N-1)$-entries are convergent. Using the same argument above, up to passing to a subsequence, we may assume further $a(i)_N \to a_N$. Choose $\sigma(N)$ so that $y_n|a(\sigma(N))_n - a_n|^2 < 1/2^{-N}$ for $n = 1, \dots N$. We claim that $a(\sigma(i)) \to a := (a_n)$ in $\ell_y(\mathbb{Z})$. Indeed, first note that by Fatou's lemma, $\|a\|_x \leq 1$. Given $\epsilon > 0$, choose N so that $y_n/x_n < \epsilon/4$ when $|n| \geq N$. Then

$$\|a(\sigma(i)) - a\|_y^2 \leq \sum_{|n|<N} y_n|a(\sigma(i))_n - a_n|^2 + \epsilon/2 \leq i/2^{-i} + \epsilon/2.$$

When i is large enough, this is bounded by ϵ. Hence $a(\sigma(i)) \to a$ as desired.

(ii) For each $s \in \mathbb{R}$, define a sequence of weights $w_n^{(s)} = (1+n^2)^s$. Given a distribution on S^1, we can calculate its Fourier series, as the functions $\exp(inx)$ are smooth functions on

S^1. Let $H^s(S^1)$ be the space of distributions u so that the coefficients of its Fourier series $\hat{u} = (a_n) \in \ell_{w^{(s)}}(\mathbb{Z})$, and set $\|u\|_{H^s} = \|\hat{u}\|_{w^{(s)}}$. We claim taking Fourier series gives an isometry between $H^s(S^1)$ and $\ell_{w^s}(\mathbb{Z})$. The only issue is surjectivity, but this is immediate from the fact that the coefficients (a_n) of Fourier series of a smooth function on S^1 satisfy $\sum n^{2k}|a_n|^2 < \infty$. The assertion in (ii) then follows from (i). □

Problem 58. (The mean value property.) Prove, using the following outline, that if f is harmonic on $\mathbb{R}^n$, then $f(x)$ is equal to the average of f over $B(x,r)$.

(Outline: let $g_r(x) = \chi_{B(0,r)}(x)/\operatorname{vol}(B(0,r))$, so $\int g_r = 1$. There is a compactly supported continuous function $h_{r,s}$ such that $\Delta h_{r,s} = g_r - g_s$. Then $(f * g_{r,s}(x)) = 0$. Letting $s \to 0$, we have $(f * g_s)(x) \to f(x)$ while $(f * g_r)(x)$ gives the average of f over $B(x,r)$.)

Solution. We follow the outline. Define $g_r(x) = \chi_{B(0,r)}(x)/\operatorname{vol}(B(0,r)) = C_n\chi_{[0,r)}(|x|)/r^n$, where C_n is a constant depending only on dimension. We first try to solve $\Delta h_r = g_r$, where $h_r(x) = f_r(|x|)$ is a radial function. We also require h_r to be C^1 with Lipschitz first partial derivatives. In this case, $f_r(y)$ is C^1, $f_r'(0) = 0$ and f_r' is Lipschitz. Then $\Delta h_r(x) = \frac{1}{y^{n-1}}\frac{d}{dy}(y^{n-1}f_r')(|x|) = C_n\chi_{[0,r)}(|x|)/r^n$. We have then

$$y^{n-1}f_r'(y) = \begin{cases} \frac{C_n y^n}{nr^n} & 0 \le y < r \\ \frac{C_n}{n} & y \ge r \end{cases},$$

and hence

$$f_r(y) = \begin{cases} C + C_{n,r}y^2 & 0 \le y < r \\ C + C'_{n,r} + C_n y^{2-n} & y \ge r \end{cases}$$

with y^{2-n} replaced by $\log(y)$ when $n-2$. Here C is an arbitrary constant, $C_{n,r}, C'_{n,r}$ depend on n, r, but C_n depends only on n. Hence by a suitable choice of C, we may assume $f_r(y) = C_n y^{2-n}$ when $y \ge r$. Now define $h_{r,s} = h_r - h_s$ for $r > s$. Then $\Delta h_{r,s} = g_r - g_s$ and $h_{r,s}$ is supported on $B(0,r)$.

Hence $f * (g_r - g_s) = f * \Delta h_{r,s} = \Delta f * h_{r,s} = 0$. For any x, $(f * g_s)(x) \to f(x)$ as $s \to 0$. On the other hand, $(f * g_r)(x)$ gives the average of f over $B(x,r)$. This implies the mean value theorem. □