Math 212a: Advanced Real Analysis

Suggested solutions to Homework 2

Problem 9. Let $X = l^\infty(\mathbb{N})$ be the Banach space of bounded real sequences $x = (x_0, x_1, \ldots)$ and let $T(x)_i = x_{i+1}$. Let X^* be the dual of X with the weak* topology. Define $\phi_n \in X^*$ by $\phi_n(x) = (1/n)\sum_0^{n-1} x_i$.
(a) Show that ϕ_n has an accumulation point in $\psi \in X^*$. (b) Show that $\psi(T(x)) = \psi(x)$, and $\psi(x) = \lim x_i$ if this limit exists. (c) Show there is no subsequence of ϕ_n that converges to ψ.

Solution. (a) Each ϕ_n has norm ≤ 1. As the unit ball in X^* is compact in the weak* topology by Alaoglu's theorem, the sequence contains an accumulation point.

(b) Fix $x \in X$. For any integer $k \ge 1$, consider the following weak open set of X^*:

$$O_{x,k} := \{\phi \in X^* : |\phi(x) - \psi(x)| < 1/k, |\phi(T(x)) - \psi(T(x))| < 1/k\}.$$

As ψ is an accumulation point of $\{\phi_n\}$, there exists infinitely many n such that $\phi_n \in O_{x,k}$. Choose n_k large enough so that $\phi_{n_k} \in O_{x,k}$ and $|\phi_{n_k}(x) - \phi_{n_k}(T(x))| = |x_0 - x_{n_k}|/n_k \le 2\|x\|_\infty/n_k < 1/k$. Then by definition of $O_{x,k}$, $\phi_{n_k}(T(x)) \to \psi(T(x))$ as $k \to \infty$. Moreover as

$$|\phi_{n_k}(T(x)) - \psi(x)| \le |\phi_{n_k}(T(x)) - \phi_{n_k}(x)| + |\phi_{n_k}(x) - \psi(x)| < 2/k$$

we also have $\phi_{n_k}(T(x)) \to \psi(x)$ as $k \to \infty$. This gives $\psi(x) = \psi(T(x))$ as desired.

Note that we also have $\phi_{n_k}(x) \to \psi(x)$ as $k \to \infty$ in the construction above, and we may always choose inductively that $\{n_k\}$ is increasing. When $\{x_i\}$ has a limit, $\phi_n(x)$ tends to this limit, and so does any subsequence. Hence $\psi(x) = \lim x_i$ should this limit exist.

(c) Suppose a subsequence $\phi_{n_k} \to \psi$ weakly. Consider the following sequence x of real numbers. The first n_1 entries are all 1. Choose k_1 such that $n_{k_1} > 10n_1$, and set the entries between n_1 and n_{k_1} all 0. Inductively choose k_j such that $n_{k_j} > 10n_{k_{j-1}}$ and assign 1 or 0 to entries between $n_{k_{j-1}}$ and n_{k_j} according to whether j is even or odd. Then the sequence $\phi_{n_k}(x)$ does not converge, a contradiction! □

Problem 10. Give an example of a sequence of continuous functions $f_n \in C[0,1]$ with $\|f_n\| = 1$ but $f_n \to 0$ weakly.

Solution. Let f be a nonnegative continuous function supported on $[0,1]$, with $\sup f = 1$. Define $f_n(x) = f(2^n x - 1)$ for $n \ge 1$. Note that f_n is supported on $[1/2^n, 1/2^{n-1}]$, and

$\|f_n\| = \sup_{[0,1]} |f_n| = 1$. Moreover $f_n \to 0$ pointwise. Let $\mu \in M[0,1] = (C[0,1])^*$ be a Borel measure on $[0,1]$, and write $\mu = \mu_+ - \mu_-$ into its positive and negative components. Since $|f_n| \leq 1$ and the constant function 1 is integrable with respect to μ_+, μ_-, by Dominated Convergence Theorem, $\mu_+(f_n) \to 0$ and $\mu_-(f_n) \to 0$. Thus $\mu(f_n) \to 0$. Therefore $f_n \to 0$ weakly, as desired. □

Problem 14. Let $K \subset C[0,1]$ be the set of all continuous functions satisfying $|f(x)| \leq 1$ and $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [0,1]$. (a) Prove that K is a compact, convex set. (b) What are its extreme points?

Solution. (a) Convexity follows from triangle inequality, and compactness follows from Arzela-Ascoli theorem.

(b) $f \in K$ is an extreme point if and only if $\|f\|_\infty = 1$ and $|f'(x)| = 1$ almost everywhere on $O_f := [0,1] \backslash \{x : |f(x)| = 1\}$. Note that $f \in K$ is absolutely continuous, hence its derivative exists a.e.

Assume f is an extreme point. Clearly $\|f(x)\|_\infty = 1$. By the Lipschitz condition, $|f'(x)| \leq 1$ whenever it exists. Assume it is not true that $|f'(x)| = 1$ almost everywhere on O_f. As O_f is open in $[0,1]$, we can write it as a countable union of open disjoint intervals (except that they may contain the end points 0 or 1). Then there exists a subset $A \subset O_f$ of positive measure such that $|f'(x)| \leq c_1 < 1$, and we may even assume $A \subset (a,b) \subset O_f$. Choose ϵ small enough so that $A \cap (a+\epsilon, b-\epsilon)$ is of positive measure, and suppose $\sup_{(a+\epsilon,b-\epsilon)} |f| \leq c_2 < 1$. Set $c = \max\{c_1, c_2\}$. Hence there exists an interval $I := (a+\epsilon, b-\epsilon)$ and $B := A \cap (a+\epsilon, b-\epsilon)$ of positive measure on which $|f'(x)| \leq c$ and $|f(x)| \leq c$ for all $x \in B$. Finally, choose $t \in I$ such that $B_1 := B \cap (a+\epsilon, t)$ and $B_2 := B \cap (t, b-\epsilon)$ has equal measure.

Set $u(x) = f'(x) + (1-c)\chi_{B_1} - (1-c)\chi_{B_2}$ and $v(x) = f'(x) - (1-c)\chi_{B_1} + (1-c)\chi_{B_2}$, and $g(x) = f(0) + \int_0^x u(y)dy$ and $h(x) = f(0) + \int_0^x v(y)dy$. Clearly $f = (g+h)/2$. Moreover, $|u(x)| \leq 1$ and $|v(x)| \leq 1$ almost everywhere, hence g, h are Lipschitz with constant 1. Finally, f, g, h agree outside I, and on I, $|g(x) - f(x)| = |h(x) - f(x)| \leq (1-c)\mu(B) \leq 1 - c$ and hence $|g(x)| \leq |f(x)| + 1 - c \leq 1$ over I and similarly for h. Thus $g, h \in K$, and f is not an extreme point, a contradiction!

Conversely, assume $f \in K$ satisfies the conditions specified above. Write $f = \lambda g + (1-\lambda)h$ with $\lambda \in (0,1), g, h \in K$. Over the set $\{x : |f(x)| = 1\}$, clearly we must have $f = g = h$. Over the set O_f, $f' = \lambda g' + (1-\lambda)h'$, and again we must have $f' = g' = h'$. As $\{x : |f(x)| = 1\}$ is nonempty, use any point there as a base point and integrate, we have $f = g = h$ over $[0,1]$. □

Problem 16. Show that the group G of affine automorphisms $g : \mathbb{R} \to \mathbb{R}$ has a right invariant measure and a left invariant Haar measure, but they are not proportional. (Here G consists of the maps $g(x) = ax + b$ with $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$.)

Solution. Note that a right- (or left-) invariant Haar measure, should it exist, is unique up to multiplication by a positive constant. Using the coordinate a, b described as in the problem, we claim $\frac{1}{a^2} da \wedge db$ gives a left-invariant Haar measure, and $\frac{1}{|a|} da \wedge db$ gives a right-invariant Haar measure. These are easy to check by considering their pullbacks under the left action $L_{(a,b)}(\alpha, \beta) = (a\alpha, a\beta + b)$ and the right action $R_{(a,b)}(\alpha, \beta) = (a\alpha, b\alpha + \beta)$. □

Problem 17. Show that $\ell^1(\mathbb{N})$ is a dual space. That is, exhibit a Banach space X such that X^* is isometric to $\ell^1(\mathbb{N})$.

Solution. Let $X = c_0$, the space of real sequences with limit 0, equipped with the sup norm. It is easy to see that c_0 is a closed subspace of $\ell^\infty(\mathbb{N})$ and hence a Banach space.

It is easy to see that we have an isometric embedding $\ell^1(\mathbb{N}) \hookrightarrow (c_0)^*$, using the pairing between elements of $\ell^1(\mathbb{N})$ and $c_0 \subset \ell^\infty(\mathbb{N})$. It remains to prove surjectivity of this embedding. Let $f \in (c_0)^*$. Let $a_i = f(e_i)$. We claim $f = (a_i)$ as bounded linear operators on c_0. Clearly $(a_i) \in \ell^1(\mathbb{N})$, as the partial sums $\sum_{i=1}^n |a_i| \leq \|f\|$. Let $x = (x_i) \in c_0$. Then the partial sequences $x^{(n)} = (x_1, \ldots, x_n, 0, 0, \ldots)$ tends to x in the sup norm (note that this fails in $\ell^\infty(\mathbb{N})$), as $\lim_{i\to\infty} |x_i| = 0$. Hence

$$f(x) = \lim_{n\to\infty} f(x^{(n)}) = \lim_{n\to\infty} \sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{\infty} a_i x_i$$

as desired. □

Problem 19. Let $X \cong \mathbb{R}^{\mathbb{N}}$ be the vector space of all real sequences $a = (a_0, a_1, a_2, \ldots)$ with the product topology. (a) Show that X^* can be identified with the subspace of X where $a_i = 0$ for all i sufficiently large, using the pairing

$$\langle a, b\rangle = \sum_i a_i b_i.$$

(b) Show that if $a(n)$ is a sequence in X^*, and $a(n) \to 0$ in the weak* topology, then there is a single N such that $a_i(n) = 0$ for all $i > N$.

Solution. (a) Let us denote the subspace of X in question by $\mathbb{R}^\infty$. Clearly any element in $\mathbb{R}^\infty$ gives a continuous linear functional on X, using the pairing stated in the problem. On the other hand, take $f \in X^*$, and set $a_i = f(e_i)$ where e_i is the sequence with 1 at the i-the place and 0 elsewhere. We claim $a_i = 0$ for i sufficiently large. Then as partial sequences approximate elements in X, we have the desired result. Suppose, to the contrary, $a_i \neq 0$ for infinitely many i. Extract a nonzero subsequence $\{a_{i_k}\}$. Then $x_k = \frac{1}{a_{i_k}} e_{i_k} \to 0$ in the product topology, while $f(x_k) = 1$, contradicting the fact that f is continuous.

(b) Suppose otherwise; then we can choose a subsequence $a(n_k)$ such that the index i_k of the last nonzero term in $a(n_k)$ is increasing. We construct the following sequence $b = (b_0, b_1, \ldots)$ inductively. Set $b_0, \ldots, b_{i_0-1}$ to be zero and $b_{i_0} = 1/a_{i_0}(n_0)$. Inductively, set $b_{i_{k-1}+1}, \ldots, b_{i_k-1}$ to be zero and $b_{i_k} = 1 - \sum_{j=0}^{i_{k-1}} a_j(n_k) b_j$. Then we have $\langle a(n_k), b\rangle = 1$, contradicting the fact that $a(n_k) \to 0$ weakly. □