Homework 5

Official Solutions

Math 154

10.2 Being dealt three hands counts as three independent trials. The probability of not getting an ace $P\{N\}$ in a given trial is $p = \binom{48}{13}/\binom{52}{13} = \frac{48!}{13!(48-13)!}\frac{13!(52-13)!}{52!} = \frac{39\cdot38\cdot37\cdot36}{52\cdot51\cdot50\cdot49} \approx 0.304$. Since the three trials are independent, we can just multiply their probabilities together $P\{NNN\} = p^3 \approx \boxed{0.028}$

10.3 The equivalent problem is to find the length of a series in which the probability of not finding a 7 less than $\frac{1}{10}$. We can consider number in the series an independent trial, the event of not finding a 7 has probability $\frac{9}{10}$. The probability of no 7 in n trials is then $(\frac{9}{10})^n$. We want this quantity to be less than $\frac{1}{10}$:

$$\begin{aligned}(\frac{9}{10})^n &\leq \frac{1}{10}\\ n\log(\frac{9}{10}) &\leq \log(\frac{1}{10})\\ n &\geq \log(\frac{1}{10})/\log(\frac{9}{10}) \approx 21.8 \qquad (1)\end{aligned}$$

So the series of numbers needs to be at least $\boxed{22}$ digits long.

10.4 We can equivalently require that the probability of not being dealt four aces $P\{N\}$ in n trials to be less than $\frac{1}{2}$. In any given dealing, $P\{N\} = 1 - P\{A\}$, where $P\{A\}$ is the probability of being dealt four aces. There are $\binom{48}{9}$ ways of choosing the remaining cards in the four ace hand, and $\binom{52}{13}$ possible hands. $P\{N\} = 1 - \binom{48}{9}/\binom{52}{13}$. Since we have independent dealings, we just multiply the probabilities of each trial to get the probability of n trials:

$$\begin{aligned}P\{NNN\ldots\} &\leq \frac{1}{2}\\ P\{N\}^n &\leq \frac{1}{2}\\ \left(1 - \binom{48}{9}/\binom{52}{13}\right)^n &\leq \frac{1}{2}\\ n &\geq \log\left(\frac{1}{2}\right)/\log\left(1 - \binom{48}{9}/\binom{52}{13}\right)\\ n &\geq 262.104 \qquad (2)\end{aligned}$$

So we would need to deal at least $\boxed{263}$ hands. To solve this for some player, we need to recalculate $P\{N\}$ to be the probability that no players is dealt 4 aces. Lets consider the case in which player 1 has 4 aces. The number of ways of arranging the rest of the cards in his hand is $\binom{48}{9}$. The number of ways of choosing cards for the rest of the players is $\binom{39}{13}\cdot\binom{26}{13}$. The total number of ways of dealing out the cards is given by $\binom{52}{13}\cdot\binom{39}{13}\cdot\binom{26}{13}$. Thus, the probability that player 1 has 4 aces is $\binom{48}{9}/\binom{52}{13}$. We can argue that for every way we can deal player 1 four aces, there is an equivalent way of dealing four aces to each other player, so the probability of any player being dealt 4 aces is $4 \cdot \binom{48}{9}/\binom{52}{13}$. Our requirement above becomes:

$$\left(1 - 4 \cdot \binom{48}{9}/\binom{52}{13}\right)^n \leq \frac{1}{2}$$
$$n \geq \log\left(\frac{1}{2}\right) / \log\left(1 - 4\binom{48}{9}/\binom{52}{13}\right)$$
$$n \geq 65.265 \tag{3}$$

So we would need to deal at least $\boxed{66}$ hands to ensure that the probability that somebody gets four aces is greater than $\frac{1}{2}$.

10.11 We have $N = 500$ pages and $m = 500$ misprints on those pages. We can model this by assuming that on each page there are n letters and that each letter represents an independent trial of having a misprint. The chance of a misprint for any given letter can be estimated by $p = \frac{m}{nN}$; by the law of large numbers, this estimate will become better as n increases. If we consider just a single page, our total number trials is simply n so the quantity $\lambda = np = \frac{m}{N} = 1$. Since we assume a sufficiently large number of letters on each page, we can assume the misprints are distributed according to the Poisson distribution, hence we estimate the probability of k misprints on any given page by $p(k;1) = e^{-1}/k!$. The probability of having at least 3 misprints is $1 - p(0;1) - p(1;1) - p(2;1) = 1 - e^{-1}\frac{5}{2} \approx \boxed{.0803}$

10.15 Since p is very small we can assume royal flush appears in poker under Poisson distribution. By (5.4), $p(0;\lambda) \approx e^{-\lambda}$. In our case $\lambda = 1$, hence $np = 1$ and we get $n = 649,740$. In conclusion for $n = 649,740$ the probability of getting no royal flushes is approximately $\frac{1}{e}$, and for $n \geq 649,740$ the probability is $\leq \frac{1}{e}$.

10.19 We have two independent trials of n tosses. If $p(k)$ is the probability that a trial resulted in k heads, the probability that our two trials have the same number of heads is given by $\sum_{k=0}^{n} p(k)p(k)$. In our case, the probability $p(k)$ is given by the binomial distribution with $p = \frac{1}{2}$:

$$p(k) = \binom{n}{k} p^k q^{n-k} = \binom{n}{k}\left(\frac{1}{2}\right)^n \tag{4}$$
$$\sum_{k=0}^{n} p(k)p(k) = \left(\frac{1}{2}\right)^{2n} \sum_{k=0}^{n} \binom{n}{k}^2 \tag{5}$$
$$\sum_{k=0}^{n} p(k)p(k) = \left(\frac{1}{2}\right)^{2n} \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} \tag{6}$$
$$\sum_{k=0}^{n} p(k)p(k) = \left(\frac{1}{2}\right)^{2n} \binom{2n}{n} \tag{7}$$
$$\tag{8}$$

Note that the last identity is true because both sides represent the number of subsets with n elements of a set with $2n$ elements, $S = a_1, a_2, \ldots, a_{2n}$. Every such subset will include

k elements from $a_1, a_2, \ldots, a_n$ and the rest of $n-k$ from $a_{n+1}, a_{n+2}, \ldots, a_{2n}$, for some k between 0 and n. Summing over k we get all the possible subsets and hence the identity is correct.
As some students noted, when n goes to infinity we can approximate the result by using Stirling's formula and we get $\frac{1}{\sqrt{\pi n}}$.
An alternative solution was given by Abdullah Kanee, by using random walks. Consider two random walks each of length n defined as follows: the first random walk assigns $+1, -1$ to obtaining heads or tails respectively (consider this the first person). The second random walk assigns $+1, -1$ to tails or heads respectively. This problem then reduces to finding the probability that a return to the origin occurs at epoch $2n$ when we consider a random walk of length $2n$ where the first n steps are defined as eariler and correspond to the tosses of the first person. Similarly, the second half of the random walk corresponds to the tosses of the second person and the required probability is $u_{2n} = \frac{\binom{2n}{n}}{2^n}$.

10.29 The $p(k;\lambda)$ are given by $e^{-\lambda}\frac{\lambda^k}{k!}$. We want to show that $p(k;\lambda)$ is maximized when k is the largest integer not exceeding λ. To do this we'll consider the ratio

$$\frac{p(k+1;\lambda)}{p(k;\lambda)} = \frac{\lambda^{k+1}k!}{\lambda^k (k+1)!} = \frac{\lambda}{k+1} \tag{9}$$

If $k+1 \leq \lambda$ then the term $p(k+1;\lambda)$ will be greater than or equal to the term $p(k;\lambda)$ since their ratio is greater than or equal to 1. if $k+1 > \lambda$ the term $p(k+1;\lambda)$ will be less than the term $p(k;\lambda)$, since their ratio will be less than 1. If r is the largest integer that doesn't exceed λ, we are guaranteed that all terms $p(j;\lambda)$ with $j > r$ will be less than $p(r;\lambda)$ since j will necessarily be an integer exceeding λ. We are also guaranteed that all terms $p(i;\lambda)$ with $i < r$ will be less than or equal to $p(r;\lambda)$ since every $i+1$ must be less than or equal to λ. Thus, $p(r;\lambda)$ must have a maximum value in the distribution.

10.42 Let us start with the definition of the binomial distribution $b(k;n,p) = \binom{n}{k}p^k q^{n-k}$. Thus we have

$$\textstyle\sum_{v=0}^{k} \binom{n_1}{v} p^v q^{n_1-v} \binom{n_2}{k-v} p^{k-v} q^{n_2-(k-v)}$$

Now we expand out the sum:

$$\binom{n_1}{0}p^0 q^{n_1}\binom{n_2}{k}p^k q^{n_2-k} + \binom{n_1}{1}p^1 q^{n_1-1}\binom{n_2}{k-1}p^{k-1}q^{n_2-(k-1)} + ... + \binom{n_1}{k}p^k q^{n_1-k}\binom{n_2}{0}p^0 q^{n_2}$$

Now we simplify each term:

$$\binom{n_1}{0}\binom{n_2}{k}p^k q^{n_1+n_2-k} + \binom{n_1}{1}\binom{n_2}{k-1}p^k q^{n_1+n_2-k} + ... + \binom{n_1}{k}\binom{n_2}{0}p^k q^{n_1+n_2-k}$$

Now we factor out all the $p^k q^{n_1+n_2-k}$ and get:

$$p^k q^{n_1+n_2-k}\left(\binom{n_1}{0}\binom{n_2}{k} + \binom{n_1}{1}\binom{n_2}{k-1} + ... + \binom{n_1}{k}\binom{n_2}{0}\right)$$

From II, (6.4) we have that:

$$\binom{n_1}{0}\binom{n_2}{k} + \binom{n_1}{1}\binom{n_2}{k-1} + ... + \binom{n_1}{k}\binom{n_2}{0} = \binom{n_1+n_2}{k}$$

Therefore:

$$\binom{n_1+n_2}{k}p^k q^{n_1+n_2-k}$$

Which is just $b(k; n_1 + n_2, p)$. And this completes the verification that

$$\sum_{v=0}^{k} b(v; n_1, p)b(k-v; n_2, p) = b(k; n_1 + n_2, p)$$

Probabilistically, this is saying that finding the probability of k successes in some $n = n_1 + n_2$ Bernoulli trials where the probability of success is p is equivalent to finding the probability of finding exactly k successes among two disjoint subpopulations of trials of sizes n_1 and n_2. Since the trials in these subpopulations are independent, we can multiply the probabilities of finding i successes in n_1 and $k - i$ successes in n_2. Since i can range from 0 to k, we need to sum all of these products together to get the total probability of finding k successes among the two populations.