

Math 154: Problem set 1

Solutions

Problem I.8.4

Most students had problems with figuring the sample space. The sample space for a coin tossed until the same result occurs twice in succession is:

$$S = \{HH, HTT, HTHH, HTHTT, \dots\} \cup \{TT, THH, THTT, THTHH, \dots\}$$

The probability of any outcome with n flips is $1/2^n$. We also need to include the infinite sequences $HTHTHTH\dots$ and $THTHTHT\dots$ with probabilities 0. More formally, we can say:

$$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup \{TT, HH, (TH)^\infty, (HT)^\infty\}, \text{ where}$$

$$A_1 = \{(TH)^k H | k \in N^*\}, A_2 = \{(HT)^k T | k \in N^*\}, A_3 = \{H(TH)^k H | k \in N^*\}, A_4 = \{T(HT)^k T | k \in N^*\}$$

The event that the experiment ends before the 6th toss is

$$A = \{2, 3, 4, 5\}$$

which has probability

$$p(A) = p(2) + p(3) + p(4) + p(5) = 1/2 + 1/4 + 1/8 + 1/16 = 15/16$$

The event that the experiment takes an even number of tosses is

$$A = \{2, 4, 6, 8, \dots\}$$

which has probability

$$\begin{aligned} p(A) &= \sum_{j=1}^{\infty} 1/2^{2j-1} = \sum_{j=0}^{\infty} 1/2^{2j+1} = 1/2 * \sum_{j=0}^{\infty} 1/2^{2j} \\ &= 1/2 * \sum_{j=0}^{\infty} 1/4^j = 1/2 * \frac{1}{1-1/4} \\ &= 2/3 \end{aligned}$$

Problem I.8.15

a) I claim $(A \cup B)(A \cup B') = A$.

Let $x \in (A \cup B)(A \cup B')$. $\rightarrow x \in (A \cup B)$ and $x \in (A \cup B')$. If $x \notin A$ then $x \in B$ and

$x \in B' \rightarrow x \in BB' = \emptyset$. Contradiction, hence $x \in A$.

Reverse, let $x \in A$. $\rightarrow x \in A \cup B$ and $x \in A \cup B'$. Hence $x \in (A \cup B)(A \cup B')$. By double inclusion we get that $(A \cup B)(A \cup B') = A$.

b) I claim $(A \cup B)(A' \cup B)(A \cup B') = AB$

By part a) we have $(A \cup B)(A' \cup B) = B$ so we can substitute it above and obtain

$$(A \cup B)(A' \cup B)(A \cup B') = B(A \cup B').$$

We are left to prove that $B(A \cup B') = AB$.

Let $x \in B(A \cup B')$. $\rightarrow x \in B$ and $x \in A \cup B'$. Since $x \notin BB' = \emptyset$, we obtain $x \in B$ and $x \in A$, hence $x \in AB$.

Reverse, let $x \in AB$. Then $x \in B$ and $x \in A$, hence $x \in (A \cup B')$. Overall we get $x \in B(A \cup B')$ and by double inclusion $(A \cup B)(A' \cup B)(A \cup B') = AB$.

c) I claim $(A \cup B)(B \cup C) = B \cup AC$.

Let $x \in (A \cup B)(B \cup C)$.

$$\rightarrow x \in A \cup B \rightarrow x \in A \text{ or } x \in B$$

$$\rightarrow x \in B \cup C \rightarrow x \in B \text{ or } x \in C$$

Suppose $x \notin B$ then $x \in A$ and $x \in C$, hence $x \in B \cup AC$.

Reverse now, let $x \in B \cup AC \rightarrow x \in B$ or $(x \in A$ and $x \in C)$. $\rightarrow x \in (A \cup B)(B \cup C)$. By double inclusion we are done again.

Problem II.10.1

The English alphabet has 26 letters.// a) If each person has 1 surname and 2 given names then there are 3 initials in each abbreviation and we get 26^3 different ones.

b) If each person has at most two given names then the abbreviation has 2 or 3 letters. There are 26^3 different 3-letters abbreviation and 26^2 different 2-letters abbreviation, which add up to $26^2 + 26^3$.

c) We need to add up the case when a person has 3 given names to case b). Overall we get $26^2 + 26^3 + 26^4$ different abbreviations.

Problem II.10.3

There are n pieces whose 2 numbers are the same. There are $\binom{n}{2}$ pieces for whose 2 numbers are different. Hence, totally we get $n + \binom{n}{2} = \frac{n(n+1)}{2}$ different pieces.

Problem II.10.4

There are $n!$ different arrangements.

a) Let A be the event that 1, 2 appear as neighbors in order 12.

For A to occur, 1 can be in spot $a_1, a_2 \dots a_{n-1}$ and the placement of 2 is completely determined for the one of 1. Hence, there are $n - 1$ choices for the placement of 1 and the other $n - 2$ elements can be arranged in $(n - 2)!$ ways.

$$\rightarrow P(A) = \frac{(n - 1)(n - 2)!}{n!} = \frac{1}{n}$$

b) Let B be the event that 1, 2, 3 appear as neighbors in order 123. For B to occur, 1 can be in spot $a_1, a_2 \dots a_{n-2}$ and the placements of 2 and 3 are completely determined by that of 1. Hence, there are $n - 2$ choices for the placement of 1 and the other $n - 3$ elements can be arranged in $(n - 3)!$ ways.

$$\rightarrow P(B) = \frac{(n-2)(n-3)!}{n!} = \frac{1}{n(n-1)}$$

Problem II.10.5

The probability that A loses, throws no aces is $(5/6)^6$, about 0.335.

The probably that B loses, throws one ace or less is (by summing the first 2 terms of the binomial)

$$\begin{aligned} & (5/6)^{12} + 12(5/6)^{11}(1/6) \\ & = (5/6)^{12} + 2(5/6)^{11} \end{aligned}$$

which is about 0.381.

So A is more likely to win.

Problem II.10.9

There are n^n ways to put n balls into n cells.

Consider the case when exactly 1 cell is empty. There are n ways to choose the empty cell. For each choice of the empty set, we must put n balls into $(n - 1)$ cells so that all the cells are not empty. Thus, in each case we must have 2 balls in one cell and one ball in each of the other $n - 2$ cells. There are $\binom{n}{2}$ ways to choose the two balls that will be put together. For eacy pick there are $(n - 2)!$ ways to arrange the remaining $n - 2$ balls. The total number of ways to arrange n balls in n cells so that only one cell remains empty is:

$$n * \binom{n}{2} * (n - 2)! * (n - 1), \text{ hence } P = \frac{\binom{n}{2} * n!}{n^n}$$

Problem II.12.1

a) For $n \geq 1$, by reverse binomial expansion,

$$\begin{aligned} & \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots \\ & = (1 - 1)^n \\ & = 1 \end{aligned}$$

b) For $n \geq 1$,

$$\begin{aligned} & \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} \dots \\ & = \sum_{k=1}^n k \binom{n}{k} \\ & = \sum_{k=1}^n \frac{kn!}{k!(n-k)!} \\ & = n \sum_{k=1}^n \frac{(n-1)!}{(k-1)!(n-k)!} \end{aligned}$$

$$\begin{aligned}
&= n \sum_{k=0}^{n-1} \frac{(n-1)}{(k)!((n-1)-k)!} \\
&= n(1+1)^{n-1} \\
&= n2^{n-1}
\end{aligned}$$

Note that an alternative solution is to just express the derivative of $2^n = (1+1)^n$ in two ways.

c) For $n \geq 2$,

$$\begin{aligned}
&\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \dots \\
&= \sum_{k=1}^n (-1)^{k-1} k \binom{n}{k}
\end{aligned}$$

By the same manipulations as in 1b,

$$\begin{aligned}
&= n \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)}{(k)!((n-1)-k)!} \\
&= n(1-1)^{n-1} \\
&= 0
\end{aligned}$$

Note that an alternative solution is to just express the derivative of $(1-1)^n$ in two ways.

d) For $n \geq 2$,

$$\begin{aligned}
&2 * 1 \binom{n}{2} + 3 * 2 \binom{n}{3} + 4 * 3 \binom{n}{4} + \dots \\
&= \sum_{k=2}^n k(k-1) \binom{n}{k} \\
&= \sum_{k=2}^n \frac{k(k-1)n!}{k!(n-k)!} \\
&= n(n-1) \sum_{k=2}^n \frac{(n-2)!}{(k-2)!(n-k)!} \\
&= n(n-1) \sum_{k=0}^{n-2} \frac{(n-2)!}{k!((n-2)-k)!} \\
&= n(n-1)(1+1)^{n-2} \\
&= n(n-1)2^{n-2}
\end{aligned}$$

Note that an alternative solution is to just express the second derivative of 2^n in two ways.

Problem II.12.6

We show for any natural number $n \geq 0$, for any $a \geq 1$,

$$\sum_{v=0}^n (-1)^v \binom{a}{v} = (-1)^n \binom{a-1}{n}$$

by induction on n

This is true when $n = 0$ because:

$$\begin{aligned} & \sum_{v=0}^0 (-1)^v \binom{a}{v} \\ &= \binom{a}{0} \\ &= 1 \\ &= (-1)^0 \binom{a-1}{0} \end{aligned}$$

Assume that it is true for $n = k$. Then:

$$\begin{aligned} & \sum_{v=0}^{k+1} (-1)^v \binom{a}{v} \\ &= (-1)^{k+1} \binom{a}{k+1} + \sum_{v=0}^k (-1)^v \binom{a}{v} \\ &= (-1)^{k+1} \binom{a}{k+1} + (-1)^k \binom{a-1}{k} \\ &= (-1)^k \left(-\binom{a}{k+1} + \binom{a-1}{k} \right) \end{aligned}$$

Using the identity that $\binom{a}{k+1} = \binom{a-1}{k+1} + \binom{a-1}{k}$:

$$\begin{aligned} &= (-1)^k \left(-\binom{a-1}{k+1} - \binom{a-1}{k} + \binom{a-1}{k} \right) \\ &= (-1)^k \left(-\binom{a-1}{k+1} \right) \\ &= (-1)^{k+1} \binom{a-1}{k+1} \end{aligned}$$

Therefore, it is true for $n = k + 1$. So by induction it is true for all $n \geq 0$.