Mathematical Probability: Final

Due 4pm, Thursday, 5 May 2011.

Write your answers neatly on separate paper, stapled together, with your name on the first page. Clearly indicate (e.g. with a box) the final answer to each problem. Answers can include factorials and binomial coefficients.

All work should be your own. Refer only to your class notes and the course text by Feller. The use of calculators is permitted.

Hand in your completed work to McMullen's mailbox outside 325 Science Center by 4 pm on Thursday, 5 May 2011.

1. A roomful of n gamblers consists of a Texans and b lawyers, including c Texan lawyers. (Thus $a + b = n + c$). Each gambler rolls one fair die. The total value turning up on the Texans' dice is A and on the lawyers' dice is B.

   (a) Calculate the correlation coefficient $\rho(A, B)$ in terms of a, b, and c.

   (b) Show $\rho(A, B) = 1$ if and only if $a = b = c$, and $\rho(A, B) = 0$ if and only if $c = 0$.

2. An unknown number of counterfeit quarters are deposited in Cambridge parking meters. An inspector uses a random sample of N coins to estimate the percentage P of counterfeits.

   (a) Approximately how large a sample is needed to guarantee that, with probability 50%, the estimate of P will be correct to within one percentage point?

   (b) A reliable informant tells us that at most one coin in ten is counterfeit. Now what is the answer to (a)?

3. Let X and Y be independent random variables, with X exponentially distributed, $E(X) = 1$, and with Y uniformly distributed in $[0, 2]$.

   (a) Compute mean and standard deviation for $S = X - Y$.

   (b) Compute the density function $f(s)$ for S.

4. Independent cosmic particles arriving at a detector have random energies $X_1, X_2, \ldots$ and random spins $S_1, S_2, \ldots$; each spin is ± 1 with equal probability.

   Let $K =$ the least index such that $X_K > X_1$ (the first moment a new record energy is observed).

   Let $N =$ the least index such that $\sum_1^N S_i = 0$ (the first moment an equal number of plus and minus spins are observed).

   (a) Show K and N are almost surely finite.

   (b) Show $E(K) = E(N) = \infty$.

   (c) Which event tends to require a longer wait? For example, is it more likely that $K > 10^6$ or that $N > 10^6$?

5. A foundering sub is one mile off shore, at the location $(a, 1)$ in $\mathbb{R}^2$ with a unknown. The crew fires torpedos in random directions, which are observed to hit the shoreline $y = 0$ at the points $(X_i, 0)$, $i = 1, 2, 3 \ldots$. (Some miss the shore entirely.) How can one use the values $(X_1, \ldots, X_n)$ to make an accurate guess for the value of a? Justify your answer.

6. At each epoch, a random walker takes 2 steps forward or 1 step back with equal probability. What are the chances that he returns infinitely often to his starting position? Justify your answer.

7. Let P be a point in the northern hemisphere of the earth, chosen randomly with respect to surface area.

   (a) What is the expected latitude L of P?

   (b) Give a formula for the distribution function $F(t) = P(L < t)$.

   (c) Are the latitude and longitude of P independent random variables?