Topology

Math 131: Homework 10
Due Tuesday, 12 November 2013

Below, the sphere will be considered as a set $S^{n-1} \subset \mathbb{R}^n$, and $x \mapsto -x$ denotes the antipodal map.

1. Let G be a connected finite graph where every vertex has even degree (it is incident to an even number of edges). Prove there exists a path that starts and ends at the same vertex of G, and crosses all the edges of G exactly once. (Hint: consider the longest circuit in G and use induction.)

2. The *Möbius band* B is the quotient of $[0,1] \times [0,1]$ by the relation $(0, y) \sim (1, 1-y)$. Show that the cylinder $S^1 \times [0,1]$ is a covering space of B.

3. Let X/A denote the quotient space of X where all points in $A \subset X$ are identified to a single point. Prove that $\mathbb{R}^2/A$ is homeomorphic to $\mathbb{R}^2$ when $A = B^2$ or $A = [0,1] \times \{0\}$, but not when $A = S^1$.

4. Give an example of 3 bounded sets of positive area $A, B, C \subset \mathbb{R}^2$ that cannot be simultaneously cut in half by a line.

5. Prove that an open set in $\mathbb{R}^2$ cannot be homeomorphic to an open set in $\mathbb{R}^3$. (Hint: use the Borsuk–Ulam theorem).

6. Suppose $f : S^2 \to S^2$ is continuous and $f(x) \neq f(-x)$ for all $x \in S^2$. Show that f is surjective.

7. Show that if S^1 is covered by two closed sets A and B, then one of them contains a pair of antipodal points. (Hint: consider the function $f(x) = d(x, A) - d(x, B)$.)

8. Prove that if S^2 is covered by 3 closed sets, then one of them contains a pair of antipodal points.