Homework 10
Algebra II

1. Given a square, is it possible to construct (with ruler and compass) a pentagon with the same area?

2. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{5})$, and let $t = \sqrt{2} + \sqrt{5}$.

   (i) Prove that $\sqrt{5} \notin \mathbb{Q}(\sqrt{2})$, and hence $[K : \mathbb{Q}] = 4$.

   (ii) Show that $K = \mathbb{Q}(t)$.

   (iii) Find the minimal polynomial for t over $\mathbb{Q}$.

3. Let $K = \mathbb{Q}(i)$ and let $p(x) = x^4 - 5$.

   (i) Prove that $p(x)$ is irreducible in $K[x]$.

   (ii) Prove that $L = K(5^{1/4})$ is the splitting field of $p(x)$ over K.

   (iii) Prove there is an automorphism $\alpha \in \mathrm{Gal}(L/K)$ such that $\alpha(5^{1/4}) = i \cdot 5^{1/4}$.

   (iv) Prove that $\mathrm{Gal}(L/K)$ is isomorphic to $\mathbb{Z}/4$.

4. Prove that $t = 2\cos\pi/9$ is a root of $p(x) = x^3 - 3x - 1$.

5. Let p be an odd prime. Show that the minimal polynomial for $\cos(2\pi/p)$ in $\mathbb{Q}[x]$ has degree $(p-1)/2$. (Hint: use the fact that $[\mathbb{Q}(\zeta_p) : \mathbb{Q}] = p-1$.)

6. (Bonus problem.) Suppose we are given the points $\mathbb{Z} \cup i\mathbb{Z} \subset \mathbb{C}$. What points in $\mathbb{C}$ can we then construct, using only a straightedge?