Math 122, Solution Set No. 9

1 6.5.2

Note if $|G| = 4$, then $G \cong C_4$ or $G \cong D_2 \cong V_4$ So assume $p \neq 2$. Then by the Sylow Theorems, $\exists$ a subgroup C_2 of order 2 and a unique (therefore normal) subgroup C_p of order p. So $C_2 \cap C_p = e$ and so by Artin 2.8.6 $G = C_pC_2$. So as in class, $G \cong C_p \ltimes_\varphi C_2$, where $\varphi : C_2 \to Aut(C_p) \cong C_{p-1}$. If φ is trivial, then we have $G \cong C_p \times C_2 \cong C_{2p}$. The only other automorphism of C_p of order dividing 2 is $\varphi : x \mapsto x^{-1}$. In this case we have G is generated by x, y with $x^p = y^2 = e$ and $xy = yx^{-1}$ and in this case $G \cong D_p$.

2 6.5.4

(a) Let $|G| = 55$. By the Sylow Theorems, we have that there is a subgroup of G isomorphic to C_5, and a unique (therefore normal) subgroup isomorphic to C_{11}. As in 6.5.2 this implies $G \cong C_{11} \ltimes C_5$, i.e. G is generated by x, y with $x^{11} = y^5 = e$ and $yxy \in C_{11}$ $e \therefore yxy = x^r, r \in 1, \ldots, 10$.

(b) Note $x = y^5xy^5 = x^{r^5} \Rightarrow r^5 = 1 \pmod{11}$. It is easily checked that $2^5 = 6^5 = 7^5 = 8^5 = 10^5 = 10 \pmod{11}$ and so r cannot be any of these integers.

(c) Note also $1^5 = 3^5 = 4^5 = 5^5 = 9^5 = 1 \pmod{11}$ and so r can take these values. If $r = 1$ then we have $G \cong C_{11} \times C_5 \cong C_{55}$. If $r = 3$ then we have $G = \langle x, y | x^{11} = y^5 = e \text{ and } yxy = x^3 \rangle$. If $r \in \{4, 5, 9\}$, then G is isomorphic to this case by the map that sends x to x and y to y^n, where $n \in \{2, 3, 4\}$.

3 6.6.2

$(34)(123)(45)(34) = (241)(35)$

4 6.6.3

Note $qp = p(qp)p^{-1}$ and so they have cycles of equal length.

5 6.6.4

Note first that the order of a permutation is the least common multiple of the lengths of its component cycles (an easy lemma). So, (a) (12345) has order 5, and (12345)(67) has order 10. Any element of S_7 is the least common multiple of the length of its cycles, and so (b) the largest possible order is 12. So (a) there is no element of order 15.

6 6.6.5

Note that the cycle $(a_1 \cdots a_k) = (a_k) \cdots (a_1)$ and therefore has parity $k - 1$. So if we write a permutation as a product of cycles, each cycle of even (respectively odd) length will have sign -1 (resp. 1). Since the sign is multiplicative, this implies $\text{sign}(\sigma) = (-1)^n$, where n is the number of cycles of even length in the cycle decomposition of σ.

7 6.6.11

Let $\sigma = (12)$. Note C_σ has order 6, and so $|Z(\sigma)| = 24/6 = 4$. Since e, (12), (34), and (12)(34) stabilize σ, this is the entire centralizer.

8 6.6.14

(a) Two permutations are conjugate iff their cycle decompositions have the same orders. It is easy to check that in S_5 there are 24 5-cycles, 30 4-cycles, 20 3-cycles, 10 2-cycles, 20 products of disjoint 2- and 3-cycles, 15 products of 2 disjoint 2-cycles, and the identity. So the class equation is: 120 = 1+ 10 + 15 + 20 + 20 + 24 + 30. (b) Note that as in 6.6.5, we can easily check that the elements in A_5 are the 5-cycles, 3-cycles, identity, and products of disjoint transpositions. Note that for a 5-cycle σ, $C_\sigma \subset S_5$ has order 24 and so $Z(\sigma) \leq S_5$ has order 5. In fact, it is clear that $Z(\sigma) = e, \sigma, \sigma^2, \sigma^3, \sigma^4 \subset A_5$ and so in A_5, $|C_\sigma| = 60/5 = 12$. Similar computations show that the conjugacy class of a 3-cycle or product of disjoint transpositions is the same in S_5 and A_5. Therefore the class equation for A_5 is the same as the icosahedral group, i.e. 60 = 1 + 12 + 12 + 15 + 20.

9 6.6.15

Note for $a < b$, (ab) = (b-1, b) ... (a+1, a+2) (a, a+1) (a+1, a+2) ... (b-1, b). As in 6.6.5, any cycle is the product of transpositions and so since any permutation is the product of cycles, the symmetric group S_n is generated by the elements (12), (23) ... (n-1, n).