Math 114 Homework 3 Solutions

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October 2, 2014

1. We will prove the stronger statement that $m(\limsup E_n) \geq c$ as well, given that all of the individual sets have more than measure c. Recalling the definition of lim sup, we have

$$\limsup E_n = \bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} E_n,$$

the set of all points which appear in infinitely many E_n. Letting $F_n = \bigcup_{n\geq N} E_n$, we see that F_n is a nested sequence of measurable sets, and $m(F_n) \geq m(E_n) > c$. Thus by continuity of measure, we have

$$m(\limsup E_n) = \lim(m(F_n)) \geq c.$$

2. We will prove the stronger statement that $K - K = [-1, 1]$. We begin by showing $K_n - K_n = [-1, 1]$ by induction, where K_n is the n-th approximation to the Cantor set K defined by $K_0 = [0, 1]$ and

$$K_{n+1} = \frac{1}{3}\Big(K_n \cup (K_n + 2)\Big)$$

\is defined recursively. Now, given $K_n - K_n = [-1, 1]$, we have

$$\begin{aligned} 3(K_{n+1} - K_{n+1}) &= (K_n - K_n) \cup (K_n - K_n + 2) \cup (K_n - K_n - 2) \\ &= [-1, 1] \cup [1, 3] \cup [-3, -1] \\ &= [-3, 3], \end{aligned}$$

as desired. Finally, pick any $x \in [-1, 1]$. By the above, there exists $k_n \in K_n$ for which $k_n + x \in K_n$ as well, for each $n \in \mathbb{N}$. The sequence $\{k_n\}$ lies in the compact interval $[0, 1]$, and so it must have a convergent subsequence $\{k_{n_j}\}$ which converges to some $k \in \cap K_n = K$. It is immediate that $k + x \in K$ as well since it is the limit point of the sequence $\{k_{n_j} + x\}$. Thus $x \in K - K$ as desired.

3. We begin with the function $h(x) = \phi(x) + x$, where $\phi(x)$ is the Cantor-Lebesgue function. As we showed in lecture, h is a homeomorphism $[0, 1] \to [0, 2]$ which maps the Cantor set $K \subset [0, 1]$ bijectively to a set $S = h(K) \subset [0, 2]$ of measure $m(S) = 1$. Theorem 17 from Royden and Fitzpatrick, due to Vitali, shows that any set of positive outer measure has a nonmeasurable subset, so in particular we can pick $T \subset S$

nonmeasurable, and define $L = h^{-1}(T)$ to be the corresponding subset of K. Now K is measure zero, so L is measurable and the indicator function χ_L is a measurable function.

But h is a homeomorphism, so it has a continuous inverse h^{-1}. We will show that $\chi_L \circ h^{-1}$ is not a measurable function. In particular, pick a Borel set B containing 1 but not 0, so that $\chi_L^{-1}(B) = L$. Then, $(\chi_L \circ h^{-1})^{-1}(B) = h(L) = T$ is nonmeasurable, so the function $\chi_L \circ h^{-1}$ cannot be measurable, as desired.

4. Let S be the set of all (open) rectangles $I \times J$ with rational endpoints. Clearly S injects into $\mathbb{Q}^4$ so it is countable. As in the one-dimensional case, we claim that any open set $U \subset \mathbb{R}^2$ is exactly equal to

$$U = \bigcup_{I\times J \in S, I\times J \subseteq U} I \times J,$$

the union taken over all rational rectangles contained in U. To see this, note that every point $x \in U$ is in the union because S forms a basis for the Euclidean topology on $\mathbb{R}^2$.

5. We would like to show that for any open set $U \subset \mathbb{R}$, the set

$$S = \{x | h(f(x), g(x)) \in U\}$$

is measurable. By continuity, $h^{-1}(U)$ is some open set $V \subset \mathbb{R}^2$. Using the previous problem, we can write $V = \bigcup I_i \times J_i$ for some countable collection of open rectangles $I_i \times J_i$ in the plane. It follows that

$$S = \bigcup_i (f^{-1}(I_i) \cap g^{-1}(J_i)),$$

and it follows by the measurability of f and g that S is a countable union of measurable sets and thus measurable.

6. The Lipschitz condition is between continuity and differentiability in strength. We note that it is necessary for part (i) but given (i), only continuity is required for part (ii).

(i) If $m(A) = 0$ then there exists for every $\epsilon > 0$ a countable collection of intervals $\{I_i\}_{i\in\mathbb{N}}$ covering A which satisfy $\sum m(I_i) < \epsilon$. We claim that $m(f(I_i)) < Lm(I_i)$ for any interval, which proves the result. For any two points $a, b \in I_i$, $|a - b| < m(I_i)$, and so $L|f(a) - f(b)| < Lm(I_i)$. Thus $f(I_i) \subset I_i'$ for some interval I_i' of length $Lm(I_i)$, proving the inequality.

(ii) We will use the property that the image of a compact set under a continuous function f is still compact. Using part (i), we can remove a set of measure 0 from E without affecting the conclusion. In particular, since any measurable set contains an F_σ set of the same measure, we can assume that $E = \bigcup F_i$ is a countable union of closed sets. Each F_i can be written as a countable union

$$F_i = \bigcup_{n\in\mathbb{N}} (F_i \cap [-n, n])$$

of compact sets, and so E can as well. Thus $f(E)$ is also a countable union of compact sets and therefore measurable.

7. Once we show part (i) for a closed subset $F \subset E$, part (ii) follows trivially since $[0, 1]$ has the same cardinality as $\mathbb{R}$.

   (i) We can always pick a closed subset $F \subset E$ of positive measure. Given F, define $f : [0, 1] \to [0, 1]$ as the function

$$f(x) = \frac{m(F \cap [0, x])}{m(F)},$$

   which counts the proportion of F up to x. To see that f is Lipschitz, we note that given $x < y \in [0, 1]$,

$$|f(x) - f(y)| = \frac{m(F \cap (x, y])}{m(F)} \le \frac{m((x, y])}{m(F)} = \frac{1}{m(F)}|x - y|,$$

   proving that f is Lipschitz with $L = 1/m(F)$. On any point $x \notin F$ we can find an open interval $I \ni x$ with $I \cap F = \emptyset$, and so f is constant on this entire interval and its derivative vanishes at x. Finally, f is a continuous function with $f(0) = 0$, $f(1) = 1$, and $f(x) \in [0, 1]$ for all x, the image $f([0, 1])$ is certainly $[0, 1]$. But if $y \in [0, 1]$ and $f^{-1}(y) \ni x$, then we can pick, by virtue of F being closed, some rightmost point $x' \in F$ before x for which $f(x') = y$ as well. It follows that $f(E) = [0, 1]$.

8. That the set A of Borel functions is closed under composition is immediate. As A is a subset of the vector space of all functions $\mathbb{R} \to \mathbb{R}$, it suffices to show that A is closed under addition and scalar multiplication; checking other vector space properties like commutativity, associativity, and distributivity are unnecessary. Since scalar multiplication $c\cdot$ is a Borel function, its composition with another Borel function by another is still Borel. It remains to show that A is closed under addition. Given f, g Borel functions, consider the inverse image of a half-infinite interval

$$(f + g)^{-1}((a, +\infty)) = \{x | f(x) + g(x) > a\}.$$

   Because these half-infinite intervals generate the Borel σ-algebra it suffices to prove that the above set is Borel. In fact, we can write it as

$$\{x | f(x) + g(x) > a\} = \bigcup_{q \in \mathbb{Q}} (\{x | f(x) > a + q\} \cap \{x | g(x) > -q\}),$$

   and since f, g are Borel and the countable union of Borel sets is Borel we are done.