

MATH 114 HOMEWORK 10 SOLUTIONS

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1. Note that ℓ^1 is separable, as the sequences with rational entries, with only finitely many nonzero entries form a countable dense subspace. However, we have shown earlier that L^∞ is not separable. Since separability is preserved under isomorphisms, ℓ^1 and L^∞ are not isomorphic.
2. (a) T is bounded by Bessel's inequality.
(b) $L^2[0, \pi]$ contains $C[0, \pi]$, which is a dense subset of $L^1[0, \pi]$, so $L^2[0, \pi]$ must be dense in $L^1[0, \pi]$.
(c) The image of $L^2[0, \pi]$ under T is contained in ℓ^2 because $L^2[0, \pi]$ is a Hilbert space, so $f = \langle f, e_n \rangle e_n$. This implies $\langle f, e_n \rangle = a_n$ are square summable.
(d) Clearly T sends $L^1[0, \pi]$ into ℓ^∞ . Since c_0 contains ℓ^2 the preimage of c_0 under T contains $L^2[0, \pi]$, but T is continuous, which means that the preimage must be closed. Hence, it must be $L^1[0, \pi]$ which shows that T sends $L^1[0, \pi]$ into c_0 .
3. We must show the kernel is trivial. Suppose f is in the kernel i.e. $\int_0^\pi f(x)e_n(x)dx = 0$. Then, $\int_0^\pi f(x)g(x)dx = 0$ is true for all $g \in \text{span } e_n$. We are given that $\text{span } e_n$ is dense in $C[0, \pi]$. Suppose $g \in C[0, \pi]$. We may choose $g_n \rightarrow g$ in $C[0, \pi]$ such that $g_n \in \text{span } e_n$. This means $g_n \rightarrow g$ uniformly, so $\exists M$ constant such that $g_n \leq g + M$ for all n . Note that $f(g + M)$ is integrable since f is integrable and $g + M$ is bounded, so since $f g_n \leq f(g + M)$ we may use the Dominated Convergence theorem to conclude that $\int_0^\pi f(x)g(x)dx = 0$ for all $g \in C[0, \pi]$. Now suppose g is an indicator function. We may use Lusin's theorem to choose $g_n \rightarrow g$ pointwise such that $g_n \leq g$. Hence (again by the Dominated Convergence theorem) we have $\int_0^\pi f(x)g(x)dx = 0$. Since this is true for all indicator functions g , we must have $f = 0$. Therefore the kernel is trivial i.e. T is injective.
4. Suppose T were surjective. Since T is bounded and a bijection, by previous problems we have $T^* : \ell^1 \rightarrow L^\infty[0, \pi]$ is bounded and a bijection. By the bounded inverse theorem $(T^*)^{-1}$ is bounded. Thus, T^* is bounded and hence an isomorphism between two spaces we have shown (in problem 1) are not isomorphic. This is a contradiction, therefore T is not surjective.

Remark. Many people forgot that for an operator T to be an isomorphism T^{-1} must be bounded as well.

5. X is compact by Tychonoff's theorem. Let $F(X) \subset C(X)$ be the set of functions that depend only on finitely many coordinates. It is easy to check that $F(X)$ is an algebra and separates points. By Stone-Weierstrass we have the desired result.
6. Let $f = \|\cdot\|_2$ be the ℓ^2 norm. It is clear that f is continuous on X , because f is continuous on ℓ^2 . However, f cannot depend on finitely many points: if it did, this would contradict the fact that given any distinct $x, y \in X$, $\|x - y\|_2 > 0$ (a property of the norm).
7. S is clearly compact. It is easy to see that the space of separable functions is an algebra and separates points. By Stone-Weierstrass we have the desired result.