Math 114 Take-Home Final

Analysis II: Measure, Integration and Banach Spaces
Due by 5 pm, Wednesday, 10 December 2014
Email to `ctm@math.harvard.edu` with "Math 114" as the Subject line.

Instructions. Put your name on the first page of your solutions. Aim for clear, concise, answers.

All work should be your own. Refer only to your own class notes, the course text by Royden and Fitzpatrick, and the materials on the course web page. All problems carry equal weight; there are 9 problems total.

1. Let us say $f : [0,1] \to \mathbb{R}$ is *pretty continuous* if the points where f is continuous form a dense subset of $[0,1]$. Show that the sum of two pretty continuous functions is also pretty continuous.

2. (i) Suppose $(na_n) \in \ell^2(\mathbb{Z})$. Show that $(a_n) \in \ell^1(\mathbb{Z})$.

   (ii) Let $f \in C(S^1)$ be a continuous function with a continuous first derivative $f'(x)$. Prove that the Fourier series of f converges uniformly on S^1.

3. Prove or disprove: A generic continuous function $f \in C[0,1]$ achieves its maximum at a unique point $x \in [0,1]$.

4. Let X, Y and Z be Banach spaces and let $T : X \times Y \to Z$ be a map. Suppose that for every $x \in X$, the map $T_x : Y \to Z$ given by $T_x(y) = T(x,y)$ is a bounded linear operator. Similarly, suppose the maps $U_y : X \to Z$ given by $U_y(x) = T(x,y)$ are also bounded linear operators. Prove there is a constant M such that

   $$\|T(x,y)\| \le M \cdot \|x\| \cdot \|y\|.$$

   (Hint: start by showing that $X_M = \{x : \|T_x\| \le M\}$ is a closed subset of X.)

5. (i) Prove that any open subset of $\mathbb{R}$ is homeomorphic to a complete metric space.

   (ii) Prove that the irrational numbers, $J = \mathbb{R} - \mathbb{Q}$ with the induced topology, are homeomorphic to a complete metric space.

6. Let $f_a(x) = \exp(iax)\exp(-x^2/2) \in L^2(\mathbb{R})$.

   (i) Show that the functions $f_a(x), a \in \mathbb{Z}$, do not span a dense subset of $L^2(\mathbb{R})$.

   (ii) Show that the functions $f_a(x), a \in \mathbb{R}$, *do* span a dense subset of $L^2(\mathbb{R})$.

   (Hint: Let $f(x) = g(x)\exp(-x^2/2)$ be a smooth function supported in $[-a,a]$. Given $b > a$, express $g(x)$ as a Fourier series on $[-\pi b, \pi b]$, of the form $g(x) = \sum a_k \exp(i(k/b)x)$. Then show that for $b \gg a$ and N large, $\exp(-x^2/2)\sum_{-N}^{N} a_k \exp(i(k/b)x)$ is a good approximation to $f(x)$.)

7. (Continuation.) Prove that the functions $f_n(x) = x^n \exp(-x^2/2)$, $n \geq 0$, span a dense subspace of $L^2(\mathbb{R})$. You may use Problem 6 and the fact that for even integers $n \geq 0$, we have

$$\langle f_0, f_n \rangle = \int_{-\infty}^{\infty} x^n \exp(-x^2)\, dx = \frac{1 \cdot 3 \cdots (n-1)\sqrt{\pi}}{2^{n/2}}.$$

8. Which of the following assertions are true? (List the letters for the ones which are.)

   (a) The Fourier transform $\widehat{f}(k)$ is well–defined for all $f \in L^{3/2}(S^1)$.

   (b) If $m(E) < \infty$, then polynomials are dense in $L^1(E)$.

   (c) If a and b are sequences in $\ell^2(\mathbb{Z})$, then the convolution $a * b$, defined by

   $$(a * b)_n = \sum_{i+j=n} a_i b_j,$$

   is also in $\ell^2(\mathbb{Z})$.

   (d) If $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, then the convolution $f * g$ is continuous.

   (e) Let $f : [0,1] \to [0,1]$ be a measurable function. Then

   $$\sin\left(\int_0^1 f(x)\, dx \right) \geq \int_0^1 \sin(f(x))\, dx.$$

9. Which of the following assertions are true? (List the letters for the ones which are.)

   (a) The set of Liouville numbers $L \subset \mathbb{R}$ satisfies $L + L = \mathbb{R}$. Here

   $$L = \{x \in \mathbb{R} - \mathbb{Q} \,:\, \forall n > 0\, \exists p/q \,\text{such that}\, |x - p/q| < 1/q^n\}.$$

   (b) Any uncountable subset of $L^1(\mathbb{R})$ has an accumulation point.

   (c) The sequence of functions $K_n(x) = \exp(-nx^2)$ gives an approximation to the identity in $L^1(\mathbb{R})$.

   (d) If $T : X \to Y$ is a bijective linear map between Banach spaces, then T or T^{-1} is continuous.

   (e) The Hilbert space $L^2(\mathbb{R})$ has a countable orthonormal basis.