Math 114 Take-Home Final

Analysis II: Measure, Integration and Banach Spaces
Due by 4 pm, Friday, 11 January 2008
Hand in to staff in the Math Dept. main office,
325 Science Center

Instructions. Write your answers neatly on separate paper, stapled together, with your name on the first page. All work should be your own. Refer only to class notes (your own and those online) and the course text by Royden. All problems carry equal weight. Problems continue on the back of the page.

1. Let us say $f : [0,1] \to \mathbb{R}$ is *pretty continuous* if points where f is continuous form a dense subset of $[0,1]$. Show that the sum of two pretty continuous functions is also pretty continuous.

2. A function $f : [0,1] \to \mathbb{R}$ is Hölder continuous if it satisfies

$$|f(x) - f(y)| \le M|x - y|^\alpha$$

for some $M, \alpha > 0$. Give an example of a Hölder continuous function which is not of bounded variation.

3. Show that the set of irrationals $J = \mathbb{R} - \mathbb{Q}$, with the induced topology, is homeomorphic to a complete metric space.

4. We say $f_n \to f$ *monotonely* on $[0,1]$ if $f_n \to f$ pointwise and $f_1 \le f_2 \le \cdots$ or $f_1 \ge f_2 \ge \cdots$. A function f is of *Baire class* 0 if f is continuous, and of Baire class $n+1$ if it is a monotone limit of functions of Baire class n.

   Show any bounded measurable function $g : [0,1] \to \mathbb{R}$ agrees with a function f of Baire class 2 outside a set of measure zero.

   Show the same statement is false for f of Baire class 1.

5. Let X be a normal topological space. State and prove a theorem characterizing those sets $Z \subset X$ which can arise as zero sets of continuous functions $f : X \to \mathbb{R}$.

6. Let X be an infinite-dimensional Banach space and let $E = \{x \in X : 1 \le \|x\| \le 2\}$. What is the weak closure of E? What happens if X is finite-dimensional? Justify your answers.

7. Let $f : X \to Y$ be a bounded linear operator between Banach spaces. Show that either $f(X) = Y$ or $f(X)$ has first category in Y.

8. Let $f : X \to Y$ be a bounded linear operator between Banach spaces. Show that f is an isometry (a bijection which preserves the norm) iff its adjoint $f^* : Y^* \to X^*$ is an isometry. (Here $f^*(\phi) = \phi \circ f$.)

9. (a) Suppose $(na_n) \in \ell_2(\mathbb{Z})$. Show that $(a_n) \in \ell_1(\mathbb{Z})$.

   (b) Let $f \in C(S^1)$ be a continuous function with a continuous first derivative $f'(x)$. Prove that the Fourier series of f converges uniformly on S^1.

10. Prove there is a sequence $(a_n) \in c_0(\mathbb{Z})$ (meaning $a_n \to 0$ as $|n| \to \infty$) that does *not* arise as the Fourier coefficients of a function $f \in L^1(S^1)$.