MATH 101 SOLUTIONS PROBLEM SET 3

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1. Suppose A and B are finite sets, with $|A| = a$ and $|B| = b$.
   (i) Give the definition of a relation between A and B.
   (ii) How many possible relations are there between A and B?

Solution. (i) A relation R between A and B is a subset $R \subset A \times B$.

(ii) There are 2^{ab} possible relations between A and B. Recall that $A \times B$ is the set $\{(x, y) : x \in A, y \in B\}$. As there are a elements in A and b elements in B, then $A \times B$ has $a \cdot b$ elements. We know from class that for a set of size n, there are 2^n possible subsets. Therefore, there are 2^{ab} possible subsets of $A \times B$, or equivalently, there are 2^{ab} possible relations between A and B.

□

2. (i) Prove that for any function $f : X \to Y$, we have

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

for all $A, B \in \mathcal{P}(Y)$.

(ii) Show by an example that it is *not* true in general that $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \in \mathcal{P}(X)$.

(iii) What natural hypothesis on f *does* imply that $f(A \cap B) = f(A) \cap f(B)$?

Before solving a problem like this, it can be helpful to understand what A and B are, and what it means for functions with A and B as inputs to be equal. Recall that for a set Y, the power set $\mathcal{P}(Y)$ is the set of all subsets of Y, and so A and B being in $\mathcal{P}(Y)$ means that A and B are each some subset of Y. And in problem (i), the notation $f^{-1}(A \cap B)$ refers to the set of points in X that f maps to $A \cap B$, where $A \cap B$ is just some subset of Y. And notice that I said $f^{-1}(A \cap B)$ is a set of points; since we're proving $f^{-1}(A \cap B)$ is equal to another set $f^{-1}(A) \cap f^{-1}(B)$, we will have to prove that these sets are subsets of one another. Here is a picture:

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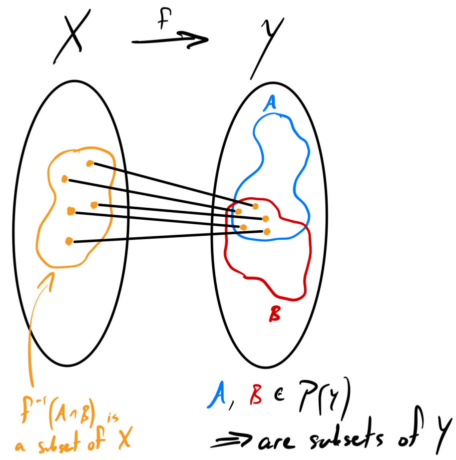
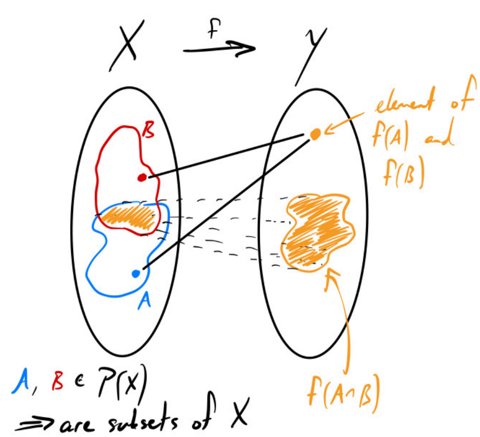


Solution. (i) To prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, we will show that these sets are subsets of one another. And to show that some set U is a subset of another set V, we show that if we take an arbitrary element x in U, then this x must also exist in V.
We first show $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$. Suppose $x \in f^{-1}(A \cap B)$. Then $f(x) \in A \cap B$, which means $f(x)$ is in A as well as in B. Thus, $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$, and so $x \in f^{-1}(A) \cap f^{-1}(B)$. Therefore $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$.
We now show $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$. Suppose $x \in f^{-1}(A) \cap f^{-1}(B)$. Then $f(x) \in A$ and $f(x) \in B$. So $f(x)$ is an element of $A \cap B$, which means $x \in f^{-1}(A \cap B)$. Therefore $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$.
As we have shown $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$, then it follows that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(ii) Before giving an example, we can again use a picture to help us understand how to construct such a function. Now, we have $A, B \in \mathcal{P}(X)$, which means A and B are subsets of X. It could be the case that an element in $A \setminus B$ and an element in $B \setminus A$ map to the same output in Y (where this output in Y is not in $f(A \cap B)$—see image below), and this would give us a situation where $f(A \cap B) \neq f(A) \cap f(B)$.



Let $X = \{0, 1, 2\}$ and let $Y = \{3, 4\}$. Assign

$$f(0) = 3$$
$$f(1) = 3$$
$$f(2) = 4$$

Let $A = \{0, 2\}$ and let $B = \{1, 2\}$. Then

$$\begin{aligned} f(A) &= f(\{0,2\}) = \{3,4\} \\ f(B) &= f(\{1,2\}) = \{3,4\} \\ f(A \cap B) &= f(\{2\}) \quad = \{4\} \end{aligned}$$

And we can see that $f(A) \cap f(B) = \{3, 4\} \neq \{4\} = f(A \cap B)$, as desired.

(iii) If f were injective, then $f(A \cap B) = f(A) \cap f(B)$.
We first show $f(A \cap B)$ is always a subset of $f(A) \cap f(B)$. Suppose $y \in f(A \cap B)$. Then $y \in f(A)$ and $y \in f(B)$, and so $y \in f(A) \cap f(B) \implies f(A \cap B) \subset f(A) \cap f(B)$.
Now we show that $f(A) \cap f(B)$ is a subset of $f(A \cap B)$ if f is injective. Suppose $y \in f(A) \cap f(B)$. Then $y \in f(A)$ and $y \in f(B)$. If f is not injective, then it could be the case that $f(x_1) = y$ and $f(x_2) = y$ for $x_1 \in A \setminus B$ and $x_2 \in B \setminus A$. In this case, $y \notin f(A \cap B)$ if some other point $x_3 \in A \cap B$ does not map to y as well. But if f *is* injective, then if we have some $x \in A$ for which $f(x) = y$, this must be the same $x \in B$ for which $f(x) = y$, and so x must be in $A \cap B$, and therefore y must be in $f(A \cap B)$. Thus, if f is injective, $f(A) \cap f(B) \subset f(A \cap B)$.
So if f is injective, $f(A \cap B) = f(A) \cap f(B)$.

□

3. Let $i, j \in \mathbb{N}$ be natural numbers. Considering i and j as sets, let

$$A = i - j = \{x \in i : x \notin j\}.$$

When is $A \in \mathbb{N}$? When A is a natural number, what number is it?

Solution. Recall that a natural number n written as a set is $\{0, 1, 2, \ldots, n-1\}$, starting with $0 = \emptyset$. We consider the set A in two cases: when $j \geq i$, and when $i > j$.
First suppose $j \geq i$. Then if some natural number n is in the set i, it must also be in the set j, and so $i \subset j$. Therefore, $A = \{\} = \emptyset = 0$. So A is a natural number in this case, and in particular it is 0.
Now suppose $i > j$. Then if some natural number n is in the set j, it must also be in the set i, and so we have $A = \{j, j+1, \ldots, i-1\}$. If $j = 0$, then $A = \{0, 1, \ldots, i-1\} = i \in \mathbb{N}$, and so A is the natural number i in this case. If $0 < j < i$, then we don't have $0 \in A$, and therefore A is not a natural number in this case. □

4. Let $\mathbb{Z} = \mathbb{N} \times \mathbb{N}/\sim$, where $(a, b) \sim (c, d)$ if $a + d = b + c$. Noting that $(a - b)(c - d) = (ac + bd) - (bc + ad)$ we define

$$(a, b) * (c, d) = (ac + bd, ad + bc).$$

Prove that if $(a', b') \sim (a, b)$, then $(a', b') * (c, d) \sim (a, b) * (c, d)$. (This shows that multiplication is well-defined on $\mathbb{Z}$.

Note: if this construction of addition and multiplication on $\mathbb{Z}$ is unfamiliar, you can review page 16 of Curt's notes here.

Solution. Assume $(a', b') \sim (a, b)$. Then by our definition of $\sim$, we have $a' + b = b' + a$. Let this sum be n, and let c, d be some two numbers in $\mathbb{N}$. Then we can write

$$cn + dn = cn + dn,$$

and therefore

$$c(a' + b) + d(b' + a) = c(b' + a) + d(a' + b).$$

We can distribute[1] c and d to get

$$(a'c + bc) + (b'd + ad) = (b'c + ac) + (a'd + bd),$$

which by commutativity and associativity of natural numbers under addition is

$$(a'c + b'd) + (ad + bc) = (a'd + b'c) + (ac + bd).$$

But here we have something of the form $a + d = b + c$, and so we can write our equation in the form $(a, b) \sim (c, d)$ as

$$(a'c + b'd, a'd + b'c) \sim (ac + bd, ad + bc).$$

But by our definition of $*$ above, we can rewrite this equation as

$$(a', b') * (c, d) = (a, b) * (c, d),$$

and therefore $(a', b') \sim (a, b) \implies (a', b') * (c, d) = (a, b) * (c, d)$. ☐

5. Prove that if A and B are finite sets, then $A \cup B$ is finite. (Note: A and B may overlap.)

Solution. Let A and B be finite sets. Then there exist natural numbers a and b such that $|A| = a$ and $|B| = b$, which is to say there are a elements in A and b elements in B. Suppose toward a contradiction that $A \cup B$ is infinite. Then there exists a bijective function $f : \mathbb{N} \to A \cup B$. To construct this function, map a elements in $\mathbb{N}$ to elements in A, and then map b more elements in $\mathbb{N}$ to B. Given that we have only mapped $a + b$ elements of $\mathbb{N}$ and that there exists a natural number greater than $a + b$, there must an unmapped element n in $\mathbb{N}$. But we cannot map n to any element in A (else we will fail to be injective), and we cannot map n to any element in B (for the same reason). So n cannot map to anywhere in $A \cup B$, and so we cannot take all natural numbers to $A \cup B$

[1]We will learn more about *why* we can do this operation when we get to abstract algebra—just keep the word *ring* in your mind for now :)

unless we have at least two inputs mapping to the same output. Therefore, there does not exist a bijective function from $\mathbb{N}$ to $A \cup B$—contradiction. It must have been the case that $A \cup B$ was finite. □

6. Prove that if $F \subset \mathbb{N}$ is a finite set, then $|\mathbb{N} - F| = |\mathbb{N}|$.

Solution. We will use two facts from class: (1) for sets A and B, if there exists an injective function $f : A \to B$, then $|A| \leq |B|$, and (2) if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. So we want to show that there exists an injective function from $\mathbb{N} - F$ to $\mathbb{N}$ and another injective function from $\mathbb{N}$ to $\mathbb{N} - F$ in order to show that these sets are the same size.
First consider $g : \mathbb{N} - F \to \mathbb{N}$ defined by $g(n) = n$, which is injective because if $g(n) = g(n')$, then $n = n'$. Therefore $|\mathbb{N} - F| \leq |\mathbb{N}|$.
Now consider $h : \mathbb{N} \to \mathbb{N} - F$. Since F is finite, we know $|F| = n \in \mathbb{N}$, and so we can write $F = \{f_1, f_2, \ldots, f_n\}$ for $f_i \in F$. And since F is a subset of the natural numbers, then each f_i is a natural number. Therefore, there must be some f_j that is greater than all other f_i in F. Define $h(n) = n + f_j + 1$ so that we map 0 to the natural number greater than all natural numbers within F, map 1 to the next natural number, map 2 to the next one, and so on. The function h is injective because if $h(n) = h(n')$, then $n + f_j + 1 = n' + f_j + 1 \implies n = n'$. Therefore $|\mathbb{N}| \leq |\mathbb{N} - F|$.
Since we have shown $|\mathbb{N} - F| \leq |\mathbb{N}|$ and $|\mathbb{N}| \leq |\mathbb{N} - F|$, then $|\mathbb{N} - F| = |\mathbb{N}|$. □

7. Prove that if $|A| = |B|$ then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

Solution. Assume $|A| = |B|$. Then there exists a bijection $f : A \to B$. We want to show there exists a bijection $g : \mathcal{P}(A) \to \mathcal{P}(B)$ in order to show $|\mathcal{P}(A)| = |\mathcal{P}(B)|$. Note that g maps elements of $\mathcal{P}(A)$ to elements of $\mathcal{P}(B)$—or, equivalently, g maps subsets of A to subsets of B. Let A' be a subset of A. Define $g(A') = \{f(a) : a \in A'\}$.
We first show g is injective. Assume there are subsets $A' \subset A$ and $A'' \subset A$ such that $g(A') = g(A'')$. For injectivity, we want to show that this assumption implies that the sets A' and A'' are equal, so we prove that $A' \subset A''$ and $A'' \subset A'$. Consider any $a \in A'$. Then $f(a) \in g(A'')$ because $f(a) \in g(A')$ by definition, and $g(A') = g(A'')$ by our assumption. And since we know f is a bijection, then f has an inverse, and in particular $f^{-1}(f(a)) = a \in A''$. Therefore $A' \subset A''$. We can repeat this argument for any a in A'' to prove $A'' \subset A'$. Thus, $A' = A''$, and so g is injective.
We now show g is surjective. Consider an arbitrary element B' in $\mathcal{P}(B)$, and remember that $B' \subset B$ by definition of belonging to the power set of B. We want to show that there is some $A' \in \mathcal{P}(A)$ such that $g(A') = B'$. Since f is surjective, we know for any element $b \in B$ there is an element $a \in A$ such that $f(a) = b$, and therefore for a subset $B' \subset B$ there is a subset $A' \subset A$ such that $f(A') = B'$. And since f is bijective, we can write the inverse $A' = f^{-1}(B')$. Therefore, for an arbitrary B' we have an $A' = f^{-1}(B')$ such that $g(A') = g(f^{-1}(B)) = B'$. □

8. Let A be a set with an equivalence relation $\sim$. Show there is an 'election' function $f : A \to A$ such that for all $x, y \in A$,
   (i) $f(x) \sim x$
   (ii) if $x \sim y$ then $f(x) = f(y)$. (One can think of $f(x)$ as the 'president' of the country whose citizens are the equivalence class $[x]$.)

Solution. By the axiom of choice, for any set A there is a function $c : \mathcal{P}(A) - \{\emptyset\} \to A$ such that $c(B) \in B$ for all subsets $B \subset A$. Consider $f(x) = c([x]) \in [x]$. Recall that $[x]$ denotes the equivalence class of x.

(i) From class, we know $[x] = \{x' \in A : x' \sim x\}$ which means that the equivalence class $[x]$ is the set of all elements in A similar to x by the equivalence relation. Therefore, since we have defined $f(x) \in [x]$, then we have $f(x) \sim x$.

(ii) Assume $x \sim y$. We know from class that any two similar elements belong to the same equivalence class, i.e. $[x] \sim [y]$. Therefore, we have $f(x) = c([x]) = c([y]) = f(y)$, as desired.

□