Final 2022: Solutions
Sets, Groups and Knots

1. Show that if sets A, B and C satisfy

$$(A \cap C) \cup B = A, \quad (A \cup C) \cap B = C \quad \text{and} \quad (A \cup B) = C,$$

then $A = B = C$.

Answer. These three equations imply $B \subset A$, $C \subset B$ and $A \subset C$ respectively, so $A = B = C$.

2. (i) Prove that for all integers $k \geq 1$, the set $\mathbb{N}^k$ is countable. In other words, $|\mathbb{N}^k| = |\mathbb{N}|$.

(ii) Let $F \subset \mathcal{P}(\mathbb{N})$ be the collection of all *finite* subsets of $\mathbb{N}$. Prove that F is countable.

Answer. (i) Proof by induction on k: we know that $|\mathbb{N}^2| = |\mathbb{N}|$, and that if $|A| = |B|$ then $|A \times C| = |B \times C|$. The statement is clear for $k = 1$. For the inductive step: if we know $|\mathbb{N}^k| = |\mathbb{N}|$, then

$$|\mathbb{N}^{k+1}| = |\mathbb{N}^k \times \mathbb{N}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}^2| = |\mathbb{N}|.$$

(ii) Let $\mathcal{P}_k(A) = \{B \subset A : |B| = k\}$. There is a surjective map $\mathbb{N}^k \to \mathcal{P}_k(\mathbb{N})$, so $\mathcal{P}_k(\mathbb{N})$ is countable by (i). Since a countable union of countable sets is countable, $F = \bigcup_k \mathcal{P}_k(\mathbb{N})$ is countable.

(ii') Alternatively, define a map $f : F \to \mathbb{N}$ by $f(A) = \sum_{n \in A} 2^n$. The base 2 digits of $f(A)$ that are 1 label the elements of A, so f is injective and thus F is countable. In fact f is a bijection!

3. Let $G \cong A_4$ be the symmetry group of a tetrahedron T, and let A denote the 4 vertices of T. Then G acts on transitively on A. It also acts on the set of 16 ordered pairs of vertices, $A \times A$, by $g \cdot (a_1, a_2) = (g \cdot a_1, g \cdot a_2)$.
(i) Compute the number of orbits of G acting on $A \times A$.
(ii) Compute the number of elements in each orbit.

Answer. (i) There are two orbits: the pairs with $a_1 = a_2$, and the pairs with $a_1 \neq a_2$. This is because the stabilizer of a given vertex, a_1, acts transitively on the remaining 3 vertices.

(ii) The orders are 4 and 12.

4. Let $A_5 \subset S_5$ be the alternating group on 5 symbols.
(i) Find a pair of elements $\tau_1, \tau_2 \in A_5$ that generate a subgroup isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.
(ii) Give examples of elements $\sigma_3, \sigma_5 \in A_5$ of order 3 and 5 respectively.
(iii) Show that the 4 elements τ_1, τ_2, σ_3 and σ_5 generate A_5.
(iv - Bonus) Show that the 2 elements σ_3 and σ_5 also generate A_5.

Answers. (i) $\{\tau_1, \tau_2\} = \{(12)(34), (13)(24)\}$ will work, since $\tau_1\tau_2 = (14)(23)$ also has order 2.

(ii) $\sigma_3 = (123)$ and $\sigma_5 = (12345)$.

(iii) Let G be the group generated by τ_1, τ_2, σ_3 and σ_5. Then $|G|$ is divisible by 4 (the order of the subgroup generated by τ_1 and τ_2), 3 and 5 (the orders of σ_3 and σ_5). Thus G is divisible by $60 = |A_5|$, so $G = A_5$.

(iv- Bonus) Since σ_5 is the stabilizer of a face F, it suffices to show that $G = \langle \sigma_5, \sigma_3 \rangle$ acts transitively on the set of 12 faces of the dodedecahedron. This is readily verified geometrically.

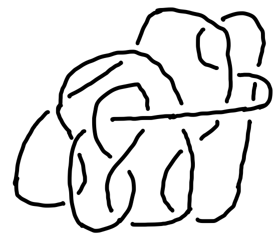
5. Let $D_{12} = \langle r, f \ : \ r^6 = f^2 = (rf)^2 = e \rangle$ denote the dihedral group of order 12. Every element of D_{12} can be expressed in the form $r^i f^j$, where $0 \le i < 6$ and $0 \le j < 2$.
(i) Find all elements of order two in D_{12}.
(ii) Show that these elements are of 3 different types, if we consider conjugate elements of D_{12} to be of the same type.
(iii) Describe these 3 types in terms of symmetries of the hexagon.

Answer. (i) The elements of order 2 are $r^i f$, $i = 0, 1, 2, 3, 4, 5$, and r^3.

(ii) The conjugacy classes are $(f, r^2 f, r^4 f)$, $(rf, r^3 rf, r^5 f)$ and (r^3).

(iii) The first conjugacy classes corresponds to flips around lines joining pairs of opposite vertices of the hexagon. The second, to flips around lines joining the midpoints of opposite sides. The last, to a rotation of the hexagon by 180 degrees.

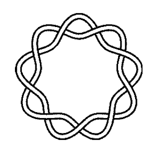
6. Write down the Jones polynomial of the knot shown below.



Answer. This knot diagram can be simplified — it is just the figure eight knot. Thus $V(K) = t^{-2} - t^{-1} + 1 - t + t^2$.

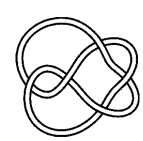
7. Prove that the fundamental group G_K of the knot $K = 9_1$ is nonabelian.

Answer. This knot diagram can be 3 colored, so there is a surjective homomorphism from G_K to S_3. Since S_3 is nonabelian, so is G_K.



9_1

8. Let K be the knot projection 5_2. Compute $w(K)$, $\langle K \rangle$, $X(K)$ and $V(K)$. You are allowed to use calculations of $\langle L \rangle$ for other knots and links L that are contained in the course notes.



5_2

Answer.

$w(K) = -5$ (all crossings are unsafe).

$\langle K\rangle = A^9 - A^5 + A - 2A^{-3} + A^{-7} - A^{-11}$

$X(K) = -A^{24} + A^{20} - A^{16} + 2A^{12} - A^8 + A^4$.

$V(K) = -t^{-6} + t^{-5} - t^{-4} + 2t^{-3} - t^{-2} + t^{-1}$.

To compute the bracket polynomial, in the diagram for 5_2, resolve the crossing at the right: we then get

$$\langle K\rangle = A\langle 4_1\rangle + A^{-1}\langle H''\rangle,$$

where H'' is the Hopf link with two unsafe crossings added by Reideimster move I. Thus $\langle H''\rangle = (-A^3)^{-2}\langle H\rangle$. To complete the computation, use the formulas $\langle 4_1\rangle = A^8 - A^4 + 1 - A^{-4} + A^8$ and $\langle H\rangle = -A^4 - A^{-4}$ given in the course notes.

9. Mark each of the following assertions True (T) or False (F).

   (a) **F.** The sets $\emptyset$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^2$ all have different cardinality. (We have $|\mathbb{R}| = |\mathbb{R}^2|$.)

   (b) **F.** You cannot construct a bijection between $\mathbb{R}$ and $2^{\mathbb{N}}$ without using the Axiom of Choice. (This can be done using binary expansions and the Schröder–Bernstein theorem.)

   (c) **F.** For any set A, $\{B : A \subset B\}$ is a set. (For example, if $A = \emptyset$ then B could be any set; but the set of all sets does not exist.)

   (d) **F.** The symmetry group of a cube is isomorphic to a subgroup of the symmetry group of a dodecahedron. (The first group is S_4, with $|S_4| = 24$, and this does not divide 60, the order of the second group.)

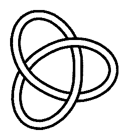
   (e) **F.** If A is a normal subgroup of B and B is a normal subgroup of C, then A is a normal subgroup of C. (For example, let a, b be (12)(34) and (13)(24) in $C = S_4$, let $A = \langle a\rangle$, and let $B = \langle a, b\rangle$. Then A is normal in $B \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, and B is normal in C, but A is not normal in C. (If it were, a would commute with every element of S_4, which it doesn't.)

   (f) **T.** Any group of order 541 is abelian. (Since 541 is prime.)

   (g) **F.** The permutations (123457) and (123) generate S_7. (These permutations are both even, so the group they generate contains none of the odd elements of S_7.)

   (h) **F.** The group $G = \langle a, b : a^2 = b^2 = e\rangle$ has order 4. (This is the infinite dihedral group; e.g. the element $r = ab$ has infinite order in G.)

(i) **F.** The Jones' polynomial $V(K)$ is invariant under Reidemeister moves II and III, but not I. (The Jones' polynomial is a knot invariant, so it is also invariant under I.)

(j) **F.**  No matter how you orient this trefoil knot diagram, its writhe is 3.
(The writhe is -3).