Symmetric power functoriality for modular forms

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Joint work with Jack Thorne
Functoriality

Here is a naive form of Langlands functoriality. Suppose we have the following inputs:

- $F$ a number field
- $G$ a split reductive algebraic group over $F$, with Langlands dual group $\hat{G}/\mathbb{C}$
- $r : \hat{G} \to \text{GL}_N$ a representation

The functoriality principle predicts a transfer $\pi \mapsto r(\pi)$ from automorphic representations of $G(\mathbb{A}_F)$ to automorphic representations of $\text{GL}_N(\mathbb{A}_F)$.

Description at unramified places:

- $\pi_v$ unramified $\leftrightarrow c_v(\pi)$ a semisimple conjugacy class in $\hat{G}(\mathbb{C})$
- $r(\pi)_v \leftrightarrow r(c_v(\pi)) \in \text{GL}_N(\mathbb{C})$.

Example

$G = \text{GL}_2$, $r = \text{Sym}^n : \hat{G} = \text{GL}_2 \to \text{GL}_{n+1}$ symmetric powers of standard representation
Symmetric power functoriality

Let $F$ be a number field and let $\pi$ be a cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$.

Conjecture (Langlands)

For each $n \geq 1$, there exists an automorphic representation $\text{Sym}^n \pi$ of $\text{GL}_{n+1}(\mathbb{A}_F)$, as predicted by the functoriality principle.

If the conjecture holds for $n$, we say that $\text{Sym}^n \pi$ is automorphic.

Description at unramified places:

- $\pi_v \leftrightarrow c_v(\pi) = \text{diag}(\alpha_v, \beta_v) \in \text{GL}_2(\mathbb{C})$
- $(\text{Sym}^n \pi)_v \leftrightarrow \text{diag}(\alpha_v^n, \alpha_v^{n-1} \beta_v, \ldots, \beta_v^n) \in \text{GL}_{n+1}(\mathbb{C})$.

In general:

- Use the local Langlands correspondence to define $(\text{Sym}^n \pi)_v$ at all places $v$.
- Take restricted tensor product to define a representation $\text{Sym}^n \pi$ of $\text{GL}_{n+1}(\mathbb{A}_F)$.
- Conjecture is that $\text{Sym}^n \pi$ is automorphic.
Symmetric power $L$-functions

Satake parameters: $\pi_v \leftrightarrow c_v(\pi) = \text{diag}(\alpha_v, \beta_v) \in \text{GL}_2(\mathbb{C})$

$$L(\pi, \text{Sym}^n, s) = (\cdots) \prod_v \prod_{\text{unram}} (1 - \alpha_v^i \beta_v^j q_v^{-s})^{-1}$$

The conjecture predicts that $L(\pi, \text{Sym}^n, s) = L(\text{Sym}^n \pi, s)$ is automorphic. In particular, if $\text{Sym}^n \pi$ is cuspidal:

**Consequences**

- $L(\pi, \text{Sym}^n, s)$ has an (entire) analytic continuation to $s \in \mathbb{C}$, and a functional equation
- $L(\pi, \text{Sym}^n, s)$ is non-vanishing on $\text{Re}(s) = 1$
- Ramanujan–Petersson: $c_v(\pi) \in U(2)$
- Sato–Tate conjecture for $\pi$: equidistribution of $c_v(\pi)$
Some previous work

**Theorem**

$\text{Sym}^n \pi$ is automorphic for:

- $n = 2$ (Shimura, Gelbart, Jacquet 1975–78)
- $n = 3, 4$ (Kim, Shahidi 2002-03).

To go further we restrict to algebraic automorphic representations. So we assume:

- $F$ is a totally real field
- $\pi$ is a cohomological cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$ (i.e. has something to do with holomorphic Hilbert modular forms of regular weight)

For each prime $p$ and isomorphism $\iota : \mathbb{C} \cong \overline{\mathbb{Q}}_p$, there is an associated Galois representation $\rho_{\pi,p} : G_F = \text{Gal}(\overline{F}/F) \to \text{GL}_2(\overline{\mathbb{Q}}_p)$ with $\rho_{\pi,p}(\text{Frob}_v) \sim \iota(c_v(\pi))$.

From this point of view, the symmetric power functoriality conjecture becomes:

**Conjecture**

For each $n \geq 1$, the $n + 1$-dimensional Galois representation $\text{Sym}^n \rho_{\pi,p}$ is automorphic: $\exists$ an aut. rep. $\text{Sym}^n \pi$ with $\text{Sym}^n \rho_{\pi,p}(\text{Frob}_v) \sim \iota(c_v(\text{Sym}^n \pi))$. 
Some previous work

$F$ totally real, $\pi$ a cohomological cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$. Suppose $\pi$ is non-CM (\implies Sym$^n\rho_{\pi,p}$ is irreducible for all $n$).

**Theorem** (Barnet-Lamb, Gee, Geraghty 2011, Clozel, Harris, Shepherd-Barron, Taylor)

*For each $n \geq 1$, there exists a finite Galois, totally real, extension $F'/F$ such that Sym$^n\rho_{\pi,p}|_{G_{F'}}$ is (cuspidal) automorphic.*

**Consequences (using Brauer’s theorem)**

- $L(\pi, \text{Sym}^n, s)$ has a *meromorphic* continuation to $s \in \mathbb{C}$, and a functional equation
- $L(\pi, \text{Sym}^n, s)$ is non-vanishing on $\text{Re}(s) = 1$
- Sato–Tate conjecture for $\pi$

Proof strategy goes back to work of Taylor in two-dimensional case: combines potential automorphy of the residual representation

$$\text{Sym}^n\overline{\rho}_{\pi,p}: G_F \to \text{GL}_{n+1}(\overline{\mathbb{F}}_p)$$

with automorphy lifting theorems.
Some previous work

What about automorphy?

**Theorem (Clozel, Thorne 2014–17)**

Sym$^n \pi$ is automorphic for $n \leq 8$.

Idea is to exploit reducibility of Sym$^n$ in characteristic $p \leq n$:

**Example**

To prove Sym$^8 \pi$ exists, they show that Sym$^8 \rho_{\pi,7}$ is automorphic. The residual representation decomposes as:

$$\text{Sym}^8 \overline{\rho}_{\pi,7} \cong (\sigma \overline{\rho}_{\pi,7} \otimes \overline{\rho}_{\pi,7}) \oplus (\det \overline{\rho}_{\pi,7})^2 \otimes \text{Sym}^4 \overline{\rho}_{\pi,7}.$$

Tensor product functoriality and Langlands’s theory of Eisenstein series implies automorphy of Sym$^8 \overline{\rho}_{\pi,7}$ (not cuspidal automorphy).

Then apply automorphy lifting theorem for residually reducible representations (Thorne, requires level raising congruence).
Main theorem

Now we specialise further, assuming $F = \mathbb{Q}$. As before, $\pi$ is a cuspidal cohomological automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$ without CM, so it is (up to twist) associated to a cuspidal Hecke eigenform, holomorphic of weight $k \geq 2$.

Theorem (N., Thorne 2019)

Suppose $\pi$ has no supercuspidal local factors. Then $\text{Sym}^n \pi$ is automorphic for all $n \geq 1$.

- Some key ingredients are provided by our joint work with Patrick Allen and the PhD thesis of Christos Anastassiades.
- The theorem includes the case of all level 1 cuspidal Hecke eigenforms, in particular $\Delta$.
- Automorphy of $\text{Sym}^n \pi$ can be used to control the error term in the Sato–Tate conjecture for $\pi$ (Kumar Murty, Bucur, Kedlaya, Rouse, Thorner).
Strategy of proof

A Seed points
For every $n$, there exists a level 1 cuspidal Hecke eigenform $f$ with $\text{Sym}^n f$ automorphic.

B Propagation in $p$-adic families
Vague version: suppose $f$ and $g$ are two cuspidal Hecke eigenforms of level $N$, which lie in an irreducible $p$-adic family of modular forms (for some prime $p \nmid N$). Then $\text{Sym}^n f$ is automorphic if and only if $\text{Sym}^n g$ is automorphic.

C ‘Ping pong’
Use A and B to show that $\text{Sym}^n f$ automorphic for every $f$ of level 1.

D Killing ramification
Reduce general case to level 1 (uses B again).
For every \( n \), there exists a level 1 cuspidal Hecke eigenform \( f \) with \( \text{Sym}^n f \) automorphic.

- Idea is similar to Clozel–Thorne; we choose \( f \) to be congruent to a CM form modulo a prime \( p \), so \( \bar{\rho}_{f,p} \) has dihedral image and \( \text{Sym}^n \bar{\rho}_{f,p} \) decomposes into 1- and 2-dimensional representations (for any \( p, n \)).

- So \( \text{Sym}^n \bar{\rho}_{f,p} \) comes from an automorphic representation \( \Pi \) of \( \text{GL}_{n+1}(\mathbb{A}_\mathbb{Q}) \) (not cuspidal).

- An automorphy lifting theorem (Allen, N., Thorne) can be applied to show that \( \text{Sym}^n \rho_{f,p} \) is automorphic, if we can find a suitable level raising congruence (mod \( p \)) between \( \Pi \) and a cuspidal aut. rep. \( \Pi' \).
For every $n$, there exists a level 1 cuspidal Hecke eigenform $f$ with $\text{Sym}^n f$ automorphic.

- **Level raising I**: a theorem of Anastassiades allows us to conclude when $n = 2^r - 1$. In fact, it allows us to reduce to the case where $n + 1$ is odd.

- **Level raising II**: to handle the odd-dimensional case the argument is quite involved. We use the endoscopic classification of automorphic representations of unitary groups (Arthur, Mok, Kaletha, Minguez, Shin, White, Moeglin, Waldspurger).

- Input for level raising II comes from the modular representation theory of a finite unitary group $U(3, q) = U_3(\mathbb{F}_{q^2}/\mathbb{F}_q)$.

- If $p | (q^2 - q + 1)$ there is a congruence mod $p$ between a unipotent cuspidal and ‘stable’ cuspidal representation of $U(3, q)$. 

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Propagation in $p$-adic families

Fix a level $N$ and a prime $p \nmid N$.

- $\mathcal{E}_p(N)$ is the Coleman–Mazur eigencurve: a $p$-adic analytic space, equidimensional of dimension 1.
- It contains a (Zariski) dense set of points corresponding to pairs $(f, \alpha_p)$, $f \in S_k(\Gamma_1(N))$ a Hecke eigenform, $\alpha_p$ one of the entries in the Satake parameter $c_p(f) = \text{diag}(\alpha_p, \beta_p)$.
- More generally, if $f \in S_k(\Gamma_1(Np^r))$ is a Hecke eigenform, for some $r \geq 1$, with non-zero $U_p$ eigenvalue $\alpha_p$, there is a corresponding point of $\mathcal{E}_p(N)$.
- Comes with a map $w : \mathcal{E}_p(N) \to \mathcal{W}$ to a weight space
- $\mathcal{W}$ is a union of one dimensional open discs, $w(f, \alpha_p)$ is determined by weight and $p$-part of character of $f$.

Suppose $(f, \alpha_p)$ and $(f', \alpha'_p)$ lie in a common irreducible component of $\mathcal{E}_p(N)$ (+ technical hypotheses), then automorphy of $\text{Sym}^n f$ implies automorphy of $\text{Sym}^n f'$.
Suppose \((f, \alpha_p)\) and \((f', \alpha'_p)\) lie in a common irreducible component of \(E_p(N)\) (+ technical hypotheses), then automorphy of \(\text{Sym}^n f\) implies automorphy of \(\text{Sym}^n f'\).

\[
\begin{array}{ccc}
E_p(N) & \rightarrow & E_{GL_{n+1},p} \\
\downarrow i & & \downarrow j \\
X_{2,p}^{fs} & \xrightarrow{\text{Sym}^n} & X_{n+1,p}^{fs}
\end{array}
\]

\(E_{GL_{n+1},p}\): generalisation of \(E_p(N)\), dense set of points corresponding to (polarized) cohomological automorphic representations of \(GL_{n+1}(\mathbb{A}_\mathbb{Q})\).

\(X_{m,p}^{fs}\): space of ‘finite slope’, \(m\)-dimensional, (polarized) \(p\)-adic representations of \(G_{\mathbb{Q}}\). Galois theoretic counterpart to \(E_{GL_m,p}\). Constructed by Kisin for \(m = 2\). Higher rank case is work of Bellaïche, Chenevier, Hellmann, Kedlaya, Pottharst, Xiao, R. Liu . . . .
Suppose \((f, \alpha_p)\) and \((f', \alpha'_p)\) lie in a common irreducible component of \(\mathcal{E}_p(N)\) (+ technical hypotheses), then automorphy of \(\text{Sym}^n f\) implies automorphy of \(\text{Sym}^n f'\).

Start with \((f, \alpha_p) = x_0 \in \mathcal{E}_p(N)\). Automorphy of \(\text{Sym}^n f\) implies that \(\text{Sym}^n (i(x_0)) \in \text{im}(j)\).

Goal: show that the Zariski closed subspace \(\mathcal{E}_p(N)^{n-Aut} = (\text{Sym}^n \circ i)^{-1}(\text{im}(j)) \subset \mathcal{E}_p(N)\) contains an open neighbourhood of \(x_0\).

Suffices to show that \(j\) is a local isomorphism at \(\text{Sym}^n (i(x_0))\).
Following Kisin, Bellaïche–Chenevier, we show that \( j : E_{GL_{n+1},p} \hookrightarrow X_{n+1,p}^{fs} \) is a local isomorphism at \( \text{Sym}^n(i(x_0)) \) using the vanishing of an adjoint Bloch–Kato Selmer group \( \text{H}^1_f(\mathbb{Q}, \text{ad}((\text{Sym}^n(\rho_{f,p})))) \).

**Theorem (N., Thorne)**

*For any non-CM cuspidal Hecke eigenform \( f \) of weight \( \geq 2 \), any prime \( p \), any \( n \geq 1 \), \( \text{H}^1_f(\mathbb{Q}, \text{ad}((\text{Sym}^n(\rho_{f,p})))) = 0 \).*

- Results like this have been obtained by Wiles, Weston, Kisin \( (n = 1) \), Allen, Breuil–Hellmann–Schraen.
- The novelty here is that there is no assumption on the image of the residual representation \( \bar{\rho}_{f,p} \). Like all these authors (except Weston!), we use the Taylor–Wiles method.
- A new idea in recent work of Lue Pan is crucial for removing assumptions on the residual image.
We fix an $n$. We want to show that all level 1 cuspidal Hecke eigenforms $f$ have $\text{Sym}^n f$ automorphic.

Picking a prime $p$ and a $U_p$-eigenvalue, we have a starting ‘seed point’ $(f_0, \alpha_p)$ in $\mathcal{E}_p(1)$ with $\text{Sym}^n f_0$ automorphic, and we can now move around in two different ways, preserving automorphy of $\text{Sym}^n$:

- Move along an irreducible component in $\mathcal{E}_p(1)$.
- Switch Satake parameter: move from $(f_0, \alpha_p)$ to $(f_0, \beta_p)$.

To show that we can reach all level 1 eigenforms this way, we need some information about the geometry of $\mathcal{E}_p(1)$.

To this end, we specialise to $p = 2$ and use a beautiful theorem of Buzzard and Kilford.
Buzzard and Kilford explicitly describe the eigencurve $\mathcal{E}_2(1)$ over an annulus $\mathcal{W}^b$ in the weight space $\mathcal{W}$.

Each component $X_i$ maps isomorphically to $\mathcal{W}^b$.

On $X_i$, the slope $v_p(\alpha_p)$ of a point is $iw_p(w)$.

Starting from a point $(f, \alpha)$ of $\mathcal{E}_2(1)$, find a sequence of moves which lands us in $X_1$. 

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Thank you!