Modularity for self-products of elliptic curves over function fields

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The Tate conjecture

This project is a meditation on Tate’s conjecture and its consequences, so let’s start there.

Let $k$ be a finitely generated field, and let $X$ be a smooth projective variety over $k$.

**Conjecture**

The cycle class map

$$A^i(X) \otimes \mathbb{Q}_\ell \to H^{2i}(X_{k^s}, \mathbb{Q}_\ell)(i)^{\text{Gal}(k^s/k)}$$

is surjective.

Here $A^i(X)$ is the group of algebraic cycles of $X$ of codimension $i$. 
The Tate conjecture and BSD

**Theorem (Tate, Milne)**

Assume the Tate conjecture. Let $E$ be an elliptic curve over a function field $K$. Then the (refined) BSD conjecture holds for $E$.

The elliptic curve $E/K$ corresponds to an elliptic fibration $\mathcal{E} \to X$. The Tate conjecture gets applied to $\mathcal{E}$. 

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Let $X$ and $Y$ be two smooth projective varieties over $k$ of the same dimension $d$. Suppose there exists a Galois-equivariant map

$$f : H^d(X_{k^s}, \mathbb{Q}_\ell) \rightarrow H^d(Y_{k^s}, \mathbb{Q}_\ell) .$$

This $f$ produces a Tate class in $H^d(X \times Y)$. The Tate conjecture for $X \times Y$ implies the existence of a correspondence $Z \rightarrow X \times Y$ which induces $f$. We write such a correspondence as $X \dashrightarrow Y$. 
Let $E/\mathbb{Q}$ be an elliptic curve. What does it mean for $E$ to be modular?

(Analytic modularity) Wiles et al.: There exists a cuspidal modular form $f$ for which $L(E, s) = L(f, s)$.

Let $N$ be the conductor of $E$ (and of $f$). Analytic modularity implies that $H^1(E_{\mathbb{Q}}, \mathbb{Q}_\ell)$ appears as a direct summand of $H^1(X_0(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$.

The Tate conjecture (a theorem of Faltings) implies the existence of a uniformization $X_0(N) \to E$ (geometric modularity).
The Tate conjecture and “modularity” for $E/\mathbb{Q}$

For an elliptic curve $E/\mathbb{Q}$, analytic modularity + Tate conjecture for divisors $\implies$ geometric modularity, i.e. existence of $f : X_0(N) \to E$.

This is the starting point for attacks on BSD. Let $K/\mathbb{Q}$ be an imaginary quadratic extension (satisfying the Heegner hypothesis relative to $E$, so that $L(E/K, 1) = 0$); then there exists a Heegner point $\xi_K \in \text{Jac} X_0(N)(K)$. Can push this into $E$ to obtain a point $P_K \in E(K)$.

**Theorem (Gross-Zagier)**

$L'(E/K, 1) = h(P_K)$ up to an explicit nonzero constant. Thus $E/K$ has analytic rank 1 if and only if $P_K$ is non-torsion.

**Theorem (Kolyvagin)**

If $P_K$ is non-torsion then $E/K$ has Mordell-Weil rank 1 (and Sha is finite). Thus analytic rank 1 $\implies$ Mordell-Weil rank 1.
The Tate conjecture and “modularity” for $E/Q$

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Our luck runs out if $L'(E/K, 1) = 0$, though; then $P_K$ is torsion, and we currently have no “modular” source of points in $E(K)$, which we expect should have rank $\geq 2$. 
$X/F_q$ be a smooth geometrically connected projective curve with function field $K$. What does it mean for an elliptic curve $E/K$ to be modular?

Analytic modularity: There exists an automorphic representation $\pi$ of $\text{GL}_2/K$ such that $L_v(\pi, s) = L_v(E, s)$ for all closed points $v$ of $X$. (In fact this is a polynomial in $q^{-s}$, if $j(E) \not\in F_q$.)

Geometric modularity: Choose a place $\infty$ at which $E$ has split multiplicative reduction, and let $A = H^0(X \setminus \{\infty\}, \mathcal{O}_X)$.

The analogue of the modular curve $X_0(N)$ is the Drinfeld modular curve $X_0^\infty(N)$, the moduli space of rank two Drinfeld $A$-modules with level structure. Here $N \subset X$ is an effective divisor.
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If $\mathcal{C}_\infty$ is the completion of an algebraic closure of $K_\infty$, then we have an isomorphism of rigid spaces $X_0^\infty(N)^{\text{an}}_{\mathcal{C}_\infty} \cong \mathcal{H}/\Gamma_0(N)$, where $\mathcal{H} = \mathbf{P}^{1,\text{an}}_{\mathcal{C}_\infty} \setminus \mathbf{P}^1(K_\infty)$ is Drinfeld’s upper half-plane.

Theorem (Drinfeld)

$H^1(X_0^\infty(N)_{K^s}, \mathbb{Q}_\ell)$ contains $H^1(E_{K^s}, \mathbb{Q}_\ell)$ as a representation of $\text{Gal}(K^s/K)$. 
Therefore by the Tate conjecture for divisors on abelian varieties (Zarhin):

**Theorem (Geometric modularity for $E/K$)**

There exists a uniformization $X_0^\infty(N) \to E$ defined over $K$.

Once again, there are “Drinfeld-Heegner points” $\xi_{K'}$ for a quadratic extension $K'/K$, which can be pushed into $E$ to obtain points $y_{K'}$.

**Theorem (Brown, Ulmer, Yun-Zhang)**

$L'(E/K', 1) = \mathrm{ht}(y_{K'})$ up to an explicit nonzero constant. Therefore if $E/K'$ has analytic rank 1, it has Mordell-Weil rank 1. (Tate had already observed that $\mathrm{rk}_{\mathrm{an}}(E) \geq \mathrm{rk}_{\mathrm{MW}}(E)$, so there is no need for a Kolyvagin-type theorem.)

If $L'(E/K', 1) = 0$, then we expect $\mathrm{rk}_{\mathrm{MW}}(E/K') \geq 3$, but the uniformization seems to be of no help constructing points of $E(K')$. 
Shtukas and their moduli spaces

In the function field setting, there exists a notion of shtukas with multiple legs, which currently does not exist over number fields. Recall our curve $X/F_q$.

**Definition**

Let $S/F_q$ be a scheme, and let $P, Q: S \to X$. An Drinfeld $X$-shtuka over $S$ is a pair $(\mathcal{F}, \phi)$, where:

- $\mathcal{F}$ is a vector bundle over $X \times_{F_q} S$
- $\phi: (\text{id} \times \text{Fr}_{S})*\mathcal{F} \dashrightarrow \mathcal{F}$ is a rational map, which is an isomorphism away from the graphs $\Gamma_P, \Gamma_Q \subset X \times_{F_q} S$.

We require that $\phi$ have a “simple pole” at $P$ and a “simple zero” at $Q$. These are the legs of the shtuka.

In this talk, our vector bundles will have rank 2.
Shtukas and their moduli spaces

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Let $\text{Sht}^2$ be the moduli stack of Drinfeld $X$-shtukas ($2 = \text{number of legs}$). This is a Deligne-Mumford stack. The projection $\text{Sht}^2 \to X \times X$ (sending a shtuka to its pair of legs) is relative dimension $2$.

It is also possible to add level structures, e.g. $\text{Sht}^2_0(N)$ for an effective divisor $N \subset X$ (this means $\Gamma_0(N)$-level structure).
Now let $E/K$ be a non-isotrivial elliptic curve of conductor $N$. Let $E \to X$ be the corresponding elliptic surface. Then $E \times_{\mathbb{F}_q} E \to X \times_{\mathbb{F}_q} X$ is a (relative) surface.

We also have the surface $\text{Sht}^2_0(N) \to X \times_{\mathbb{F}_q} X$.

**Expectation**

There exists a nontrivial algebraic correspondence $\text{Sht}^2_0(N) \dashrightarrow E \times_{\mathbb{F}_q} E$.

By Drinfeld, the cohomology of $H^1(E) \otimes H^1(E) \subset H^2(E \times E)$ appears in $H^2(\text{Sht}^2_0(N))$; by the Tate conjecture there should exist an algebraic correspondence inducing this.
There exists a nontrivial algebraic correspondence $\text{Sht}_0^2(N) \to \mathcal{E} \times_{\mathcal{F}_q} \mathcal{E}$.

We might call such an $E$ “2-modular”.

The big questions are then:

1. Can we find examples of $E/K$ which are 2-modular?
2. If $E$ is 2-modular, can we use the uniformization by $\text{Sht}_0^2(N)$ to solve BSD for $E$?
Equations for spaces of shtukas

It’s hard to write down an equation for $Sht_0^2(N)$, since it’s not of finite type. But it’s not hard to write down equations for truncations of $Sht_0^2(N)$, where you impose some condition on the vector bundle $F$.

Let $Sht_0^{2,\text{triv}}(N) \subset Sht_0^2(N)$ denote the open subspace where $F$ is trivial pointwise on $S$. Then $Sht_0^{2,\text{triv}}(N)$ is a dense open subset of a connected component of $Sht_0^2(N)$.

**Theorem (de Frutos Fernández)**

Let $X = \mathbb{P}^1_{F_q}$. Then $Sht_0^{2,\text{triv}}(N)$ is...

- a rational surface, if $\deg N = 3$.
- an elliptic surface, if $\deg N = 4$.
- a K3 elliptic surface, if $N = (0) + 2(1) + (\infty)$ or $N = 3(0) + (\infty)$ and $q = 2$. 
An example: $X = \mathbb{P}^1_{\mathbb{F}_2}, \ N = (0) + 2(1) + (\infty)$.

Let’s look at the case $X = \mathbb{P}^1_{\mathbb{F}_2}, \ N = (0) + 2(1) + (\infty)$. There is a unique cuspidal automorphic form for $\text{GL}_2$ at this level, and it corresponds to an elliptic curve

$$E_t: \ y^2 + (t + 1)xy = x^3.$$

Meanwhile, $\text{Sht}^{2,\text{triv}}(N)$ is a K3 elliptic surface of rank 18, defined over $\mathbb{F}_2(P, Q)$, with equation

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2,$$

where

$$a_1(t) = (P + 1)(Q + 1)t$$
$$a_2(t) = (P + 1)(Q + 1)t(t + P)(t + Q)$$
$$a_3(t) = (P + 1)(Q + 1)t(t + P)(t + Q)(t + 1)(t + PQ)$$
An example: $X = \mathbb{P}^1_{\mathbb{F}_2}$, $N = (0) + 2(1) + (\infty)$. 

Elkies observed that $(0, 0)$ is a 6-torsion section of $\text{Sht}^2_{0, \text{triv}}(N) \to \mathbb{P}^1_t$, and that in fact $\text{Sht}^2_{0, \text{triv}}(N)$ is the universal K3 elliptic surface with 6-torsion section.

**Theorem (Elkies)**

*Working over $\mathbb{F}_2(P, Q)$, there exists a finite-to-one map from $\text{Sht}^2_{0, \text{triv}}(N)$ onto the Kummer surface $\text{Km}(E_P \times E_Q)$.***

Recall that $\text{Km}(E_P \times E_Q)$ is the desingularization of $(E_P \times E_Q)/[-1]$. It is a K3 elliptic surface of rank 18. The cartesian diagram

$$
\begin{array}{ccc}
Z & \rightarrow & E_P \times E_Q \\
\downarrow & & \downarrow \\
\text{Sht}^2_0(N) & \rightarrow & \text{Km}(E_P \times E_Q)
\end{array}
$$

now shows that $E_t$ is 2-modular!
Heegner-Drinfeld points

Let $E/K$ be an elliptic curve over a function field, and assume that $E$ is 2-modular, so that we have a uniformization $f : \text{Sht}^2_0(N) \to \mathcal{E} \times \mathcal{E}$ over $X \times X$.

Let $K'/K$ be a quadratic extension satisfying the Heegner hypothesis. This time $L(E/K', s)$ is even. Then there exists a Heegner-Drinfeld cycle $\xi_{K'} \in A^2(\text{Sht}^2_0(N)_{X \times X'})$; this is essentially the locus of shtukas with “CM by $K'$”. Let $x_{K'} = f(\xi_{K'})$, so that $x_{K'} \in A^2(\mathcal{E}' \times \mathcal{E}')$, where $\mathcal{E}' = \mathcal{E} \times_X X'$.

Theorem (Yun-Zhang)

We have $L^{(2)}(E/K', 1) = h(x_{K'})$ up to an explicit nonzero constant.

This is true regardless of whether $L(E/K', 1)$ is 0 or not!! There is a similar formula for higher derivatives.
Theorem (Yun-Zhang)

For the Heegner-Drinfeld cycle $x_{K'} \in A^2(\mathcal{E}' \times \mathcal{E}')$, we have $L^{(2)}(E/K', 1) = h(x_{K'})$ up to an explicit nonzero constant.

Let’s suppose $\text{rk}_{\text{an}}(E/K') = 2$. The theorem says that $x_{K'} \in A^2(\mathcal{E}' \times \mathcal{E}')$ is nonzero. On the other hand, BSD would have us believe that there exist independent sections $R_1, R_2 \in X' \to \mathcal{E}'$ spanning the Mordell-Weil group, and that $L^{(2)}(E/K', 1)$ should relate to the regulator $\det \langle R_i, R_j \rangle$.

This suggests that, up to a constant:

$$x_{K'} = R_1 \otimes R_2 - R_2 \otimes R_1 \in A^1(\mathcal{E}') \otimes A^1(\mathcal{E}') \subset A^2(\mathcal{E}' \times \mathcal{E}').$$

Yun-Zhang already imply that $x_{K'}$ is “alternating in the two legs”, but this is not enough to imply that $x_{K'}$ belongs to $A^1(\mathcal{E}') \otimes A^1(\mathcal{E}')$.

$E(K')$ is a quotient of $A^1(\mathcal{E}')$, so this would be a way of constructing points of Mordell-Weil.
There is a feature of this story we have not yet leveraged. The surface $Sht^2_0(N) \to X \times X$ has a *partial Frobenius structure*. This means an endomorphism $\Phi_1$ of $Sht^2_0(N)$ making the diagram commute:

\[
\begin{array}{ccc}
Sht^2_0(N) & \xrightarrow{\Phi_1} & Sht^2_0(N) \\
\downarrow & & \downarrow \\
X \times X & \xrightarrow{Fr_X \times id} & X \times X.
\end{array}
\]

Similarly for $\Phi_2$, and $\Phi_1 \Phi_2 = \Phi_2 \Phi_1$ equals absolute Frobenius.

The product $\mathcal{E} \times \mathcal{E}$ has an obvious Frobenius structure, namely $\Phi_1 = Fr_\mathcal{E} \times id$ and $\Phi_2 = id \times Fr_\mathcal{E}$.
Equivariance under Partial Frobenius

Under the Tate conjecture, there exists a correspondence

$$f : \text{Sht}^2_0(N) \rightarrow \mathcal{E} \times \mathcal{E}$$

over $X \times X$. Both source and target have partial Frobenius structure; a stronger version of Tate’s conjecture would predict that we could choose $f$ to be equivariant under partial Frobenius.

This is a stronger condition on $E$, call it “2-modular + EPF”.

**Theorem**

Assume that $E/K$ is 2-modular + EPF. If $\text{rk}_{\text{an}}(E/K') = 2$, then $\text{rk}_{\text{MW}}(E/K') = 2$.

You can formulate this with 2 replaced by anything.
Caveats

Theorem

Assume that $E/K$ is 2-modular + EPF. If $\text{rk}_{\text{an}}(E/K') = 2$, then $\text{rk}_{\text{MW}}(E/K') = 2$.

1. Concerning the two elliptic curves (with $q = 2$) which are known to be 2-modular: I don’t know if they are 2-modular + EPF.

2. We are assuming something stronger than Tate’s conjecture, to imply something (BSD) that Tate’s conjecture already implies.
The use of Drinfeld’s lemma

Suppose $E/K'$ has rank 2. Consider our Heegner-Drinfeld cycle $y_{K'} = f_*(\xi_{K'})$. It is a formal sum of irreducible subschemes $Z \subset \mathcal{E} \times \mathcal{E}$, where the projection $Z \to X \times X$ is generically étale. The trouble was, we could not factor this through $Z \to R_1 \times R_2$, where $R_1, R_2 \subset \mathcal{E}$ are two curves.

But now assume that the uniformization $f$ is EPF. Then each $Z$ has a partial Frobenius structure, compatible with $\mathcal{E} \times \mathcal{E}$.

*Drinfeld’s lemma* says that $Z \cong (\tilde{X}_1 \times \tilde{X}_2)/H$, where $\tilde{X}_i \to X$ are Galois branched covers, and $H \subset \text{Gal}(\tilde{X}_1/X) \times \text{Gal}(\tilde{X}_2/X)$. (PFE is what you need to get $\pi_1(X \times X) \cong \pi_1(X) \times \pi_1(X)$ to work.)

Our cycle $Z$ is the image of a map $\tilde{X}_1 \times \tilde{X}_2 \to \mathcal{E} \times \mathcal{E}$, which is equivariant for the canonical PF structures.
We have a map \( f = (f_1, f_2) : \tilde{X}_1 \times \tilde{X}_2 \to \mathcal{E}' \times \mathcal{E}' \) which commutes with partial Frobenii. Applied to \( f_1 \), this means that
\[
\tilde{X}_1 \times \tilde{X}_2 \xrightarrow{f_1} \mathcal{E}'
\]
comutes. This implies (use here that \( \tilde{X}_2 \) is connected) that \( f_1 : \tilde{X}_1 \times \tilde{X}_2 \to \mathcal{E}' \) factors through a map \( \tilde{X}_1 \to \mathcal{E}' \), say with image \( Z_1 \subset \mathcal{E}' \).

Similarly, \( f_2 \) has image \( Z_2 \subset \mathcal{E}' \), and \( Z = Z_1 \times Z_2 \).
This argument shows that \([x_{K'}]\) lies in \( E(K') \otimes E(K')\). Now suppose \( \text{rk}_{an}(E/K') = 2 \), so that \([x_{K'}]\) is nontrivial (Yun-Zhang). Yun-Zhang’s arguments show that \([x_{K'}]\) is antisymmetric, so that it lies in \( \wedge^2 E(K') \). So \( \text{rk}_{MW}(E/K') \geq 2 \), and thus \( = 2 \).
Theorem

Assume that $E/K$ is 2-modular + EPF. If $\text{rk}_{\text{an}}(E/K') = 2$, then $\text{rk}_{\text{MW}}(E/K') = 2$.

Moral: assume a strong (PFE) version of the Tate conjecture (and in particular assume BSD). Let $E/K'$ have rank $r$. Then the Heegner-Drinfeld cycles coming from $\text{Sht}_{0}^{r}(N)$ ($r$-legged shtukas) span the one-dimensional space $\wedge^{r}E(K')$.

Thank you for listening!