REVIEW OF MIXED HODGE STRUCTURES
ZIJIAN YAO

Abstract. We discuss examples of mixed Hodge structures in this talk, which will be further generalized to mixed Hodge modules. Preliminary draft

Contents

1. The weight filtration for open varieties 1
2. Mixed Hodge structures on open varieties 6
3. Degenerating/limit Hodge structures 8
References 8

1. The weight filtration for open varieties

Let $X$ be a smooth proper variety over $\mathbb{C}$, and let $D = \bigcup D_i$ be a strict normal crossing divisor. Let $\overline{X}$ be the log scheme obtained from $X$ with the divisorial log structure coming from $D$. As a log scheme, $\overline{X}$ is log smooth over $\mathbb{C}$ (equipped with the standard (pre-)log structure $\mathbb{N} \to \mathbb{C}$). Let $U = X \setminus D$ be the complete of $D$, which as a subscheme of $\overline{X}$ is equipped with the trivial log structure. What is relevant for us is the logarithmic de Rham complex $\omega^\bullet_{\overline{X}} = (\mathcal{O}_{\overline{X}} \to \omega^1_{\overline{X}} \to \omega^2_{\overline{X}} \to \cdots)$. Here $\omega^1_{\overline{X}}$ is generated by $\Omega^1_X$ and \( \frac{\text{dlog} x_i}{x_i} \) where the normal crossing divisor $D$ is (locally) given by the equation $\prod x_i = 0$.\(^1\)

Notation. We fix the following notation for the divisor $D = \bigcup_{i \in I} D_i$.

- Write $j : U \hookrightarrow X$ (resp. $i : D \hookrightarrow X$) for the open (resp. closed) immersion of schemes.
- For each subset $K \subset I$, we let $D_K = \cap_{i \in K} D_i$ denote the intersection the branches $D_i$ indexed by $K$ — it is a closed subscheme of $X$ of codimension $k = |K|$. Write $i_K : D_K \hookrightarrow X$ for the closed immersion.
- For each $k \leq |I|$, define $D^{(k)} = \bigcup_{|K| = k} D_K$ to be the disjoint union of $D_K$, where $K$ has size $k$. Let $i^{(k)} : D^{(k)} \to X$ be the corresponding map.
- As in the previous lecture, $X^\infty$ denotes the $C^\infty$ (= smooth real)-manifold, with $\mathcal{O}_X^\infty$ the sheaf of smooth $\mathbb{R}$-valued functions. Let $\mathcal{A}^k$ denote the complexified $C^\infty$ k-forms,

\(^1\) In fact we only care about the differential forms on $\overline{X}$. From this point of view the knowledge of log schemes is not at all necessary.

\(^2\) The log differential $\omega^1$ is sometimes denoted as $\Omega^1(\log D)$ in the literature.
namely $A^k = \Omega^k_{X,\infty} \otimes_{\mathbb{R}} \mathbb{C}$.

Write $A^k := \Gamma(X, \mathcal{A}^k)$ for the space of global $\mathbb{C}$-valued smooth $k$-forms.

Let $\mathcal{A}_U$ be the restriction of $\mathcal{A}$ to $U$, then there is a quasi-isomorphism

$$Rj_* \mathcal{C} \xrightarrow{\sim} Rj_* \mathcal{A}_U^* \cong j_* \mathcal{A}_U^* \cong \omega_{X}^*,$$

which means that the (singular) cohomology of the open variety $U$ can be computed by

$$H^k(U, \mathcal{C}) \cong H^k(X, \omega_{X}^*).$$

The Hodge Filtration. On the log de Rham complex $\omega_{X}^*$ we have the usual Hodge filtration $\text{Fil}^\bullet$, namely the decreasing filtration given by

$$\text{Fil}^p \omega^* = \left(0 \to \cdots \to 0 \to \omega_X^p \to \omega_X^{p+1} \to \cdots \right).$$

This gives rise to the usual Hodge to de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \omega_X^p) \Rightarrow H^{p+q}(X, \omega_{X}^*).$$

As in the smooth case (see previous lecture), this spectral sequence degenerates at the first page. In [1], Deligne proves this by proving something stronger about complexes with two filtrations.

Weight filtration. In our setup, $\omega_{X}^*$ comes with an even more interesting filtration — namely the weight filtration $W^\bullet$, which is given as follows:

(i). For $m < 0$, define $W_m \omega_X^p = 0$.

(ii). For $m \geq 0$, let

$$W_m \omega_X^p = \begin{cases} \omega_X^p & \text{if } p \leq m \\ \omega_X^m \wedge \Omega^{p-m} & \text{if } p > m \end{cases}$$

In other words, the $m$-filtrant of the weight filtration is given by differential forms with at most $m$ log poles along the branches in $D$. In other words, we have

$$W_m \omega^* = \left(\mathcal{O}_X \to \omega^1 \to \cdots \to \omega^m \to \omega^m \wedge \Omega^1 \to \omega^m \wedge \Omega^2 \to \cdots \right)$$

$$W_{m+1} \omega^* = \left(\mathcal{O}_X \to \omega^1 \to \cdots \to \omega^m \to \omega^m \wedge \omega^{m+1} \wedge \Omega^1 \to \omega^{m+1} \wedge \Omega^2 \to \cdots \right)$$

Note that $W^\bullet$ is an increasing filtration. For convenience in the computation of spectral sequences (associated to a decreasing filtration), we reindex to define

$$W^k \omega^* := W_{-k} \omega^*$$

Now the associated spectral sequence becomes

$$E_1^{p,q} = H^{p+q}(\text{gr}^p \omega^*) = H^{p+q}(\text{gr}^{-p} \omega^*) \Rightarrow H^{p+q}(X, \omega_{X}^*).$$

This is our weight spectral sequence.

---

3Recall that the decomposition $\mathcal{A}^1 = \mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1}$ induces the familiar decomposition $\mathcal{A}^k = \oplus_{p+q=k} \mathcal{A}^{p,q}$, where $\mathcal{A}^{p,q}$ consists of forms of type $(p,q)$. 
The residue map. The first step to understand the weight spectral sequence is to understand the graded pieces of the weight filtration.

**Lemma 1.1.** There is an isomorphism of complexes

$$\text{Res} : \text{gr}_m \omega^\bullet = W_m \omega^\bullet / W_{m-1} \omega^\bullet \longrightarrow \iota^{(m)} \ast (\Omega^\bullet_{D(m)})$$

where $\iota^{(m)} : D(m) \to X$ is the map defined above in the Notation section. The map is given by the residue map, as the name suggests.

**Proof.** We first define the map Res as follows. For each local section (say on $V \subset X$)

$$\alpha = \sum_{K,L} \alpha_{K,L} \frac{dz_L}{z_K} \in W_m \omega^\bullet$$

where the subset $K \subset I$ in the summation has size $|K| \leq m$, we send $\alpha$ to $(\text{Res} \alpha)_{K,|K|=m} := (2\pi i)^m \sum L \alpha_{K,L} \frac{dz_L}{z_K} |_{D_K \cap V}$.

Clearly Res vanishes on $W_{m-1} \omega^\bullet$, since it only collects the terms where there are exactly $m$ log-poles.

If locally $z_i' = z_i f_i$ is another parametrization of the branch $D_i$, then we have

$$\frac{dz_i'}{z_i'} = \frac{dz_i}{z_i} + df_i$$

where $f_i$ is invertible, so the difference between $\frac{dz_i'}{z_i'}$ and $\frac{dz_i}{z_i}$ is a local section in $\Omega^1$. This shows that Res is well-defined (independent on the choice of local parametrization).

It remains to show that Res is an isomorphism — injectivity is clear so we sketch the argument for surjectivity. Suppose that for some index subset $K \subset I$ of size $m$, we have a (local) differential $\beta$ on $D_K$, namely

$$\beta = \sum_L \gamma_L d\frac{z_L}{z_K} |_{V \cap D_K} \in \iota^{(m)} \ast (\Omega^\bullet_{D_K})$$

Then as $dz_L$ is a holomorphic differential form on (some open subscheme of) $D_K$, it extends to a holomorphic form $d\frac{z_L}{z_K}$ locally on $X$. Then we just take $\tilde{\beta} = \sum_L (2\pi i)^{-m} \beta_L d\frac{z_L}{z_K}$. □

The weight spectral sequence. Via Lemma 1.1, the $E_1$ page of the weight spectral sequence becomes

$$wE_1^{p,q} = \bigoplus_{K,|K|=m} H^{p+q}(\text{gr}_{-p} W^\bullet \omega^\bullet) = \bigoplus_{K,|K|=m} H^{p+q}(\iota^{(m)} \ast \Omega^\bullet_{D_K})$$

Now the differentials $\rightarrow wE_1^{p-1,q} \xrightarrow{d_1} wE_1^{p,q} \xrightarrow{d_1} wE_1^{p+1,q} \rightarrow \cdots$ on the first page of the spectral sequence becomes

$$\bigoplus_{K,|K|=m} H^{2p+q}(D_K, \Omega^\bullet_{D_K}) \xrightarrow{d_1} \bigoplus_{L,|L|=m-1} H^{2p+q+2}(D_L, \Omega^\bullet_{D_L})$$

$$\bigoplus_{K,|K|=m} H^{2p+q}(D_K, \mathbb{C}) \xrightarrow{d_1} \bigoplus_{L,|L|=m-1} H^{2p+q+2}(D_L, \mathbb{C})$$
Here each $D_K$ has codimension $-p$ and $D_L$ has codimension $(-p) - 1$. Clearly if $K \not\subseteq L$ then the corresponding differential is 0. The question is, in the case $K \subseteq L$, what is this differential? It turns out that in this case $d_1$ agrees with the Gysin map.

**Remark** (Aside on Gysin maps). Recall from [6, § 5] that for a closed immersion $i : D \hookrightarrow X$ from a closed subscheme $D$ of dimension $d = n - c$ into a scheme $X$ of dimension $n$, the Gysin map is given by the composition

\[ H^k(D, \mathbb{C}) \xrightarrow{i_*} H^{2n-2d+k}(X, \mathbb{C}) \xrightarrow{} H^{k+2c}(X, \mathbb{C}) \]

Under the isomorphism (see [6, §4] for notations)

\[ H^*(X, \mathbb{C}) \cong H^*(X, 

the Gysin map $i_*$ has the following description. Let $\alpha$ be a closed smooth $k$-form on $D$ representing its class in $H^k(D, \mathbb{C})$, $i_*(\alpha)$ is represented by a closed $(k + 2c)$-form on $X$ satisfying

\[ \langle \alpha, i^* \beta \rangle = \langle i_* \alpha, \beta \rangle \]

for every closed $(2d - k)$-form $\beta$ on $X$. In other words, for each $\beta$ one has

\[ \int_D \alpha \wedge i^* \beta = \int_X i_* \alpha \wedge \beta. \]

**Proposition 1.2.** Retain the setup from above, for $p \leq -1$, the differential

\[ d_1 : W_{E_{1}^{p, q}} = \bigoplus_{K, |K| = -p} H^{2p+q}(D_K, \mathbb{C}) \longrightarrow W_{E_{1}^{p+1, q}} = \bigoplus_{L, |L| = -p-1} H^{2p+q+2}(D_L, \mathbb{C}) \]

on the first page of the weight spectral sequence is given by 0 for those $K \not\subseteq L$, and (up to a sign) by the Gysin map for $K \subseteq L$.

**Sketch.** To simplify notations, let us assume that $D$ consists of a single branch, so we have $U \xrightarrow{\iota} X \xleftarrow{\iota} D$. The relevant weight filtrations are the following

\[
\begin{align*}
W^1 &= W_{-1} = 0 \\
W^0 &= W_0 = (\mathcal{O}_X \to \Omega^1 \to \Omega^2 \to \cdots) = \Omega_X^* \\
W^{-1} &= W_1 = (\mathcal{O}_X \to \omega^1 \to \omega^1 \wedge \Omega^1 \to \cdots) = \omega_X^*
\end{align*}
\]

(note that $\omega^1 \wedge \Omega^k \cong \omega^{k+1}$ by the assumption on $D$). The differential $d_1$ in this case is therefore the connecting homomorphism of the cohomology of the following exact sequence of complexes

\[
\begin{align*}
0 \longrightarrow \text{gr}_{W}^0 \omega^* &\longrightarrow W^{-1}/W^1 \longrightarrow \text{gr}_{W}^{-1} \omega^* \longrightarrow 0 \\
\Omega_X^* &\longrightarrow \omega_X^* \xrightarrow{\text{Res}} i_* \Omega_D^*[-1]
\end{align*}
\]

\[ ^4 \text{well it is a map from } H^*(Y, \mathbb{C}) \to H^{*+2}(Y', \mathbb{C}) \text{ where } Y \subset Y' \text{ is of codimension 1 – what else could it be?} \]
The idea is to connect the short exact sequence above with smooth differential forms, which then can be linked to the Gysin map via the remark above. To this end, one can prove that, if $\mathcal{A}^\bullet(\log D)$ denotes the sheaf of $C^\infty$ section generated by $\mathcal{A}_X^\bullet$ and $\frac{dx}{x}$ where $x$ is a local defining equation for $D$, then there is a quasi-isomorphism $\omega_X^\bullet \xrightarrow{\sim} \mathcal{A}_X^\bullet(\log D)$. There is a similar residue map $\text{Res} : \mathcal{A}_X^\bullet(\log D) \to \iota_*\mathcal{A}_D^\bullet$ and we denote the kernel by $\tilde{A}$. Then we have the following diagram

$$
\begin{array}{ccc}
\Omega_X^\bullet & \xrightarrow{\text{Res}} & \iota_*\Omega_D^\bullet[-1] \\
\downarrow & & \downarrow \\
\tilde{A}_X^\bullet & \xrightarrow{\text{Res}} & \mathcal{A}^\bullet(\log D) \to \iota_*\mathcal{A}_D^\bullet[-1]
\end{array}
$$

The left vertical is in fact a quasi-isomorphism as the other two are, thus the inclusion

$$\mathcal{A}^\bullet \hookrightarrow \mathcal{A}^\bullet(\log D)$$

is also a quasi-isomorphism. As the bottom sequence consists of fine sheaves, their cohomology is just the cohomology of the smooth sections (which we denote using $\mathcal{A}^\bullet(\log D)$ etc., as in [6]). Now we want to see the connection homomorphism on cohomology more explicitly. Let $\phi : T \to D$ be a tubular neighborhood of the divisor $D$ in $X$ (as smooth manifolds), so we have $X \leftarrow T \xrightarrow{\phi} D$. Now suppose $\alpha \in H^{q-2}(D, \mathbb{C}) \cong H^{q-2}(\mathcal{A}_D^\bullet)$ is a closed (global) $C^\infty$-form on $D$, then $\xi_\alpha := (2\pi i)^{-1}\pi^*\alpha \wedge \frac{dx}{x}$ is a smooth form on $T$. We extend $\pi^*\alpha$ smoothly to a form $\tilde{\alpha}$ on $X$ and let $\tilde{\xi}_\alpha = (2\pi i)^{-1}\tilde{\alpha} \wedge \frac{dx}{x} \in \mathcal{A}_X^\bullet(\log D)$. By construction we know that $\text{Res}(\tilde{\xi}_\alpha) = \alpha$. Note that the connection homomorphism $\delta$ on cohomology (hence the corresponding differential $d_1$ on $\mathbb{W}E_1$) sends $\alpha$ to $d\xi_\alpha$, so it remains to show that $[d\xi_\alpha] = (-1)^q \iota_*[\alpha] \in H^q(X, \mathbb{C})$.

By the Remark above, we want to show that for any closed $C^\infty$-form $\beta$ of degree $2n - q$ on $X$, we have $\langle \alpha, \iota^\beta \rangle = (-1)^q \langle d\xi_\alpha, \beta \rangle$, which amounts to showing

$$\int_D \alpha \wedge \iota^\beta = (-1)^{q+1} \int_X d\tilde{\xi}_\alpha \wedge \beta.$$  

Let $T_\epsilon$ be the tubular neighborhood of $D$ of radius $\epsilon$ when $\epsilon$ is small enough. The RHS can be computed as (using Stokes theorem and the residue theorem)

$$(-1)^{q+1} \int_X d\tilde{\xi}_\alpha \wedge \beta = (-1)^{q+1}(2\pi i)^{-1} \lim_{\epsilon \to 0} \int_{X-T_\epsilon} d\tilde{\alpha} \wedge \frac{dx}{x} \wedge \beta = (-1)(2\pi i)^{-1} \lim_{\epsilon \to 0} \int_{X-T_\epsilon} d(\tilde{\alpha} \wedge \beta) \wedge \frac{dx}{x} = (2\pi i)^{-1} \lim_{\epsilon \to 0} \int_{\partial T_\epsilon} \alpha \wedge \beta \wedge \frac{dx}{x} = LHS$$

**Corollary 1.3.** Equip $\mathbb{W}E_1^{p,q} \cong \oplus_K H^{2p+q}(D_K, \mathbb{C})$ the pure Hodge structure of weight $2p + q$ (coming from the Hodge decomposition of the singular cohomology of each of the smooth variety $D_K$), then the differential $d_1$ is a morphism of pure Hodge structures from weight $2p + q$ to $2p + q + 2$. In particular, $d_1$ is a strict morphism of filtered modules with respect to the Hodge filtration on both sides.

**Proof.** This follows from [6, §5].
2. Mixed Hodge structures on open varieties

The lemma of two filtrations. In this discussion suppose we are given a triple \((K, W, F)\) — a bounded cochain complex \(K\) equipped with two biregular filtrations \(W\) and \(F\), where \(F\) is a decreasing filtration while \(W\) is increasing. Following Deligne, we write \(E_r(K, W)\) for the spectral sequence induced by the \(W\)-filtration on \(K\) (after reindexing, like what we did to the weight filtration on the complex of log differentials \(\omega^\bullet\)).

The filtration \(F\) on \(K\) induces 3 filtrations on \(E_r(K, W)\), which we now describe.

The direct filtration. The direct filtration \(F^\bullet_d\) on \(E_r(K, W)\) is defined to be the image

\[
F^p_d := \text{im} \left( E_r(F^p K, W) \to E_r(K, W) \right)
\]

The second direct filtration. The second direct filtration \(F^\bullet_{d^*}\) on \(E_r(K, W)\) is

\[
F^p_{d^*} := \ker \left( E_r(K, W) \to E_r(K/F^p K, W) \right).
\]

In fact, this filtration \(F^p_{d^*}\) agrees with \(F^p_d\) on \(E_0\) (also on \(E_1\)).

The recurrent filtration. The recurrent filtration \(F^\bullet_r\) is defined inductively. On \(E_0(K, W)\) it is defined as \(F^\bullet_{d^*}\). Suppose \(F^\bullet_r\) has been defined on the \(k\)-th page \(E_k(K, W)\), then we use the definition of \(E^\bullet_{k+1}\) as a sub-quotient of \(E^\bullet_k\), namely

\[
\ker(d_k) \hookrightarrow E^\bullet_k \quad \downarrow \quad E^\bullet_{k+1}
\]

Now \(F^\bullet_r\) on \(E_k(K, W)\) induces a sub-filtration on \(\ker d_k\) which in turn induces a quotient-filtration on \(E_{k+1}\).

Lemma 2.1 (Deligne). The differential \(d_r\) is always compatible with the direct and second direct filtrations \(F^\bullet_d\) and \(F^\bullet_{d^*}\).

For the recurrent filtration, Deligne then proves the following

Theorem 2.2 (Deligne). Notation and setup as above. We have the following

1. One always has

\[
F^p_d \subset F^p_r \subset F^p_{d^*}
\]

on \(E^a_{m}^b\), for both \(m\) finite or infinite.

2. If for all \(r \leq r_0\), each \(d_r\) on \(E_r(K, W)\) is strictly compatible with the recurrent filtration \(F^\bullet_r\), then

2a. \(E_r(F^p K, W) \to E_r(K, W) \to E_r(K/F^p K, W)\) is exact (in the middle).

2b. For all \(r \leq r_0 + 1\) we have

\[
F^\bullet_d = F^\bullet_r = F^\bullet_{d^*}.
\]

So in particular, \(d_{r_0 + 1}\) is also compatible with the recurrent filtration \(F^\bullet_r\) on \(E_{r_0 + 1}\).
Complement of normal crossing and mixed Hodge structures. Now back to our previous discussion of the weight spectral sequence

$$wE^p_q = \bigoplus_{K \mid |K| = -p} H^{2p+q}(D_K, \Omega^*_D) \Rightarrow H^{p+q}(X, \omega^*) \cong H^{p+q}(U, \mathbb{C}).$$

We apply the lemma of two filtrations to the Hodge filtration and the weight filtration on $\omega^*_X$. We observe that the residue isomorphism

$$W_m \omega^*/W_{m-1} \omega^* \xrightarrow{\sim} \iota_\ast \Omega^*_{D(m)}$$

identifies the recurrent filtration (which is the Hodge filtration on the quotient complex of sheaves) on the LHS with the Hodge filtration on the RHS up to a shift. More precisely, in the isomorphism

$$H^{p+q}(\text{gr} W_p \omega^*) \cong H^{2p+q}(D(-p), \mathbb{C})$$

The recurrent/direct filtration on the LHS is obtained by the Hodge filtration on the RHS by shifting degree $-p$, which endows it a pure Hodge structure of weight $q$ (which is identified with the pure Hodge structure on RHS of weight $2p + q$ shifted by $(-p, -p)$). In other words, $wE^p_q$ is pure Hodge structure of weight $q$.

The upshot is the following — the differential $d_1$ is a strict morphism, which is in fact a morphism of pure Hodge structures of weight $q$, from $wE^p_q$ to $wE^{p+1}_q$. Now by the lemma of two filtrations (Theorem 2.2), we know that $d_2 : E_2^{p,q} \rightarrow E_2^{p+2,q-1}$ is a morphism of pure Hodge structures ($d_2$ is compatible with complex conjugation), where $E_2^{p,q}$ inherits the pure Hodge structure of weight $q$ from

$$\ker(d_1) \hookrightarrow E_1^{p,q}$$

while $E_2^{p+2,q-1}$ has weight $q - 1$. Now this forces $d_2 = 0$. In fact one can show that all $d_r = 0$ for $r \geq 2$. We summarize as follows:

**Theorem 2.3.** Set up as above ($U \hookrightarrow X \twoheadrightarrow D$), the cohomology $H^\ast(U, \mathbb{C}) \cong H^\ast(X, \omega^*_X)$ is equipped with a Hodge filtration $F$ and a weight filtration $W$ as discussed above. We claim that

1. The Hodge to de Rham spectral sequence degenerates at $E_1$.
2. The weight spectral sequence degenerates at $E_2$.
3. On the graded pieces on cohomology by the weight filtration

$$wE^p_q = \text{gr} W_p H^{p+q}(U, \mathbb{C})$$

the Hodge filtration $F^\ast$ induces a pure Hodge structure of weight $q = (p + q) - p$.

In other words, we have equipped $H^\ast(U, \mathbb{C})$ with a mixed Hodge structure of weight 0. More generally, let us state the following definition.

**Definition 2.4.** An integral (resp. rational) mixed Hodge structure is a tuple

$$(V_\mathbb{Z} (resp. V_\mathbb{Q}), W_\ast, F^\ast),$$
where $V\mathbb{Z}$ is a finite free $\mathbb{Z}$-module (resp. finite dimensional $\mathbb{Q}$-vector space), $W_\bullet$ an increasing filtration on $V\mathbb{Z}$ (resp. $V\mathbb{Q}$), and $F^\bullet$ a decreasing filtration on $V \otimes \mathbb{C}$, such that the induced filtration $F^\bullet$ on $\text{gr}^W_r(V)$ defines a pure Hodge structure of weight $k + r$.

**Remark.** In fact $H^\bullet(U, \mathbb{C})$ is a MHS over $\mathbb{Q}$ -- we have not mentioned why the weight filtration is defined over $\mathbb{Q}$, which follows from identifying the weight spectral sequence $E^\bullet$ to the Leray spectral sequence $E^\bullet_{-1}$ (for the inclusion $j : U \to X$).

**Remark.** Deligne actually proves a stronger theorem -- namely he starts with a smooth variety $U$ over $\mathbb{C}$ and then compactifies it as the complement of a normal crossing divisor of a smooth variety $X$ (Nagata + resolution of singularity). He then shows that the MHS one puts on $H^\bullet(U, \mathbb{C})$ is independent of the compactification.

3. Degenerating/limit Hodge structures

To be added.

REFERENCES


E-mail address: zijian.yao.math@gmail.com

Department of Mathematics, Harvard University.