Existence of an exotic plane in an acylindrical 3-manifold

Yongquan Zhang

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Abstract

In this short note, we give an example of an exotic plane, i.e. a geodesic plane in a convex cocompact acylindrical hyperbolic 3-manifold that is closed and nonempty in the interior of the convex core but not closed in the whole manifold. This example illuminates when generalizations of Ratner’s theorem hold and when they fail for hyperbolic manifolds of infinite volume.

1 Introduction

This paper is a contribution to the study of the topological behavior of geodesic planes in a geometrically finite hyperbolic 3-manifold.

Hyperbolic 3-manifolds. Given a hyperbolic manifold $M \cong \Gamma \backslash \mathbb{H}^3$, where the Kleinian group $\Gamma \subset \text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C})$ has limit set $\Lambda \subset S^2$, the convex core of $M$ is given by

$$\text{core}(M) = \Gamma \backslash \text{hull}(\Lambda) \subset M,$$

where $\text{hull}(\Lambda) \subset \mathbb{H}^3$ is the convex hull of $\Lambda$. Equivalently, $\text{core}(M)$ is the smallest closed convex subset of $M$ so that inclusion induces an isomorphism on $\pi_1$. We say $M$ is convex cocompact if $\text{core}(M)$ is compact, geometrically finite if some neighborhood of $\text{core}(M)$ has finite volume. Let $M^*$ be the interior of $\text{core}(M)$.

Geodesic planes in $M$. A geodesic plane $P$ in $M$ is an isometric immersion $f : \mathbb{H}^2 \to M$. We often identify the map $f$ with its image $f(\mathbb{H}^2)$ and call the latter a geodesic plane as well. Given a geodesic plane $P$ in $M$, set $P^* = M^* \cap P$.

Considerable research has been devoted to understanding the closure of a geodesic plane $P$ in a geometrically finite hyperbolic 3-manifold $M$. When $M$ has finite volume, $P$ is either closed or dense. This is established in the work of Shah [Sha] and follows from the much more general results of Ratner on unipotent flows [Rat]. In [MMO2], McMullen, Mohammadi and Oh extend this remarkable rigidity to geodesic planes in the convex core of a certain family of hyperbolic 3-manifolds of infinite volume:

Theorem 1.1 ([MMO2]). Let $M$ be a convex cocompact, acylindrical, hyperbolic 3-manifold. Then any geodesic plane $P$ with $P^* \neq \emptyset$ is either closed or dense in $M^*$. 

1
The restriction to $M^*$ naturally prompts the following question in [MMO2]: if $P^*$ is closed in $M^*$, is $P$ closed in $M$? We give an answer to this question:

**Theorem 1.2.** There exists a convex cocompact, acylindrical, hyperbolic 3-manifold $M = \Gamma \backslash \mathbb{H}^3$ and a geodesic plane $P$ in $M$ so that $P^*$ is closed and nonempty in $M^*$ but $P$ is not closed in $M$.

For simplicity, we call such a geodesic plane $P$ an exotic plane. This 3-manifold $M$ comes from the one-parameter family of acylindrical orbifolds $M(t)$ constructed in [Zha] with $t = t_0 \approx 1.202$. We remark that since any hyperbolic orbifold has a manifold cover of finite degree, and all properties of interest are preserved under finite covers, we may work with orbifolds instead of manifolds.

For an exotic plane $P$, $\overline{P}$ is not a closed submanifold of $M$. Theorem 1.2 suggests that for a generalization of Ratner’s theorem in the context of convex cocompact acylindrical hyperbolic 3-manifolds, the interior of the convex core, instead of the whole manifold, may indeed be the proper setting, as in [MMO2].

**The orbit of an exotic circle.** An exotic circle of $M = \Gamma \backslash \mathbb{H}^3$ is the circle at infinity of any lift to $\mathbb{H}^3$ of an exotic plane in $M$. Figure 1 gives some visualizations of the example in Theorem 1.2. Figure 1a depicts the limit set of $\Gamma$, with a circle $C$ corresponding to the exotic plane $P$ marked in red, and Figure 1b shows the orbit of $C$ under $\Gamma$ (and thus contains all the exotic circles corresponding to $P$). We remark that the limit set $\Lambda$ in Figure 1a is an example of a Sierpiński curve.

As a matter of fact, when $M$ is convex cocompact of infinite volume, it is acylindrical if and only if $\Lambda$ is a Sierpiński curve.

![Figure 1](attachment:fig1.png)

(a) Limit set $\Lambda$ of $\Gamma$  
(b) Orbit of a circle $C$ corresponding to $P$

**Figure 1:** The acylindrical orbifold $M = \Gamma \backslash \mathbb{H}^3$ and an exotic plane $P$ in $M$

In Figure 1b, if a sequence of circles $C_i$ converges (in the Hausdorff topology on $S^2$) and is not eventually constant, then either the radius of $C_i$ tends to 0, or $C_i$ tends to an circle in the orbit of

\[1\text{ A Sierpiński curve is a compact subset $\Lambda$ of the 2-sphere $S^2$ such that $S^2 - \Lambda = \cup_i D_i$ is a dense union of Jordan disks with diam($D_i$) → 0 and $D_i \cap D_j = \emptyset$ for all } i \neq j.\]
\( C' \) (this is a circle contained in the closure of a component of the domain of discontinuity). This follows from our discussion of the geometry of \( P \) (see below), and is quite visible from Figure 1b.

**Geometry of \( P \), \( P^* \) and \( \overline{P} \).** We briefly describe here the geometry of the exotic plane \( P \) in Theorem 1.2, its restriction to the convex core \( P^* \), and its closure \( \overline{P} \). \( P \) is a nonelementary, convex cocompact surface with one infinite end; its restriction to \( \text{core}(M) \) cuts the infinite end into a crown with two tips. In particular \( P^* \) has finite volume. The two tips of the crown wraps around and tends to the bending geodesic on the boundary of \( \text{core}(M) \). Finally, \( \overline{P} = P \cup P' \), where \( P' \) is a closed geodesic plane contained in the infinite end of \( M \). As a matter of fact, \( P' \) is a cylinder whose core curve is precisely the bending geodesic. Below is a picture of \( P \) near the convex core boundary in the quotient of \( M \) by a reflection symmetry. We remark that the behavior of \( P \) near the convex core boundary only depends on the geometry of the corresponding quasifuchsian manifold; see Props. 2.2 and 3.3 and Figure 5 for details.

![Figure 2: A cross section near the convex core boundary, in a plane orthogonal to \( P \). Here \( \eta \) denotes the bending geodesic.](image)

These observations lead naturally to the following open questions: for a geodesic plane \( P \) in a convex cocompact acylindrical hyperbolic 3-manifold \( M \) so that \( P^* \) is nonempty and closed, (1) does \( P^* \) always have finite volume? (2) is \( \overline{P} \) always the union of \( P \) and finitely many closed elementary surfaces contained in the ends of \( M \)? It is possible that both of these questions may be answered by constructing an example where the bending lamination on the boundary of \( \text{core}(M) \) is non-atomic.

**Notes and references.** In the prequel [MMO1] to [MMO2], the authors studied the special case where \( M \) is also assumed to have Fuchsian ends, and established a complete classification of geodesic planes in \( M \): they are either closed, dense in \( M \), or dense in an infinite end. In particular, the closure of any geodesic plane is a closed submanifold of \( M \), and no exotic plane exists. Theorem 1.1 has also recently been extended to geometrically finite acylindrical manifolds in [BO].

In the cylindrical case, Ratner’s theorem can fail badly. For example, a geodesic plane in a quasifuchsian manifold can have wild, non-manifold, even fractal closure, as explained in [MMO1, Appx. A].

This paper is organized as follows. Section 2 is devoted to an example of exotic planes in a family of quasifuchsian manifolds. We then leverage the quasifuchsian example to construct an acylindrical example, giving a proof of Theorem 1.2 in Section 3. As mentioned above, the
behavior of the acylindrical example near the convex core boundary is exactly the same as that of the quasifuchsian example. Finally, we make some explicit calculations about the example in Section 4.

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2 A quasifuchsian example

In this section, we give an example of exotic planes in a family of quasifuchsian manifolds.

Fix an integer \(n \geq 3\). Let \(R\) be a quadrilateral in the extended complex plane whose sides are either line segments or circular arcs, and whose interior angles are all \(\pi/n\). Reflections in the sides of \(R\) generate a discrete subgroup of \(\text{PSL}(2, \mathbb{C})\), and let \(\Pi_R\) be its index 2 subgroup of orientation preserving elements. We note that \(\Pi_R\) is a quasifuchsian group, and the corresponding quasifuchsian orbifold \(N_R := \Pi_R \backslash \mathbb{H}^3\) is homotopic to a sphere with 4 cone points of order \(n\).

Denote by \(C_i, 1 \leq i \leq 4\) the circles on which the four sides of \(R\) lie, so that \(C_1\) and \(C_3\) contain opposite sides. The corresponding hyperbolic planes in \(\mathbb{H}^3\) are also denoted by \(C_i\). To construct \(N_R\), one can take two copies of the infinite “tube” bounded by these four planes and identify corresponding faces. A fundamental domain for \(\Pi_R\) is thus two copies of the tube, although for most discussions we consider the full reflection group with fundamental domain simply the tube.

Two copies of the geodesic segment perpendicular to both \(C_1\) and \(C_3\) glue up to a closed geodesic \(\xi\) in \(N_R\), and similarly the other two planes give a closed geodesic \(\eta\). Let \(2 \cosh^{-1}(s)\) and \(2 \cosh^{-1}(t)\) be the lengths of \(\xi\) and \(\eta\) respectively. Then

**Proposition 2.1.** For any quadrilateral \(R\), we have \((s - 1)(t - 1) \leq 4 \cos^2(\pi/n)\), with equality if and only if \(\Pi_R\) is Fuchsian. When the inequality is strict, the convex core of \(N_R\) is bent along \(\xi\) on one side and \(\eta\) on the other.

This proposition is a combination of Prop. 4.1 and Lemma 4.3 in [Zha]. See Figure 3 for some visualizations of an example with \(n = 3\). Note that Figure 3a is drawn so that \(R\) is centered at the origin, and symmetric across real and imaginary axes. The end points of a lift \(\tilde{\eta}\) of \(\eta\) then lie on the imaginary axis; the corresponding hyperbolic element, also denoted by \(\tilde{\eta}\), is given by reflections across \(C_2\) and then \(C_4\). Similarly, the end points of a lift \(\tilde{\xi}\) of \(\xi\) lie on the real axis, see Figure 4a.

Let \(\Lambda = \Lambda_R\) be the limit set of \(\Pi_R\). Take the circle \(C = C_R\) passing through the end points of \(\tilde{\xi}\) (marked by a pair of blue dots in Figure 4a) and the repelling fixed point \(p\) of \(\tilde{\eta}\) (marked by a red dot in Figure 4a); see the red circle in Figure 4a. It is easy to see that \(p\) is an isolated point in \(C \cap \Lambda\). As \(C\) is stabilized by reflections in \(C_1\) and \(C_3\), it also passes through the orbit of \(p\) under the group generated by these reflections. It is clear that \(C \cap \Lambda\) consists of points in this orbit together with the end points of \(\xi\). We have

**Proposition 2.2.** \(C\) is an exotic circle.

\[^2\text{See www.geomview.org}\]
Proof. For simplicity, we will suppress all subscript $R$. Since $C$ passes through the repelling fixed point of $\tilde{\eta}$, the sequence $\tilde{\eta}^n \cdot C$ tends to a circle $C'$ passing through both end points of $\tilde{\eta}$; in particular, the geodesic plane $P$ determined by $C$ is not closed in $N$, and accumulates on the plane $P'$ determined by $C'$. This is clearly visible in Figure 4b, where the orbit of $C$ under $\Pi$ is drawn.

It remains to show $P^*$ is closed in $N^*$. The hyperbolic plane $\tilde{P}$ in $\mathbb{H}^3$ determined by $C$ is divided into two half plane by $\tilde{\xi}$. One half descends to a half cylinder contained in the bottom end of $N$; indeed, this half plane is stabilized by $\tilde{\xi}$ and a fundamental domain for the action of this hyperbolic element is compact in $\mathbb{H}^3 \cup \Omega$, where $\Omega$ is the domain of discontinuity of $\Pi$, and thus our assertion follows from proper continuity of the action.

For the other half, again since it is stabilized by $\tilde{\xi}$, it suffices to consider a fundamental domain under the action of this hyperbolic element. As a matter of fact, we may even consider a fundamental domain under the full reflection group. One choice of this is the portion of the half plane sandwiched between $C_1$ and $C_3$ in Figure 5a. Let $p'$ and $p''$ be the images of $p$ under reflections across $C_1$ and $C_3$ respectively. The geodesic with end points $p$ and $p'$ descends to a complete geodesic contained in the convex core boundary, and same for the geodesic with end points $p$ and $p''$. Hence to understand $P^*$ we only need to consider the portion mentioned above bounded between the geodesics $pp'$ and $pp''$; see Figure 5a.

This portion is divided by orbits of Faces 3 and 5 into countably many pieces, each one compact and descends to a piece contained entirely in $\text{core}(N)$. Moreover, the maximum distance of a point
The slice in (b) →

(a) A fundamental domain inside $\tilde{P}$

(b) A perpendicular slice of $\tilde{P}$

Figure 5: Some visualizations for the proof of Prop. 2.2

It is also easy to understand the behavior of $P$ in $N$. Note that the limit plane $P'$ described in the proof above is closed, and $P$ only accumulates on this plane. These assertions can be proved using similar arguments to the proof above. In particular we have $\overline{P} = P \cup P'$. Finally, the proof above also gives a clear picture of the topology of $P^*$; indeed, we described a fundamental domain under the action of the full reflection group (the gray region in Figure 5a). Two pieces of this gives a crown with two tips, and $P^*$ is precisely the interior of this crown properly immersed in $N^*$.

3 An acylindrical example

In this section, we briefly review the acylindrical orbifolds constructed in [Zha], which are covered by quasifuchsian orbifolds described in the last section. We then construct an exotic plane in one of these acylindrical orbifolds by projecting down the corresponding quasifuchsian example.

The example manifold and its deformations. A compact three manifold with boundary $M$ is acylindrical if it has incompressible boundary and every essential cylinder in $M$ is boundary parallel. One can obtain explicit examples of hyperbolic 3-manifolds by gluing faces of hyperbolic
polyhedra (with desired properties) via hyperbolic isometries. See [Thu §3.3], [PZ], [Fri] and [Gas] for some acylindrical examples.

Along the same idea, we construct a hyperbolic polyhedron given by the combinatorial data encoded in the Coxeter diagram below. Let $\tilde{Q}$ be the hyperbolic polyhedron with an infinite end obtained by extending across Face 1 to infinity; see Figure 6c. Reflections in all faces of $\tilde{Q}$ generate a discrete subgroup of hyperbolic isometries; a subgroup of index 2 gives an acylindrical hyperbolic 3-orbifold with Fuchsian end. We can deform $\tilde{Q}$ by pushing closer or pulling apart Faces 3 and 5, fixing the dihedral angles, and obtain deformations of the orbifold. This gives [Zha Thm 1.2]:

![Coxeter diagram](image)

(a) The Coxeter diagram

![Polyhedron](image)

(b) The corresponding polyhedron

![Hyperbolic polyhedron](image)

(c) The hyperbolic polyhedron in the unit ball model

Figure 6: Combinatorial data and visualization of the polyhedron

**Theorem 3.1.** For each $t \in \left(1, \frac{5 + \sqrt{39}}{3}\right)$, there exists a unique hyperbolic polyhedron $\tilde{Q}(t)$ so that the hyperbolic distance between Faces 3 and 5 is $\cosh^{-1}(t)$. The corresponding hyperbolic orbifold $M(t)$ is acylindrical convex cocompact, whose convex core boundary is totally geodesic if and only if $t = 2$.

See [Zha Figure 3] for some samples of the deformation. One way to explicitly construct $M(t)$ is to take two copies of $\tilde{Q}(t)$ and identify corresponding faces. A fundamental domain for the corresponding Kleinian group is thus two copies of $\tilde{Q}(t)$, although for most discussions we often consider the full reflection group with fundamental domain simply $\tilde{Q}(t)$.

It is clear from construction that the quasifuchsian orbifold $N(t)$ corresponding to the boundary of $M(t)$ is an example of those described in Section 2. Consistent with the notations there, let $\eta$ be the simple closed geodesic in $M(t)$ coming from two copies of the geodesic segment orthogonal to both Faces 3 and 5, and $\xi$ be that from Faces 2 and 4. We have [Zha Thm 1.3]:

7
Theorem 3.2. The convex core boundary of $M(t)$ is bent along $\eta$ for $t \in (1, 2)$, and $\xi$ for $t \in (2, (5 + \sqrt{39})/3)$, with hyperbolic lengths $l_\eta(t) = 2 \cosh^{-1}(t)$ and $l_\xi(t) = 2 \cosh^{-1}(s)$, where $s = \phi(t)$ is an explicit monotonic function. Bending angles $\lambda_\eta(t)$ and $\lambda_\xi(t)$ can also be explicitly calculated. On the other side of the corresponding quasifuchsian orbifold, the boundary is bent along $\xi$ for $t \in (1, 2)$ and $\eta$ for $t \in (2, (5 + \sqrt{39})/3)$, and lengths and angles can also be similarly calculated.

![Figure 7: A schematic picture of core($M(t)$) for $t \in (1, 2)$. Note that a copy core($N(t)$) is embedded in the polyhedron](image)

We refer to [Zha] for proofs of these statements and explicit calculations.

Existence of an exotic plane. We wish to make use of the quasifuchsian example in Section 2 to construct an acylindrical example. In particular, if we take the same plane $P(t)$ in $N(t)$ and project it down to $M(t)$ via the covering map $N(t) \to M(t)$ of infinite degree, we have to make sure that the cylinder contained in the bottom end of $N(t)$ projects to a closed surface in $M(t)$. For this, we have

**Proposition 3.3.** There exists $t = t_0 \in (1, 2)$ so that the image of $P(t_0)$ under the projection $N(t_0) \to M(t_0)$ is an exotic plane.

**Proof.** For this, we look at how the plane intersects a fixed fundamental domain, the polyhedron $\hat{Q}(t)$. A lift $\hat{P}(t)$ of the plane $P(t)$ passes through the geodesic segment orthogonal to Faces 2 and 4, so it is orthogonal to both faces as well. When $t \to 2$, $\hat{P}(t)$ tends to Face 1; when $t \to 1$, $\hat{P}(t)$ tends to a plane orthogonal to Face 6, dividing the polyhedron $\hat{P}(1)$ into two equal parts. By continuity, for some subinterval of $(1, 2)$, $\hat{P}(t)$ intersects the edge shared by Faces 7 and 8. Another continuity argument guarantees the existence of a $t = t_0 \in (1, 2)$ so that $\hat{P}(t_0)$ intersects this edge orthogonally. This implies that $\hat{P}(t_0)$ intersects both Faces 7 and 8 orthogonally, and disjoint from Face 5. See Figure 8 for a schematic picture of $\hat{P}(t_0) \cap \hat{Q}(t_0)$.

Clearly, as $\hat{P}(t_0)$ is orthogonal to Faces 2, 4, 7 and 8, the half plane that projects to a cylinder in the bottom end in $N(t_0)$ further projects down to an orbifold surface in $M(t_0)$, with one geodesic boundary component, two cone points of order 2 and one cone point of order 3. The other half behaves exactly as that in the quasifuchsian example (recall that a copy of core($N(t_0)$) is embedded in core($M(t_0)$)), see Figure 7, so it is indeed an exotic plane in $M(t_0)$.

Theorem 1.2 is then a direct consequence of Prop. 3.3.

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4The top end of $N(t)$ the the infinite end shared with $M(t)$, and the other end is the bottom end.
4 Computations

In this section, we calculate the explicit value of \( t_0 \) predicted in the previous section. We will refer to calculations in [Zha, §6] freely.

We can uniquely determine a hyperbolic plane in \( \mathbb{H}^3 \) using its unit normal in the hyperboloid model. Set \( u = \sqrt{(t+1)/2} \). One choice of normals for Faces 2 and 4 may be

\[
\left( \frac{u - \sqrt{4u^2 + 4v^2 - 3 - 4u^2v^2}}{2u^2 - 2}, \pm v, 0, \frac{-1 + u\sqrt{4u^2 + 4v^2 - 3 - 4u^2v^2}}{2u^2 - 2} \right)
\]

where \( v = \frac{3u + \sqrt{(u^2+2)(16u^2-3)}}{8u^2-2} \). Correspondingly, the normals for Faces 7 and 8 are

\[
\left( \frac{u\sqrt{3} - \sqrt{u^2 + 2}}{2u^2 - 2}, \mp \frac{\sqrt{3}}{2}, \frac{\sqrt{3} u\sqrt{u^2 + 2} - u\sqrt{3}}{2u^2 - 2} \right).
\]

Suppose the unit normal to the plane \( \tilde{P} \) is \((x_0, x_1, x_2, x_3)\). Then this vector is orthogonal to the normals listed above. Therefore \( x_1 = 0 \) and

\[
x_2 = \frac{\sqrt{u^2 + 2} - u\sqrt{3}\sqrt{4u^2 + 4v^2 - 3 - 4u^2v^2}}{\sqrt{3}(-1 + u\sqrt{4u^2 + 4v^2 - 3 - 4u^2v^2})} x_0, \quad x_3 = \frac{u - \sqrt{4u^2 + 4v^2 - 3 - 4u^2v^2}}{-1 + u\sqrt{4u^2 + 4v^2 - 3 - 4u^2v^2}} x_0.
\]

On the other hand, the sequence of planes \( \tilde{\eta}^n. \tilde{P} \rightarrow \tilde{P}' \), where \( \tilde{P}' \) has unit normal \((\frac{1}{\sqrt{u^2 - 1}}, 0, 0, -\frac{u}{\sqrt{u^2 - 1}})\). This can be calculated using the formula in [Zha, §6.2] for the hyperbolic element \( \tilde{\eta} \), and the fact that the circle \( C' \) corresponding to \( \tilde{P}' \) passes through the fixed points of \( \tilde{\eta} \) and is symmetric across the imaginary axis. Since \( \tilde{P} \) is tangent to \( \tilde{P}' \) at infinity, the inner product of their unit norms is 1 (or -1, but we can always change the orientation of \( \tilde{P} \)), so

\[
-x_0 \frac{1}{\sqrt{u^2 - 1}} - x_3 \frac{u}{\sqrt{u^2 - 1}} = 1.
\]

Hence we have

\[
x_0 = \frac{1 - u\sqrt{4u^2 + 4v^2 - 3 - 4u^2v^2}}{\sqrt{u^2 - 1}},
\]

\[
x_2 = \frac{-\sqrt{u^2 + 2} + u\sqrt{3}\sqrt{4u^2 + 4v^2 - 3 - 4u^2v^2}}{\sqrt{3}\sqrt{u^2 - 1}},
\]

\[
x_3 = \frac{\sqrt{4u^2 + 4v^2 - 3 - 4u^2v^2} - u}{\sqrt{u^2 - 1}}.
\]
Since \(-x_0^2 + x_2^2 + x_3^2 = 1\), we have
\[
4u^2 + 4v^2 - 3 - 4u^2v^2 - \frac{2\sqrt{3}}{3}u\sqrt{u^2 + 2\sqrt{4u^2 + 4v^2 - 3 - 4u^2v^2}} + \frac{u^2 + 2}{3} = 0,
\]
and thus
\[
9(4u^2 + 4v^2 - 3 - 4u^2v^2)^2 + (u^2 + 2)^2 + 6(4u^2 + 4v^2 - 3 - 4u^2v^2)(u^2 + 2)
- 12u^2(u^2 + 2)(4u^2 + 4v^2 - 3 - 4u^2v^2) = 0.
\]
Plugging in the expression of \(v\) in terms of \(u\), the left hand side gives
\[
\frac{u^2 - 1}{(4u^2 - 1)}(f(u) + g(u)\sqrt{16u^4 + 29u^2 - 6})
\]
where
\[
f(u) = -625 + 11153u^2 - 53284u^4 + 65632u^6 - 38720u^8 + 22144u^{10} - 9216u^{12},
g(u) = 900u - 7092u^3 + 9072u^5 - 4032u^7 + 1152u^9.
\]
Thus we have
\[
0 = f(u)^2 - g(u)^2(16u^4 - 29u^2 - 6)
= (4u^2 - 1)^4(-625 + 944u^2 - 976u^4 + 576u^6)
(-625 + 3586u^2 - 6585u^4 + 3112u^6 - 1360u^8 + 576u^{10})
= \frac{1}{4}(1 + 2t)^4(-325 + 200t - 28t^2 + 72t^3)(-625 - 2330t - 3237t^2 + 916t^3 + 20t^4 + 72t^5)
\]
Set \(h(t) := -625 - 2330t - 3237t^2 + 916t^3 + 20t^4 + 72t^5\). Then \(h''(t) = -6474 + 5496t + 240t^2 + 1440t^3 > 0\) when \(t \in (1, 2)\). So \(h(t)\) attains maximum at \(t = 1\) or \(t = 2\). But \(h(1) = -5184 < 0\) and \(h(2) = -8281 < 0\), so \(h(t) < 0\) for \(t \in (1, 2)\). To find \(t \in (1, 2)\) satisfying the equation above, we must thus solve
\[
-325 + 200t - 28t^2 + 72t^3 = 0.
\]
This polynomial has a unique real root
\[
t = t_0 = \frac{1}{54} \left( 7 - \left( \frac{25515\sqrt{773} - 654761}{2} \right)^{1/3} \right)^{1/3} + \left( \frac{25515\sqrt{773} + 654761}{2} \right)^{1/3} \approx 1.202,
\]
in \((1, 2)\) as desired.

References


