1. Overview

My research is in arithmetic algebraic geometry. One of my primary interests is the arithmetic properties of Calabi-Yau varieties, and my work so far has included two aspects of this: the Attractor Conjecture coming from string theory, and Calabi-Yau varieties over finite fields. In what follows, unless otherwise specified, by Calabi-Yau we mean the following:

Definition 1.0.1. A smooth proper variety $X$ of dimension $n$ over a field $k$ is said to be Calabi-Yau (CY) if the canonical bundle $\Omega^n$ is trivial.

The study of Calabi-Yau varieties in dimensions one and two is familiar territory for number theorists, since these consist of elliptic curves, abelian surfaces, and K3 surfaces. One of the reasons that these are amenable to study is that their moduli spaces are Shimura varieties, which roughly speaking are quotients of hermitian symmetric domains by arithmetic groups. Shimura varieties come with a rich theory, the most relevant for us is that of a collection of special points on the Shimura variety known as CM points. Perhaps the most notable application of CM points is the construction of canonical models of Shimura varieties over number fields due to Deligne. On the other hand, for Calabi-Yau varieties in dimensions 3 and higher, the situation is much more mysterious: the moduli spaces are no longer Shimura varieties in general, and we no longer have the tools mentioned above at our disposal.

1.1. The Attractor Conjecture and related themes. In 1998 a striking conjecture (the Attractor Conjecture) was made by Greg Moore in his study of the attractor mechanism in string theory; it postulates the existence of many special points on the moduli spaces of Calabi-Yau varieties, analogous to CM points on Shimura varieties. See Conjecture 2.1.2 for the precise statement. Much of my work has been to explore various aspects of this conjecture, which I will now describe.

In a joint work with Arnav Tripathy, we give an infinite family of counterexamples to this conjecture in higher dimensions (see Theorem 2.1.4), giving strong evidence that the conjecture is not true as stated. We also find several new exceptional examples for which the conjecture holds, where the moduli spaces are indeed Shimura varieties. The condition of having Shimura moduli is in fact well motivated physically, being equivalent to having no quantum corrections; this implies for example that genus zero Gromov-Witten invariants vanish.

It is therefore natural to guess that Conjecture 2.1.2 holds if and only if the moduli space is a Shimura variety. Although few Shimura varieties are known to parametrize Calabi-Yau threefolds, many do underly a Calabi-Yau variation of Hodge structures (CYVHS), which is equally good for the purposes of Conjecture 2.1.2. These CYVHS have been studied independently by mathematicians and physicists. I have proven the Attractor Conjecture in this setup, subsuming all cases where the conjecture was previously known to be true: see Theorem 2.1.2. Moreover, this result gives new simple parametrizations of (certain) CM points on many Shimura varieties, including exceptional ones.

It turns out that these CYVHS on Shimura varieties are related to several areas of number theory. The first is arithmetic invariant theory in the sense of Bhargava; see Section 2.3 for details on this; the second is automorphic forms; in Theorem 2.6.2 we show that certain black hole wavefunctions agree with special functions found by A. Pollack in his theory of Fourier expansion for quaternionic discrete series automorphic forms.

Physics suggests another variant of the attractor mechanism, the non-BPS case. For the CYVHS cases mentioned above, I have shown that the mechanism gives totally geodesic (real) submanifolds of (roughly) half dimension instead of points; for the first non-trivial case I have given a conceptual, geometric description of these points: see Theorem 2.4.3. In general I believe that the attractor moduli spaces are generalizations of real quadratic geodesics in the upper half plane, as well as analogues of the “moduli of real abelian varieties” studied by Goresky-Tai.

Attractors in higher dimensions have also been considered recently by W. Yang [Yan20] who made a conjecture on the Hodge structures of the Fermat Calabi-Yau varieties, discovered through numerical computations. I have verified this conjecture: see Theorem 2.5.1.

1.2. Calabi-Yau varieties over finite fields. We now come to the second aspect of my work on Calabi-Yau varieties, namely that of Calabi-Yau varieties over finite fields. To begin with, in [Jos14] K. Joshi formulated a criterion for the liftability (since not all Calabi-Yau threefolds are liftable!) of Calabi-Yau threefolds. I
have found counterexamples to this conjecture by considering the *Godeaux Calabi-Yau threefolds*, at the same time answering an outstanding question of van der Geer-Katsura: see Theorem 3.1.2 and Corollary 3.3.1

Furthermore, I have shown that the Godeaux Calabi-Yau threefolds shed new light on the Beauville-Bogomolov decomposition in positive characteristic, answering a question of Patakfalvi-Zdanowicz: see Theorem 3.2.2.

I have also studied the phenomenon of Hodge-de Rham degeneration in characteristic $p$. In particular, I answer a question posed by Bhatt, as well as give another example to a question of Illusie and Ogus: see Theorem 3.4.3 and the remarks following it. These are done by considering an extremely interesting Calabi-Yau threefold constructed by Cynk-van Straten. Related to this, I have also answered an outstanding question of Ekedahl: see Theorem 3.4.2.

One of the main tools I use to obtain these results is integral $p$-adic Hodge theory, in particular the effect of torsion in cohomologies, for example the recent work of Bhatt-Morrow-Scholze, as well as older results of Caruso and Faltings. Using similar ideas, I have suggested possible examples of derived equivalent Calabi-Yau threefolds over finite fields with different Hodge numbers. Over $\mathbb{C}$, the invariance of Hodge numbers under derived equivalences is a theorem of Popa-Schnell [PS11], and in positive characteristic the question was studied by Antieau-Bragg [AB19].

Finally, I have proven a Calabi-Yau analogue of Bombieri-Mumford’s theorem on *Enriques surfaces*, which says that there is a classification of the latter into three distinct types in characteristic two. See Theorem 3.6.1 for details on this.

2. The Attractor Conjectures and its variants

2.1. Counterexamples to the Attractor Conjecture.

**Definition 2.1.1.** A Calabi-Yau $n$-fold is an attractor variety (or attractor in short) if there exists $\gamma \in H^n(X, \mathbb{Q})$ such that $\gamma \perp H^{n-1,1}$. If a point on a moduli space of Calabi-Yau varieties corresponds to an attractor variety, we will refer to it as an attractor point. Following the physics literature, we will refer to $\gamma$ as the charge vector.

**Conjecture 2.1.2** (due to Greg Moore when $n = 3$). *If $X$ is an attractor variety then it is defined over $\overline{\mathbb{Q}}$.***

**Remark 2.1.3.** The condition in Definition 2.1.1 implies that attractor points are discrete in moduli space, which is evidently necessary for Conjecture 2.1.2 to hold. Indeed, this is one of the motivations for making this definition.

In joint work with Arnav Tripathy we give an infinite family of counterexamples to this conjecture, using tools from transcendental number theory and assuming a standard conjecture in *unlikely intersection* theory.

**Theorem 2.1.4** ([Lam20a Theorem 1.1.3]). *Assuming the Zilber-Pink conjecture, in each odd dimension except for $3, 7, 9$ the generalized Attractor Conjecture is false: more precisely, in these dimensions, the set of points for which the attractor points are defined over $\overline{\mathbb{Q}}$ is not Zariski dense. For the dimensions $n = 3, 7, 9$ we obtain further examples where the conjecture holds: in each of these cases the moduli space is a Shimura variety, and the attractor points are CM points.*

**Remark 2.1.5.** In the counterexamples we consider, it is straightforward to check that the attractor points are dense even topologically. Thus Theorem 2.1.4 shows that there are many attractor points not defined over $\overline{\mathbb{Q}}$.

To prove Theorem 2.1.4 we consider the Dolgachev Calabi-Yau varieties whose moduli spaces embed into Shimura varieties since the Hodge structures are controlled by that of certain associated curves. Using a transcendence result due to Shiga-Wolfart, we show that the Attractor Conjecture would imply that inside the Shimura varieties the moduli spaces intersect other sub-Shimura varieties in a bigger subspace than expected. The Zilber-Pink conjecture then implies that the moduli spaces of the Dolgachev Calabi-Yau varieties are themselves Shimura varieties, giving a contradiction for $n \neq 3, 5, 7$.

The exceptional cases of $n = 3, 7, 9$ are beautiful in their own right, and the curves $C$ in these examples show up in the study of tilings on flat surfaces [ES18]. The case $n = 3$ considered above also provides a new example for which the original Attractor Conjecture holds, though we feel that this is due to the property of having Shimura moduli, rather than the property of being in dimension 3: indeed the fact that the conjecture also holds for Dolgachev Calabi-Yau varieties in dimensions 7 and 9 seems to be evidence for this. To sum up, we believe Theorem 2.1.4 gives strong indications that the original Attractor Conjecture is false as stated.

2.2. The Attractor Conjecture for variations of Hodge structures. The dichotomy in Theorem 2.1.4 then suggests that the Attractor Conjecture holds if and only if the moduli space is a Shimura variety. Very few families of Calabi-Yau varieties are known to be parametrized by Shimura varieties; on the other hand, every Hermitian symmetric domain carries an (essentially unique) Calabi-Yau variation of Hodge structure (CYVHS), by which we mean a variation of Hodge structure with the same Hodge numbers as one coming from a Calabi-Yau variety. We therefore obtain CYVHS over many Shimura varieties. These were constructed by Gross [Gro94], [SZ10], and FL+13. We will refer to these as Gross’ VHS.
Remark 2.2.1. For most of these VHS, it is an open problem to show that they come from geometry. The most notorious is the example of the 56-dimensional VHS over the $E_7$ hermitian symmetric domain. In fact, it seems likely that this weight 3 VHS does not come from an honest Calabi-Yau threefold, but rather as a piece of the motive of some higher dimensional variety.

We may now formulate and study the Attractor Conjecture in these cases. To that end I have proven the following theorem, which covers all cases in which the Attractor Conjecture was previously known to be true.

Theorem 2.2.2 ([Lam20f, Theorem 1.2]). For each of the weight 3 VHS of CY type constructed by Gross and Sheng-Zuo, the attractor points are CM points. In other words, there exists a set theoretic map (which is the map “take the attractor point of a charge vector”) 

$$\text{Att} : V(\mathbb{Z})/G(\mathbb{Z}) \to \text{Sh},$$

whose image is (strictly) contained in the set of CM points. Here $G$ denotes the group for the Shimura variety, and $V$ is the representation of $G$ giving rise to the CYVHS.

Remark 2.2.3. As a result of Theorem 2.2.2, we obtain explicit parametrizations of (certain) CM points on Shimura varieties of $E_6$ and $E_7$ type; we are not aware of any previous such results.

We give an idea of the proof of Theorem 2.2.2. The key is to show that the attractor condition implies $H^{1,0} \oplus H^{0,3} \subset V \otimes C$ is actually a rational sub Hodge structure. To show this amounts to computing explicit periods, which we do using special geometry, a set of special coordinates well adapted to the situation first discovered by Strominger [Str90]. This seems to be the first appearance of these special coordinates in the proof of a mathematical result.

2.3. Relations to arithmetic invariant theory. Spaces similar to $V(\mathbb{Z})/G(\mathbb{Z})$ in Theorem 2.2.2 have been studied by Bhargava and his collaborators, and more relevant for us, by Pollack [Pol16]. Indeed, Pollack shows that these orbits parametrize quadratic rings $S$ along with a fractional ideal inside $S \otimes J$. The fraction field of $S$ is exactly the CM field obtained in 2.2.2. The following connects Pollack’s result to ours:

Proposition 2.3.1. The fraction field of $S$ is precisely the CM field in Theorem 2.2.2.

We plan to investigate further the relation between the more refined information, namely the integrality as well as the fractional ideals and the Hodge structures obtained here.

Furthermore, J. Thorne has constructed an intriguing map which seems to be a function field version of the map $\text{Att}$.

Theorem 2.3.2 ([Tho19]). Let $C/\mathbb{F}_q$ be a curve and $\text{Spec}\mathcal{O} \subset C$ an affine model. There is a group $G/C$, a free module $V$ over $\mathcal{O}$, and a map

$$V(\mathcal{O})/G(\mathcal{O}) \to \text{Bun}_C(C)(\mathbb{F}_q),$$

whose image equidistributes. Furthermore the integral orbits $V(\mathcal{O})/G(\mathcal{O})$ parametrizes 2-Selmer elements of elliptic curves over $C$.

We would like to view Theorem 2.2.2 as giving an arithmetic analogue of Thorne’s map: indeed, in Theorem 2.3.2, the domain of the map is the analogue of the vector space of charge vectors, and the target is the analogue of a Shimura variety. We will explore this analogy in future work. For example, I hope to study the following conjecture as well as its applications to arithmetic statistics:

Conjecture 2.3.3. For each of the cases in Theorem 2.2.2, we can define a function field analogue of $\text{Att}$ similar to Thorne’s map [4], whose image equidistributes.

Remark 2.3.4. F. Denef and M. Douglas [DD05], motivated by string theory again, have also formulated conjectures on the distribution of attractor points.

2.4. Other variants of the attractor mechanism. From the point of view of physics, the study of attractors has many variants, and in this section we mention one such. In general, black holes fall into two categories: BPS and non-BPS, and so far we have focused on the former. This distinction in the VHS of CY type cases may be seen as follows: as before we have an action of $G$ on $V$, and it turns out that there is precisely one invariant quartic form $I_4$ on $V$.

Definition 2.4.1. A charge vector $\gamma \in V$ is called BPS if it satisfies $I_4(\gamma) > 0$, and non-BPS otherwise.

Remark 2.4.2. In a specific example which we describe momentarily, the distinction is precisely that of binary cubic forms with negative discriminant (three real roots) versus those with positive discriminant (exactly one real root).
Non-BPS attractors have also been defined in the physics literature: it is no longer cut out by a Hodge theoretic condition, but rather it is defined by minimizing a real valued function depending on the charge vector $\gamma$. There is also a variant known as 5d attractors, and again these can be BPS or non-BPS. I have explicitly computed these attractor points (spaces):

**Theorem 2.4.3 ([Lam20d] Theorems 1.6, 1.8, 1.10]).**

1. The non-BPS attractors are totally geodesic subspaces inside the Shimura variety $S_h$.
2. In the example of the so-called $t^3$-model, the attractor points (both BPS and non-BPS) are given by Julia's covariant map.
3. In the case of 5d BPS attractors, the points are analogues of (zero-dimensional) Kudla-Millson cycles.

We describe the second part of Theorem 2.4.3. In this case the group is $\text{SL}_2$, and the representation is given by $V = \text{Sym}^3\text{Std}$, where $\text{Std}$ denotes the standard representation. The space $V$ can equivalently be viewed as the space of binary cubic forms, and as mentioned above the corresponding charge vector is BPS if we have positive discriminant, and non-BPS otherwise. Julia’s covariant map is an $\text{SL}_2$-equivariant map from the space of binary cubic forms to the upper half plane $\mathfrak{h}$ which we now describe. For a binary cubic form $Q = aX^3 + bX^2Y + cXY^2 + dY^3$, consider the associated polynomial

$$p = ax^3 + bx^2 + cx + d.$$ 

The roots of $p$ give three points $\alpha, \beta, \gamma \in \mathbb{P}^1(\mathbb{C})$. Viewing $\mathfrak{h}$ as the disk sitting in the interior of $\mathbb{P}^1(\mathbb{C})$, the Julia map sends $Q$ to the center of mass of $\alpha, \beta, \gamma$. Theorem 2.4.3 says that this gives the attractor point (in the BPS and non-BPS cases). This gives the first conceptual description of these attractor points, as far as I am aware.

The non-BPS attractor spaces are also reminiscent of moduli spaces of real abelian varieties studied by Goresky-Tai [GT03b] [GT03a]. We also intend to study this connection in detail in the future.

2.4.1. **Future work.** There are many other variants of the attractor mechanism that we have not mentioned. For example, there are theories with “more supersymmetry”, where one can again define attractors; I intend to analyze these in the future.

2.5. A conjecture of Wenzhe Yang. Higher dimensional attractor varieties has also been studied in an interesting recent work by Wenzhe Yang [Yan20]. In particular, by performing numerical calculations, he made several conjectures about the Hodge structures of Calabi Calabi-Yau hypersurfaces, which is the $n$-dimensional Calabi-Yau hypersurface $X$ inside $\mathbb{P}^{n+1}$ defined by the equation

$$X_0^{n+2} + X_1^{n+2} + \cdots + X_n^{n+2} = 0.$$ 

**Theorem 2.5.1 ([Lam20d] Theorem 1.1]).** The middle Hodge structure of $X$ admits a particular decomposition, confirming Yang’s conjecture.

For the precise statement we refer the reader to [Lam20d]. The proof uses crucially a beautiful inductive structure of Fermat hypersurfaces found by Katsura-Shioda [SK79].

2.6. Fourier expansion of quaternionic discrete series modular forms. In [GW96] Gross-Wallach studied automorphic forms in the quaternionic discrete series, which exist for a $\mathbb{Q}$-group $G$ only if $G(\mathbb{R})$ takes a special form. The groups $G(\mathbb{R})$ include the infinite series $\text{SO}(4, n), \text{SU}(2, n)$, as well as five other exceptional groups including a form of $E_8$; this list is closely related to the list of groups defining the Shimura varieties in Section 2.2. We will refer to these as quaternionic groups from now on. We will see that the attractor mechanism is closely related to quaternionic discrete series representations.

An important point is that the symmetric spaces $G/K$ no longer have hermitian structure, and are in some sense the next natural class to consider when one wants to go beyond Shimura varieties. Although it is still possible to define the Fourier coefficients of automorphic forms on these groups through representation theory, the theory of Fourier expansion into genuine functions as in the case of classical modular forms was lacking until recently. This was accomplished by Pollack recently, and in order to state this result we fix some notation. Any theory of Fourier expansion requires a choice of parabolic $P$, and then the Fourier modes are indexed by characters of its unipotent subgroup, while the Fourier coefficients are functions on the symmetric space of the Levi.

Coming back to the quaternionic groups, there is a natural choice of parabolic called the Heisenberg parabolic, whose Levi subgroups are precisely the groups attached to the Shimura varieties in Section 2.2. For example, the Levi associated to $\text{SO}(4, n)$ is $\text{SO}(2, n)$, the Levi associated to (the form of) $E_8$ is (a form of) $E_7$, and so

$${\text{exponent } n \text{ should be thought of as the nth power of the exponential map } G_n \to G_m. \text{ The Fourier coefficients are then functions on the symmetric space associated to the Levi (in this case they are just numbers).}}$$
Theorem 2.6.1 ([Pol20]). For each quaternionic group \( G \) (with the exception of the \( SU(2,n) \)-family) there exist certain explicit (non-holomorphic!) functions \( K_\gamma \) (see below for the indexing set \( \{ \gamma \} \)) on \( \mathcal{H} \) and a theory of Fourier expansions for automorphic forms of \( G \) in the quaternionic discrete series.

In what follows we will refer to these special functions as Pollack’s functions. I have shown that these functions agree with certain special wavefunctions which show up in the physical theories from [22].

Theorem 2.6.2. Pollack’s functions coincide with the black hole wavefunctions constructed in [NPV07].

Roughly, the wavefunctions are constructed as contour integrals on a twistor space, which gives rise to their non-holomorphicity. As an application we have the following

Corollary 2.6.3. A similar recipe gives rise to a theory of Fourier expansions for the quaternionic discrete series for the groups \( SU(2,n) \), which seems to not have been addressed by Pollack.

2.6.1. Future work. There are other variants of these supergravity theories whose moduli spaces are more general symmetric spaces: one could therefore hope that the black hole wavefunctions in these theories could lead to a theory of Fourier expansion for these groups.

3. Calabi-Yau varieties over finite fields

Since the formulation of the Weil conjectures, we have known that the study of varieties over finite fields, along with their cohomologies, is extremely rich. The theory of abelian varieties and K3 surfaces over finite fields is well developed since the pioneering work of Deuring, Tate, Deligne and many others, and it is therefore natural to ask what can be said or expected about Calabi-Yau varieties over finite fields. Here very little is known, and it is unclear what answers to expect for many basic questions. I will describe several aspects of this which I have studied; inevitably, due to our current lack of knowledge, much of this work is concerned with computations in specific interesting examples. In this section we will further demand that a Calabi-Yau variety satisfies the condition \( H^1(X,\mathcal{O}) = 0 \).

3.1. A conjecture of K. Joshi. We first discuss the question of lifting varieties from characteristic \( p \) to characteristic zero. For abelian varieties as well as K3 surfaces, liftings always exist; on the other hand, the same is not true for Calabi-Yau threefolds, and the first non-liftable example was due to Hirokado [Hir99]. In [Jos14], K. Joshi formulated a conjecture giving a criterion for lifting Calabi-Yau threefolds:

Conjecture 3.1.1 ([Jos14] Conjecture 7.7.1). Let \( X \) be a smooth proper Calabi-Yau threefold over a finite field \( k \). Then \( X \) lifts to characteristic zero if and only if

1. \( H^0(X,\Omega^1) = 0 \), and
2. \( X \) is classical.

Here classical essentially means that \( H^2(X,\mathbb{Q}) \neq 0 \). Guided by results from integral \( p \)-adic Hodge theory, I have exhibited a moduli space of counterexamples to Conjecture 3.1.1.

Theorem 3.1.2 (, mcy] There exists a family of counterexamples to Conjecture 3.1.1 in characteristic \( p = 5 \).

Sketch of proof of Theorem 3.1.2. The Calabi-Yau threefolds we use are the Godeaux Calabi-Yau threefolds constructed by Kim-Reid (c.f. [Rei19] as well as forthcoming work of Kim-Reid [RK]). Since their work has not appeared in print, I have also constructed the family of varieties that I need: see [Lam20c][Appendix A]. I have given a minimal construction of what is needed for Theorem 3.1.2 and the work of Kim-Reid gives a lot more. Our counterexamples \( X \) have the form

\[
X = Y/\mu_p,
\]

where \( Y \subset \mathbb{P}^4 \) is a hypersurface, and the \( \mu_p \)-action is free; moreover the construction can be done over \( \mathbb{Z}_p \). Now the generic fiber of \( X \) (i.e. over \( \mathbb{Q}_p \)) has 5-torsion in its Betti cohomology, and by the recent work of Bhattacharya-Brion [BMS18], this implies that the de Rham cohomology of the special fiber is non-zero. Actually, an older result of Caruso is more useful for us here and we will not need the full force of [BMS18]. In any case, the Hodge cohomologies of the special fiber are therefore non-zero, and we need to arrange for \( H^0(X_p,\Omega^1) \) to be non-zero, as well as for \( H^1(X_p,\mathcal{O}) \) to vanish. One then checks that taking a quotient by \( \mu_p \) has the desired effect.

Remark 3.1.3. Note that we have only addressed one direction of Joshi’s conjecture: it would be interesting to study the other direction, namely the statement that conditions (1) and (2) imply liftability.
3.2. Beauville-Bogomolov decomposition in positive characteristic. It turns out that the Godeaux Calabi-Yau threefolds also shed new light on the analogue of the Beauville-Bogomolov decomposition in positive characteristic. Over the complex numbers this result says that $K$-trivial varieties decompose into a product of varieties of three types: abelian, hyperkähler and Calabi-Yau, and is a powerful tool in analyzing $K$-trivial varieties in characteristic zero. The important question of finding an analogue in positive characteristic was pursued recently by Patakfalvi-Zdanowicz [PZ19]. Along the way, they posed the following Question, the answer to which would give us an idea of the best version of Beauville-Bogomolov decomposition one could hope for in positive characteristic.

**Question 3.2.1** ([PZ19 Question 13.6]). Let $k$ be an algebraically closed field of characteristic $p > 0$. Does there exist a finite $k_p$-quotient $X \to Y$ such that $X$ is a singular projective Gorenstein $K$-trivial variety and $Y$ is a smooth, weakly ordinary, projective $K$-trivial variety over $k$ with $\bar{q}(Y) = 0$? Here $\bar{q}(Y)$ denotes the augmented irregularity, namely

$$\bar{q}(Y) := \max\{\dim \text{Alb}_V, |Y' \to Y| \text{ finite } \text{étale}\}.$$

I have shown that the Godeaux Calabi-Yau threefolds (which featured also as the counterexamples to Joshi’s conjecture; see Section 3.1) provides a positive answer to Question 3.2.1.

**Theorem 3.2.2** (Theorem 1.2 of [Lam20b]). The generic (multiplicative) Godeaux Calabi-Yau threefold $Y$ and its 5-fold cover $X$ satisfy the conditions in Question 3.2.1.

**Remark 3.2.3.** One can construct many more examples satisfying the conditions of Question 3.2.1 by taking similar quotients.

3.3. A question of van der Geer-Katsura. In their study of heights of Calabi-Yau varieties over finite fields, van der Geer-Katsura [GK03, Section 7] asked whether Calabi-Yau threefolds in positive characteristic can have non-vanishing global 1-forms and 2-forms (since we are assuming $h^{01} = 0$ they certainly do not in characteristic zero). Theorem 3.1.2 gives a positive answer to the first part of their question.

**Corollary 3.3.1.** Calabi-Yau threefolds can have global 1-forms in positive characteristic.

3.4. Hodge-de Rham degeneration. In the last section, we saw the following phenomenon: $X$ is a Calabi-Yau threefold over $\mathbb{Z}_p$ such that the Hodge number $h^{10}$ is zero for the generic fiber, and is 1 for the special fiber. We will refer to this (for general $h^{ij}$’s) as the phenomenon of jumping Hodge numbers.

As alluded to above, torsion in Betti cohomology of the generic fiber will certainly lead to jumping of some Hodge numbers, and one may wonder whether this is the only situation in which this phenomenon occurs. Since the Hodge-de Rham (Hdr) spectral sequence degenerates in characteristic zero, if the Hdr spectral sequence does not degenerate for the special fiber, then we would most likely have an example where the jumping of Hodge number is not due to torsion in Betti cohomology. Indeed, I have proven that such an example exists, by computing all the cohomologies of a Calabi-Yau threefold constructed by Cynk-van Straten (referred to as the CvS Calabi-Yau threefold in the sequel): this is a Calabi-Yau threefold $X$ over a local ring $\mathcal{O}$ of mixed characteristic $(0, 5)$. We denote by $X_5$ (resp. $X[1/5]$) its special fiber (resp. generic fiber). The following Hodge numbers were computed by Cynk-van Straten:

$$h^{2,1}(X[1/5]) = 0, \ h^{2,1}(X_5) = 1.$$

**Theorem 3.4.1** ([Lam20c Theorem 1.5]). The crystalline cohomologies of $X_5$ are torsion free (and therefore Betti cohomology of $X[1/5]$ is also torsion free); furthermore, the Hdr spectral sequence does not degenerate at the $E_1$-page.

This theorem answers a question posed by Bhatt [Bha17 p.48 2(c)], who asked whether there exists a variety $X$ over $\mathcal{O}_{\mathbb{Z}_p}$ whose étale and crystalline cohomologies were torsion free, but the Hodge cohomologies have torsion. Theorem 3.4.1 gives, as far as I know, the first example of a variety whose Hdr spectral sequence does not degenerate, and such that there is no (crystalline or étale) torsion in cohomology: it is also liftable to a ring with ramification degree 2, giving another example that Hdr degeneration goes wrong as soon as there are no liftings over $W$, giving further examples to a question of Illusie and Ogus [Lan95]. I have also obtained a number of other properties of this variety (as well as another similar Calabi-Yau threefold constructed by Cynk-van Straten): for example, it is supersingular, in the sense that its Artin-Mazur formal group has infinite height. These results may be found in [Lam20c, Theorem 1.5]. The proof of Theorem 3.4.1 uses integral $p$-adic Hodge theory, as well as some classical algebraic topology.

I have also shown the non-degeneration of the Hdr spectral sequence for several other families of Calabi-Yau threefolds. In particular, the following answers a question of Ekedahl [Eke03]

**Theorem 3.4.2** ([Lam20c Corollary 1.7]). For the Hirokado threefold (alluded to above as the first example of a Calabi-Yau threefold admitting no characteristic zero lifts), the Hdr spectral sequence does not degenerate at the $E_1$-page.
Remark 3.4.3. One of the reasons that Calabi-Yau varieties form such a nice class of varieties is that, in characteristic zero, their deformation is unobstructed. As the Cynk-van Straten example shows, this is no longer true in positive characteristic. On the other hand, a recent work of Brantner-Mathew [BM19] provides a general framework to study deformations in positive characteristic, putting the latter on an almost equal footing as the characteristic zero story. One may therefore wonder whether their theory can shed light on deformations of Calabi-Yau varieties in positive characteristic; in any case, Calabi-Yau varieties seem like the natural starting point to try to apply the theory of [BM19].

3.5. Hodge numbers of derived equivalent Calabi-Yau threefolds. Since the formulation of the homological mirror symmetry by Kontsevich, much of algebraic geometry has been devoted to the study of the derived category \( D_{coh}^b(X) \) of a variety \( X \). One recurring theme is the question of how much information is contained in the derived category: for example, Popa-Schnell [PS11] have shown that for threefolds in characteristic zero, Hodge numbers are derived invariants. That is, for smooth proper threefolds \( X, Y \) over \( \mathbb{C} \) satisfying
\[
D^b_{coh}(X) \cong D^b_{coh}(Y),
\]
\( X \) and \( Y \) have the same Hodge numbers.

The same question in positive characteristic was studied recently by Antieau-Bragg [AB19], where they show the same result for threefolds under certain restrictions: in particular, they assume that the crystalline cohomologies of \( X \) and \( Y \) are torsion free. Inspired by the ideas in the previous section, I have suggested possible examples showing that Hodge numbers should not be derived invariants in general.

Our idea is as follows: as in the previous section, since torsion in Betti cohomology contribute to Hodge numbers in the special fiber, and the former is not invariant under derived equivalences (known by the work of Addington [Add17]), Hodge numbers should not be derived invariant in general. Implementing this strategy is work in progress, since the varieties in question have not been shown to exist rigorously (though they have been studied at a physical level of rigor [Sch06; DGS09]); these Calabi-Yau threefolds seem to be close analogues of Addington’s examples. We state the following

**Conjecture 3.5.1.** The Calabi-Yau threefolds in [Sch06] can be constructed rigorously, and (generically) have good reduction at primes \( p \) dividing the torsion in its Betti cohomology. Furthermore, the derived equivalences between these Calabi-Yau threefolds extend to characteristic \( p \).

This conjecture would imply that Hodge numbers are not derived invariants in characteristic \( p \).

To conclude this section, note that it is also possible for torsion in Betti cohomology to give rise to extra directions of deformations of Calabi-Yau threefolds in positive characteristic (since the latter is governed by the cohomology group \( H^1(\Omega^2) \)). To that end I have speculated on how these extra directions could arise as Moret-Bailley-type families: see [Lam20c, Section 9.2] for details.

3.6. Calabi-Yau varieties associated to Enriques surfaces. Oguiso-Schöer found a recipe to generate Calabi-Yau varieties of any even dimension, starting with an Enriques surface \( S \), which roughly are quotients of K3 surfaces by an involution. Indeed, they show that the Hilbert scheme of points \( \text{Hilb}^n(S) \) are Calabi-Yau for all \( n \geq 2 \); this should be compared to the construction of hyperkähler varieties by taking \( \text{Hilb}^n \) of K3 surfaces.

I have studied this infinite family of Calabi-Yau varieties in positive characteristic. In fact, the most interesting prime for us is \( p = 2 \), since Bombieri-Mumford discovered a rich theory of Enriques surfaces in characteristic 2: they found that they come in three distinct types, classical, singular, and supersingular. In each case, the Enriques surface \( S \) admits a double cover
\[
\pi: T \to S
\]
which is K3-like, and the map \( \pi \) is a torsor for a group scheme of order 2, with the group depending on which type \( S \) is. I have generalized this result to the varieties \( \text{Hilb}^n(S) \):

**Theorem 3.6.1 ([Lam20d, Theorem 1.1]).** The varieties \( \text{Hilb}^n(S) \) admit canonical simply connected Calabi-Yau-like double covers, which are torsors of the same type as the K3-like double cover of \( S \).

3.6.1. Future work. It should be possible to prove a similar result for other moduli of sheaves on Enriques surfaces, for example the Calabi-Yau varieties constructed by Sacca [Sac12], as well as starting with bi-elliptic surfaces, the last case of surfaces of Kodaira-dimension zero.

**References**


