ON CY SYMPLECTIC ASSOCIATED TO ENRIQUES SURFACES

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Abstract. In this paper we investigate a family of Calabi-Yau varieties found by Oguiso-Schr"oer in positive characteristic.

1. Introduction

In [OS11], Oguiso-Schr"oer constructed, in characteristic zero, Calabi-Yau varieties in every even dimension starting from an Enriques surface. Recall that an Enriques surface is one of the four classes of surfaces of Kodaira dimension 0: roughly, they are quotients of K3 surfaces by a fixed point free involution. Just as Hilbert schemes of points of K3 surfaces produce hyperk"ahler manifolds, one could ask what kind of variety we get by taking $Y := \text{Hilb}^n(S)$ for $S$ an Enriques surface, and this is precisely the construction of Oguiso-Schr"oer. Somewhat surprisingly, they found that $Y$ is in fact Calabi-Yau: more precisely, the universal cover $X$ of $Y$ is irreducible and Calabi-Yau, i.e. its holonomy is $\text{SU}(2n)$.

In the present work we begin the investigation of these Calabi-Yau varieties in positive characteristic. In fact, in characteristics $\neq 2$, all the known statements about $\text{Hilb}^n(S)$ are true; on the other hand, as in the case of Enriques surfaces, the story becomes much richer when $p = 2$: therefore in this paper we view $p = 2$ as being most interesting. Our main result, stated somewhat vaguely, is the following (for the precise statement see Theorem 3.1):

Theorem 1.1. The varieties $\text{Hilb}^n(S)$ admit canonical simply connected Calabi-Yau-like double covers, which are torsors of the same type as $S$.

We also prove several more results, including the compatibility with the characteristic zero versions of Oguiso-Schr"oer.

We now give some brief context to Theorem 1.1. Recall that Enriques surfaces come in three different types in characteristic 2: in terms of the K3(-like) cover $T \to S$ which is a $G$-torsor for a group scheme of order 2, the types are classified by which group $G$ is. By the classification of order 2 group schemes over fields of characteristic 2, there are three possibilities: if $G \cong \mu_2$, then $S$ is called classical; if $G \cong \mu_2$ then singular, and finally if $G \cong \alpha_2$ then $S$ is supersingular. Note that the covers $T$ will not be smooth in the classical and supersingular cases (essentially because smooth K3s cannot admit global non-vanishing vector fields), and hence they are referred to as K3-like, in that their cohomologies are the same as that of a smooth K3; in particular the dualizing sheaf is trivial. Theorem 1.1 therefore says that we have the same trichotomy (namely classical, singular, supersingular)
2. Recollections on Enriques surfaces

In this section we provide some facts about Enriques surfaces, especially in characteristic 2, as well as their K3-like covers. For details we refer the reader to [BM76] or [CD89].

**Definition 2.1.** An Enriques surface is a smooth, projective, and minimal surface $S$ of Kodaira dimension zero and $b_2 = 10$. Here $b_i$ denotes the $i$th (étale) Betti number.

Bombieri-Mumford [BM76] discovered the striking phenomenon that Enriques surfaces come in three distinct types in characteristic 2.

**Definition 2.2.** For an Enriques surface $S$ in characteristic 2, the Hodge number $h^{01}$ is either 0 or 1. $S$ is called classical if $h^{01} = 0$. On the other hand, if $h^{01} = 1$, and the action of Frobenius on $H^1(S, \mathcal{O}_S)$ is non-trivial, then $S$ is called singular. Finally, if $h^{01} = 1$ and the action of Frobenius is trivial, then $S$ is called supersingular.

**Theorem 2.3** (Bombieri-Mumford). Over a field of characteristic 2, each Enriques surface $S$ admits a cover $T \to S$ where $T$ is a $G$-torsor over $S$, for $G$ an order 2 group scheme. The group $G \cong \mu_2$ (resp. $\mathbb{Z}/2$, $\alpha_2$) if $S$ is classical (resp. singular, supersingular). Moreover, the surface $T$ is K3-like, in that

$$H^i(T, \mathcal{O}_T) \cong \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ 1 & \text{if } i = 2 \end{cases}$$

and the dualizing sheaf $\omega_T$ is trivial. When $S$ is singular, $T$ is smooth, and when $S$ is classical or supersingular, $T$ is not smooth.

3. Proof of the main theorem

**Theorem 3.1.** Let $S$ be an Enriques surface in characteristic 2. Then for all $n \geq 2$, $Y := \text{Hilb}^n(S)$ has a double cover $X$ which is Calabi-Yau, in the sense that its dualizing sheaf is trivial. Furthermore the covering $X \to Y$ is the same type as the canonical K3-type double cover of $S$: that is, they are both $G$-torsors for $G$ being one of $\mu_2$, $\mathbb{Z}/2$ or $\alpha_2$. Moreover, the cover $X$ is simply connected.

For any lift $\tilde{S}/R$ of $S$ over a local ring $R$ with residue field $k$, the cover $X$ lifts to $R$, and the generic fiber is precisely the double cover given by Oguiso-Schröer.

**Remark 3.2.** Note that, as in the case of the K3-like cover of $S$, the variety $X$ is probably not smooth in general.

We recall some facts about symmetric products from [BK07]. We have the following diagram

$$\begin{array}{ccc}
\text{Hilb}^n(S) & \xrightarrow{\gamma} & \mathcal{S}^{(n)} \\
\downarrow & & \\
S^n & \xrightarrow{\pi} & S^{(n)}
\end{array}$$

(1)
where $S^{(n)}$ is the $n$th symmetric product of $S$, with $\pi$ being the natural quotient map, and $\gamma$ the Hilbert-Chow morphism.

**Lemma 3.3.**

(a) The map

$$\mathcal{L} \mapsto \mathcal{L}^{(n)} := (\pi_\ast \mathcal{L}^{\boxtimes n}) \mathcal{S}_n$$

gives an injection $\text{Pic}(S) \to \text{Pic}(S^{(n)})$.

(b) There is a natural isomorphism

$$H^0(S^{(n)}, \mathcal{L}^{(n)}) \cong H^0(S, \mathcal{L}).$$

(c) Furthermore, we have

$$\omega_{S^{(n)}} = \omega_S^{(n)},$$

where for a variety $Z$, $\omega_Z$ denotes the canonical sheaf of $Z$.

**Proof.** These are all stated in [OS11], though some without proofs; here we provide proofs in the cases where we were not able to find appropriate references.

The fact that the map (2) sends line bundles to line bundles is [BK07, Lemma 7.1.1(iv)]. Now (b) follows from the fact that the statement is true for the trivial line bundle $\mathcal{O}$ (essentially by the construction of the symmetric power) and arbitrary varieties, and by covering $S$ by affine opens where $\mathcal{L}$ trivializes and gluing. This proves statement (b).

For the injectivity of $\mathcal{L} \mapsto \mathcal{L}^{(n)}$, suppose $\mathcal{L}$ gets sent to the trivial line bundle. By (b) we have a non-trivial section

$$s \in H^0(S, \mathcal{L}),$$

and similarly

$$t \in H^0(S, \mathcal{L}^{-1}).$$

Moreover the section

$$s \otimes t \in H^0(S, \mathcal{L} \otimes \mathcal{L}^{-1}) \cong H^0(S, \mathcal{O}_S)$$

is the identity, and therefore nowhere vanishing. Hence $s$ is nowhere vanishing as well, and we conclude that $\mathcal{L}$ is trivial as required. This concludes the proof of (a). Finally, (c) is [BK07, Lemma 7.1.7].

**Lemma 3.4.** Suppose $S$ is a classical Enriques surface. Then the line bundle $\gamma^\ast \omega_{S^{(n)}}$ is non-trivial.

**Proof of Claim.** Suppose it is trivial. Then we have

$$\pi_\ast \pi^\ast \omega_{S^{(n)}} \cong \pi_\ast \mathcal{O}_{\text{Hilb}^n(S)} \cong \mathcal{O}_{S^{(n)}},$$

and by adjunction we have a non-trivial map

$$\omega_{S^{(n)}} \to \mathcal{O}_{S^{(n)}}.$$  

Dualizing, we get a non-trivial map (here we use the fact that $\omega_{S^{(n)}}$ is 2-torsion)

$$\mathcal{O}_{S^{(n)}} \to \omega_{S^{(n)}},$$

and therefore $H^0(\omega_{S^{(n)}}) \neq 0$. Since $H^0(\omega_S) = H^0(\omega_{S^{(n)}})$, the latter is also non-trivial. However, for a classical Enriques surface $S$, the Hodge number $h^{20} = 0$ (c.f. [Ill79, Proposition 7.3.8]), so we have a contradiction. Therefore we must have $\gamma^\ast \omega_{S^{(n)}}$ non-trivial, as claimed.  

Lemma 3.5. Suppose $S$ is a singular or supersingular Enriques surface, and let $H$ denote the image of the map 
\[ \text{Pic}(S) \to \text{Pic}(S^{(n)}) \]
which is isomorphic to $\mu_2$ is $S$ is singular, and $\alpha_2$ if it is supersingular. Then the pullback map 
\[ \gamma^* : H \to \text{Pic}(\text{Hilb}^n(S)) \]
is injective.

Proof. We show that the map above is injective on tangent spaces. The unique non-zero element in $H(F[\epsilon]$ (with $\epsilon^2 = 0$) corresponds to a line bundle $L$ over $S^{(n)} \times_F F[\epsilon]$ (we also make the same definition for any scheme $T$ over $F$, as well as sheaves over them). Now consider the map 
\[ \gamma[\epsilon] : \text{Hilb}^n(S)[\epsilon] \to S^{(n)}[\epsilon], \]
and suppose for the sake of contradiction that $\gamma[\epsilon]^* L$ is trivial on $\text{Hilb}^n(S)[\epsilon]$. Since $\gamma_* O \cong O$, by adjunction, we have a map 
\[ L \to O_{S^{(n)}[\epsilon]}, \]
whose reduction mod $\epsilon$ is the identity, and dualizing we obtain a section $s$ of $L$, whose reduction mod $\epsilon$ is nowhere vanishing. Therefore $s$ itself is nowhere vanishing, and hence $L$ is trivial, which is a contradiction. Therefore the pullback back $\gamma^*$ is injective when restricted to $H$, as required. □

We record the following lemma of Bombieri-Mumford for later use:

Lemma 3.6 (Proposition 9 of [BM76]). Suppose $G$ is a finite group scheme. Then for any $G$-torsor $X \to Y$, 
\[ g^* \omega_Y \cong \omega_X. \]
Here $\omega_Z$ denotes the dualizing sheaf.

Now we may prove Theorem 3.1

Proof of Theorem 3.1. Regardless of the type of $S$, $\omega_{\text{Hilb}^n(S)}$ is always 2-torsion (though possibly trivial), since by Lemma 3.3 
\[ \omega_{S^{(n)}} = \omega_{S}^{(n)}, \]
and we have $\omega_{S}^{\otimes 2} \cong O_S$, as well as the fact that the map $\gamma$ is a crepant resolution. Furthermore, $\omega_S$ is trivial if $S$ is singular or supersingular, and therefore the same is true of $\omega_{\text{Hilb}^n(S)}$ in these cases. On the other hand, Lemma 3.4 implies $\omega_{\text{Hilb}^n(S)}$ is non-trivial.

Now in each of the cases we have an injection 
\[ \text{Pic}(S) \to \text{Pic}(\text{Hilb}^n(S)), \]
which gives a non-trivial double cover of $\text{Hilb}^n(S)$, of the same type as the canonical K3-like cover of $S$; let 
\[ \pi : X \to Y \]
denote the covering map. Now in the classical case, the 2-torsion element in $\text{Pic}(\text{Hilb}^n(S))$ giving rise to this cover is precisely $\omega_{\text{Hilb}^n(S)}$, and hence 
\[ \pi^* \omega_Y \]
is trivial, and by Lemma 3.6, \( \pi^* \omega_Y \cong \omega_X \), and hence \( X \) has trivial dualizing sheaf. For \( S \) singular or supersingular, \( \omega_S \) is already trivial, and hence the same is true of \( \omega_Y \) and \( \omega_X \).

Now we come to the fundamental groups of \( Y = \text{Hilb}^n(S) \) and its double cover \( X \). The variety \( Y \) admits lifts to characteristic zero since \( S \) itself does: let \( \tilde{S} \) be a lift of \( S \), and let \( \tilde{Y} \) denote \( \text{Hilb}^n(\tilde{S}) \). Since

\[
\pi_1(Y) \cong \mathbb{Z}/2,
\]

we have that \( \pi_1^d(Y) \) is either trivial or \( \mathbb{Z}/2 \). In the case that \( S \) is singular, we have exhibited a non-trivial étale double cover of \( Y \), namely \( X \), and hence \( \pi_1^d(Y) \cong \mathbb{Z}/2 \) and \( X \) is simply connected.

Now suppose \( S \) is either classical or supersingular, and that \( Y \) has an étale double cover. We can lift this to a double cover of \( \tilde{Y} \), which corresponds to a subgroup

\[
\mu_2 \subset \text{Pic}(\tilde{Y}).
\]

Now over the generic fiber the non-trivial element in this \( \mu_2 \) subgroup must be the canonical bundle \( \omega \). However, by considering the composition

\[
\text{Pic}(\tilde{S}) \to \text{Pic}(\text{Hilb}^n(\tilde{S})) \to \text{Pic}(\tilde{Y})
\]

as before, and using the fact that \( \omega_{\tilde{S}} \) gets sent to the canonical bundle, our arguments above showed that the scheme theoretic closure of the subgroup generated by

\[
\omega \in \text{Pic}(\tilde{Y}[1/2])
\]

(here for a scheme \( Z \) over \( R \) we denote by \( Z[1/2] \) the fiber product of \( Z \) and \( R[1/2] \)) must have \( \mathbb{Z}/2 \) as its special fiber in the classical case, and \( \alpha_2 \) in the supersingular case, giving a contradiction. Therefore \( Y \) is simply connected in these cases, as is \( X \) since the covering map \( X \to Y \) is inseparable in these cases.

Finally we prove the statement about lifts of \( S \) and its associated varieties. Suppose we have a lift \( \tilde{S} \) of \( S \) over a local ring \( R \); then \( \text{Hilb}^n(S) \) has an obvious lift. Furthermore, the map

\[
\text{Pic}(S) \to \text{Pic}(\text{Hilb}^n(S))
\]

naturally lifts as well, and is again an injection by the same argument as before. Therefore the image of the lift

\[
\text{Pic}(\tilde{S}) \to \text{Pic}(\text{Hilb}^n(\tilde{S}))
\]

map gives an order 2 subgroup of \( \text{Pic}(\text{Hilb}^n(\tilde{S})) \), and hence a torsor for the Cartier dual of this group scheme, whose special fiber is simply the cover \( X \) from before. Its generic fiber is precisely the cover constructed by Oguiso-Schröer, since in the generic fiber we may identify this order 2 group scheme as

\[
\mathbb{Z}/2 \to \text{Pic}(\text{Hilb}^n(\tilde{S})[1/2]),
\]

where the non-trivial element of \( \mathbb{Z}/2 \) is sent to the canonical bundle. \( \Box \)

References


