PRO-EXCISION OF ALGEBRAIC $K$-THEORY

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Abstract. This is an expository article on the pro-excision theorem of algebraic $K$-theory for abstract blow-up squares, due to Kerz-Strunk-Tamme [KST18].

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0.1. **Excision theorems.**

0.1.1. Algebraic $K$-theory grew out of a series of elaborate invariants attached to a ring and has found numerous applications in algebraic geometry and number theory. Given a ring $R$, the abelian groups $K_i(R)$ are difficult to compute in general. Hence one hopes that calculation of $K_i(R)$ can be reduced to the $K$-groups attached to other, potentially simpler rings. A basic tool in algebraic topology that allows one to calculate cohomology groups is the Mayer-Vietoris sequence, so one may ask for an analogue that works for algebraic $K$-groups.

0.1.2. A prototype of the Mayer-Vietoris sequence arises from Zariski coverings of schemes. Let us take $X$ to be a nice (e.g. quasi-compact and quasi-separated) scheme together with an open covering $X = U \cup V$, and let us say that the algebraic $K$-theory of commutative rings has been extended to such schemes. Then one obtains a long exact sequence:

$$
\cdots \to K_i(X) \to K_i(U) \oplus K_i(V) \to K_i(U \cap V) \to \cdots
$$

A more conceptual way to phrase this result uses the $K$-theory spectrum $K(X)$ attached to $X$. The abelian group $K_i(X)$ is recovered as the $i$th homotopy group of $K(X)$. The Mayer-Vietoris sequence can be seen as coming from a homotopy Cartesian diagram of spectra:

$$
\begin{array}{ccc}
K(U) & \to & K(U \cap V) \\
\downarrow & & \downarrow \\
K(V) \to & & K(V)
\end{array}
$$

We thus obtain the Zariski **excision** theorem: algebraic $K$-theory, regarded as a functor, turns the following Cartesian square of schemes into a homotopy Cartesian square of spectra:

$$
\begin{array}{ccc}
U \cap V & \to & U \\
\downarrow & & \downarrow \\
V & \to & X
\end{array}
$$

(0.1)

0.1.3. Zariski coverings form a very coarse Grothendieck topology on the category of schemes. In order to gain more computational power, one needs excision for a finer class of coverings. Among the flat topologies, it is possible to upgrade the excision theorem to Nisnevich covers in place of (0.1). These excision results are in turn used to prove that the $K$-theory spectrum, as opposed to the individual $K$-groups, satisfies Nisnevich descent. However, the same excision statement is already false for étale coverings.

0.1.4. In another direction, one can look for excision results for certain proper coverings. We call a Cartesian square of schemes:

$$
\begin{array}{ccc}
E & \to & \bar{X} \\
\downarrow & & \downarrow \\
Y & \overset{i}{\to} & X
\end{array}
$$

(0.2)

an **abstract blow-up** square if

(a) $i$ is a closed immersion; and

(b) $p$ is a proper map that induces an isomorphism $ar{X} \setminus E \xrightarrow{\sim} X \setminus Y$.

---

1In this article, all rings (or simplicial rings) are assumed to be commutative and unital.
Excision for abstract blow-up squares is a strong property. A theorem of D.-C. Cisinski \cite{Ci13} shows that the $\mathbb{A}^1$-localization of algebraic $K$-theory $L_{\mathbb{A}^1}K$ turns (0.2) into a homotopy Cartesian square of spectra. Using this theorem, M. Kerz and F. Strunk \cite{KS16} gave a quick (and clever) proof of the vanishing of the $i$th $L_{\mathbb{A}^1}K$-group of a Noetherian scheme $X$ of finite Krull dimension, for all $i < -\dim(X)$.

0.1.5. Unfortunately, excision for abstract blow-up squares fails for algebraic $K$-theory without $\mathbb{A}^1$-localization. As we shall see, a functor that satisfies excision for abstract blow-up squares cannot detect the difference between a scheme $X$ and its reduced subscheme $X_{\text{red}}$, whereas algebraic $K$-theory captures information about non-reduced structures. This leads one to suspect that in order to have any form of excision for proper covers, one must make adjustments that take into account the non-reduced structures.

0.1.6. Given an abstract blow-up square (0.2), denote by $Y^{(n)}$ (resp. $E^{(n)}$) the $n$th infinitesimal neighborhood of $Y$ inside $X$ (resp. $E$ inside $\tilde{X}$). The main result that we will explain in this article is the following theorem of M. Kerz, F. Strunk, and G. Tamme \cite{KST18}: given an abstract blow-up square (0.2), the following square of pro-spectra is homotopy Cartesian:

$$
\begin{array}{ccc}
K(X) & \longrightarrow & \lim_n K(E^{(n)}) \\
\downarrow & & \downarrow \\
K(\tilde{X}) & \longrightarrow & \lim_n K(Y^{(n)})
\end{array}
$$

(0.3)

We emphasize that the limit here is taken in the $\infty$-category of pro-spectra and not that of spectra. Using this pro-excision theorem, the same quick proof yields the vanishing of $K_i(X)$ for all Noetherian schemes $X$ of finite Krull dimension and $i < -\dim(X)$, settling a conjecture of C. Weibel that had been open for more than 30 years.

0.1.7. We should mention that the idea of pro-excision, as well as the host of insights that led to its proof, has origins in the works of A. Suslin, M. Wodzicki, G. Cortiñas, T. Geisser, L Hesselholt, M. Morrow, and many others. Although we do not specify their contributions in the introduction, this lineage of ideas will become visible in the main body of the text. On the other hand, the work of Kerz-Strunk-Tamme brings a new and powerful input, which makes an essential use of derived algebraic geometry as developed by J. Lurie. More concretely, derived structures enter the picture in the following ways:

(a) The excision result of G. Cortiñas, C. Haesemeyer, M. Schlichting and C. Weibel \cite{CHSW08} for blow-ups at regularly immersed centers can be generalized to derived locally complete intersections. This gives us a large class of blow-up squares where excision holds on the nose. One can then use such squares to approximate an abstract blow-up square.

(b) The ring-theoretic pro-excision theorem of T. Geisser, L. Hesselholt \cite{GH06} and Morrow \cite{Mo18} holds word-for-word for derived rings. Combined with a classical Artin-Rees argument, one deduces pro-excision for abstract blow-up squares (0.2) where $p$ is a finite morphism between derived schemes$^2$. This makes the approximation argument in (a) possible, and in particular, allows the passage from derived schemes back to classical ones.

0.2. “Autor del Quixote”.

$^2$Although the complement $\tilde{X}\setminus E \xrightarrow{\sim} X\setminus$ is still required to be classical.
0.2.1. This expository article has roughly the same length and content as the original proof of [KST18]. Moreover, we do not claim to have made any mathematical contribution—most proofs of this article are mere paraphrases of loc.cit. It then begs the question: why does this article exist?

0.2.2. The answer is that we make some arguments more formal, as an attempt to render the proof conceptually simpler. We now list the main parts in which the present article differs from [KST18]:

(a) In this article, we use the language of (derived) ind-schemes. This provides us with a more robust framework to work with formal completions and their $K$-theory pro-spectra.

(b) We dispense of any use of the “derived exceptional divisor” $D$ (c.f. Definition 3.1 of loc.cit.) in the proof of the pro-excision theorem, and only work with the “semi-derived exceptional divisor” $E$, which we believe to be the correct notion of the exceptional divisor for a derived blow-up.

(c) The original proof features prominently a particular sequence of ideals $(a_1^n, \cdots, a_r^n)$ defining the formal completion of a closed subscheme with ideal $(a_1, \cdots, a_r)$. We remove the use of this sequence whenever it is possible, the only exception being the “trick” that deduces pro-excision for classical blow-up squares from derived ones.

(d) The original proof of excision for derived blow-ups (Theorem 3.7 of loc.cit.) repeats a large part of the proof of classical blow-ups at regularly immersed centers by Cortiñas-Haesemeyer-Schlichting-Weibel. We show, instead, that it follows formally from the latter using the Lurie tensor product for stable co-complete $\infty$-categories. The same goes for the proof of the projective bundle formula for derived schemes.

0.2.3. Because of the expository nature of this article, we try to be as self-contained as possible. Whenever it is reasonable to include a proof or a sketch of ideas for a standard fact, we choose to do so. A lot of the arguments from Suslin-Wodzicki [SW92] are also spelled out in detail as they were used to prove the ring-theoretic pro-excision theorem in §2.

0.3. Acknowledgements. The author thanks Mike Hopkins for his minor-thesis supervision and many helpful conversations. It was also a pleasure to attend the workshop on pro-excision organized by Ben Antieau, Elden Elmanto, and Jeremiah Heller in May 2018; the author thanks them sincerely for this great opportunity and their hospitality. Along with them, Marc Hoyois patiently answered many of the author’s questions about $K$-theory.

1. Algebraic $K$-theory

In this section, we collect general facts about stable $\infty$-categories and define $K$-theory via its universal property, following Blumberg-Gepner-Tabuada [BGT13]. Then we turn to a preliminary study of the $K$-theory of quasi-compact, quasi-separated derived schemes. We show that it behaves very much like the $K$-theory of classical schemes: it satisfies Zariski descent, the projective bundle formula, the Bass fundamental theorem, etc.

Then we study derived blow-up squares. Its definition is rigged in a way that allows one to formally transport the excision results of Cortiñas-Haesemeyer-Schlichting-Weibel [CHSW08] for classical blow-ups at regularly immersed centers to the derived context. Finally, we state the main pro-excision theorem of Kerz-Strunk-Tamme [KST18].

1.1. Stable $\infty$-categories.
1.1.1. Recall that an ∞-category $\mathcal{C}$ is stable if it satisfies the following conditions:

(a) $\mathcal{C}$ has a zero object;
(b) $\mathcal{C}$ has all finite limits and finite colimits;
(c) a square in $\mathcal{C}$ is a pullback if and only if it is a push-out:

\[
\begin{array}{ccc}
c & \to & c' \\
\downarrow & & \downarrow \\
d & \to & d'
\end{array}
\]

The natural functors to consider between stable ∞-categories are exact functors, i.e., those that preserve finite limits and colimits. We denote by $\text{Cat}_{\text{ex}}^\text{st}$ the ∞-category of small, stable ∞-categories together with exact functors.

1.1.2. We define $\text{Cat}_{\text{idem}}^\text{st} \hookrightarrow \text{Cat}_{\text{ex}}^\text{st}$ to be the full subcategory of idempotent-complete ∞-categories. This inclusion admits a left adjoint, sending a small stable ∞-category $\mathcal{C}$ to its idempotent completion $\mathcal{C}_{\text{idem}}$. One can show that the ∞-categories $\text{Cat}_{\text{idem}}^\text{st}$ and $\text{Cat}_{\text{ex}}^\text{st}$ have all limits and colimits (c.f. [BGT13, Corollary 4.25]).

1.1.3. Let $\text{Cat}_{\text{cont}}^{\text{co-comp}}$ denote the ∞-category of co-complete ∞-categories together with continuous (i.e. colimit-preserving) functors between them. We may also restrict our attention to the full subcategory $\text{Cat}_{\text{cont}}^{\text{st,co-comp}}$ of stable ones. Note that a continuous functor between stable co-complete ∞-categories is automatically exact. The process of ind-completion defines a functor:

$$\text{Ind} : \text{Cat}_{\text{ex}}^{\text{st, idem}} \to \text{Cat}_{\text{cont}}^{\text{st, co-comp}}$$ (1.1)

which identifies $\text{Cat}_{\text{ex}}^{\text{st, idem}}$ with the 1-full subcategory of $\text{Cat}_{\text{cont}}^{\text{st, co-comp}}$ whose:

(a) objects are compactly generated stable co-complete ∞-categories;
(b) morphisms are continuous functors $F : \mathcal{C} \to \mathcal{D}$ preserving compact objects.

Indeed, the inverse process is given by taking the compact objects of a co-complete stable ∞-category $\mathcal{C} \rightsquigarrow \mathcal{C}^\omega$.

1.1.4. The following criterion for the continuity of right adjoints turns out to be very useful:

**Lemma 1.1.** Suppose $\mathcal{C}, \mathcal{D}$ are stable co-complete ∞-categories. If $\mathcal{C}$ is compactly generated and $F : \mathcal{C} \to \mathcal{D}$ is a continuous functor. Then its right adjoint $F^R$ is continuous if and only if $F$ preserves compact objects.

**Proof.** It is clear that the continuity of $F^R$ implies that $F$ preserves compact objects. To prove the converse, one notes that the hypothesis implies that $F^R$ preserves filtered colimits. It remains to see that $F^R$ also preserves finite colimits, which in turn are generated by push-outs and the initial object. The functor $F^R$ preserves them because it is a limit-preserving functor between stable ∞-categories. □

---

3To be careful about set-theoretic issues, one should consider “presentable” ∞-categories. We will, however, ignore set-theoretic issues. So we do not mention “accessibility” and the word “presentable” is synonymous to “co-complete.”
1.1.5. We now define the notion of exact sequences of co-complete stable ∞-categories. Let
\[ C' \xrightarrow{F} C \xrightarrow{G} C'' \tag{1.2} \]
be a triangle of co-complete stable ∞-categories (i.e., we are supplied with a null-homotopy of the composition \( C' \to C'' \)). We say that (1.2) is an exact sequence if
(a) \( C' \to C \) is fully faithful;
(b) the map from the cofiber is an equivalence \( C/\!\!/C' \xrightarrow{\sim} C'' \);
(c) the right adjoints \( F^R \) and \( G^R \) are continuous.

Here, the cofiber is calculated in the ∞-category \( \text{Cat}_{\text{co-comp}}^{\text{st}} \).

1.1.6. One can make the above condition (b) more explicit. Indeed, since \( G \circ G^R : C'' \to C'' \) is another continuous functor whose composition with \( C \xrightarrow{G} C'' \) identifies with \( G \), the universal property of the cofiber shows that \( G \circ G^R \xrightarrow{\sim} \text{id}_{C''} \) is an equivalence. In other words, \( G^R \) is fully faithful. Treating \( C' \) and \( C'' \) both as full subcategories of \( C \), one may then show that \( C'' \) identifies with the right orthogonal \( (C')^\perp \), and \( C' \) identifies with the left orthogonal \( ^\perp (C'') \). In particular, there is a canonical triangle attached to any \( c \in C \):
\[ FF^R c \to c \to G^R G c. \tag{1.3} \]

1.1.7. A triangle of small stable ∞-categories \( C' \to C \to C'' \) is an exact sequence if their ind-completion is one in the sense of §1.1.5:
\[ \text{Ind } C' \to \text{Ind } C \to \text{Ind } C''. \]

Remark 1.2. Note that by Lemma 1.1, the condition (c) in §1.1.5 becomes superfluous. One can characterize exact sequences of small stable ∞-categories in a more intrinsic manner, without referring to their ind-completions. It is proved in [BGT13, §5] that exactness is equivalent to either of the following statements:
(a) The functor \( C' \to C \) is fully faithful and \( C/\!\!/C' \to C'' \) becomes an equivalence after idempotent completion.
(b) The homotopy categories \( \text{Ho}(C') \to \text{Ho}(C) \to \text{Ho}(C'') \) is an exact sequence of triangulated categories.

1.1.8. Recall the symmetric monoidal structure on \( \text{Cat}_{\text{cont}}^{\text{st,co-comp}} \) given by the Lurie tensor product \( \otimes \). It induces a symmetric monoidal structure on \( \text{Cat}_{\text{ex,ident}}^{\text{st}} \) in a way that makes the ind-completion functor (1.1) is symmetric monoidal. More precisely, given \( \mathcal{C}, \mathcal{D} \in \text{Cat}_{\text{ident}}^{\text{ex}} \), their tensor product is given by:
\[ \mathcal{C} \otimes \mathcal{D} := (\text{Ind } \mathcal{C} \otimes \text{Ind } \mathcal{D})^{\omega}, \]
i.e., the subcategory of compact objects in \( \text{Ind } \mathcal{C} \otimes \text{Ind } \mathcal{D} \).

1.1.9. We note the variants of the above constructions in the relative setting. Suppose \( \emptyset \in \text{Cat}_{\text{cont}}^{\text{st,co-comp}} \) is a symmetric monoidal object. A triangle:
\[ \mathcal{C}' \xrightarrow{F} \mathcal{C} \xrightarrow{G} \mathcal{C}'' \tag{1.4} \]
of \( \emptyset \)-module objects in \( \emptyset \in \text{Cat}_{\text{cont}}^{\text{st,co-comp}} \) is an exact sequence if:
(a) \( \mathcal{C}' \to \mathcal{C} \) is fully faithful;
(b) the map from the cofiber is an equivalence \( \mathcal{C}/\!\!/\mathcal{C}' \xrightarrow{\sim} \mathcal{C}'' \) in \( \emptyset \text{-Mod}(\text{Cat}_{\text{cont}}^{\text{st,co-comp}}) \);
(c) the right adjoints \( F^R \) and \( G^R \) are continuous and \( \emptyset \)-linear.

\textsuperscript{4}A priori, they are only right-lax \( \emptyset \)-linear, i.e., one has a canonical morphism \( F^R (c) \otimes o \to F^R (c \otimes o) \) which is not necessarily an isomorphism.
By the identification of $\mathcal{C}''$ as the fiber of $F^R : \mathcal{C} \to \mathcal{C}'$, we see that the $\mathcal{O}$-linearity of $G^R$ follows from that of $F^R$. Suppose $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{O}$-modules in $\text{Cat}_{\text{cont}}^{\text{st,co-comp}}$. Then we can form an $\mathcal{O}$-module $\mathcal{C} \otimes \mathcal{D}$ as the geometric realization of the bar complex:

$$\cdots \longrightarrow \mathcal{C} \otimes \mathcal{O} \otimes \mathcal{D} \longrightarrow \mathcal{C} \otimes \mathcal{D}.$$ 

1.1.10. The version for small categories is similar. Suppose $\mathcal{O} \in \text{Cat}_{\text{ex}}^{\text{st, idem}}$ is a symmetric monoidal object. Then a triangle of $\mathcal{O}$-modules $\mathcal{C}' \to \mathcal{C} \to \mathcal{C}''$ is an exact sequence if their ind-completion is an exact sequence of $\text{Ind}(\mathcal{O})$-modules in $\text{Cat}_{\text{cont}}^{\text{st,co-comp}}$. The tensor product of $\mathcal{C}, \mathcal{D} \in \mathcal{O}-\text{Mod}(\text{Cat}_{\text{ex}}^{\text{st,idem}})$ is set to be:

$$\mathcal{C} \otimes \mathcal{D} := (\text{Ind} \mathcal{C} \otimes \text{Ind} \mathcal{D})^{\omega}.$$ 

One can show that the category $\text{Ind} \mathcal{C} \otimes \text{Ind} \mathcal{D}$ is also compactly generated.

**Lemma 1.3.** Given an exact sequence $\mathcal{C}' \to \mathcal{C} \to \mathcal{C}''$ of $\mathcal{O}$-modules in $\text{Cat}_{\text{ex}}^{\text{st, idem}}$ and another $\mathcal{O}$-module $\mathcal{D}$, then the following triangle:

$$\mathcal{C}' \otimes \mathcal{D} \to \mathcal{C} \otimes \mathcal{D} \to \mathcal{C}'' \otimes \mathcal{D}$$

remains exact.

**Proof.** It suffices to show the corresponding statement for $\text{Cat}_{\text{cont}}^{\text{st,co-comp}}$. In the absolute case, this is [BGT13, Lemma 9.35]. The fact that the colimit of $\mathcal{C}' \otimes \mathcal{O} \to \mathcal{C} \otimes \mathcal{D}$ identifies with $\mathcal{C}'' \otimes \mathcal{D}$ is straightforward. To show that $\mathcal{C}' \otimes \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ remains faithfully, we use the fact that $F : \mathcal{C}' \to \mathcal{C}$ admits a continuous, $\mathcal{O}$-linear right adjoint $F^R$ and the unit $\text{id}_{\mathcal{C}'} \sim F^R F$ is an isomorphism.

**Remark 1.4.** The symmetric monoidal objects $\mathcal{O} \in \text{Cat}_{\text{cont}}^{\text{st,co-comp}}$ we will consider in this paper are all rigid in the sense of [GR17, I.1, §9]. The module categories over $\mathcal{O}$ are much better behaved than the general case. In particular, one can show that every lax $\mathcal{O}$-linear functor $\mathcal{M} \to \mathcal{N}$ of $\mathcal{O}$-module categories is in fact $\mathcal{O}$-linear, c.f. Lemma 9.3.6 of loc. cit. This makes the requirement on $\mathcal{O}$-linearity in §1.1.9(c) redundant.

### 1.2. Definition of $K$.

1.2.1. Let $\text{Sptr}$ denote the stable $\infty$-category of spectra. It is the unit object of the $\infty$-category of stable, co-complete $\infty$-categories equipped with the Lurie tensor structure. We define algebraic $K$-theory as the initial functor:

$$K : \text{Cat}_{\text{ex}}^{\text{st}} \to \text{Sptr}$$

satisfying the following properties:

(a) $K$ applied to the trivial stable $\infty$-category is equivalent to the zero spectrum;

(b) The canonical map $K(\mathcal{C}) \to K(\mathcal{C}_{\text{idem}})$ is an equivalence;

(c) $K$ preserves filtered colimits;

(d) $K$ sends exact sequences in $\text{Cat}_{\text{ex}}^{\text{st}}$ to exact triangles in $\text{Sptr}$.

**Remark 1.5.** If instead of the stable $\infty$-category $\text{Sptr}$, our target category is some triangulated category $\mathcal{T}$, then (a) would be a consequence of (d) by considering the exact sequence $\mathcal{C} \to \mathcal{C} \to 0$ for any stable $\infty$-category $\mathcal{C}$. The same argument does not apply here because for a sequence in $\text{Sptr}$ to be an exact triangle requires the additional datum of a null-homotopy, which must be supplied by the given exact sequence in $\text{Cat}_{\text{ex}}^{\text{st}}$. To put it differently, (d) only makes sense as a condition if we already have (a).
Remark 1.6. Property (b) is commonly referred to as “$K$ inverts Morita equivalence”, and (d) is referred to as “$K$ is a localizing invariant.”

1.2.2. We note that connective algebraic $K$-theory $K^{\text{conn}}$ does not satisfy property (d). One can characterize $K^{\text{conn}}$ in a similar manner by replacing (d) with the following weaker property:

(d’) $K$ sends split exact sequences in $\text{Cat}_{\text{ex}}^\text{st}$ to exact triangles in $\text{Sptr}$.

Here, a triangle in $\text{Cat}_{\text{ex}}^\text{st}$:

$$
\mathcal{E}' \xrightarrow{F} \mathcal{E} \xrightarrow{G} \mathcal{E}''
$$

is split exact if $F$ and $G$ admit exact right adjoints such that the unit (resp. counit) is an equivalence on $\mathcal{E}'$ (resp. $\mathcal{E}''$).

1.2.3. One can regard the main theorem of [BGT13, §9] as the following assertion:

**Theorem 1.7.** The functor (1.5) exists.

1.3. $K$-theory of (derived) schemes.

1.3.1. The $K$-theory of a (quasi-compact, quasi-separated) scheme $X$ can be seen as the functor (1.5) applied to the stable $\infty$-category of perfect complexes on $X$. As such, the definition generalizes immediately to derived schemes.

1.3.2. We will use $\text{Sch}$ to denote the $\infty$-category of derived schemes. It is locally modeled on the $\infty$-category of simplicial commutative rings $\text{sRing}$, although the results in this section are independent of the choice of the model. Clearly, $\text{Sch}$ contains a full subcategory $\text{Sch}^\text{cl}$ of classical schemes. We let $\text{Sch}_{\text{qc-qs}}$ (resp. $\text{Sch}^\text{cl}_{\text{qc-qs}}$) denote the full subcategory of quasi-compact quasi-separated derived (resp. classical) schemes. The algebraic $K$-theory of a (derived) scheme is the following composition of functors:

$$
K : \text{Sch}_{\text{qc-qs}}^{\text{op}} \xrightarrow{\text{Perf}} \text{Cat}_{\text{ex}}^\text{st} \xrightarrow{K} \text{Sptr}
$$

where $\text{Perf}$ associates to a (derived) scheme $\mathcal{X}$ the small stable $\infty$-category of perfect complexes on $\mathcal{X}$, and to a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ the pullback functor $f^*$.

1.3.3. We now turn to specific properties of the stable $\infty$-category $\text{Perf}(\mathcal{X})$ for a derived scheme $\mathcal{X}$. The following assertions are taken from [BZFN10] but they have a much longer history going back to the work of R. Thomason and A. Neeman.

**Lemma 1.8.** Let $\mathcal{X} \in \text{Sch}_{\text{qc-qs}}$. Then the natural functor $\text{Ind Perf}(\mathcal{X}) \rightarrow \text{QCoh}(\mathcal{X})$ is an equivalence and identifies $\text{Perf}(\mathcal{X})$ as $\text{QCoh}(\mathcal{X})^\omega$.

*Sketch of proof.* This is Proposition 3.19 of loc.cit. (the result there requires $\mathcal{X}$ to have affine diagonal but this assumption is immaterial.) The case for $\mathcal{X}$ affine is immediate, and one proves the general case by induction on the number of affines in an open affine cover of $\mathcal{X}$. The key tool in the induction step is an “extension” result of compact objects, c.f. [TT90] and [Ne92]. $\square$

**Lemma 1.9.** For any diagram $\mathcal{X} \rightarrow \mathcal{Y} \leftarrow \mathcal{Y}'$ in $\text{Sch}_{\text{qc-qs}}$, the following functor is an equivalence in $\text{Cat}_{\text{idem}}$:

$$
\text{Perf}(\mathcal{X}) \otimes_{\text{Perf}(\mathcal{Y})} \text{Perf}(\mathcal{Y}') \xrightarrow{\sim} \text{Perf}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')
$$

---

5We will generally use curly letters to denote derived schemes which are not necessarily classical.
**Sketch of proof.** By Lemma 1.8, it suffices to prove the equivalence:

\[ \text{QCoh}(\mathcal{X}) \otimes_{\text{QCoh}(\mathcal{Y})} \text{QCoh}(\mathcal{Y}') \xrightarrow{\sim} \text{QCoh}(\mathcal{X} \times \mathcal{Y}'), \]

which is essentially [BZFN10, Theorem 4.7]. The case where \( \mathcal{Y} = \text{Spec}(\mathbb{Z}) \) follows immediately from the equivalence \( A\text{-Mod} \otimes B\text{-Mod} \xrightarrow{\sim} (A \otimes B)\text{-Mod} \) for derived rings \( A \) and \( B \). In the general case, one forms a simplicial object:

\[ \cdots \text{QCoh}(\mathcal{X} \times \mathcal{Y} \times \mathcal{Y}') \xrightarrow{\sim} \text{QCoh}(\mathcal{X} \times \mathcal{Y}'). \]

This complex identifies with the bar complex of §1.1.9 so its geometric realization identifies with \( \text{QCoh}(\mathcal{X}) \otimes_{\text{QCoh}(\mathcal{Y})} \text{QCoh}(\mathcal{Y}') \). One then compares it to \( \text{QCoh}(\mathcal{X} \times \mathcal{Y}') \) by an application of the monadicity theorem. \( \square \)

1.3.4. **Zariski descent.** We will now turn to descent properties of algebraic \( K \)-theory. The most basic one is Zariski descent, as stated below. Note that its proof is very quick using the localization property of \( K \) (c.f. §1.2.1). The same argument, using the excisive property of Perf for Nisnevich covers, can be used to show that algebraic \( K \)-theory satisfies Nisnevich descent.

Note that \( K \)-theory does not satisfy étale descent.

**Lemma 1.10.** The functor \( K : \text{Sch}^{\text{op}}_{\text{qc-qs}} \to \text{Sptr} \) satisfies descent for Zariski topology.

**Proof.** Using the cd-structure of Voevodsky (c.f. [AHW17, §2.1]), the proof reduces to showing the following excision property: given a Zariski cover of \( \mathcal{X} \in \text{Sch}_{\text{qc-qs}} \) by two opens \( \mathcal{U}, \mathcal{V} \), the following diagram is (homotopy) Cartesian:

\[
\begin{array}{ccc}
K(\mathcal{X}) & \longrightarrow & K(\mathcal{U}) \\
\downarrow & & \downarrow \\
K(\mathcal{V}) & \longrightarrow & K(\mathcal{U} \cap \mathcal{V})
\end{array}
\] (1.6)

In order to show the latter statement, we note that both of the fibers of the horizontal functors in the following commutative diagram:

\[
\begin{array}{ccc}
\text{Perf}(\mathcal{X}) & \longrightarrow & \text{Perf}(\mathcal{U}) \\
\downarrow & & \downarrow \\
\text{Perf}(\mathcal{V}) & \longrightarrow & \text{Perf}(\mathcal{U} \cap \mathcal{V})
\end{array}
\]

identify with the objects of \( \text{Perf}(\mathcal{X}) \) which are set-theoretically supported on \( \mathcal{X} \setminus \mathcal{U} \). Hence (1.6) is (homotopy) Cartesian by property (d) of the definition of \( K \) in §1.2.1. \( \square \)

**Remark 1.11.** Even though connective \( K \)-theory is not a localizing invariant, it also satisfies Zariski (or Nisnevich) descent because the functor \( \Omega^\infty : \text{Sptr} \to \text{Spc} \) preserves limits.

1.3.5. **Projective bundle formula.** For any \( \mathcal{X} \in \text{Sch} \), we may associate the relative projective space \( \mathbb{P}^r_{\mathcal{X}} := \mathbb{P}^r_{\mathbb{Z}} \times \mathcal{X} \). For each integer \( l \), we consider the morphism:

\[
\text{Perf}(\mathcal{X}) \xrightarrow{p^*} \text{Perf}(\mathbb{P}^r_{\mathcal{X}}) \otimes_{\mathcal{O}(-l)} \text{Perf}(\mathbb{P}^r_{\mathcal{X}}),
\] (1.7)

where \( p : \mathbb{P}^r_{\mathcal{X}} \to \mathcal{X} \) is the projection map and \( \mathcal{O}(-l) \) denotes the pullback of the same-named twisting sheaf on \( \mathbb{P}^r_{\mathbb{Z}} \).
Lemma 1.12. Suppose \( \mathcal{X} \in \text{Sch}_{\text{qc-qs}} \). Then for \( l = 0, \cdots, r \), the collection of functors (1.7) defines an equivalence:

\[
\bigoplus_{l=1}^{r} K(\mathcal{X}) \xrightarrow{\sim} K(\mathbb{P}^r_{\mathcal{X}}).
\] (1.8)

Proof. In the absolute case \( \mathbb{P}^r_{\mathcal{Z}} \), we have a filtration of \( \text{Perf}(\mathbb{P}^r_{\mathcal{Z}}) \) by stable full subcategories tensored over \( \text{Perf}(\mathcal{Z}) \):

\[
\text{Perf}(\mathcal{Z}) \xrightarrow{\sim} \text{Perf}^0(\mathbb{P}^r_{\mathcal{Z}}) \hookrightarrow \text{Perf}^1(\mathbb{P}^r_{\mathcal{Z}}) \hookrightarrow \cdots \hookrightarrow \text{Perf}^r(\mathbb{P}^r_{\mathcal{Z}}) = \text{Perf}(\mathbb{P}^r_{\mathcal{Z}})
\]

where \( \text{Perf}^l(\mathbb{P}^r_{\mathcal{Z}}) \) is generated by \( \mathcal{O}(-l), \mathcal{O}(-l+1), \cdots, \mathcal{O} \). The analogously defined functor (1.7) gives us a splitting for each \( l \geq 1 \):

\[
\text{Perf}(\mathcal{Z}) \xrightarrow{(1.7)} \text{Perf}(\mathbb{P}^r_{\mathcal{Z}}) \rightarrow \text{Perf}^l(\mathbb{P}^r_{\mathcal{Z}})/\text{Perf}^{l-1}(\mathbb{P}^r_{\mathcal{Z}}).
\]

Now, the lower sequence is exact as \( \text{Perf}(\mathcal{Z}) \)-module categories (c.f. §1.1.9). Hence we may apply \( - \hat{\otimes} \text{Perf}(\mathcal{X}) \) to the above diagram to obtain the following diagram

\[
\text{Perf}(\mathcal{X}) \xrightarrow{(1.7)} \text{Perf}(\mathbb{P}^r_{\mathcal{X}}) \rightarrow \text{Perf}^l(\mathbb{P}^r_{\mathcal{X}})/\text{Perf}^{l-1}(\mathbb{P}^r_{\mathcal{X}}),
\]

where the lower sequence remains exact by Lemma 1.3. Applying \( K \) to these stable \( \infty \)-categories while using the fact that \( K \) is a localizing invariant gives the isomorphism (1.8). \( \square \)

1.3.6. Bass fundamental theorem. One deduces from the projective bundle formula a way to express the \((n-1)\)th algebraic \( K \)-group of \( \mathcal{X} \in \text{Sch}_{\text{qc-qs}} \) using only the \( n \)th algebraic \( K \)-group of \( \mathcal{X}, \mathcal{X} \times \mathbb{A}^1_\mathcal{Z} \), and \( \mathcal{X} \times \mathbb{G}_m \).

Lemma 1.13. Suppose \( \mathcal{X} \in \text{Sch}_{\text{qc-qs}} \). Then for each integer \( n \) we have an exact sequence:

\[
0 \rightarrow K_n(\mathcal{X}) \rightarrow K_n(\mathcal{X}[t]) \oplus K_n(\mathcal{X}[t^{-1}]) \rightarrow K_n(\mathcal{X}[t, t^{-1}]) \rightarrow K_{n-1}(\mathcal{X}) \rightarrow 0.
\] (1.9)

Proof. Consider the Zariski covering of \( \mathbb{P}^1_{\mathcal{X}} \) by \( X[t] \) and \( X[t^{-1}] \). The descent result Lemma 1.10, together with Lemma 1.12, shows that we have a homotopy Cartesian square:

\[
K(\mathcal{X}) \oplus K(\mathcal{X}) \xrightarrow{(\varphi_0, \varphi_1)} K(\mathbb{P}^1_{\mathcal{X}}) \rightarrow K(\mathcal{X}[t])
\]

\[
\downarrow
\]

\[
K(\mathcal{X}[t^{-1}]) \rightarrow K(\mathcal{X}[t, t^{-1}])
\]

where \( \varphi_0, \varphi_1 \) come from the functors (1.7) for \( l = 0, 1 \). We note that \( \varphi_1 \) is null-homotopic and \( \varphi_0 \) has a left inverse. This shows that the homotopy groups associated to the above diagram decomposes into (1.9) for each \( n \). \( \square \)

1.4. Blow-ups.
1.4.1. Let \( X \in \textbf{Sch}_{qc-qs}^{cl} \) be a classical scheme. For a classical closed subscheme \( Y \hookrightarrow X \), we may form the blow-up square:

\[
\begin{array}{ccc}
E & \longrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
\]

which is Cartesian (only in the classical sense!) and \( E \) is an effective Cartier divisor in \( \tilde{X} \). We recall the result [CHSW08, Proposition 1.5]:

**Lemma 1.14.** Suppose \( Y \hookrightarrow X \) is regularly immersed of codimension \( d \). Then there exist filtrations of \( \text{Perf}(E) \) and \( \text{Perf}(\tilde{X}) \) by \( \text{Perf}(X) \)-submodules together with functors between them:

\[
\begin{array}{ccccccccc}
\text{Perf}(X) & \xrightarrow{p^*} & \text{Perf}^0(\tilde{X}) & \xrightarrow{} & \text{Perf}^1(\tilde{X}) & \xrightarrow{} & \cdots & \xrightarrow{} & \text{Perf}^{d-1}(\tilde{X}) & \xrightarrow{} & \text{Perf}(\tilde{X}) \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
\text{Perf}(Y) & \xrightarrow{} & \text{Perf}^0(E) & \xrightarrow{} & \text{Perf}^1(E) & \xrightarrow{} & \cdots & \xrightarrow{} & \text{Perf}^{d-1}(E) & \xrightarrow{} & \text{Perf}(E)
\end{array}
\]

such that the successive quotients are identified. □

1.4.2. The most basic example of a regular immersion of co-dimension \( d \) is the embedding of the origin inside \( \mathbb{A}_d^d \). We denote the exceptional divisor of this blow-up by \( P \):

\[
P \longrightarrow \tilde{\mathbb{A}}_d^d
\]

1.4.3. We now generalize Lemma 1.14 to the derived setting. Recall that a closed immersion \( Y \hookrightarrow X \) in \( \textbf{Sch}_{qc-qs} \) is quasi-smooth of co-dimension \( d \) if Zariski locally on \( X \), we have a morphism \( X \rightarrow \mathbb{A}_d^d \) and an identification \( Y \cong X \times 0 \). It is easy to see that a closed immersion \( Y \hookrightarrow X \) of classical schemes is quasi-smooth if and only if it is a regular immersion.

1.4.4. A commutative square of derived schemes:

\[
\begin{array}{ccc}
\mathcal{E} & \longrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
\mathcal{Y} & \longrightarrow & \tilde{X}
\end{array}
\]

is called a derived blow-up square if Zariski locally on \( \tilde{X} \), there exists a morphism \( \tilde{X} \rightarrow X \) where \( X \) is a classical scheme and (1.13) is the (derived) base change of a classical blow-up square (1.10) at a regularly immersed center. In particular, the closed immersion \( \mathcal{Y} \hookrightarrow \tilde{X} \) in a derived blow-up square must be quasi-smooth. Via Zariski gluing, one can also show:

**Lemma 1.15.** Every quasi-smooth closed immersion \( \mathcal{Y} \hookrightarrow \tilde{X} \) fits into a derived blow-up square (1.13). □

**Remark 1.16.** We emphasize that a derived blow-up square (1.13) is not derived Cartesian in general. This does not mean that our definition of derived blow-up squares is inadequate: in fact, one expects blow-ups to replace a closed immersion of virtual codimension \( d \) by one of virtual codimension 1. If (1.13) was derived Cartesian, it would imply that the closed immersions \( \mathcal{Y} \hookrightarrow \tilde{X} \) and \( \mathcal{E} \hookrightarrow \tilde{X} \) are of the same virtual codimension.
1.4.5. We now show that the calculation of perfect complexes in Lemma 1.14 extends to derived blow-up squares and thereby proves the excision of algebraic $K$-theory for such squares.

**Lemma 1.17.** Given a quasi-smooth closed immersion $Y \hookrightarrow X$ in $\text{Sch}_{qc-qs}$, the $K$-theory functor takes the derived blow-up square:

$$\begin{array}{ccc}
\mathcal{E} & \longrightarrow & \tilde{X} \\
\downarrow & & \downarrow \overset{p}{\longrightarrow} \\
\mathcal{Y} & \longrightarrow & X
\end{array}$$

(1.14)
to a (homotopy) Cartesian square of spectra.

**Proof.** The question is Zariski local on $X$, so we may assume that the diagram (1.14) arises as the (derived) base change along $f : X \to X$ of a classical blow-up square (1.10) at regular center. Thus we may apply $- \hat{\otimes} \text{Perf}(X)$ to the commutative diagram (1.11) to obtain:

$$\begin{array}{cccc}
\text{Perf}(X) & \overset{p^*}{\longrightarrow} & \text{Perf}^{0}(\tilde{X}) & \overset{\cdot}{\longrightarrow} & \text{Perf}^{1}(\tilde{X}) & \overset{\cdot}{\longrightarrow} & \cdots & \overset{\cdot}{\longrightarrow} & \text{Perf}^{d-1}(\tilde{X}) & \overset{\sim}{\longrightarrow} & \text{Perf}(\tilde{X}) \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
\text{Perf}(Y) & \overset{\cdot}{\longrightarrow} & \text{Perf}^{0}(E) & \overset{\cdot}{\longrightarrow} & \text{Perf}^{1}(E) & \overset{\cdot}{\longrightarrow} & \cdots & \overset{\cdot}{\longrightarrow} & \text{Perf}^{d-1}(E) & \overset{\sim}{\longrightarrow} & \text{Perf}(E).
\end{array}$$

Here, we have used Lemma 1.3 to guarantee that this is still a morphism between filtered stable $\infty$-categories and the quotients are identified. Therefore, an application of $K$ turns all commutative squares in the above diagram into Cartesian squares of spectra. \qed

1.5. **Abstract blow-up squares.**

1.5.1. A commutative square in $\text{Sch}_{cl}$ is called an abstract blow-up square if it is Cartesian (in the classical sense) and the following conditions are satisfied:

$$\begin{array}{ccc}
E & \longrightarrow & \tilde{X} \\
\downarrow & & \downarrow \overset{p}{\longrightarrow} \\
Y & \longrightarrow & X
\end{array}$$

(†)

(a) The map $i$ is a closed immersion;
(b) The map $p$ is proper and induces an isomorphism $\tilde{X} \setminus E \sim X \setminus Y$.

1.5.2. Algebraic $K$-theory does not satisfy excision for abstract blow-up squares. We will now provide an example. Indeed, one instance of an abstract blow-up square (†) is the following one for any $X \in \text{Sch}_{cl}$:

$$\begin{array}{ccc}
X_{\text{red}} & \longrightarrow & X_{\text{red}} \\
\downarrow & & \downarrow \\
X_{\text{red}} & \longrightarrow & X
\end{array}$$

Thus any prestack $E$ satisfying excision for abstract blow-up squares must be nil-invariant, i.e., $E(X) \sim E(X_{\text{red}})$. We claim that algebraic $K$-theory is not nil-invariant. For example, for a field $k$, we have

$$K_{1}(k[\varepsilon/\varepsilon^{2}]) = (k[\varepsilon]/\varepsilon^{2})^{\times} \neq k^{\times}.$$  

On the other hand, after imposing $A^{1}$-invariance, algebraic $K$-theory does satisfy excision for abstract blow-up squares, c.f. the main theorem of [Ci13]. This variant of algebraic $K$-theory is called homotopy $K$-theory, and its excision property is used by Kerz-Strunk [KS16] to prove Weibel’s conjecture for homotopy $K$-theory.
1.5.3. Motivated by Weibel’s conjecture for $K$-theory, one seeks a weaker version of excision for abstract blow-up squares. The idea of Kerz-Strunk-Tamme [KST18] is that one can obtain a version of the excision theorem if one takes into account the formal neighborhoods of $Y$ and $E$. Indeed, this “pro-excision” theorem suffices for the purpose of proving Weibel’s conjecture.

1.5.4. We will now state their main theorem, where one replaces the $K$-theory spectrum of the scheme $Y$ (resp. $E$) by the pro-spectrum associated to its infinitesimal thickenings $Y^{(n)}$ (resp. $E^{(n)}$) as closed subschemes of $X$ (resp. $\tilde{X}$).

**Theorem (Kerz-Strunk-Tamme).** Given an abstract blow-up square (†), the following diagram of pro-spectra is (homotopy) Cartesian:

$$
\begin{array}{ccc}
K(X) & \longrightarrow & \lim_n K(E^{(n)}) \\
\downarrow & & \downarrow \\
K(\tilde{X}) & \longrightarrow & \lim_n K(Y^{(n)})
\end{array}
$$

(1.15)

We emphasize that the limit in (1.15) is taken in the category $\text{Pro(Sptr)}$ and not $\text{Sptr}$. In the main body of the text, this theorem will re-appear as Theorem 3.3 where we use the language of ind-schemes to give a more natural formulation.

2. **Pro-excision of simplicial rings**

The first step in obtaining pro-excision for abstract blow-up squares is to prove a pro-excision theorem for simplicial rings. In this section, we will state this ring-theoretic result, and give a proof which mostly draws on ideas of Suslin and Wodzicki. It is the content of §4 of the main text of Kerz-Strunk-Tamme [KST18].

2.1. **Statement of pro-excision.**

2.1.1. We use the convention that a ring refers to a commutative, unital ring, and similarly for simplicial rings. We let $\text{sRing}$ denote the $\infty$-category of simplicial rings; it can be regarded as the localization of the simplicial model category of simplicial rings at its weak equivalences. The $K$-theory functor of §1.3 gives rise to a functor, again denoted by $K$:

$$K : \text{sRing} \rightarrow \text{Sptr}, \quad R \mapsto K(\text{Spec}(R)).$$

Furthermore, it extends as a functor preserving co-filtered limits:

$$K : \text{Pro(sRing)} \rightarrow \text{Pro(Sptr)}.$$  

(2.1)

2.1.2. Let $R$ be a discrete, Noetherian ring together with an ideal $J$. Suppose $A \rightarrow B$ is a morphism of Noetherian simplicial $R$-algebras. Note that we do not assume that $A$ or $B$ is of finite type over $R$. We will prove the following theorem of pro-excision of $K$-theory.

**Theorem 2.1.** Suppose the following diagram of pro-simplicial rings is homotopy Cartesian:

$$
\begin{array}{ccc}
A & \longrightarrow & \lim_n A \otimes_R R/J^n \\
\downarrow & & \downarrow \\
B & \longrightarrow & \lim_n B \otimes_R R/J^n.
\end{array}
$$

(2.2)

Then it remains homotopy Cartesian after applying the $K$-theory functor (2.1).
2.1.3. We will give a reformulation of the homotopy Cartesian-ness of (2.2) by computing the homotopy groups of the fibers along horizontal maps.

**Lemma 2.2.** There is a canonical equivalence:

$$\lim_n J^n \pi_i(A) \sim \pi_i \text{Fib}(A \to \lim_n A \hat{\otimes}_R R/J^n)$$

*Proof.* We note that $\lim_n J^n \pi_i(A)$ is the fiber of the morphism $\pi_i(A) \to \lim_n \pi_i(A) \hat{\otimes}_R R/J^n$. Therefore, it suffices to produce a canonical isomorphism:

$$\lim_n \pi_i(A) \hat{\otimes}_R R/J^n \sim \lim_n \pi_i(A \hat{\otimes}_R R/J^n) \quad (2.3)$$

We consider the spectral sequence

$$E^2_{pq} = \text{Tor}_q(\pi_p(A), R/J^n) \Rightarrow \pi_{p+q}(A \hat{\otimes}_R R/J^n). \quad (2.4)$$

Note that the flatness of completion $R \to R_\hat{}$ shows that $\lim_n \text{Tor}_q(\pi_p(A), R/J^n) = 0$ whenever $q \geq 1$. Hence the limit of (2.4) degenerates and we obtain the desired isomorphism (2.3). \qed

Therefore, the homotopy Cartesian-ness of (2.2) is equivalent to saying that the map $A \to B$ induces isomorphisms:

$$\lim_n J^n \pi_i(A) \sim \lim_n J^n \pi_i(B), \text{ for each } i \geq 0. \quad (2.5)$$

**Remark 2.3.** The isomorphisms (2.5) are the original formulation of the pro-excision theorem in [KST18], see Theorem 4.11 of *loc.cit*.

2.1.4. Suppose $A \to B$ is a morphism of discrete $R$-algebras. Then (2.5) is simply the isomorphism of pro-$R$-modules:

$$\lim_n J^n A \sim \lim_n J^n B \quad (2.6)$$

We caution the reader that the inverse of the map (2.6) is not in general given by a levelwise map $J^n B \to J^n A$. However, for each $n$, there exists some $m$ together with a map $J^m B \to J^n A$ such that the following two compositions are the canonical embedding:

$$J^m A \xrightarrow{\text{can}} J^m B \xrightarrow{\text{can}} J^n A \xrightarrow{\text{can}} J^n B.$$ 

In particular, this shows that the radicals of $J^n A$ and $J^n B$ are isomorphic.

2.2. Prelude 1: Milnor squares.

2.2.1. We will now take a closer look at the discrete case, and instead of pro-excision we will ask the more naïve question of whether excision holds on the nose. Historically, this was the first investigation on excision of algebraic $K$-theory. We include this discussion also because the result on non-positive $K$-groups below will enter the proof of Theorem 2.1.
2.2.2. Let $R \to R'$ be a map of discrete rings. Suppose $I$ is an ideal in $R$ which maps isomorphically onto an ideal $I'$ of $R'$, or equivalently, the following commutative square:

\[
\begin{array}{c}
R \\ -
\downarrow \\
R' \\ -
\end{array} \quad \begin{array}{c}
\text{onto} \\ \\
R/I \\ -
\downarrow \\
R'/I' \\ -
\end{array}
\]

is Cartesian. Such squares are called Milnor squares.

**Remark 2.4.** One can think of the hypothesis (2.2) as the pro-analogue of Milnor squares (in the case of simplicial rings).

2.2.3. We investigate the result of applying the $K$-theory functor to (2.7). The square:

\[
\begin{array}{c}
K(R) \\ -
\downarrow \\
K(R') \\ -
\end{array} \quad \begin{array}{c}
\text{onto} \\ \\
K(R/I) \\ -
\downarrow \\
K(R'/I') \\ -
\end{array}
\]

is in general not homotopy Cartesian. However, it behaves like one for homotopy groups in degrees $\leq 1$.

**Lemma 2.5.** Given a Milnor square (2.7), the following sequence is exact:

\[
K_1(R) \to K_1(R') \oplus K_1(R/I) \to K_1(R'/I') \to \cdots \to K_{-i}(R) \to K_{-i}(R') \oplus K_{-i}(R/I) \to K_{-i}(R'/I') \to \cdots
\]

**Sketch of proof.** One first proves that the sequence is exact up to $i = 0$ by explicit calculation (see [We13, Theorem III.2.6]). Then one uses Lemma 1.13 to show that the exactness continues for all non-positive $K$-groups. □

2.2.4. An equivalent way to phrase Lemma 2.5 is by considering the relative $K$-theory spectra

\[
K(R; I) := \text{Fib}(K(R) \to K(R/I))
\]

and similarly for $K(R'; I')$. By considering the long exact sequence on homotopy groups associated to this fibration, one sees that the exactness of the Mayer-Vietoris sequence amounts to the equivalences:

\[
K_i(R; I) \cong K_i(R'; I'), \quad \text{for } i \leq 0
\]

for each Milnor square (2.7).

2.2.5. We turn to yet another re-formulation of Lemma 2.5, which explains the presence of non-unital rings in the study of excision for algebraic $K$-theory. Recall that any non-unital ring $I$ has a unitalization $\tilde{I} := I \times \mathbb{Z}$, constructed using the natural $\mathbb{Z}$-action on $I$. This construction supplies a left adjoint to the forgetful functor $\text{Ring} \to \text{Ring}^{\text{non-unital}}$. In particular, given any ring $R$ containing $I$ as an ideal, we canonically obtain a Milnor square:

\[
\begin{array}{c}
\tilde{I} \\ -
\downarrow \\
R \\ -
\end{array} \quad \begin{array}{c}
\text{onto} \\ \\
\mathbb{Z} \\ -
\downarrow \\
R/I \\ -
\end{array}
\]

Defining $K(I) := \text{Fib}(K(\tilde{I}) \to K(\mathbb{Z}))$, we see that there is a canonical map

\[
K(I) \to K(R; I)
\]

(2.10)

Therefore, we may rephrase the isomorphism (2.9) as:
Lemma 2.6. Let $I$ be a non-unital ring. For each ring $R$ containing $I$ as an ideal, the map (2.10) is an equivalence on $\pi_i$ for $i \leq 0$. □

Informally, one may summarize the content of Lemma 2.6 as saying that the non-positive relative $K$-groups depend only on the ideal $I$.

2.3. Prelude 2: Suslin-Wodzicki excision.

2.3.1. We include another historical note that specifies precisely when (2.8) is homotopy Cartesian, due to A. Suslin and M. Wodzicki. Although the excision results explained here will not be used directly, the proof of Theorem 2.1 follows more or less the same strategy.

2.3.2. Since Lemma 2.5 (and its variants) already tells us that non-positive algebraic $K$-theory satisfies excision, we will focus on the positive $K$-groups. For these $K$-groups, one has an explicit model given by Quillen’s $+$-construction. This construction can be extended to non-unital rings, which will be explained in detail in §2.4. For now, let us simply say:

- The infinite loop space $\Omega^\infty K(I)$ is equivalent to $K_0(I) \times B\text{GL}(I)^+$, where $\text{GL}(I)$ is a group canonically attached to $I$ (and agrees with the usual one when $I$ is unital).

2.3.3. Let $I$ be a (discrete) non-unital ring. We say that $I$ satisfies excision for algebraic $K$-theory if for any (discrete) non-unital ring $R$ containing $I$ as an ideal, the triangle:

$$K(I) \to K(R) \to K(R/I)$$

is a fiber sequence of spectra. Indeed, this is equivalent to asking (2.10) to be an equivalence, or asking (2.8) to be homotopy Cartesian for any Milnor square.

2.3.4. The main result of Suslin-Wodzicki [SW92] characterizes those discrete non-unital rings which satisfy excision. For simplicity, we will state their answer for $\mathbb{Q}$-algebras even though the original paper also handles the absolute case. The answer of loc.cit. is that such rings must “look like” a unital ring. Indeed, recall that to any $I$ we can attach its bar complex:

$$\cdots \to I \otimes I \otimes I \to I \otimes I$$

(2.11)

We call $I$ homologically unital if the augmentation map of (2.11) to $I$ is an equivalence. Every unital ring is homologically unital, but the converse is not true.

Theorem 2.7. The following are equivalent for a discrete, non-unital $\mathbb{Q}$-algebra $I$:

(a) $I$ satisfies excision for algebraic $K$-theory;
(b) $I$ is homologically unital.

2.3.5. The direction (a) $\implies$ (b) is contained in a previous work of Wodzicki. The authors of [SW92] deduce (b) $\implies$ (a) via an intermediate condition. To state this condition, we consider the embedding:

$$\text{GL}(I) \hookrightarrow \widetilde{\text{GL}}(I) := \text{GL}(I) \times M_{\infty,1}(I)$$

(2.12)

where $\widetilde{\text{GL}}(I)$ identifies with the colimit of matrices:

$$
\begin{pmatrix}
\text{GL}_n & M_{n,1} \\
0 & 1
\end{pmatrix}
$$

The intermediate condition is:

(a') The relative homology $H_q(\widetilde{\text{GL}}(I); \text{GL}(I))$ along (2.12) vanishes.

The implication (b) $\implies$ (a') uses the “Volodin spaces” and is the content of [SW92, §2 and 6].
2.3.6. For the implication \((a') \implies (a)\), one introduces a \textit{simple} space \(F(A; I)\), for any ring \(A\) containing \(I\) as an ideal, whose homotopy groups compute the relative \(K\)-groups for degrees \(\geq 1\). Let us spell out its definition: suppose \(\mathbb{GL}(A/I)\) is the image:

\[
\text{GL}(A) \to \mathbb{GL}(A/I) \hookrightarrow \text{GL}(A/I).
\]

Then \(F(A; I)\) is set to be the fiber of \(B \text{GL}(A)^+ \to B \mathbb{GL}(A/I)^+\). The content of §1 of \textit{loc.cit} is essentially the following chain of implications:

\[
\text{vanishing of } H_\bullet(\mathbb{GL}(I); \text{GL}(I)) \implies \text{GL}(A) \text{ acts trivially on } H_\bullet(\text{GL}(I)) \implies \text{The homology groups of } B \text{GL}(I) \text{ and } F(A; I) \text{ agree} \implies I \text{ satisfies excision}
\]

The simplicity of \(F(A; I)\) allows us to reduce a question about homotopy groups to one about homology groups.

2.4. The plus-construction for simplicial rings.

2.4.1. We will now review the plus-construction for a simplicial ring \(A\). This will give us an explicit model for the infinite loop space of \(K(A)\). As the proof of Theorem 2.1 is an adaptation of Suslin-Wodzicki excision, it relies heavily on this model.

2.4.2. Let \(\text{sMon}\) denote the \(\infty\)-category of simplicial monoids. A simplicial monoid \(M\) is \textit{grouplike} if the monoid \(\pi_0(M)\) is a group. Thus grouplike simplicial monoids form a full subcategory \(\text{sMon}_{\text{grp-like}}\) of \(\text{sMon}\). We recall that a simplicial monoid \(M\) has a classifying space \(BM\) and the canonical map:

\[
M \to \Omega BM
\]

is an equivalence when \(M\) is grouplike.

2.4.3. Recall that any discrete non-unital ring \(I\) has a unitalization \(\tilde{I} := I \rtimes \mathbb{Z}\), c.f. §2.2.5. We can associate a group \(\text{GL}(I)\) to \(I\) by the exact sequence:

\[
1 \to \text{GL}(I) \to \text{GL}(\tilde{I}) \to \text{GL}(\mathbb{Z}) \to 1.
\]

This construction identifies with the usual one when \(I\) has a unit.

2.4.4. We define a functor:\(^\text{6}\)

\[
\text{GL} : \text{sRing}_{\text{non-untl}} \to \text{sMon}_{\text{grp-like}}
\]

by the fiber product of simplicial monoids for all \(I \in \text{sRing}_{\text{non-untl}}\):

\[
\begin{array}{ccc}
\text{GL}(I) & \longrightarrow & M(I) \\
\downarrow & & \downarrow \\
\text{GL}(\pi_0 I) & \longrightarrow & M(\pi_0 I)
\end{array}
\]

The result \(\text{GL}(I)\) is grouplike since its \(\pi_0\) identifies with the group \(\text{GL}(\pi_0 I)\).

\(^6\)Our notation \(\text{GL}\) defers from that of the main text [KST18], where the same construction is denoted by \(\tilde{\text{GL}}\). We shall make similar notational modifications in order to stick closer to [SW92].
2.4.5. Recall that for a connected space \( S \), the plus construction produces a space \( S^+ \) together with a morphism \( i : S \to S^+ \) characterized by the following properties:
(a) the map \( \pi_1(S) \to \pi_1(S^+) \) identifies with the abelianization map \( \pi_1(S) \to \pi_1(S)^{ab} \);
(b) for each local system \( \mathcal{L} \) over \( S^+ \), the canonical map on homology groups is an isomorphism:
\[
H_*(S; i^* \mathcal{L}) \xrightarrow{\sim} H_*(S^+; \mathcal{L}).
\]
We apply the plus construction to the classifying space \( B \text{GL}(I) \) to obtain a space \( B \text{GL}(I)^+ \). The following is \([\text{BGT}13, \text{Lemma 9.39}]\):

**Lemma 2.8.** Let \( A \in \text{sRing} \). Then \( K_0(\pi_0(A)) \times B \text{GL}(A)^+ \) represents the infinite loop space associated to the connective \( K \)-theory spectrum \( K^\text{conn}(A) \). \( \square \)

2.4.6. We now compare the \( K \)-theory of a simplicial ring \( A \) with that of \( \pi_0(A) \).

**Lemma 2.9.** Let \( A \in \text{sRing} \). Then the canonical map \( K(A) \to K(\pi_0 A) \) induces an isomorphism on \( \pi_i \) for all \( i \leq 1 \).

**Proof.** We note that Lemma 2.8, together with the isomorphisms:
\[
\pi_1(B \text{GL}(A)^+) \xrightarrow{\sim} \pi_1(B \text{GL}(A))^{ab}
\]
\[
\xrightarrow{\sim} \pi_0(\text{GL}(A))^{ab} \xrightarrow{\sim} \text{GL}(\pi_0 A)^{ab},
\]
shows that \( K(A) \to K(\pi_0 A) \) defines an isomorphism for \( \pi_0 \) and \( \pi_1 \). We then use the exact sequence (1.9) associated to both \( X = \text{Spec}(A) \) and \( \check{X} = \text{Spec}(\pi_0 A) \) to obtain the isomorphisms for \( \pi_i \) with \( i \leq 1 \). \( \square \)

**Remark 2.10.** We caution the reader that the analogous statement for derived non-affine schemes is false.

2.5. **Homological considerations in the spirit of Suslin-Wodzicki.**

2.5.1. The proof of pro-excision follows roughly the same strategy as Suslin-Wodzicki excision. According to the outline in §2.3.6, we must first obtain an unconditional statement about the vanishing of certain relative homology groups. We will then introduce the pro-analogues of \( B \text{GL}(I) \) and \( F(A; I) \) and prove that their homology groups agree.

2.5.2. Recall the set-up of Theorem 2.1: \( R \) is a discrete Noetherian ring together with an ideal \( J \), and \( A \to B \) is a morphism of Noetherian simplicial \( R \)-algebras. The proof of Theorem 2.1 proceeds by induction on the number of generators of \( J \) and the following technical statements only enter the base case where \( J = (r) \) is monogenic. We will therefore focus on this case. By replacing \( A \) with an equivalent simplicial \( R \)-algebra, we may assume that the image of \( r \) is an element \( a \in A_0 \) that acts injectively on each \( A_i \).\(^7\) Under this hypothesis, we have \( A \otimes_R R/r^n \xrightarrow{\sim} A/a^n \) and therefore the pro-system \( \lim_n a^n A \) represents the fiber:
\[
\lim_n a^n A \to A \to \lim_n A \otimes_R R/r^n.
\]
We will now proceed to prove a bunch of (highly model-dependent) results concerning the following set up:
\[ I \text{ is a simplicial non-unital ring and } a \in I_0 \text{ acts injectively on each } I_i. \quad (*) \]
which will eventually be applied to \( I = a^n A \).

\(^7\) This replacement can be obtained as follows: regard \( A \) as a simplicial \( R[x] \)-algebra via the map \( R[x] \to R \), \( x \mapsto r \). Then we find a cofibrant replacement of \( A \) in the model category \( R[x]-\text{Alg}^{\Delta^{op}} \). The flatness of each \( A' \) over \( R[x] \) implies that the image of \( x \) acts as a non-zerodivisor.
2.5.3. The idea for proving the following unconditional statement comes from Suslin [Su95].

Lemma 2.11. Under the set-up (*), the following map vanishes:

\[ H_i(\overline{\text{GL}}(aI); \text{GL}(aI)) \to H_i(\overline{\text{GL}}(I); \text{GL}(I)). \]

Sketch of proof. The statement is saying that the map \( I \to aI \) induces the zero endomorphism on \( H_\bullet(\overline{\text{GL}}(I); \text{GL}(I)) \). Since \( \overline{\text{GL}}(I) = \text{GL}(I) \ltimes M_{\infty, 1}(I) \), it suffices to show that:

\[ \text{GL}(I) \ltimes M_{\infty, 1}(I) \xrightarrow{(\text{id}, a)} \text{GL}(I) \ltimes M_{\infty, 1}(I) \]

induces the zero endomorphism on \( H_\bullet(\overline{\text{GL}}(I); \text{GL}(I)) \). This will follow from showing that for any integers \( m, n \geq 1 \), the map \( M_{n, 1}(I) \xrightarrow{a} M_{n+m, 1}(I) \), together with the natural inclusion \( \text{GL}_n(I) \hookrightarrow \text{GL}_{n+m}(I) \), induces the zero map:

\[ H_i(\text{GL}_n(I) \ltimes M_{n, 1}(I); \text{GL}_n(I)) \to H_i(\text{GL}_{n+m}(I) \ltimes M_{n+m, 1}(I); \text{GL}_{n+m}(I)). \]  

(2.13)

for all \( i \leq m \). In the case where \( I \) is discrete, this statement is [Su95, Theorem 3.5] and the proof there adapts to the simplicial case. More precisely, one first notes that (2.13) factors through:

\[ H_i(\text{GL}_n(I) \ltimes M_{n, 1}(I); \text{GL}_n(I)) \to H_i(\text{GL}_n(I) \ltimes \tilde{M}_{n, m}(I); \text{GL}_n(I)) \]

(2.14)

where \( \tilde{M}_{n, m} \) is the semi-direct product (where \( T_m \) denotes triangulated matrices):

\[ \left( \begin{array}{cc} 1_n & M_{n, m} \\ 0 & T_m \end{array} \right) \ltimes M_{n+m, 1} \]

and (2.14) is induced from the map \( M_{n, 1}(I) \xrightarrow{a} \tilde{M}_{n, m}(I) \). The latter map gives rise to a morphism of simplicial abelian groups using the usual bar construction:

\[ C_\bullet(B M_{n, 1}(I); \mathbb{Z}) \xrightarrow{a} C_\bullet(B \tilde{M}_{n, m}(I); \mathbb{Z}) \]  

(2.15)

One interprets \( H_\bullet(\text{GL}_n(I) \ltimes M_{n, 1}(I); \mathbb{Z}) \) as the homology groups of \( \text{GL}_n(I) \) with coefficients in \( C_\bullet(B M_{n, 1}(I); \mathbb{Z}) \) (and similarly for \( H_\bullet(\text{GL}_n(I) \ltimes \tilde{M}_{n, 1}(I); \mathbb{Z}) \)). Thus, the vanishing of (2.14) will follow from comparing the simplicial abelian groups (2.15). In the case where \( I \) is discrete, one can construct a null-homotopy of (2.15) up to degree-\( m \) simplices—since the construction is functorial in the pair \((I, a)\), the same holds for general \( I \).  

One then deduces from this null-homotopy the vanishing of (2.14) for \( i \leq m \). \( \square \)

2.5.4. We now deduce from Lemma 2.11 a statement about the triviality of \( \text{GL}(\mathbb{Z}) \)-action.

Lemma 2.12. Under the set-up (*), the \( \text{GL}(\mathbb{Z}) \)-action on \( H_\bullet(\text{GL}(I)) \) is trivial on the image of \( H_\bullet(\text{GL}(aI)) \to H_\bullet(\text{GL}(I)) \).

Proof. Denote by \( i : \text{GL}(I) \hookrightarrow \overline{\text{GL}}(I) \) and \( p : \overline{\text{GL}}(I) \to \text{GL}(I) \) the canonical embedding and projection. Their composition is the identity endomorphism on \( \text{GL}(I) \). Hence the long exact sequence of homology groups decomposes into split short exact sequences:

\[
0 \to H_i(\text{GL}(I); \mathbb{Z}) \xrightarrow{p_*} H_i(\overline{\text{GL}}(I); \mathbb{Z}) \to H_i(\text{GL}(I), \text{GL}(I); \mathbb{Z}) \to 0
\]

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\[ 8 \text{The explicit formula for this construction is unimportant. We refer the interested reader to [KST18, Lemma 4.16] for the construction of this “m-homotopy.”} \]
The map $aI \to I$ gives rise to a map between short exact sequences which preserves the splitting:

\[
0 \to \operatorname{H}_i(\operatorname{GL}(aI); \mathbb{Z}) \xrightarrow{i_*} \operatorname{H}_i(\widetilde{\operatorname{GL}}(aI); \mathbb{Z}) \to \operatorname{H}_i(\operatorname{GL}(aI), \operatorname{GL}(aI); \mathbb{Z}) \to 0
\]

\[
0 \to \operatorname{H}_i(\operatorname{GL}(I); \mathbb{Z}) \xrightarrow{i_*} \operatorname{H}_i(\widetilde{\operatorname{GL}}(I); \mathbb{Z}) \to \operatorname{H}_i(\operatorname{GL}(I), \operatorname{GL}(I); \mathbb{Z}) \to 0
\]

Lemma 2.11 shows that the last vertical map vanishes. Hence we obtain an isomorphism between the images of the first two vertical maps for each $i \geq 0$:

\[
i_* : \operatorname{H}_i(\operatorname{GL}(I); \mathbb{Z})_{\text{from } aI} \xrightarrow{\sim} \operatorname{H}_i(\widetilde{\operatorname{GL}}(I); \mathbb{Z})_{\text{from } aI} \tag{2.16}
\]

We now perform the argument of [SW92, Proposition 1.5] with (2.16) in place of the hypothesis (AH$^\circ$) of loc.cit. More precisely, we let $E_{i,j}(1) \in \operatorname{GL}(\mathbb{Z})$ denote the elementary matrix with 1 at the $(i, j)$-entry. We shall show that $E_{i+1,1}(1)$ acts trivially on $\operatorname{H}_\bullet(\widetilde{\operatorname{GL}}(I); \mathbb{Z})_{\text{from } aI}$.

Indeed, write $E_{i,\infty}(1) \in \operatorname{GL}(\mathbb{Z})$ for the element

\[
E_{i,\infty}(1) = \begin{pmatrix} 1_n & e_i \\ 0 & 1 \end{pmatrix}
\]

where $e_i \in M_{\infty,1}(\mathbb{Z})$ is the $i$th column vector. Consider the following commutative diagram, where the vertical maps are conjugation by the respective matrices in $\operatorname{GL}(\mathbb{Z})$ and $\operatorname{GL}(\mathbb{Z})$:

\[
\begin{array}{ccc}
\operatorname{GL}(I) & \xrightarrow{P} & \widetilde{\operatorname{GL}}(I) \\
\downarrow \text{id} & & \downarrow \text{Ad } E_{i,\infty}(1) \\
\operatorname{GL}(I) & \xrightarrow{P} & \widetilde{\operatorname{GL}}(I)
\end{array}
\]

and the embedding $\tilde{j}$ is given by:

\[
\begin{pmatrix} \alpha & u \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ u & \alpha \end{pmatrix}, \quad \text{where } \alpha \in \operatorname{GL}(I), \ u \in M_{\infty,1}(I).
\]

It follows from (2.16) that $p_* : \operatorname{H}_\bullet(\widetilde{\operatorname{GL}}(I); \mathbb{Z})_{\text{from } aI} \xrightarrow{\sim} \operatorname{H}_\bullet(\operatorname{GL}(I); \mathbb{Z})_{\text{from } aI}$ is an isomorphism. Hence conjugation by $E_{i,\infty}(1)$ acts trivially on $\operatorname{H}_\bullet(\widetilde{\operatorname{GL}}(I); \mathbb{Z})_{\text{from } aI}$, and conjugation by $E_{i+1,1}(1)$ acts trivially on the image of $\operatorname{H}_\bullet(\widetilde{\operatorname{GL}}(I); \mathbb{Z})_{\text{from } aI}$ under $\tilde{j}_*$. Since the morphism:

\[
\operatorname{GL}(I) \xrightarrow{j} \operatorname{GL}(I), \quad \alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \tag{2.17}
\]

factors through $\widetilde{\operatorname{GL}}(I) \xrightarrow{j} \operatorname{GL}(I)$, we see that it suffices to show that (2.17) induces an injective map on each of the groups $\operatorname{H}_\bullet(\operatorname{GL}(I); \mathbb{Z})$ (hence on the subgroups $\operatorname{H}_\bullet(\operatorname{GL}(I); \mathbb{Z})_{\text{from } aI}$). To prove the latter statement, we consider the commutative diagram:

\[
\begin{array}{ccc}
\operatorname{GL}_n(I) & \to & \operatorname{GL}(I) \\
\downarrow \text{Ad } \varepsilon_n & & \downarrow j \\
\operatorname{GL}(I) & \xrightarrow{\text{Ad } \varepsilon_n} & \operatorname{GL}(I)
\end{array}
\]

where both the top and left arrows are the canonical embedding. This shows that the kernel of the composition:

\[
\operatorname{H}_\bullet(\operatorname{GL}_n(I); \mathbb{Z}) \to \operatorname{H}_\bullet(\operatorname{GL}(I); \mathbb{Z}) \xrightarrow{j_*} \operatorname{H}_\bullet(\operatorname{GL}(I); \mathbb{Z})
\]

vanishes. Thus the kernel of $j_*$, being the (filtered) colimit of these kernels, also vanishes. This proves that $E_{i+1,1}(1)$ acts trivially on $\operatorname{H}_\bullet(\widetilde{\operatorname{GL}}(I); \mathbb{Z})_{\text{from } aI}$. 


Using a similar argument (where we replace \( \overrightarrow{\text{GL}}(I) \) by \( \text{GL}(I) \times M_{1,\infty}(I) \)) we can show that \( E_{1,j+1}(1) \) acts trivially on \( H_*(\text{GL}(I); \mathbb{Z})_{\text{from } aI} \). Note that \( \text{GL}(\mathbb{Z}) \) is generated by:

\[ E_{1,j+1}(1), \, E_{1,j+1}(1) \in \text{GL}(\mathbb{Z}), \, \text{and} \, -1 \in \text{GL}_1(\mathbb{Z}) \]

all of which act trivially on \( H_*(\text{GL}(I); \mathbb{Z})_{\text{from } aI} \). The lemma follows. \( \Box \)

**Lemma 2.13.** Under the setup \((*)\), the group \( \text{GL}(\mathbb{Z}) \) acts trivially on \( \lim_n H_*(\text{GL}(a^nI)) \).

**Proof.** Generalities show that for \( \text{GL}(\mathbb{Z}) \) to act trivially on \( \lim_n H_*(\text{GL}(a^nI)) \), it suffices that for each \( n \), there exists \( \bar{n} \geq n \) such that \( \text{GL}(\mathbb{Z}) \) acts trivially on the image of the map:

\[ H_*(\text{GL}(a^nI)) \to H_*(\text{GL}(a^{\bar{n}}I)). \]

Lemma 2.12 shows that we may take \( \bar{n} \) to be \( n + 1 \). \( \Box \)

2.5.5. We now apply the above results to \( I \) being an ideal of the form \( a^nA \) where \( A \) is a (unital) simplicial ring.

**Lemma 2.14.** Suppose \( A \) is a (unital) simplicial ring satisfying \((*)\) for \( I = A \). Then the simplicial group \( \text{GL}(A) \) acts trivially on \( \lim_n H_*(\text{GL}(a^nA)) \).

**Proof.** Fix \( i \geq 0 \). For each \( m \geq 1 \), we will show that \( \text{GL}_m(A) \) acts trivially on \( H_i^{\text{GL}}(A) \). Indeed, by Lemma 2.13, for each \( n \geq 0 \) there is some \( \bar{n} \geq n \) such that \( \text{GL}(\mathbb{Z}) \) acts trivially on the image of

\[ H_i(\text{GL}(a^nA)) \to H_i(\text{GL}(a^{\bar{n}}A)). \quad (2.18) \]

We consider the commutative diagram:

\[
\begin{array}{ccc}
\text{GL}_m(a^nA) \times \text{GL}_m(a^{\bar{n}}A) & \longrightarrow & \text{GL}(a^nA) \\
\text{swap} & & \text{Ad}\sigma, \quad \sigma := \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix} \\
\text{GL}_m(a^nA) \times \text{GL}_m(a^{\bar{n}}A) & \longrightarrow & \text{GL}(a^nA)
\end{array}
\]

The triviality of \( \text{GL}(\mathbb{Z}) \)-action shows that the image of \( H_i(\text{GL}_m(a^{\bar{n}}A)) \oplus 0 \) inside \( H_i(\text{GL}(a^nA)) \) identifies with that of \( 0 \oplus H_i(\text{GL}_m(a^{\bar{n}}A)) \). It follows that \( \text{GL}_m(A) \) also acts trivially on the image of \( (2.18) \). \( \Box \)

2.5.6. We now construct a connected space \( F(A; I) \) which is nilpotent and calculates the relative \( K \)-groups of \( (A, I) \) of degree \( \geq 1 \). Consider the image:

\[ \text{GL}(\pi_0A) \to \overrightarrow{\text{GL}}(\pi_0(A/I)) \hookrightarrow \text{GL}(\pi_0(A/I)). \]

Define \( \overrightarrow{\text{GL}}(A/I) \) to be the fiber product:

\[
\begin{array}{ccc}
\overrightarrow{\text{GL}}(A/I) & \longrightarrow & \text{GL}(A/I) \\
\downarrow & & \downarrow \\
\overrightarrow{\text{GL}}(\pi_0(A/I)) & \longrightarrow & \text{GL}(\pi_0(A/I)).
\end{array}
\]

Then we set \( F(A; I) \) to be the fiber of \( B \text{GL}(A)^+ \to B\overrightarrow{\text{GL}}(A/I)^+ \).

**Lemma 2.15.** The space \( F(A, I) \) is nilpotent and \( \pi_i F(A; I) \xrightarrow{\sim} K_i(A; I) \) for all \( i \geq 1 \).
**Proof.** Recall the isomorphisms:

\[ \pi_1(B_{\text{GL}}(A)^+) \cong \pi_1(B_{\text{GL}}(A))^{ab} \cong \text{GL}(\pi_0(A))^{ab} \]

and the analogous ones for \( A/I \) instead of \( A \). By construction, \( B_{\text{GL}}(A/I)^+ \) is the covering space of \( B_{\text{GL}}(A)^+ \) corresponding to the subgroup of its fundamental group which is the image of:

\[ \text{GL}(\pi_0(A))^{ab} \to \text{GL}(\pi_0(A/I))^{ab}. \]

Therefore, the projection \( B_{\text{GL}}(A/I)^+ \to B_{\text{GL}}(A/I)^+ \) induces an isomorphism on \( \pi_i \) for \( i \geq 2 \), so \( \pi_i F(A, I) \cong K_i(A; I) \) in this range; furthermore, the long exact sequence:

\[
\cdots \to \pi_1 F(A, I) \to \pi_1 B_{\text{GL}}(A^+) \to \pi_1 B_{\text{GL}}(A/I^+) \to \pi_0 F(A, I) \to 1 \]

shows that \( \pi_1 F(A, I) = K_1(A; I) \) and \( \pi_0 F(A, I) \) is trivial. Finally, \( F(A, I) \) is nilpotent since it is the homotopy fiber of nilpotent spaces. \( \square \)

2.5.7. Consider the following map between fiber sequences:

\[
B_{\text{GL}}(a^n A) \to B_{\text{GL}}(A) \to B_{\text{GL}}(A/a^n) \]

(2.19)

\[
F(A; a^n A) \to B_{\text{GL}}(A)^+ \to B_{\text{GL}}(A/a^n)^+ \]

**Lemma 2.16.** The morphism \( \varphi \) induces an equivalence on homology \( \lim_n H_\ast(\ast; \mathbb{Z}) \).

**Proof.** We regard the homology groups of the fiber \( \lim_n H_\ast(\text{GL}(a^n A)) \) as a local system on the base \( \lim_n B_{\text{GL}}(A/a^n) \). The monodromy action of (the limit of)

\[ \pi_1(B_{\text{GL}}(A/a^n)) \cong \pi_0(\text{GL}(A/a^n)) \cong \text{GL}(\pi_0(A/a^n)) \]

on \( \lim_n H_\ast(\text{GL}(a^n A)) \) is induced from the conjugation action of \( GL(\pi_0 A) \), which is trivial by Lemma 2.14. Hence \( \lim_n H_\ast(\text{GL}(a^n A)) \) is the trivial local system on \( \lim_n B_{\text{GL}}(A/a^n) \). On the other hand, \( \lim_n H_\ast(F(A; a^n A)) \) is also the trivial local system on \( \lim_n B_{\text{GL}}(A/a^n)^+ \) since this is a fibration of infinite loop spaces. The lemma thus follows from comparing the Serre spectral sequences associated to the fibrations in (2.19). \( \square \)

2.6. **Proof of pro-excision.**

2.6.1. In order to make the induction argument work, we need to prove a more general version of Theorem 2.1 for pro-simplicial \( R \)-algebras. In other words, we are given a morphism \( A_m \to B_m \) of pro-systems of finite type simplicial \( R \)-algebras. The hypothesis is that the following diagram is homotopy Cartesian:

\[
\begin{array}{ccc}
\lim_m A_m & \to & \lim_m A_m \otimes_R R/J^n \\
\downarrow & & \downarrow \\
\lim_m B_m & \to & \lim_m B_m \otimes_R R/J^n.
\end{array}
\]

(2.20)
and our goal is the (2.20) remains homotopy Cartesian after applying the $K$-theory functor. For notational simplicity, we will prove that relative $K$-theory pro-spectra are isomorphic:

$$\lim_{m,n} K(A_m; J^n) \to \lim_{m,n} K(B_m; J^n)$$

where these pro-spectra are the fibers of the horizontal maps in (2.20) after applying $K$.

2.6.2. We first show that (2.21) is an isomorphism on $\pi_i$ for $i \leq 0$. Indeed, we note that in this range, we may apply Lemma 2.9 to the simplicial rings $A_m$ and $A_m \otimes R/J^n$ to conclude:

$$\lim_{m,n} K_i(A_m; J^n) \simeq \lim_{m,n} K_i(\pi_0 A_m; J^n).$$

Therefore, we may assume $A_m$ and $B_m$ are systems of discrete $R$-algebras. We thus obtain isomorphisms for $i \leq 0$:

$$\lim_{m,n} K_i(A_m; J^n) \simeq \lim_{m,n} K_i(J^n A_m) \simeq \lim_{m,n} K_i(J^n B_m) \simeq \lim_{m,n} K_i(B_m; J^n)$$

using Lemma 2.6 and the identification $\lim_{m,n} J^n A_m \simeq \lim_{m,n} J^n B_m$ of pro-systems of ideals.

2.6.3. To prove the isomorphism on relative $K$-groups of degrees $i \geq 1$, we proceed by induction on the number of generators of $J$. Let us first deal with the base case $J = (r)$. According to the discussion in §2.5.2, we may assume that the images of $r$ in $(A_m)_0, (B_m)_0$ are elements $a_m, b_m$ which act as non-zerodivisors on each $(A_m)_i, (B_m)_i$. We are thus reduced to showing the isomorphism:

$$\lim_{m,n} K_i(A_m; a^n_m A_m) \simeq \lim_{m,n} K_i(B_m; b^n_m B_m)$$

The hypothesis shows that the map:

$$B GL(a^n_m A_m) \to B GL(b^n_m B_m)$$

induces a homotopy equivalence of pro-spectra. Using Lemma 2.16, we obtain an equivalence of pro-groups:

$$\lim_{m,n} H_*(F(A_m; a^n_m A_m); \mathbb{Z}) \simeq \lim_{m,n} H_*(F(B_m; b^n_m B_m); \mathbb{Z}).$$

Using the fact that $F(A_m; a^n_m A_m)$ and $F(B_m; b^n_m B_m)$ are nilpotent spaces (Lemma 2.15), we deduce that the homotopy groups of their pro-spectra are equivalent.

2.6.4. We now handle the induction step. Suppose $J = (J, r)$ where $J$ is an ideal with less generators than $\tilde{J}$. We first show that the hypothesis for $\tilde{J}$ implies the hypothesis for $(r)$, i.e., the following diagram is also homotopy Cartesian:

$$\begin{array}{ccc}
\lim_{m} A_m & \to & \lim_{m,n} A_m \otimes R/r^n \\
\downarrow & & \downarrow \\
\lim_{m} B_m & \to & \lim_{m,n} B_m \otimes R/r^n.
\end{array}$$

9The fact that homological equivalence between nilpotent spaces induces homotopy equivalence is the content of Whitehead’s theorem. The pro-version of the theorem is also true.
Indeed, for any simplicial ring \( \lim_n A_m \) and any multiplicatively closed system \( S \subset R \), we let \( S_i^{-1} \lim_m A_m \) denote the final object in the \( \infty \)-category of simplicial rings over \( \lim_n A_m \) on which each \( f \in S \) acts by equivalence. Tautologically, for \( \tilde{S} \subset S \) we have:

\[
S_i^{-1} \lim_m A_m \cong S_i^{-1} \lim_m A_m.
\]  

(2.24)

Note that we have a fiber sequence:

\[
S_i^{-1} \lim_m A_m \to \lim_m A_m \otimes_{R} R/J^n
\]

where \( \tilde{S} := \bigcap_{p \not\supset J} R \setminus p \). Our hypothesis is equivalent to the identification of \( \tilde{S}_i^{-1} \lim_m A_m \) with \( \tilde{S}_i^{-1} \lim_m B_m \). Using (2.24) for \( S := \bigcap_{j(r)} R \setminus p \), we see that \( \tilde{S}_i^{-1} \lim_m A_m \) identifies with \( \tilde{S}_i^{-1} \lim_m B_m \), i.e., (2.23) is homotopy Cartesian. Thus (2.20) factors into the composition of two homotopy Cartesian squares of simplicial \( R \)-algebras:

\[
\begin{array}{ccc}
\lim_m A_m & \longrightarrow & \lim_m A_m \otimes_{R} R/r^n \\
\downarrow & & \downarrow \\
\lim_m B_m & \longrightarrow & \lim_m B_m \otimes_{R} R/r^n
\end{array}
\]

The monogenic case shows that the left square is sent to a homotopy Cartesian square under \( K \). The induction hypothesis, applied to the map of simplicial \( R/r^n \)-algebras \( \lim_m A_m \otimes_{R} R/r^n \to \lim_m B_m \otimes_{R} R/r^n \) and the image of \( J \) in \( R/r^n \) shows that the right square is also sent to a homotopy Cartesian square under \( K \). Therefore (2.20) remains homotopy Cartesian after applying \( K \). \( \square \)

3. Pro-excision of schemes

In this section, we first study (derived) ind-schemes, which will provide us a flexible framework to work with schemes alongside with their formal completions. Then we (re)state the pro-excision theorem in terms of \( K \)-theory of ind-schemes. In the rest of this section, we give a complete proof of the theorem.

3.1. \( K \)-theory of ind-schemes.

3.1.1. We will use the term \emph{ind-scheme} to mean strict, ind-(derived)-schemes. These are prestacks \( \mathcal{X} \) which admit a presentation as a filtered colimit:

\[
\mathcal{X} \sim \colim_{i \in I} \mathcal{X}_i,
\]

where the transition maps \( \mathcal{X}_i \to \mathcal{X}_j \) are closed immersions. We emphasize that the above colimit is taken in the category of prestacks. Ind-schemes form a full subcategory \( \text{IndSch} \) of prestacks\(^{10}\).

3.1.2. An ind-scheme \( \mathcal{X} \) is \emph{ind-affine} if it can be represented as a colimit (3.1) where each \( \mathcal{X}_i \) is an affine scheme. In this case, we may regard \( \mathcal{X} \) as defined by the pro-simplicial ring \( \mathcal{O}_\mathcal{X} = \lim_{i \in I} \mathcal{O}_{\mathcal{X}_i} \).

\(^{10}\)We refrain from the notation \( \text{Ind}(\text{Sch}) \) since our ind-schemes do not include all ind-objects of the category of schemes.
3.1.3. The $K$-theory functor on schemes extends as a limit-preserving functor:

$$K : \text{IndSch}^{op} \to \text{Pro}(\text{Sptr})$$

where $\text{Pro}(\text{Sptr})$ is the pro-completion of $\text{Sptr}$; this procedure gives us the notion of $K$-theory of ind-schemes. Suppose $\mathcal{X}$ is represented by $\text{colim}_{i \in I} \mathcal{X}_i$, then we have a tautological equivalence

$$K(\mathcal{X}) \sim \lim_{i \in I} K(\mathcal{X}_i).$$

3.2. Formal completions.

3.2.1. Suppose $\mathcal{Y} \to \mathcal{X}$ is a map of prestacks. We let $\mathcal{X}_\mathcal{Y}$ denote the prestack defined by:

$$\text{Maps}(S, \mathcal{X}_\mathcal{Y}) := \text{Maps}(S, \mathcal{X}) \times_{\text{Maps}(S, \mathcal{X})} \text{Maps}(S, \mathcal{Y}).$$

The prestack $\mathcal{X}_\mathcal{Y}$ is called the formal completion of $\mathcal{X}$ along $\mathcal{Y}$. We note that $\mathcal{X}_\mathcal{Y}$ is equivalent to $\mathcal{X}_{\mathcal{Y}_{\text{red}}}$, where $\mathcal{Y}_{\text{red}}$ is the reduced prestack $^{11}$ of $\mathcal{Y}$. In other words, formal completion is insensitive to the non-reduced (or derived) structure of the source.

3.2.2. We now show that $\text{IndSch}$ is closed under the operation of formal completions along closed immersions. The following result is [GR14, Proposition 6.3.1] in the DG setting but its proof is model-independent.

Lemma 3.1. Suppose $\mathcal{Y} \to \mathcal{X}$ is a closed immersion in $\text{IndSch}$. Then $\mathcal{X}_\mathcal{Y}$ is an ind-scheme.

Sketch of proof. One considers the category $\mathcal{I}$ of objects $S$ in $\text{Sch}_{qc-qs}$ which comes equipped with a closed embedding $S \to \mathcal{X}$ that factors through $\mathcal{X}_\mathcal{Y}$. The one checks:

1. the category $\mathcal{I}$ is filtered; this follows from an explicit description of its finite colimits.
2. the colimit $\text{colim}_{S \in \mathcal{I}} S$ identifies with $\mathcal{X}_\mathcal{Y}$; indeed, since $\mathcal{X}$ is the colimit of $S \in \text{Sch}_{qc-qs}$ together with a closed embedding $S \to \mathcal{X}$, we may reduce to the case where $\mathcal{X} \in \text{Sch}_{qc-qs}$. Now it is sufficient to observe that if a map $Z \to \mathcal{X}$ in $\text{Sch}_{qc-qs}$ factors through $\mathcal{X}_\mathcal{Y}$, so does its schematic image.

Thus we obtain a presentation of $\mathcal{X}_\mathcal{Y}$ as a filtered colimit of objects in $\text{Sch}_{qc-qs}$. □

3.2.3. The procedure of formal completion is functorial with respect to Cartesian squares. More precisely, given a Cartesian square of prestacks:

$$\begin{array}{ccc}
\mathcal{Y}' & \longrightarrow & \mathcal{X}' \\
\downarrow & & \downarrow \\
\mathcal{Y} & \longrightarrow & \mathcal{X}
\end{array} \,(3.2)$$

we obtain another Cartesian square of prestacks:

$$\begin{array}{ccc}
(\mathcal{X}')_\mathcal{Y} & \longrightarrow & \mathcal{X}' \\
\downarrow & & \downarrow \\
\mathcal{X}_\mathcal{Y} & \longrightarrow & \mathcal{X} \\
\end{array} \,(3.3)$$

Remark 3.2. When (3.2) is a square of classical schemes which is Cartesian only in the class sense, we still obtain a (derived) Cartesian square (3.3); indeed, this is because formal completion is only sensitive to the reduced scheme structure on the source.

---

11Regarding a prestack $\mathcal{Y}$ as a functor $(\text{Sch}_{aff})^{op} \to \text{Spc}$, its reduced prestack $\mathcal{Y}_{\text{red}}$ is the right Kan extension of the restriction of $\mathcal{Y}$ to reduced affine schemes $(\text{Sch}_{aff})^{op}$; thus $\mathcal{Y}_{\text{red}}(S)$ identifies with $\lim T$ where the limit is taken over all reduced affine schemes $T$ with a morphism $T \to S$. 

---
3.2.4. We may now state the main pro-excision theorem using the language of ind-schemes.

**Theorem 3.3** (Pro-excision). *Given an abstract blow-up square (†), the $K$-theory functor turns the following Cartesian square in $\text{IndSch}$:*

$$
\begin{array}{ccc}
\tilde{X}_E & \rightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
X_Y & \rightarrow & X
\end{array}
$$

*into a Cartesian square in $\text{Pro(Sptr)}$.\ *

3.2.5. The proof of Theorem 3.3 occupies §3.3-3.5, and will be done in successive generality as indicated in the flow chart below:

| pro-excision for | $\Rightarrow$ | pro-excision along |
| simplicial rings | $\Rightarrow$ | finite morphisms $\Rightarrow$ |
| $\Rightarrow$ | pro-excision for | pro-excision for $\Rightarrow$ |
| classical blow-ups | abstract blow-ups |

Deducing pro-excision along finite morphisms $p : \tilde{X} \to X$ from the ring-theoretic statement is essentially an Artin-Rees argument, which relates abstract blow-up squares to Milnor squares of rings, c.f. §3.3. The next step is to prove pro-excision for classical blow-up squares; it uses the previous step to reduce the proof to the case of derived blow-up squares and then appeal to the result of [CHSW08] for their excision (which holds on the nose), c.f. §3.4. This is the only step in the proof where derived algebraic geometry comes into play. Finally, the case of general abstract blow-up squares follows from the previous case using Raynaud-Gruson’s *platification par éclatement* [RG71], c.f. §3.5.

3.3. **Pro-excision along finite morphisms.**

3.3.1. Suppose:

$$
\begin{array}{ccc}
\mathcal{E} & \rightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
Y & \rightarrow & X
\end{array}
$$ (3.4)

is a square in $\text{Sch}_{\text{qc-qs}}$ satisfying the following conditions:

(a) it is Cartesian up to a nil-isomorphism;
(b) the map $i$ is a closed immersion;
(c) the map $p$ is *finite* and induces an isomorphism $\tilde{X}\setminus \mathcal{E} \xrightarrow{\sim} X\setminus Y$ of *classical* schemes.

We explain the meaning of condition (a): the square (3.4) induces a morphism $\mathcal{E} \to Y \times X$, and condition (a) means that this morphism is an isomorphism on the reduced structures.

3.3.2. The following result is a combination of Corollary 4.13 and Proposition 5.2 of [KST18].

**Lemma 3.4.** *Under the hypothesis of §3.3.1, the following diagram of pro-spectra is homotopy Cartesian:*

$$
\begin{array}{ccc}
K(\mathcal{X}) & \rightarrow & K(X_Y) \\
\downarrow & & \downarrow \\
K(\tilde{X}) & \rightarrow & K(\tilde{X}_E)
\end{array}
$$
3.3.3. We first use pro-excision of simplicial rings to reduce the proof of Lemma 3.4 to the case of classical schemes. We claim that the following relative $K$-theory pro-spectra are identified:

$$K(X, X_{\bar{y}}) \sim \rightarrow K(X^{cl}, (X^{cl})_{\bar{y}^{\sim}}).$$ (3.5)

Indeed, in proving (3.5) we may assume that $X$ is affine and $Y \hookrightarrow X$ is a quasi-smooth closed immersion. This implies that we may take some discrete ring $R$ and an ideal $I \subset R$ such that $O_X$ is an $R$-algebra and $O_Y \sim \rightarrow O_X \mathbin{\otimes}^R R/I$. Thus (3.5) will follow from pro-excision, applied to the $R$-algebra map $O_X \rightarrow \pi_0 O_X$ if we can show that $\lim_{n} I^n \pi_i(O_X) = 0$ for all $i \geq 1$.

However, since $\pi_i(O_X)$ is a coherent $\pi_0(O_X)$-module set-theoretically supported on $|Y|$, we see that $I \cdot \pi_0(O_X) \subset \pi_i(O_X)$ for sufficiently large $n$; the desired vanishing thus follows. The same argument shows that $K(\tilde{X}, X_{\bar{Y}}) \sim \rightarrow K(\tilde{X}^{cl}, (\tilde{X}^{cl})_{\bar{Y}^{\sim}})$. Hence in proving the lemma, we may assume that (3.4) is a square of classical schemes.

3.3.4. Replacing $E$ if necessary, we may further assume that the square (3.4) is Cartesian in the classical sense. Since the question is Zariski local in $X$, the hypothesis is equivalent to a push-out square of discrete rings:

$$\begin{array}{ccc}
A & \longrightarrow & A/I \\
\varphi \downarrow & & \downarrow \\
B & \longrightarrow & B/J
\end{array}$$ (3.6)

where $\varphi$ is finite and induces an isomorphism $A_f \sim \rightarrow B_f$ for every $f \in I$. We must show that (3.6) induces a Cartesian square of $K$-theory pro-spectra. The idea is approximate the situation with Milnor squares.

3.3.5. For a sufficiently large integer $N$, we have

(a) $\varphi(I^N)B \subset \varphi(A)$; indeed, consider the cokernel $L$ of the map $A \rightarrow B$ as finite $A$-modules. The hypothesis shows that $I$ acts locally nilpotently on $L$. Thus $I^N L = 0$ for some $N$. This choice of $N$ guarantees that the image of $\varphi(I^N)B$ in $L$ vanishes, so $\varphi(I^N)B \subset \varphi(A)$.

(b) $I^N \cap \text{Ker}(\varphi) = 0$; indeed, the Artin-Rees lemma shows that for some integer $k$, we have:

$$I^N \cap \text{Ker}(\varphi) \subset I^{N-k} \text{Ker}(\varphi).$$

for $N \geq k$. An argument as before shows that $I^{N-k} \text{Ker}(\varphi) = 0$ for sufficiently large $N$.

Therefore, by replacing $I$ with $I^N$ we may assume that the image of $\varphi : A \rightarrow B$ contains $J$ and the induced map $I \rightarrow J$ is injective. By factoring $\varphi$ into a surjection followed by an injection, we have a commutative diagram:

$$\begin{array}{ccc}
I & \longrightarrow & T \\
\downarrow & & \downarrow \\
A \cap J & \longrightarrow & J \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}$$

Both of the skew squares give rise to Milnor squares, whereas $T$ and $A \cap J$ define the same pro-system of ideals in $A$. The proof is thus complete by pro-excision of rings, i.e., Theorem 2.1 for discrete rings. \hspace{1cm} \Box \text{(Lemma 3.4)}

3.4. Pro-excision for classical blow-up squares.
3.4.1. Let \( Y \hookrightarrow X \) be a closed immersion in \( \text{Sch}_{\text{qc}}^{\text{cl}} \). We consider the corresponding blow-up square:

\[
\begin{array}{ccc}
E & \hookrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
Y & \hookrightarrow & X
\end{array}
\]

**Lemma 3.5.** Under the above hypothesis, the following diagram of pro-spectra is homotopy Cartesian:

\[
\begin{array}{ccc}
K(X) & \hookrightarrow & K(X_{\tilde{Y}}) \\
\downarrow & & \downarrow \\
K(\tilde{X}) & \hookrightarrow & K(\tilde{X}_{\tilde{E}})
\end{array}
\]

3.4.2. The proof of Lemma 3.5 proceeds by first proving the analogous statement for a derived blow-up \( \tilde{X} \) and the exceptional divisor \( \mathcal{E} \), and then compare the relative pro-spectrum \( K(\tilde{X}, \tilde{X}_{\tilde{E}}) \) to \( K(\tilde{X}, \tilde{X}_{\tilde{E}}) \). We recall that the map \( Y \hookrightarrow X \) is the classical underlying structure of some quasi-smooth closed embedding \( Y \hookrightarrow X \). The construction of §1.4.3 gives rise to a commutative cube:

\[
\begin{array}{ccc}
E & \hookrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
\mathcal{E} & \hookrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
Y & \hookrightarrow & X \\
\downarrow & & \downarrow \\
\mathcal{E} & \hookrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
Y & \hookrightarrow & X
\end{array}
\] (3.7)

3.4.3. We first claim that the morphism \( \iota : \tilde{X} \to \tilde{X} \) is a closed immersion. Indeed, Zariski locally on \( X \), the derived scheme \( \tilde{X} \) is the pullback of the blow-up \( \mathbb{A}^d \to \mathbb{A}^d \) at the origin along some map \( f : X \to \mathbb{A}^d \) with \( f^{-1}(0) = Y \). Thus the claim follows from:

**Lemma 3.6.** Given a closed immersion \( Z \hookrightarrow Y \) in \( \text{Sch}_{\text{cl}} \) and an ideal \( \mathfrak{a} \subset \mathcal{O}_Y \) which defines a subscheme \( Y' \subset Y \), there is a canonical closed immersion:

\[
\text{Bl}_{Z \times Y'}(Z) \hookrightarrow \text{Bl}_{Y'}(Y).
\] (3.8)

**Proof.** Suppose \( Z \hookrightarrow Y \) is represented by a surjective ring map \( \varphi : A \to B \). Then \( Z \times Y' \) is defined by the ideal \( \varphi(\mathfrak{a}) \). Thus we have a surjective map of graded algebras:

\[
\varphi^\bullet : \bigoplus_{d \geq 0} \mathfrak{a}^d \to \bigoplus_{d \geq 0} \varphi(\mathfrak{a})^d.
\]

This is the quotient by the homogeneous ideal generated by \( \mathfrak{a} \cap \text{Ker}(\varphi) \) (in degree \( d = 1 \)), which defines the closed immersion (3.8). \( \square \)

3.4.4. We note that \( \iota \) induces an isomorphism \( \tilde{X} \setminus E \simeq \tilde{X} \setminus \mathcal{E} \) of classical schemes (both identifying with \( X \setminus Y' \)). Therefore the top square of (3.7) satisfies the hypothesis alluded to in Lemma 3.4, so we obtain an equivalence:

\[
K(\tilde{X}, \tilde{X}_{\tilde{E}}) \simeq K(\tilde{X}, \tilde{X}_{\tilde{E}}).
\] (3.9)
3.4.5. We will now start the proof that the map of pro-spectra:

$$K(X, X_{\mathcal{Y}}) \to K(\overline{X}, \overline{X}_{\mathcal{Y}}) \quad (3.10)$$

is again an equivalence. This is a Zariski local question on $X$ (c.f. Lemma 1.10), so we may assume that $\mathcal{Y}$ is the pullback of the origin along some map $f : X \to \mathbb{A}^d$. Let $0^{(n)}$ denote the $n$th infinitesimal thickening of the origin in $\mathbb{A}^d$. Note that the blow-ups of $\mathbb{A}^d$ at $0^{(n)}$ are canonically identified for any $n \geq 1$; we will denote them by $\tilde{\mathbb{A}}^d$. Thus we have a classical blow-up square:

$$P^{(n)} \longrightarrow \tilde{\mathbb{A}}^d \quad (3.11)$$

$$\downarrow \quad \downarrow$$

$$0^{(n)} \longrightarrow \mathbb{A}^d$$

where $P^{(n)}$ is the exceptional divisor corresponding to $0^{(n)}$.

3.4.6. If the center $0^{(n)}$ was regularly immersed in $\mathbb{A}^d$, we would be able to obtain a derived blow-up square and get excision of $K$-theory spectra on the nose (c.f. §1.4.3). However, this is not the case for $n \geq 2$. Therefore, we will employ a different presentation of $(\mathbb{A}^d)^\sim$ that is termwise regularly immersed, and then compare the resulting $K$-theory pro-spectra. Let $0(n) \in \mathbb{A}^d$ denote the (regularly immersed) closed subscheme defined by the ideal $(x_1^n, \cdots, x_d^n)$. Consider the blow-up square:

$$P(n) \longrightarrow \tilde{\mathbb{A}}^d(n) \quad (3.12)$$

$$\downarrow \quad \downarrow$$

$$0(n) \longrightarrow \mathbb{A}^d$$

We note two quick facts:

(a) there is a canonical identification $\text{colim}_n 0(n) \simto (\mathbb{A}^d)_0$;

(b) the ideal $(x_1^n, \cdots, x_d^n)$ is a reduction of $(x_1, \cdots, x_d)^n$ by [HS06, Proposition 8.1.5]. This implies that their blow-ups are related by a finite map $q$ (c.f. Theorem 8.2.1 of loc.cit.):\[12\]

$$P^{(n)} \longrightarrow \tilde{\mathbb{A}}^d \quad (3.13)$$

$$\downarrow \quad \downarrow q$$

$$P(n) \longrightarrow \tilde{\mathbb{A}}^d(n)$$

3.4.7. The (derived) base change of the squares (3.12) and (3.13) gives rise to a commutative diagram of derived schemes:

$$\mathcal{E}^{(n)} \longrightarrow \tilde{\mathcal{X}} \quad (\text{with } q')$$

$$\downarrow \quad \downarrow$$

$$\mathcal{E}(n) \longrightarrow \tilde{\mathcal{X}}(n)$$

$$\downarrow \quad \downarrow$$

$$\mathcal{Y}(n) \longrightarrow X$$

\[12\] A subideal $J \subset I$ in a ring $R$ is called a reduction if there exists some integer $n$ for which $J \cdot I^n = I^{n+1}$. In the case where $I$ is finitely generated, an equivalent characterization is that $I$ belongs to the integral closure $\overline{J} \subset R$ (Caution: $\overline{J}$ is not the integral closure of $J$ as a subring). Note that ones has the inclusion $\overline{J} \subset \sqrt{J}$ but the converse is false in general, so $J \subset I$ being a reduction is stronger than saying that the subschemes defined by $I$ and $J$ have the same underlying set.
where \( \mathcal{Y}(n) \) defines a colimit presentation of \( X_0 \) (hence of \( X_\varphi \)) and \( q' \) is finite. Since the lower square is a derived blow-up square, we may apply §1.4.3 to obtain an isomorphism:

\[
K(X, \mathcal{Y}(n)) \xrightarrow{\sim} K(\tilde{X}(n), \mathcal{E}(n))
\]

of relative \( K \)-theory spectra. Using Lemma 3.4 for the upper square, we obtain:

\[
\lim_n K(\tilde{X}(n), \mathcal{E}(n)) \xrightarrow{\sim} \lim_n (\tilde{X}, \tilde{X}_\varphi).
\]

Therefore, we see that

\[
K(X, X_\varphi) \xrightarrow{\sim} \lim_n K(X, \mathcal{Y}(n)) \xrightarrow{\sim} \lim_n (\tilde{X}, \tilde{X}_\varphi)
\]

which is in turn identified with \( K(\tilde{X}, \tilde{X}_\varphi) \) by (3.9). □

### 3.5. Pro-excision for abstract blow-up squares.

#### 3.5.1. Suppose we are given an abstract blow-up square (†). We want to show that the corresponding square of pro-spectra:

\[
\begin{array}{ccc}
K(X) & \longrightarrow & K(X_\varphi) \\
\downarrow & & \downarrow \\
K(\tilde{X}) & \longrightarrow & K(\tilde{X}_\varphi)
\end{array}
\]

is (homotopy) Cartesian. In other words, we would like to show that the canonical map:

\[
K(X; X_\varphi) \xrightarrow{\sim} K(\tilde{X}; \tilde{X}_\varphi)
\]

(3.14)

is an isomorphism in \( \text{Pro} \text{(Sptr)} \).

#### 3.5.2. The proof proceeds by approximating an abstract blow-up square with “concrete” blow-up squares, where one has pro-excision by Lemma 3.5. We will first reduce to the case where the complements of the closed subschemes \( Y \hookrightarrow X \) and \( E \hookrightarrow \tilde{X} \) are schematically dense. Indeed, for any closed immersion \( i : Y \hookrightarrow X \) in \( \text{Sch}_{\text{qc-qs}} \), we write \( U := X \setminus Y \) and \( \overline{U} \) for its schematic closure inside \( X \). Then the square:

\[
\begin{array}{ccc}
Y \times_X \overline{U} & \longrightarrow & \overline{U} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{i} & X
\end{array}
\]

satisfies the hypothesis of §3.3.1. Hence we obtain an isomorphism of pro-spectra:

\[
K(X; X_\varphi) \xrightarrow{\sim} K(U; \overline{U})_{\mathcal{Y}(n)}.
\]

This shows that in proving (3.14) is an isomorphism, we may assume that \( U \) is schematically dense in \( X \) and \( p^{-1}U \) is schematically dense in \( \tilde{X} \).

#### 3.5.3. With this assumption, the morphism \( p : \tilde{X} \rightarrow X \) is more similar to a blow-up map at a center away from \( X \setminus Y \). However, in general \( p \) is not a blow-up or even a sequence of blow-ups. In order to bring us to the previous situation, we use a general result [Te08, Lemma 2.1.5] that shows that \( p \) is dominated by a blow-up map \( p' \). The existence of \( p' \) follows directly from Raynaud-Gruson’s theorem of platification par éclatement [RG71, Théorème 5.2.2].

**Lemma 3.7.** Let \( X \) be a Noetherian scheme and \( U \subset X \) be a schematically dense open subscheme. Suppose \( p : \tilde{X} \rightarrow X \) is a proper morphism, such that:

(a) \( p^{-1}U \) is schematically dense in \( \tilde{X} \);
(b) $p$ induces an isomorphism $p^{-1}U \xrightarrow{\sim} U$.

Then there exists a closed subscheme $Z \hookrightarrow X$ supported away from $U$ such that the blow-up $\text{Bl}_Z(X) \to X$ factors through $p : \tilde{X} \to X$.

**Sketch of proof.** We first informally describe the *platification* technique: given a quasi-coherent sheaf $\mathcal{F}$ which is flat over some open subscheme, we can blow-up away from this open so that the strict transform $\mathcal{F}^{\text{St}}$ becomes flat over the entire space (c.f. [RG71, Théorème 5.2.2] for the precise statement). We apply *platification* in the relative situation where $\mathcal{F} = \mathcal{O}_{\tilde{X}}$ is flat over $U \subset X$; the theorem supplies us with a commutative diagram:

\[
\begin{array}{ccc}
U & \xrightarrow{p} & \tilde{X} \\
\downarrow & & \downarrow \\
\tilde{X} & \xrightarrow{\text{Bl}_Z(X)} & X
\end{array}
\]

where the strict transform $(\mathcal{O}_{\tilde{X}})^{\text{St}}$ is flat over $\text{Bl}_Z(X)$. We check that the closure of $U$ inside $\tilde{X} \times \text{Bl}_Z(X)$ has structure sheaf $(\mathcal{O}_{\tilde{X}})^{\text{St}}$ and maps isomorphically onto $\text{Bl}_Z(X)$.\qed

**3.5.4.** We now return to the proof that (3.14) is an isomorphism. Applying Lemma 3.7, we obtain a closed subscheme $Z \hookrightarrow X$ supported away from $X \setminus Y$, together with a factorization $p' : \text{Bl}_Z(X) \to \tilde{X} \xrightarrow{\pi} X$. We consider the pre-images of $Y$ in $\tilde{X}' := \text{Bl}_Z(X)$ and $\tilde{X}$ (denoted by $E'$ and $E$) as well as their completions:

\[
\begin{array}{ccc}
\tilde{X}'_{E'} & \to & \tilde{X}' \\
\downarrow & & \downarrow \\
\tilde{X}_{E} & \to & \tilde{X} \\
\downarrow & & \downarrow p \\
X_{\bar{\mathcal{F}}} & \to & X
\end{array}
\]

We obtain morphisms on relative $K$-theory pro-spectra:

\[
K(X; X_{\bar{\mathcal{F}}}) \to K(\tilde{X}; \tilde{X}_{E}) \to K(\tilde{X}'; \tilde{X}'_{E'})
\]

(3.16)

We claim that it will be sufficient to show that the composition (3.16) is an isomorphism. Indeed, the Cartesian square:

\[
\begin{array}{ccc}
E' & \to & \tilde{X}' \\
\downarrow & & \downarrow \\
E & \to & \tilde{X}
\end{array}
\]

is again an abstract blow-up square where the complements $\tilde{X} \setminus E$ and $\tilde{X}' \setminus E'$ are schematically dense. This means that we may iterate the above construction to continue (3.16) to a sequence of morphisms:

\[
K(X; X_{\bar{\mathcal{F}}}) \to K(\tilde{X}; \tilde{X}_{E}) \to K(\tilde{X}'; \tilde{X}'_{E'}) \to K(\tilde{X}''; \tilde{X}''_{E''})
\]

where both compositions of subsequent maps are isomorphisms. Hence (3.14) is an isomorphism by the 2-out-of-3 property.
3.5.5. We now prove that the composition (3.16) is an isomorphism. Note that although the embedding $Z \hookrightarrow X$ may not factor through $Y$, we may assume that it does after replacing $Y$ by an infinitesimal thickening. Let $Z^{(n)}$ (respectively $Y^{(n)}$) be the $n$th infinitesimal thickening of $Z$ (respectively $Y$) inside $X$. We may thus consider the following diagram consisting of Cartesian squares:

\[
\begin{array}{ccc}
\text{Bl}_{Z^{(n)}}(Y^{(n)}) \times_{Y^{(n)}} Z^{(n)} & \rightarrow & \text{Bl}_{Z^{(n)}}(Y^{(n)}) \\
\downarrow & & \downarrow \\
\text{Bl}_{Z^{(n)}}(X) \times_{X} Z^{(n)} & \rightarrow & \text{Bl}_{Z^{(n)}}(X) \times_{X} Y^{(n)} \\
\downarrow & & \downarrow \\
Z^{(n)} & \rightarrow & Y^{(n)} \\
\end{array}
\]

where the square (3) gives rise to the composition (3.15) after taking colimit. Thus we must show that the square (3) becomes a Cartesian square of pro-spectra after taking $\lim_n K(-)$. However, we already know that the following squares satisfy this property:

(a) The composed squares (1)+(2) and (2)+(3), as implied by pro-excision for classical blow-up squares (Lemma 3.5).

(b) The square (1), as $\iota$ identifies with the closed immersion of the strict transform of $Y^{(n)}$ inside $\text{Bl}_{Z^{(n)}}(X)$ to its fiber [Stacks, 080E], so we may apply pro-excision along finite maps (Lemma 3.4).

The same property of (3) thus follows, and we have shown that (3.16) is an isomorphism. □ (Theorem 3.3)

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