# Geometric Satake (Elden) 9/24

or how I learned to change my characteristic

**Motivation:**

$$K_0(R) \cong \left(\mathrm{Mod}^{f.g.,\,proj.}, \oplus\right)^+$$

Ex: $K_0(k) \cong \mathbb{Z}$ ← iso independent of char.

NB: + denotes group completion.

"Explanation/Def'n"

$$\left(\coprod_{n\geq 0} BGL_n(R)\right)^+$$

$GL_n(R)$ is not naturally an $R$-module. however, deformation thy of $BGL_{n,R}$ recovers $R$-linear data.

## § Review of Affine Grassmannian.

fix $k$ alg. closed

$G/k$ sm aff. gp scheme & reductive

**Def'n** $\mathrm{Gr}_G(R) := \left\{(\mathcal{E}, \beta) \;\middle|\; \begin{array}{l}\mathcal{E} \text{ a } G\text{-torsor} \\ \;\downarrow \\ D_r\end{array}\right\}$

$$\beta : \mathcal{E}|_{D_r^*} \cong \underbrace{\mathcal{E}^0}_{\text{trivial}}|_{D_r^*}$$

**Thm** $Gr_G$ is ind-scheme, ind-fin. type

$\nexists$ reductive $\Longrightarrow$ ind-projective

**Rmk** $Gr_G$ is the algebraic geometers loop group.

precisely,

$$Gr_G(\mathbb{C})^{an} \simeq \Omega G$$

$\Omega G$ has two multiplications.

$\rightsquigarrow$ $\Omega G$ has an $\mathbb{E}_2$-algebra structure.

**Exercise!** for $G$ discrete w/ 2 multiplications

$\Longrightarrow$ $G$ is abelian.

## Some Geometry of $Gr_G$

$$L^+X := \underline{\hom}(\mathbb{Z}[[t]], X)$$
$$LX := \underline{\hom}(\mathbb{Z}((t)), X)$$

there is an action

$$LG \times Gr_G \xrightarrow{\text{adjust } \rho} Gr_G$$

the action is transitive w/ stab. $L^+G$

<u>Cor</u>: $[LG/L^+G]_{\text{fpqc}} \cong Gr_G$

<u>Variant</u>:

$$Gr_G \tilde{\times} Gr_G := \left\{ \begin{matrix} \mathcal{E}_1 & \mathcal{E}_2 \\ \beta_1 & \beta_2 \end{matrix} \middle| \mathcal{E}_2 \dashrightarrow \mathcal{E}_1 \dashrightarrow \mathcal{E}^0 \right\}$$

<u>convention</u>: $\dashrightarrow$ is an iso $/ D_R^*$

$$Gr_G \tilde{\times} Gr_G \xrightarrow{\text{compose}} Gr_G$$

<u>prop</u>: $Gr_G \tilde{\times} Gr_G \cong LG \times^{L^+G} LG / L^+G$

meta <u>Thm</u> (Ginzburg, Mirkovic-Vilonen, Lusztig, Beilinson-Drinfeld)

$\mathcal{M}$ an $R$-linear sheaf theory

$$\underset{\substack{| \\ \text{equivariance}}}{\mathcal{M}_{L^+G}}(Gr_G; R) \cong \operatorname{Qcoh}(B\check{G}_R)$$

## Examples

$k = \mathbb{C}$ $\quad \mathcal{M} = Shv_{an}(-; E)$ (another field)

$k = \mathbb{C}$ $\quad \mathcal{M} = Dmod$

$k = \mathbb{C}$ $\quad \mathcal{M} = Shv_{an}(-; \mathbb{Z})$

$k = \mathbb{F}_p$ $\quad \mathcal{M} = Shv_{et}(-; \uparrow)$

$\mathbb{Q}_\ell, \mathbb{Z}_\ell, \ldots$

Morally: LHS only depends on $Gr_G/\mathbb{Z}$

Strategy of the proof.

We'll refer to the LHS as $Sat_G$

(A) Define a rigid ~~symm~~ mon. structure $\star$ on $Sat_G$

(B) Define / Invoke symm. mon. fiber functor
$H^* R\Gamma : Sat_G \longrightarrow Vect$

(C) Tannakian formalism gives

$$Sat_G \simeq Rep(Aut^{\otimes}_{H^* R\Gamma})$$

## § Schubert Geometry of $Gr_G$

fix

$T \subseteq B \subseteq G$ , $X_\bullet(T) = \hom(\mathbb{G}_m, T)$

$X^\bullet(T) = \hom(T, \mathbb{G}_m)$

given a coCharacter $\mu$

$$\mu : k((t)) \longrightarrow T(k((t))) \longrightarrow G(k((t)))$$
$$t \longmapsto t^\mu$$

$\rightsquigarrow$ defines a $k$-point $t^\mu \in Gr_G$

Thm (Cartan) $G(k((t))) \cong \coprod_{\mu \in X_\bullet(T)^+} G(k[[t]])\, t^\mu\, G(k[[t]])$

rmk: this is a generalization of bruhat decomp.

rmk: tells us $k$-pts of $[L^+G \backslash LG / L^+G]$

$$\mathrm{Inv} : Gr_G \longrightarrow X_\bullet(T)^+$$

Ex: $GL_n$

$$\begin{array}{ccc} X_\bullet(T) \cong \mathbb{Z}^n & & (\alpha_1, \alpha_1 + a_2, \dots) \\ \uparrow & & \updownarrow \\ X_\bullet(T)^+ \cong \mathbb{Z} \times \mathbb{N}^{n-1} & & (\alpha_1, \cdots) \end{array}$$

$Gr_\mu$ take reduced subscheme.

<u>Def'n</u>

$$\begin{tikzcd} ? \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Gr_G \arrow[d, "\pi"] \\ * \arrow[r, "\mu"] & L^+G \backslash Gr_G \end{tikzcd}$$

$$\overline{Gr_\mu} =: Gr_{\leq \mu}$$

<u>Prop</u>: • $Gr_\mu$ is a single $L^+G$-orbit
• $Gr_\mu$ is sm quasi-proj.

## <u>§ Step A</u>

<u>Fake proof</u>

$Sat_G$ has a mon. str. from $\star$

$Gr_G$ has an $\mathbb{E}_2$-str from fusion.

$Sat_G$ aquires an $\mathbb{E}_3$-str. $\Longrightarrow$ sym. mon.

$$Gr \times Gr \xleftarrow{p} LG \times Gr \xrightarrow{q} LG \times_{L^+G} Gr \longrightarrow Gr$$

$A, B \in Sat_G$

uses the equivalence.

$$A \star B := m_!(A \tilde{\boxtimes} B)$$

NB: $q^!(- \tilde{\boxtimes} -) \cong - \boxtimes -$

**Thm** $A, B$ are perverse, then $A \star B$ is perverse.

**pf** the only non-trivial part is $m_!$

but, $m$ is semi-small!

**Def'n** $f: X \longrightarrow Y$ in $Sch^{f.t.}$ is **semismall** if $\dim(X) = \dim(X \times_Y X)$

**Thm** $Sat_G$ is semisimple in char $0$.
(the simple objects are $IC_\mu$)

$$IC_{\mu_1} \star \cdots \star IC_{\mu_n} \cong \bigoplus_\lambda V_\mu^\lambda \star IC_\lambda$$

[To be cont'd]