NOTES ON SINGULAR SUPPORT AND SPECTRAL GLUING

THE LANGLANDS SUPPORT GROUP

Contents

1. Overview

1.1. The naïve conjecture.

1.1.1. One goal of the number-theoretic Langlands theory is to attach Hecke eigenfunctions to Galois representations. Let us be given a global field F. We ask for the following association:

$$\begin{array}{c}\text{Galois representation } \sigma \\ \text{valued in } G\end{array} \quad \rightsquigarrow \quad \begin{array}{c}\text{Hecke eigenfunctions} \\ \text{on } \check{G}(\mathbb{A})\end{array}$$

where $\check{G}$ is the Langlands dual group of G.

1.1.2. To phrase this problem in geometric language, we fix a ground field k, assumed algebraically closed of characteristic zero, and let X be a smooth, proper curve over k. The role of Galois representations will be played by points of the stack LocSys_G, and instead of functions on $\check{G}(\mathbb{A})$ we will study $\mathcal{D}$-modules on $\mathrm{Bun}_{\check{G}}$.

Thus the above problem translates into an association:

$$\begin{array}{c}k\text{-points} \\ \sigma \in \mathrm{LocSys}_G\end{array} \quad \rightsquigarrow \quad \begin{array}{c}\text{Hecke eigen-}\mathcal{D}\text{-modules} \\ \text{on } \mathrm{Bun}_{\check{G}}\end{array}$$

Remark 1.1. Let us recall at this moment that LocSys_G is defined as the mapping stack $\underline{\mathrm{Maps}}(X_{\mathrm{dR}}, \mathrm{B}\,G)$, where X_{dR} is the de Rham prestack associated to X. This object is a *derived* algebraic stack, as we will study in the later parts of the semester.

1.1.3. The above problem is asymmetric, in the sense that on one hand, we are studying k-points of a certain stack, while on the other hand we are concerned with objects of a DG category. To make the problem more symmetric, we propose:

Attempt 1.2. Perhaps $\mathrm{QCoh}(\mathrm{LocSys}_G) \xrightarrow{\sim} \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_{\check{G}})$?

Under this equivalence, what used to be the Hecke eigen-$\mathcal{D}$-module associated to $\sigma \in \mathrm{LocSys}_G$ would now be the image of the skyscraper sheaf $k_\sigma \in \mathrm{QCoh}(\mathrm{LocSys}_G)$. The Hecke eigenproperty would then be a compatibility condition between this equivalence and the geometric Satake equivalence.

Date: Fall, 2018.

1.1.4. The Langlands duality is also supposed to be compatible with passage to Levi subgroups. In the geometric theory, this is the compatibility with Eisenstein series functors—for any parabolic $P \in \mathrm{Par}_G$ with Levi quotient M, the following diagram needs to commute[1]:

$$\begin{array}{ccc} \mathrm{QCoh}(\mathrm{LocSys}_M) & \longrightarrow & \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_{\check{M}}) \\ \downarrow {\scriptstyle \mathrm{Eis}^{\mathrm{Spec}}} & & \downarrow {\scriptstyle \mathrm{Eis}_!} \\ \mathrm{QCoh}(\mathrm{LocSys}_G) & \longrightarrow & \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_{\check{G}}) \end{array}$$

1.2. **Geometric Eisenstein series.**

1.2.1. Consider the diagram and the following facts:

$$\mathrm{Bun}_{\check{G}} \xleftarrow{\mathfrak{p}} \mathrm{Bun}_{\check{P}} \xrightarrow{\mathfrak{q}} \mathrm{Bun}_{\check{M}}.$$

(a) the morphism $\mathfrak{p}$ is schematic;

Remark 1.3. This follows from a general fact—given a proper map $Y \to S$ of schemes, then $\mathrm{Sect}_{/S}(S, Y)$ is representable by a scheme.

(b) the morphism $\mathfrak{q}$ is smooth;

Remark 1.4. This follows from calculation of the relative cotangent complex. In particular, the functor $\mathfrak{q}^*$ on $\mathcal{D}$-modules is well-defined.

(c) the partially defined functor $\mathfrak{p}_!$ on $\mathcal{D}$-modules is well defined on the image of $\mathfrak{q}^*$

Remark 1.5 (Lin)**.** This follows from considering Drinfeld's compactification $j : \mathrm{Bun}_{\check{P}} \hookrightarrow \overline{\mathrm{Bun}}_{\check{P}}$, and showing that $j_! \mathcal{O}_{\mathrm{Bun}_{\check{P}}}$ is ULA with respect to $\mathfrak{q}$.

In particular, we may define the functor:

$$\mathrm{Eis}_! := \mathfrak{p}_! \circ \mathfrak{q}^* : \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_{\check{M}}) \to \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_{\check{G}}).$$

It admits a continuous right adjoint $\mathfrak{q}_* \circ \mathfrak{p}^!$; hence $\mathrm{Eis}_!$ preserves compact objects, by the following:

Lemma 1.6. *Suppose we have an adjunction* $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ *of compactly generated DG categories. Then the following are equivalent:*

(a) F preserves compact objects;

(b) G is continuous. □

1.2.2. On the other hand, we have a diagram:

$$\mathrm{LocSys}_G \xleftarrow{\mathfrak{p}^{\mathrm{Spec}}} \mathrm{LocSys}_P \xrightarrow{\mathfrak{q}^{\mathrm{Spec}}} \mathrm{LocSys}_M.$$

(a) the morphism $\mathfrak{p}^{\mathrm{Spec}}$ is schematic and proper;

Question 1.7. Eh? How do you show this?

(b) the morphism $\mathfrak{q}^{\mathrm{Spec}}$ is quasi-smooth;

Remark 1.8. Once we know what quasi-smooth means, we will see that this follows from calculating the cotangent complex. For now, it will tell us that $\mathfrak{q}^*_{\mathrm{Spec}}$ on QCoh is well-defined and sends perfect complexes to perfect complexes.

The problem, however, is that $(\mathfrak{p}^{\mathrm{Spec}})_*$ on QCoh does not preserve compact objects.[2] This is the familiar fact that proper pushforward of a perfect complex may not be perfect.

[1] Technically, it should commute up to tensoring by some (cohomologically shifted) line bundle.

[2] *A priori*, the compact objects in $\mathrm{QCoh}(\mathcal{Y})$ may not identify with the perfect complexes. However, in the case $\mathcal{Y} = \mathrm{LocSys}_G$ this does happen. It's a consequence of the QCA property of LocSys_G.

1.3. Ind-coherent sheaves.

1.3.1. The *raison d'être* of ind-coherent sheaves is to remedy the non-preservation of compact objects under proper pushforward. Let us take a moment to review the basic theory. Suppose $S \in \mathbf{Sch}^{\mathrm{aff}}_{\mathrm{laft}}$ is an affine (derived) scheme locally almost of finite type, we can make sense of the full subcategory $\mathrm{Coh}(S) \hookrightarrow \mathrm{QCoh}(S)$. We let $\mathrm{IndCoh}(S)$ be the ind-completion of $\mathrm{Coh}(S)$.

Remark 1.9. Since $\mathrm{Coh}(S)$ is idempotent-complete, $\mathrm{Coh}(S)$ identifies with the full subcategory of $\mathrm{IndCoh}(S)$ of compact objects.

Note that ind-extending the inclusion $\mathrm{Coh}(S) \hookrightarrow \mathrm{QCoh}(S)$ gives rise to a functor $\Psi_S : \mathrm{IndCoh}(S) \to \mathrm{QCoh}(S)$.

1.3.2. If S is eventually coconnective, then $\mathcal{O}_S \in \mathrm{QCoh}(S)$ belongs to the full subcategory $\mathrm{Coh}(S)$. Hence, so does $\mathrm{Perf}(S)$. Ind-completing the inclusion $\mathrm{Perf}(S) \hookrightarrow \mathrm{Coh}(S)$ gives rise to an embedding $\Xi : \mathrm{QCoh}(S) \hookrightarrow \mathrm{IndCoh}(S)$, and we have an adjunction:

$$\Xi_S : \mathrm{QCoh}(S) \rightleftarrows \mathrm{IndCoh}(S) : \Psi_S$$

where the left adjoint is fully faithful.

1.3.3. Given a map $S \to S'$ in $\mathbf{Sch}^{\mathrm{aff}}_{\mathrm{laft}}$, we have *continuous* functors $f_{*,\mathrm{IndCoh}} : \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(S')$ and $f^! : \mathrm{IndCoh}(S') \to \mathrm{IndCoh}(S)$ together with base change for every Cartesian diagram.

Remark 1.10 (David)**.** The functor $f^!$ does not preserve Coh in general. Take $f : \mathrm{pt} \to S = \mathrm{Spec}(k[\epsilon]/\epsilon^2)$. Being proper, $f^!$ is right adjoint to $f_{*,\mathrm{IndCoh}}$. Thus we have:

$$f^!\mathcal{F} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{IndCoh}(S)}(k, \mathcal{F}).$$

Hence for $\mathcal{F} = k$ the skyscraper, $f^!(k)$ has nonzero cohomology in all degrees ≥ 0:[3]

$$\cdots \to 0 \to 0 \to k \xrightarrow{0} k \xrightarrow{0} \cdots$$

1.3.4. Suppose $\mathcal{Y}$ is a laft prestack. Then we set $\mathrm{IndCoh}(\mathcal{Y}) := \lim\limits_{S \to \mathcal{Y}} \mathrm{IndCoh}(S)$ where S ranges through affine schemes mapping to $\mathcal{Y}$.

Remark 1.11. As the above remark shows, we may not define $\mathrm{Coh}(\mathcal{Y})$ by an analogous formula. Instead, we have to specify $\mathrm{Coh}(\mathcal{Y})$ as the full subcategory of $\mathrm{QCoh}(\mathcal{Y})$ for which pullbacks to $S \in \mathbf{Sch}^{\mathrm{aff}}$ belongs to $\mathrm{Coh}(\mathcal{Y})$.

1.3.5. We may now re-define $\mathrm{Eis}_{\mathrm{Spec}}$ as a functor:

$$\mathrm{Eis}_{\mathrm{Spec}} : \mathrm{IndCoh}(\mathrm{LocSys}_M) \to \mathrm{IndCoh}(\mathrm{LocSys}_G)$$

and one can check that it does preserve compact objects (which form the subcategories $\mathrm{Coh}(\mathrm{LocSys}_M)$, $\mathrm{Coh}(\mathrm{LocSys}_G)$). We may now state:

Attempt 1.12. Perhaps $\mathrm{IndCoh}(\mathrm{LocSys}_G) \xrightarrow{\sim} \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_{\check{G}})$?

1.4. Langlands duality for tori.

1.4.1. Unfortunately, Attempt 1.12 already fails for $G = \mathbb{G}_m$. More generally, we will show that Attempt 1.2 is actually correct for tori. Of course, the problem reduces to the case of $\mathbb{G}_m$.

[3]Regarding k as an object of $\mathrm{QCoh}(S)$, we have $f^*(k)$ being the complex

$$\cdots \to k \xrightarrow{0} k \xrightarrow{0} 0 \to 0 \to \cdots$$

which has nonzero cohomology in all degrees ≤ 0.

1.4.2. Fix a point $x \in X$. Then we have a product decomposition:

$$\mathrm{Bun}_{\mathbb{G}_m} \xrightarrow{\sim} \mathrm{Pic}^0(X) \times \mathrm{B}\, G_m \times \mathbb{Z}.$$

Corresponding to the projection $\mathrm{LocSys}_{\mathbb{G}_m} \to \mathrm{Bun}_{\mathbb{G}_m}$, we have:

$$\mathrm{LocSys}_{\mathbb{G}_m} \xrightarrow{\sim} \widetilde{\mathrm{Pic}}^0(X) \times T^* \mathrm{B}\,\mathbb{G}_m \xrightarrow{\sim} \widetilde{\mathrm{Pic}}^0(X) \times (\mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt}) \times \mathrm{B}\,\mathbb{G}_m$$

[4]where $\widetilde{\mathrm{Pic}}^0(X)$ is the universal vectorial extension of $\mathrm{Pic}^0(X)$:

$$0 \to \mathrm{H}^0(X, \omega_X) \to \widetilde{\mathrm{Pic}}^0(X) \to \mathrm{Pic}^0(X) \to 0.$$

1.4.3. There is an equivalence of DG categories

$$\mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Pic}^0(X)) \xrightarrow{\sim} \mathrm{QCoh}(\widetilde{\mathrm{Pic}}^0(X))$$

provided by Laumon's *transformation de Fourier généralisée.*

Claim 1.13. There are canonical equivalence:

(a) $\mathcal{D}\text{-}\mathbf{Mod}(\mathrm{B}\,\mathbb{G}_m) \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt})$;

(b) $\mathcal{D}\text{-}\mathbf{Mod}(\mathbb{Z}) \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{B}\,\mathbb{G}_m)$.

Proof. (a) Consider the *smooth* map $\sigma : \mathrm{pt} \to \mathrm{B}\,\mathbb{G}_m$. Since $\mathcal{D}\text{-}\mathbf{Mod}(\mathrm{pt}) = \mathbf{Vect}$ is generated by holonomic objects, the left adjoint $\sigma_!$ to $\sigma^!$ is well-defined. One checks that this adjunction is monadic. To compute the monad $\sigma^! \sigma_!$, we use base change along the Cartesian square:

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow{\scriptstyle \sigma} \\ \mathrm{pt} & \xrightarrow{\sigma} & \mathrm{B}\,\mathbb{G}_m \end{array}$$

to see that $\sigma^! \sigma_!(k)$ is the compactly supported cohomology $\mathrm{H}^*_c(\mathbb{G}_m; \omega_{\mathbb{G}_m}) = \mathrm{H}^*(\mathbb{G}_m; k_{\mathbb{G}_m})$. This is the algebra $\mathrm{Sym}(k[1])$. Hence we have commutative squares

$$\begin{array}{ccc} \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{B}\,\mathbb{G}_m) & \xrightarrow{\sim} & \mathrm{QCoh}(\mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt}) \\ {\scriptstyle \sigma_!}\uparrow\downarrow{\scriptstyle \sigma^!} & & {\scriptstyle \pi^*}\uparrow\downarrow{\scriptstyle \pi_*} \\ \mathbf{Vect} & \xrightarrow{\sim} & \mathbf{Vect} \end{array}$$

where $\pi : \mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt} \to \mathrm{pt}$ is the projection.

(b) Both sides identifies with $\mathbb{Z}$-graded vector spaces. □

Remark 1.14 (David)**.** The DG categories $\mathrm{QCoh}(\mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt})$ and $\mathrm{IndCoh}(\mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt})$ are not equivalence even as plain DG categories. The reason is that the Hom-space between any two compact objects in $\mathrm{QCoh}(\mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt})$ lives in $\mathbf{Vect}^c$; however, $\mathrm{Hom}_{\mathrm{IndCoh}(\mathrm{pt} \underset{\mathbb{A}^1}{\times} \mathrm{pt})}(k, k)$ has unbounded cohomologies.

Remark 1.15. Under the equivalence (a), the $\mathcal{D}$-modules $\omega_{\mathrm{B}\,\mathbb{G}_m}$ passes to the skyscraper sheaf. Thus, under the Langlands duality, $\omega_{\mathrm{Bun}_{\mathbb{G}_m}}$ passes to the skyscraper at $\mathrm{triv} \in \mathrm{LocSys}_{\mathbb{G}_m}$, as one would expect.

1.5. **Singular support.**

[4]More generally, $T^* \mathrm{B}\, G \xrightarrow{\sim} (\mathrm{pt} \underset{\mathfrak{g}^*}{\times} \mathrm{pt})/G$.

1.5.1. We now return to the theory of ind-coherent sheaves, and see how it plays out for the derived scheme $S = \mathrm{pt} \underset{V}{\times} \mathrm{pt}$, where V is a vector space. The first observation is that we have an equivalence of DG categories:

$$\mathrm{IndCoh}(S) \xrightarrow{\sim} \mathrm{Sym}(V[-2])\text{-}\mathbf{Mod}.$$

We shall see that under this equivalence, the full subcategory $\Xi_S : \mathrm{QCoh}(S) \hookrightarrow \mathrm{IndCoh}(S)$ passes to $\mathrm{Sym}(V[-2])\text{-}\mathbf{Mod}_{\{0\}}$, the subcategory of modules $\mathcal{M} \in \mathrm{Sym}(V[-2])\text{-}\mathbf{Mod}$ such that $\bigoplus_{i\in\mathbb{Z}} \mathrm{H}^i(\mathcal{M})$ is set-theoretically supported on $\{0\} \subset V^*$ (as acted on by the graded algebra $\mathrm{Sym}(V)$.) In other words, every element in V acts locally nilpotently on $\bigoplus_{i\in\mathbb{Z}} \mathrm{H}^i(\mathcal{M})$.

1.5.2. The above observation shows that for every conical Zariski closed subset $\mathcal{N} \subset V^*$, we may define $\mathrm{IndCoh}_{\mathcal{N}}(S)$ as a full subcategory of $\mathrm{IndCoh}(S)$ corresponding to $\mathrm{Sym}(V[-2])\text{-}\mathbf{Mod}_{\mathcal{N}}$—modules $\mathcal{M}$ such that $\bigoplus_{i\in\mathbb{Z}} \mathrm{H}^i(\mathcal{M})$ is set-theoretically supported on $\mathcal{N}$. Thus we have:

$$\begin{array}{ccc} \mathrm{IndCoh}(S) & \xrightarrow{\sim} & \mathrm{Sym}(V[-2])\text{-}\mathbf{Mod} \\ \uparrow & & \uparrow \\ \mathrm{IndCoh}_{\mathcal{N}}(S) & \xrightarrow{\sim} & \mathrm{Sym}(V[-2])\text{-}\mathbf{Mod}_{\mathcal{N}} \end{array}$$

And $\mathrm{IndCoh}_{\{0\}}(S) = \mathrm{QCoh}(S)$.

1.5.3. The notion of singular support can be generalized to a large class of (derived) algebraic stacks. We call an algebraic stack $\mathcal{Y}$ *quasi-smooth* if $T^*_{\mathcal{Y}}$ is perfect of Tor-amplitude ≥ -1. In this case, $T_{\mathcal{Y}}$ is perfect of Tor-amplitude ≤ 1.

Remark 1.16. We shall see that a quasi-smooth scheme is just one which locally is presented by "n generators and m relations," although the word "relation" has to be interpreted in the derived sense.

1.5.4. Suppose $\mathcal{Y}$ is a quasi-smooth algebraic stack. We set $\mathrm{Sing}(\mathcal{Y})$ to be a fiber bundle over $\mathcal{Y}^{\mathrm{cl}}$ whose fiber at $S \to \mathcal{Y}^{\mathrm{cl}}$ is $\mathrm{Spec}_S \mathrm{Sym}(\mathrm{H}^1(T_{\mathcal{Y}}|_S))$.

Remark 1.17. For $\mathcal{Y} = \mathrm{pt} \underset{V}{\times} \mathrm{pt}$, we obtain $\mathcal{Y}^{\mathrm{cl}} = V^*$.

Then for any conical Zariski closed subset $\mathcal{N} \subset \mathrm{Sing}(\mathcal{Y})$, we shall be able to define a DG category $\mathrm{IndCoh}_{\mathcal{N}}(\mathcal{Y})$, sitting inside $\mathrm{IndCoh}(\mathcal{Y})$.

1.5.5. We will prove:

Lemma 1.18. *The stack* LocSys_G *is quasi-smooth.*

Furthermore, the classical stack $\mathrm{Sing}(\mathrm{LocSys}_G)$ is the moduli spaces of triples $(\mathcal{P}_G, \nabla, A)$ where $(\mathcal{P}_G, \nabla) \in \mathrm{LocSys}_G$ and A is a *flat* global section of the coadjoint bundle $\mathfrak{g}^*_{\mathcal{P}_G}$. Fixing a G-invariant identification $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$, we may regard the cone of nilpotent elements $\mathrm{Nilp} \subset \mathfrak{g}$ as a subset of $\mathfrak{g}^*$. We write

$$\mathcal{N}\mathrm{ilp} \subset \mathrm{Sing}(\mathrm{LocSys}_G)$$

for the conical Zariski closed subset where A belongs to $\mathrm{Nilp}_{\mathcal{P}_G} \subset \mathfrak{g}^*_{\mathcal{P}_G}$.

1.5.6. Finally, we may state the current form of geometric Langlands duality:

Attempt 1.19. $\mathrm{IndCoh}_{\mathcal{N}\mathrm{ilp}}(\mathrm{LocSys}_G) \xrightarrow{\sim} \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_{\check{G}})$.

This conjecture passes the Eisenstein series test, and specializes to the correct equivalence for $G = T$ a torus.

Remark 1.20 (David)**.** In the number-theoretic setting, the spectral side should be a G-valued representation of $\mathrm{Gal}(\bar{F}/F) \times \mathrm{SL}_2$ rather than just $\mathrm{Gal}(\bar{F}/F)$. Notice that the additional SL_2-factor contributes a nilpotent element $e \in \mathfrak{sl}_2 \to \mathfrak{g}$ which commutes with the $\mathrm{Gal}(\bar{F}/F)$-action. This is exactly what one may interpret geometrically as a flat, nilpotent global section of $\mathfrak{g}_{\mathcal{P}_G}$.

Remark 1.21. For $G = \mathrm{GL}_n$, Arthur has argued that the trivial function over $G(\mathbb{A})$ should correspond to the representation $\mathrm{triv} \boxtimes \mathbb{C}^n$ of $\mathrm{Gal}(\bar{F}/F) \times \mathrm{SL}_2$. Geometrically, this means that the image of ω_{Bun_n} under Langlands duality should be some kind of sheaf supported at $\mathrm{triv} \in \mathrm{LocSys}_n$, but it should not be quasi-coherent as long as $n \geq 2$.

1.6. **Main theorems (A) and (B).**

1.6.1. We can now formulate a theorem to the effect that $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_G)$ is "large enough" to contain all the Eisenstein series. We fix a Borel $B \subset G$, and let Par_G denote the (discrete) set of standard parabolics of G (including $P = G$).

Theorem 1.22 (A)**.** *As $P \in \mathrm{Par}_G$, the images of the functors:*

$$\mathrm{Eis}_{\mathrm{Spec}} : \mathrm{QCoh}(\mathrm{LocSys}_M) \to \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_G)$$

collectively generate the target category.

We will deduce this theorem from carefully studying the interaction between $\mathrm{Eis}_{\mathrm{Spec}}$ and singular support.

1.6.2. We will now formulate another theorem to the effect that $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_G)$ is "small enough" to be embedded in the extended Whittaker categories[5] for $\check{G}$.

1.6.3. Suppose $f : \mathcal{Z} \to \mathcal{Y}$ is a map of prestacks. We set

$$\mathrm{IndCoh}(\mathcal{Z})_{\mathrm{conn}\,/\mathcal{Y}} := \mathrm{IndCoh}(\mathcal{Z}_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}).$$

Remark 1.23. One may interpret $\mathrm{IndCoh}(\mathcal{Z})_{\mathrm{conn}\,/\mathcal{Y}}$ as the DG category of ind-coherent sheaves on $\mathcal{Z}$ with a connection along fibers of f.

When $\mathcal{Z}$ is quasi-smooth, we define $\mathrm{QCoh}(\mathcal{Z})_{\mathrm{conn}\,/\mathcal{Y}}$ by the Cartesian diagram:

$$\begin{array}{ccc} \mathrm{QCoh}(\mathcal{Z})_{\mathrm{conn}\,/\mathcal{Y}} & \hookrightarrow & \mathrm{IndCoh}(\mathcal{Z})_{\mathrm{conn}\,/\mathcal{Y}} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(\mathcal{Z}) & \xhookrightarrow{\Xi_{\mathcal{Z}}} & \mathrm{IndCoh}(\mathcal{Z}). \end{array}$$

Then the functor $\Psi_{\mathcal{Z}}$ induces a functor $\widetilde{\Psi}_{\mathcal{Z}} : \mathrm{IndCoh}(\mathcal{Z})_{\mathrm{conn}\,/\mathcal{Y}} \to \mathrm{QCoh}(\mathcal{Z})_{\mathrm{conn}\,/\mathcal{Y}}$.

[5]But we won't be concerned with what these categories are, inasmuch as they receive fully faithful functors from $\mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\mathrm{LocSys}_G}$.

1.6.4. As $P \in \mathrm{Par}_G$ varies, we obtain a functor

$$\mathrm{IndCoh}(\mathrm{LocSys}_G) \to \lim_{P \in \mathrm{Par}_G} \mathrm{IndCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\mathrm{LocSys}_G}$$

We would like to apply $\widetilde{\Psi}_{\mathrm{LocSys}_P}$ to each term in the limit, but these functors only lax-commute with pullbacks on $\mathrm{IndCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\mathrm{LocSys}_G}$.

Remark 1.24. We recall the formation of lax-limits of DG categories. Suppose $\mathcal{C}_i$ is a family of DG categories indexed by $i \in I$. Equivalently, we have a co-Cartesian fibration $\mathcal{C} \to I$ whose fiber at $i \in I$ is $\mathcal{C}_i$. Then $\underset{i\in I}{\text{lax-}\lim}(\mathcal{C}_i)$ is the category of *all* sections of $\mathcal{C} \to I$. We have:

$$\lim_{i\in I}(\mathcal{C}_i) \hookrightarrow \underset{i\in I}{\text{lax-}\lim}(\mathcal{C}_i)$$

consisting of *co-Cartesian* sections.

The above procedure defines a functor

$$\mathrm{IndCoh}(\mathrm{LocSys}_G) \to \lim_{P \in \mathrm{Par}_G} \mathrm{IndCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\mathrm{LocSys}_G}$$
$$\xrightarrow{\widetilde{\Psi}_{\mathrm{LocSys}_G}} \underset{P \in \mathrm{Par}_G}{\text{lax-}\lim} \mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\mathrm{LocSys}_G}.$$

Theorem 1.25 (B)**.** *This composition is fully faithful on* $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_G)$.

This is what we call the "spectral gluing" theorem. You may know that attached to $\check{G}$ and $\check{P}$ are certain "degenerate" Whittaker categories $\mathrm{Whit}(\check{G};\check{P})$, and there are fully faithful functors $\mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\mathrm{LocSys}_G} \hookrightarrow \mathrm{Whit}(\check{G};\check{P})$. Together, we obtain an embedding:

$$\underset{P \in \mathrm{Par}_G}{\text{lax-}\lim} \mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\mathrm{LocSys}_G} \hookrightarrow \underset{\check{P} \in \mathrm{Par}_{\check{G}}}{\text{lax-}\lim} \mathrm{Whit}(\check{G};\check{P}). \tag{1.1}$$

On the other hand, we conjecture that $\mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_{\check{G}})$ embeds fully faithfully in the right hand side. Hence, in order to define the Langlands transform $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_G) \to \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_{\check{G}})$, we may pass through (1.1) and compare the images of $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys}_G)$ and $\mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_{\check{G}})$ inside $\underset{\check{P} \in \mathrm{Par}_{\check{G}}}{\text{lax-}\lim} \mathrm{Whit}(\check{G};\check{P})$. This is why Theorem (B) is important for us.