1. (Stereographic projection) Let 
\[ S^n := \{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1 \} \subset \mathbb{R}^{n+1} \]
be equipped with the subset topology. That is, a set \( V \subset S^n \) is open if \( V = S^n \cap U \) for an open set \( U \subset \mathbb{R}^{n+1} \). Let \( N = (0, \ldots, 0, 1) \) be the North pole, and \( S = (0, \ldots, 0, -1) \) be the south pole. Define \( \pi_1 : S^n - \{ N \} \rightarrow \mathbb{R}^n \) (resp. \( \pi_2 : S^n - \{ S \} \rightarrow \mathbb{R}^n \)) so that \((\pi_1(p), 0)\) (resp. \((\pi_2(p), 0)\)) is the point where the line passing through \( N \) (resp. \( S \)) and \( p \) intersects the hyperplane \( \{ x_{n+1} = 0 \} \).

(a) Prove that \( \Phi := \{ (S^n - \{ N \}, \pi_1), (S^n - \{ S \}, \pi_2) \} \) is a \( C^\infty \) atlas on \( S^n \).

(b) Prove that \((S^n, \Phi)\) is a smooth submanifold on \( \mathbb{R}^{n+1} \). That is, the smooth structure defined by the \( \Phi \) coincides with the smooth structure induced on \( S^n \) as a submanifold of \( \mathbb{R}^{n+1} \).

2. Suppose \( X \) is a connected topological space. Assume that \( X \) is Hausdorff, and locally euclidean of dimension \( n \); that is, \( X \) can be covered by charts \((U_\alpha, \phi_\alpha)\) such that \( \phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n \) is a homeomorphism. The following three properties are equivalent

(a) \( X \) is second countable. That is, there is a countable collection of open sets \( \{U_i\}_{i \in \mathbb{N}} \) such that, for an open set \( W \) we can write \( W = \bigcup U_{i_k} \) for some \( i_k \). For example, \( \mathbb{R}^n \) is second countable, where the \( U_i \) can be taken to be open balls centered on rational points, and with rational radii.

(b) \( X \) is paracompact.

(c) There exist compact sets \( \{K_i\}_{i \in \mathbb{N}} \) such that \( K_i \subset \text{int}(K_{i+1}) \) and \( X = \bigcup K_i \). That is, \( X \) has a compact exhaustion.

Prove that (b) and (c) are equivalent. **Here is a “hint”.** To prove \( (b) \Rightarrow (c) \), cover \( X \) by open sets which are preimages, under \( \phi_\alpha \) of open balls (with compact closure). By paracompactness, you can take a locally finite refinement \( \{V_\alpha\}_{\alpha \in A} \) all of which have compact closure. Use these sets to construct \( K_i \) iteratively. To prove \( (c) \Rightarrow (b) \), let \( \{V_\alpha\} \) be any open cover. Since \( X \) is Hausdorff, compact sets are closed, and so \( E_{i,j} := K_i - \text{int}(K_j) \) is compact for \( j > i \). Take a finite subcover of the \( \{V_\alpha\} \) covering \( E_{i+1,i} \), and set \( W_{\alpha,i} = V_\alpha \cap \text{int}(E_{i+2,i-1}) \).

Show that the resulting collection \( \{W_{\alpha,i}\} \) is a locally finite refinement. For fun, prove the equivalence of \( (a)/(b) \) and \( (c) \).
3. If $M, N$ are connected, smooth manifolds, then the product $M \times N$ can be made into a smooth manifold using the **product manifold** structure. Given patches $(U, \phi)$ on $M$ and $(V, \psi)$ on $N$ we use $(U \times V, \phi \times \psi)$ as a patch on $M \times N$. Show that this makes $M \times N$ into a smooth manifold. To show $M \times N$ is paracompact, use the preceding problem.

4. Let $(x, y, z)$ be coordinates on $\mathbb{R}^3$. Let $Y_r$ be the set of points in $\mathbb{R}^3$ at distance $r > 0$ from the circle

\[ C = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1, y = 0\} \]

(a) Let $A = \{r \in (0, \infty) | Y_r \text{ is a submanifold of } \mathbb{R}^3\}$. Find $A$.

(b) Let $S^1$ be equipped with the smooth structure given by stereographic projection (see (1)), and let $S^1 \times S^1$ be equipped with the product manifold structure (see below). Prove that $Y_r$ is diffeomorphic to $S^1 \times S^1$ for any $r \in A$. 